A Nonlinear Eigenvalue Problem in Astrophysical Magnetohydrodynamics: Some Properties of the Spectrum

John A. Adam
Old Dominion University, jadam@odu.edu

Original Publication Citation

Adam, J. A. (1989). A nonlinear eigenvalue problem in astrophysical magnetohydrodynamics: Some properties of the spectrum. *Journal of Mathematical Physics, 30*(3), 744-756. doi:10.1063/1.528392

A nonlinear eigenvalue problem in astrophysical magnetohydrodynamics: Some properties of the spectrum

John A. Adam
Department of Mathematics and Statistics, Old Dominion University, Norfolk, Virginia 23529

(Received 11 March 1988; accepted for publication 9 November 1988)

The equations of ideal magnetohydrodynamics (MHD) with an external gravitational potential—a "magnetoatmosphere"—are examined in detail as a singular nonlinear eigenvalue problem. Properties of the spectrum are discussed with specific emphasis on two systems relevant to solar magnetohydrodynamics. In the absence of a gravitational potential, the system reduces to that of importance in MHD and plasma physics, albeit in a different geometry. This further reduces to a form isomorphic to that derived in the study of plasma oscillations in a cold plasma, Alfvén wave propagation in an inhomogeneous medium, and MHD waves in a sheet pinch. In cylindrical geometry, the relevant model equations are those for a diffuse linear pinch. The full system, including gravity, has been applied to the study of flare-induced coronal waves, running penumbral waves in sunspots, and linear wave coupling in a highly inhomogeneous medium. The structure of the so-called MHD critical layer and its contribution to the continuous spectrum is examined in detail for a model magnetoatmosphere, based on properties of the hypergeometric differential operator. The relationship of this singular region to critical layers in classical linear hydrodynamic stability theory is also discussed in the light of a specific model (in the Appendix).

I. INTRODUCTION

In this paper, a magnetoatmosphere is defined as a system, which, when linearly perturbed about a stable equilibrium, may support wave motion due to the combined restoring forces of compressibility, buoyancy, and magnetic fields.[1-3] Systems of this type have been of interest in a number of areas in solar physics: that of running penumbral waves in sunspots,[4] flare-induced coronal waves,[5] waves in "magnetic flux tubes,"[6,7] and the associated problem of coronal heating.[8,9] Indeed, there has been a resurgence of interest in the latter problem in recent years, because the outer solar atmosphere has been shown to be highly inhomogeneous inasmuch as it consists of myriads of magnetic flux tubes in complicated field configurations.[9] Such structures are of interest because they are believed to be capable of supporting waves that may be crucial in determining energy balance in the solar corona (and, by implication, the coronae of at least other stars similar in spectral type to the sun).

In the corona the effects of buoyancy and gravity-induced density stratification may often be neglected, and the ideal magnetohydrodynamic (MHD) equations form the basis for models of flux tubes and their stability.[7] Similar configurations on a smaller scale are of great importance in plasma physics, particularly in connection with nuclear fusion devices and their MHD stability.[10-12] In this area, there has also been interest in the problem of MHD stability in the presence of an external gravitational potential[13] (the Rayleigh–Taylor stability problem), not because gravitational forces *per se* play an important role in laboratory plasmas (usually they do not), but because acceleration forces that act on many plasma configurations can be simulated by a gravitational-type term in the equations.[14]

Another aspect of the formulation of the Rayleigh–Taylor problem, namely that of stable systems, is ideally suited to magnetoatmospheric wave propagation problems. In Secs. IV and VII specific attention will be paid to the spectral analysis of the differential operator arising in the study of an isothermal atmosphere permeated by a uniform horizontal magnetic field. The solution of this system has been utilized in a model of the low corona–chromosphere transition region, for comparison with observations of flare-induced coronal waves.[5] In that model, horizontally propagating disturbances were found to exist in a waveguide formed by the sudden density increase into the chromosphere below, and by the rapidly increasing Alfvén speed above in the corona.

Another topic of interest in solar physics for which models of this type are relevant is that of running penumbral waves.[4] These waves have been observed in H_α propagating horizontally outward across sunspot penumbrae. It appears that these are essentially gravity-modified magnetoacoustic waves (the ω_+ modes discussed in Sec. VI). A considerable number of papers have appeared on magnetoacoustic-gravity waves; detailed accounts and references may be found in the reviews by Thomas,[2] Campos,[15] and also the paper by Zhugzda and Dzhalilov.[16]

It should also be realized that magetoatmospheric or magnetoacoustic-gravity (MAG) systems can be supplemented by plane or rotational *shear* (and for such systems we will adopt the acronym SMAG). Some theoretical work has been carried out for SMAG waves (that is shear-modified MAG waves).[17-22] The study of shear in incompressible fluids is the basis for much of classical linear hydrodynamic stability theory, of interest in oceanography and meteorology alike.[23] So-called "critical layers" are associated with those levels z_c within the fluid such that the horizontal phase speed $\hat{c}$ is equal to the local flow speed $U(z_c)$. The topic is not only one of the relevance to geophysical fluid dynamics.[24] The study of waves in SMAG systems leads to a generalized critical layer that may be of significance in the study of Evershed flow in sunspots[18] (though it must be pointed out

 0022-2488/89/030744-13$02.50

that the physical significance of MHD critical layers is still in dispute).[15]

In general, in all these types of systems, the normal modes have a continuous as well as a discrete spectral component.[10–12,25–28] The former is not of the type associated with a spatially infinite operator domain[29] (though this may occur), but it comes from the presence of a frequency-dependent singularity in the governing ordinary differential equations in the finite (or infinite) spatial domain. Associated with this singular point (or set of such points) are singular normal modes,[30] or improper eigenfunctions.[31] The inclusion of such modes is necessary for completeness in a mathematical sense. An alternative to the normal mode approach in problems of this type is the initial value problem (IVP). Careful analysis shows, in general, that they yield identical results,[25,31] but that the IVP is often more straightforward from a physical point of view. The time-harmonic, Fourier-transformed system of partial differential equations, or the Fourier–Laplace-transformed system for the IVP, gives rise, in general, to an ordinary differential operator of the form

$$\frac{d}{dz}\left\{A(z,\hat{c})\frac{d}{dz}\right\} + B(z,\hat{c}) \tag{1.1}$$

together with such boundary conditions and/or initial conditions as may be appropriate. In (1.1) the terms $A(z,\hat{c})$ and $B(z,\hat{c})$ may be highly nonlinear in the eigenvalue $\hat{c}$. (Here $\hat{c}$ is a horizontal phase speed. In Sec. II the eigenvalue is $\omega^2 = \hat{c}^2k^2$, where k is a Fourier wavenumber.) However, for the case of incompressible shear flow A has the misleadingly simple form[30,32]

$$A(z,\hat{c}) = (U(z) - \hat{c})^2. \tag{1.2}$$

In the case of incompressible magnetic shear flow,[32]

$$A(z,\hat{c}) = \{(U(z) - \hat{c})^2 - a^2k^2\}, \tag{1.3}$$

$a(z)$ being the Alfvén speed (defined below). Clearly, MHD systems, of which (1.3) is illustrative and typical but by no means general, contain the hydrodynamic systems in the limit $a\rightarrow 0$. The difference between the solutions for the two systems is characterized by the nature of the coefficients A: in (1.2) for real $\hat{c}$, A is always non-negative, whereas in (1.3) A can change sign in the interior of the domain (see Ref. 33 for a detailed discussion of a specific example in this context). The solutions in the neighborhood of a critical layer at $z = z_c$ have been discussed elsewhere.[32,34] Another major difference between MAG problems and SMAG problems is that in the former case, as noted above, the operator $\mathbf{F}(\mathbf{r})$ is formally self-adjoint, and the eigenvalues ω^2 are therefore real, whereas in any shear-type problem the operator is formally non-self-adjoint, and eigenvalues may be complex. There is much in the literature on complex eigenvalue bounds for such systems, the most well known being Howard's celebrated "semi-circle theorem" for incompressible plane parallel shear flow under gravity.[35] There are a number of interesting modifications of the extensions to this result: further details may be found in Ref. 21.

Mathematically, the state of a magnetoatmospheric, MHD, or plasma medium can be represented as an element in Hilbert space, and the powerful theory of self-adjoint linear operators becomes available, together with many fruitful analogies in quantum mechanics.[10] Ironically, the latter subject can be regarded as more straightforward (relatively, at least!) and "classical" than when compared with the exceedingly rich structure inherent in the MHD systems discussed above.

There are several classification schemes for the spectrum of self-adjoint linear operators in Hilbert space, and, while some differences exist, for the purpose of this paper they are essentially equivalent. Two will be stated here since each one is germane to topics introduced in the main body of the paper.

In Eq. (2.11) we denote the operator $\rho_0^{-1}\mathbf{F} + \omega^2$ by

$$T_\lambda = T - \lambda I, \tag{1.4}$$

where I is the identity operator on the domain of $T = \rho_0^{-1}\mathbf{F}$ and $\lambda = -\omega^2$ here is real (since T is formally self-adjoint). If T_λ has an inverse

$$R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{1.5}$$

$R_\lambda(T)$ is called the resolvent operator.

If H is a Hilbert space and T: $D(T)\rightarrow X$, a regular value λ of T is a number (in general complex) such that (i) $R_\lambda(T)$ exists, (ii) $R_\lambda(T)$ is bounded, and (iii) $\overline{D(R_\lambda)} = X$, i.e., $R_\lambda(T)$ is defined on a set which is dense in X. The resolvent set $\rho(T)$ of T is the set of all regular values λ of T, and its complement $\sigma(T) = \mathscr{R} - \rho(T)$ in the real line is called the spectrum of T. A partition of the spectrum is as follows.[36,37]

(i) The point spectrum $P\sigma(T)$ is the set of λ such that $R_\lambda(T)$ does not exist. Equivalently, it is the set of λ such that T_λ is not injective, i.e., its null space $N(T_\lambda)$ is nontrivial. The point spectrum is thus the set of all eigenvalues.

(ii) The continuous spectrum $C\sigma(T)$ is the set of λ such that $R_\lambda(T)$ exists, its domain is dense in X, but $R_\lambda(T)$ is unbounded. Thus T_λ is an injective operator.

Returning to the original operator, the initial value problem would have the form

$$[\rho_0^{-1}\mathbf{F} + \omega^2]\boldsymbol{\phi} = \mathbf{S}, \tag{1.6}$$

where $\mathbf{S}$ contains the initial data. Here, ω^2 being in the point spectrum of $T = \rho_0^{-1}\mathbf{F}$ means that

$$[\rho_0^{-1}\mathbf{F} + \omega^2]\boldsymbol{\phi} = \mathbf{0} \tag{1.7}$$

possesses a nontrivial solution.

It is worth noting that the *discrete spectrum* $D\sigma(T)$ [often identified as being synonymous with $P\sigma(T)$] is the set of all isolated spectral points, excluding eigenvalues of infinite multiplicity. Thus $D\sigma(T)\subset P\sigma(T)$. Furthermore, if an element of $P\sigma(T)$ is not isolated from the $C\sigma(\mathrm{T})$, it is said to be in the point-continuous spectrum, $PC\sigma(T)$.

Consider the self-adjoint operator $Ty = (p(z)y')' + q(y)$, $a\leqslant z\leqslant b < \infty$. The two linearly independent solutions of

$$T_\lambda(y) = T(y) - \lambda s(z)y = 0 \tag{1.8}$$

are $\phi(z,\lambda)$ and $\psi(z,\lambda)$, and these satisfy the initial conditions

$$\begin{aligned} \phi(a,\lambda) &= -\alpha_2, & \phi'(a,\lambda) &= \alpha_1/r(a), \\ \psi(a,\lambda) &= \alpha_1, & \psi'(a,\lambda) &= \alpha_2/r(a), \end{aligned} \tag{1.9}$$

where $\alpha_1\in R$, $\alpha_2\in R$, and $|\alpha_2| + |\alpha_2|\neq 0$. Each solution of (1.8), except multiples of ψ, can be expressed as a multiple of

$$Y = \phi + m\psi, \quad m\in\mathscr{C}.$$

If, at a point $b_0 < \infty$, the boundary condition

$$\beta_1 y(b_0,\lambda) + \beta_2 r(b_0) y'(b_0,\lambda) = 0, \tag{1.10}$$

where $\beta_1 \in R$, $\beta_2 \in R$, and $|\beta_1| + |\beta_2| \neq 0$, is imposed, then for $\mathrm{Im}(\lambda) \neq 0$, as the ratio β_1/β_2 takes on all real values, m describes a circle in the complex λ plane, with center

$$S = -W(\phi,\psi;b_0)/W(\psi,\psi^*;b_0) \tag{1.11}$$

and radius

$$R = \frac{(\alpha_1^2 + \alpha_2^2)^{1/2}}{2|\mathrm{Im}(\lambda)|\int_a^{b_0} s(z)|\psi|^2\,dz}. \tag{1.12}$$

Here W is the Wronskian of its arguments and ψ^* denotes the complex conjugate of ψ. As b_0 increases with λ fixed, R decreases and then the circle either approaches a *limit circle* $C_\infty(\lambda)$ or a *limit point* $m_\infty(\lambda)$.[38] In the former case all solutions are square-integrable over (a,∞) relative to the weight function $s(z)$. In the latter case only one solution, $y = \phi + m_\infty \psi$, is so square-integrable.

The paper is arranged as follows. Having introduced areas of application and spectral theoretic terminology in the present section, Sec. II deals with the basic equations of ideal MHD, and the fundamental reduced wave equations (2.16)–(2.18) is stated. In Sec. III the so-called exponential magnetoatmospheric wave equation is expressed in canonical hypergeometric form. The spectral problem is studied in Sec. IV, including a discussion of the number of zeros of the dispersion relation, and a point of accumulation of such zeros is identified. Conditions are derived under which the end points for the operator domain are limit point or limit circle in nature (in the sense of Weyl and Titchmarsh).

In Sec. V the effect of a *finite* upper boundary in the exponential magnetoatmospheric problem is discussed. Another important case—the constant parameter magnetoatmosphere—is examined in detail via its governing dispersion relation. In Sec. VI the effect of gravity on the hierarchical ordering of characteristic frequencies is identified, as are points of accumulation for the various discrete subspectra that are present in the system. Conditions under which the discrete spectra are Sturmian and anti-Sturmian are given. In the light of these properties, the spectrum of the exponential atmosphere is discussed further, and a fuller description of the nature of the spectrum is presented in Sec. VII.

Finally, the Appendix deals briefly with other material of relevance to the paper: the comparison of the spectral analysis of a shear flow system with that in Sec. IV.

In what follows, $p_0(z)$, $\rho_0(z)$, $\mathbf{B}_0(z)$, and $\psi_0(z)$ represent the equilibrium distributions of pressure, density, magnetic field intensity, and external potential, respectively. Linear perturbations to these basic field variables have no subscript (e.g., $p, \rho, \mathbf{B}, \psi$). The velocity perturbation field is $\mathbf{v}$, the ratio of specific heats for the medium is denoted by γ, the sound speed $c_0(z) = (\gamma p_0/\rho_0)^{1/2}$, and the Alfvén speed $a_0(z) = |\mathbf{B}_0|/\rho_0^{1/2}$, $a_x = B_x/\rho_0^{1/2}$, etc. Other variables are introduced as needed.

II. THE IDEAL MAGNETOHYDRODYNAMIC EQUATIONS

The equations of ideal magnetohydrodynamics are those of momentum, induction, isentropy, and continuity, respectively:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla p - \mathbf{B}\times(\nabla\times\mathbf{B}) - \rho\,\nabla\psi, \tag{2.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{2.2}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right](p\rho^{-\gamma}) = 0, \tag{2.3}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right]\rho = -\rho\nabla\cdot\mathbf{v}. \tag{2.4}$$

The equations are linearized about a static equilibrium defined by

$$\nabla p_0 + \mathbf{B}_0\times(\nabla\times\mathbf{B}_0) = -\rho_0\,\nabla\psi_0. \tag{2.5}$$

If the external potential ψ_0 corresponds to a uniform gravitational field, as assumed below, then $\nabla\psi_0 = -\mathbf{g} = -(0,0,-g)$, and the right-hand side of Eq. (2.5) is replaced by the vector $\rho_0\mathbf{g}$.

In terms of the linear Lagrangian displacement field $\boldsymbol{\xi}(\mathbf{r},t)$, defined by

$$\mathbf{v} = \frac{\partial\boldsymbol{\xi}}{\partial t}, \tag{2.6}$$

the equation of motion of ideal MHD is obtained by integrating and eliminating all dependent variables except $\boldsymbol{\xi}$. Thus

$$\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = \rho_0^{-1}\mathbf{F}(\boldsymbol{\xi}), \tag{2.7}$$

where the "force" operator $\mathbf{F}(\boldsymbol{\xi})$ is defined as

$$\mathbf{F}(\boldsymbol{\xi}) = \nabla(\gamma p_0\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p_0) - \mathbf{B}_0\times(\nabla\times\mathbf{Q}) - \mathbf{Q}\times(\nabla\times\mathbf{B}_0) + \nabla\cdot(\rho_0\boldsymbol{\xi})\nabla\psi_0, \tag{2.8}$$

where

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0). \tag{2.9}$$

The operator $\rho_0^{-1}\mathbf{F}$ can be proved to be self-adjoint for many boundary conditions of interest in MHD.[39]

We now regard $\boldsymbol{\xi}(x,y,z,t)$ as an integral superposition of harmonic terms, with $\boldsymbol{\xi}$ and its first x derivative vanishing at infinity: thus

$$\boldsymbol{\xi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\boldsymbol{\phi}(z;k)e^{ikx}\,dk, \tag{2.10}$$

where the horizontal wave vector $\mathbf{k}$ is described with respect to oriented axes such that $\mathbf{k} = (k,0,0)$. Thus, in general, the horizontal magnetic field has components given by

$$\mathbf{B}_0 = (B_x(z), B_y(z), 0).$$

Now Eq. (2.7) becomes

$$-\rho_0\omega^2\boldsymbol{\phi} = \mathbf{F}(\boldsymbol{\phi},k). \tag{2.11}$$

Eliminating all dependent variables except ϕ_z we obtain an equation of the form[40]

$$\frac{d}{dz}\left[A(z,\omega)\frac{d\phi_z}{dz}\right] + B(z,\omega)\phi_z = 0, \tag{2.12}$$

where, after some algebra, the coefficients $A(z,\omega)$ can be expressed as

$$A(z,\omega) = \frac{\rho_0(a_x^2 + a_y^2 + c_0^2)(\omega^2 - \omega_1^2(z))(\omega^2 - \omega_2^2(z))}{(\omega^2 - \omega_3^2(z))(\omega^2 - \omega_4^2(z))} \tag{2.13}$$

and

$$B(z,\omega) = \rho_0\left\{(\omega^2 - \omega_1^2(z)) - \frac{k^2g^2(\omega^2 - \omega_1^2(z))}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)} \frac{-k^2 g}{\rho_0} \times \frac{d}{dz}\left[\frac{\rho_0(a_y^2 + c_0^2)(\omega^2 - \omega_5^2)}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)}\right]\right\}, \tag{2.14}$$

where

$$\omega_1^2(z) = a_x^2 k^2, \tag{2.15}$$

$$\omega_2^2(z) = a_x^2 c_0^2 k^2 (a_x^2 + c_0^2)^{-1}, \tag{2.16}$$

$$\omega_3^2(z) = \tfrac{1}{2}[k^2(a_x^2 + a_y^2 + c_0^2) - \Delta], \tag{2.17}$$

$$\omega_4^2(z) = \tfrac{1}{2}[k^2(a_x^2 + a_y^2 + c_0^2) + \Delta], \tag{2.18}$$

$$\omega_5^2(z) = a_x^2 c_0^2 k^2 (a_y^2 + c_0^2)^{-1}, \tag{2.19}$$

where

$$\Delta^2 = k^4(a_x^2 + a_y^2 + c_0^2)^2 - 4a_x^2 c_0^2 k^4. \tag{2.20}$$

The expressions ω_3^2 and ω_4^2 are thus seen to be the roots of the quadratic in ω^2 given by

$$\omega^4 - \omega^2 k^2(a_y^2 + a_y^2 + c_0^2) + a_x^2 c_0^2 k^4 = 0. \tag{2.21}$$

III. THE "EXPONENTIAL" MAGNETOATMOSPHERE

We now restrict ourselves further in system (2.12) by considering the special case of an isothermal atmosphere ($c_0 = c =$ constant) with a uniform magnetic field (B_x,0,0). Under these circumstances the equilibrium density field is given in terms of a constant scale height $H\,[H = -\rho_0'(z)/\rho_0(z)]$ as

$$\rho_0(z) = \rho_0(0)e^{-z/H} \tag{3.1}$$

and the Alfvén speed

$$a_x(z) = a_x(0)e^{z/2H}. \tag{3.2}$$

(Hereafter, in this section we drop the subscript x for simplicity.) This magnetoatmosphere was examined by Nye and Thomas in connection with a model of flare-induced coronal waves,[5] and by Adam[41] in a study of critical layer behavior (see also Ref. 16). Equation (2.12) simplifies considerably [on multiplying by $\rho_0^{-1}(\omega^2 - c^2k^2)$] to become the equation

$$\{\omega^2c^2 + (\omega^2 - c^2k^2)a^2(0)e^{z/H}\}\phi'' - c^2\omega^2H^{-1}\phi' + \{(\omega^2 - c^2k^2)(\omega^2 - a^2(0)e^{z/H}k^2) - g(g - c^2/H)k^2\}\phi = 0. \tag{3.3}$$

Let us make the following transformations of dependent and independent variables:

$$\phi(z) = e^{-kz}f(w), \tag{3.4}$$

$$w = w_0 e^{-z/H}, \tag{3.5}$$

where we have noted that since k^2 appears in Eq. (3.3), we can replace k by $|k|$ without loss of generality, so

$$K = |k|H, \tag{3.6}$$

and

$$w_0 = \omega^2c^2/a^2(0)(\tilde{\omega}_3^2 - \omega^2) \tag{3.7a}$$

$$= \{\beta^2(c^2k^2/\omega^2 - 1)\}^{-1}, \tag{3.7b}$$

where $\beta^2 = a^2(0)/c^2$. (See Fig. 1.)

Under these transformations Eq. (3.3) appears in the canonical form of the hypergeometric differential equation, namely,

$$w(1-w)\frac{d^2f}{dw^2} + \{c - (a+b+1)w\}\frac{df}{dw} - abf = 0, \tag{3.8}$$

or, in terms of Riemann's P symbol,

$$f = P\begin{bmatrix} 0 & \infty & 1 & \\ 0 & K+\frac{1}{2}-S & 0 & W \\ -2K & K+\frac{1}{2}+S & 0 & \end{bmatrix}. \tag{3.9}$$

Thus

$$a + b = c = 2K + 1 \tag{3.10}$$

and

$$ab = H^2\omega^2/c^2 + K + (\gamma - 1)K^2c^2/\gamma^2H^2\omega^2. \tag{3.11}$$

In Eq. (3.9), $S(K,\omega)$ is defined as [with $\mathrm{Re}(S) > 0$]

$$S = \{K^2(1 - N^2/\omega^2) + \tfrac{1}{4}(1 - \omega^2/\omega_c^2)\}^{1/2} \tag{3.12a}$$

$$= \{\tfrac{1}{4} - R(K,\omega)\}^{1/2}, \tag{3.12b}$$

where

$$R = K^2(N^2/\omega^2 - 1) + \omega^2/4\omega_c^2, \tag{3.13}$$

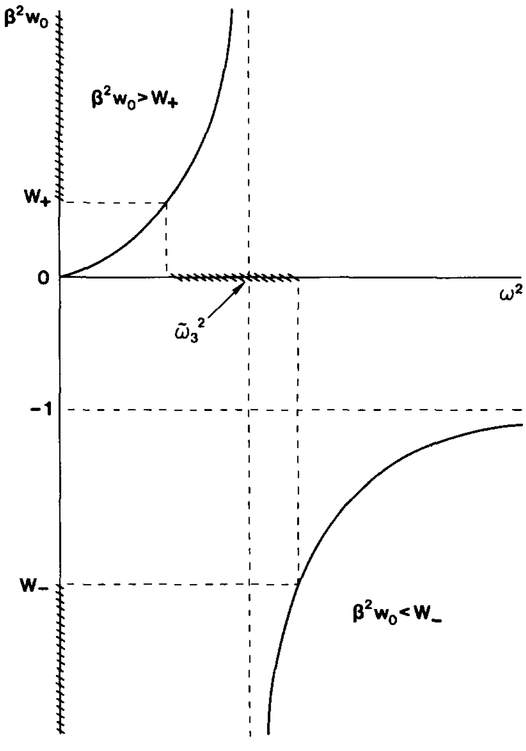

FIG. 1. The mapping $\omega^2 \to \beta w_0$ defined by Eq. (3.7b), where $\beta^2 = a^2(0)/c^2$. The expressions for $W_\pm$ are found in Sec. VII [Eq. (7.20)]. The hatched region on the ω^2 axis corresponds to the union of subacoustic and superacoustic horizontal phase speeds, which corresponds to (semi-infinite) intervals of $\beta^2 w_0$ (see Sec. VII for details).

and we have written S and R (called the *propagation number*) in terms of the Brunt–Vaisala frequency N, where

$$N^2 = -g\rho_0'/\rho_0 - g^2/c^2$$

$[=(\gamma-1)g^2/c^2$ for this isothermal atmosphere], and the acoustic cutoff frequency $\omega_c = \gamma g/2c$. (See Fig. 2 for the physical significance of S^2 and its magnetic counterpart in Fig. 3.)

In terms of S, then

$$a = k - S + \tfrac{1}{2} \tag{3.14}$$

and

$$b = K + S + \tfrac{1}{2}. \tag{3.15}$$

There has been much interest in the literature concerning the significance of the singularities $\omega^2 = \omega_1^2$ and $\omega^2 = \omega_2^2$ in the differential equation (2.12) with A given by (2.13).[15,41,42] In the system considered in this section it is the latter singularity that is of concern, namely, when

$$(a^2 + c^2)\omega^2 = a^2c^2k^2, \tag{3.16}$$

or, in terms of w, $w = 1$.

The hypergeometric differential equation possesses regular singularities at $w = 0$, 1, and ∞, and that at $w = 1$ is often referred to as the cusp singularity in the MHD literature.[43] It is in some ways analogous to the classical "critical layer" singularity in hydrodynamic stability theory. A particular example of the critical layer problem is compared and contrasted with the cusp singularity in the Appendix.

The boundary conditions for the exponential magnetoatmosphere are taken to be the simple forms required by Nye and Thomas[5]; namely, a semi-infinite ($0 \leqslant z < \infty$) medium bounded below by the rigid plane $z = 0$, with the vertical displacement remaining finite as $z \to \infty$. The general solution of Eq. (3.8) convergent in the range $|w| < 1$ is, for constants E and D,

$$f(w) = D\,{}_2F_1(a,b;c;w) + Ew^{1-c} \times {}_2F_1(a-c+1,b-c+1;2-c;w) \tag{3.17}$$

provided $1-c$ is not zero or a positive integer. Other representations of the solution will be used for $|w| > 1$. The boundedness condition at infinity implies that $f(w)$ must be regular at the origin $w = 0$, and since $1-c<0$ this means E must be zero. The other condition, $\phi(0) = 0$, yields $f(w_0) = 0$, i.e., for nontrivial f,

$${}_2F_1(a,b;c;w_0) = 0. \tag{3.18}$$

Hence we have that the roots of the eigenvalue equation correspond to the zeros of the hypergeometric function ${}_2F_1(a,b;c;w_0)$ in a plane cut along the interval $1 < w_0 < \infty$. Let us note several properties of the mapping (3.7b) (see Fig. 1): it is clearly a one-to-one mapping of the right $\hat{c}$ plane ($\hat{c} \neq c$) into the w_0 plane, with real $\hat{c}$ mapping onto real w_0 according to

$$\hat{c}\in(0^+,c^-) \to w_0 \in (0^+,\infty^-),$$

$$\hat{c}\in(c^+,\infty) \to w_0 \in (-\infty,[-\beta^{-2}]^-),$$

where $\hat{c} = \omega/k$ is a horizontal phase speed. Again, for real ω we may take $\omega > 0$ without loss of generality.

The domain of $w_0(\hat{c})$ is thus $\hat{c}\in(0^+,c^-)\cup(c^+,\infty)$, and the range is $w_0\in(-\infty,-\beta^{-2})\cup(0,\infty)$. For $w_0 \in(-\beta^{-2},0)$, $\hat{c}^2 < 0$.

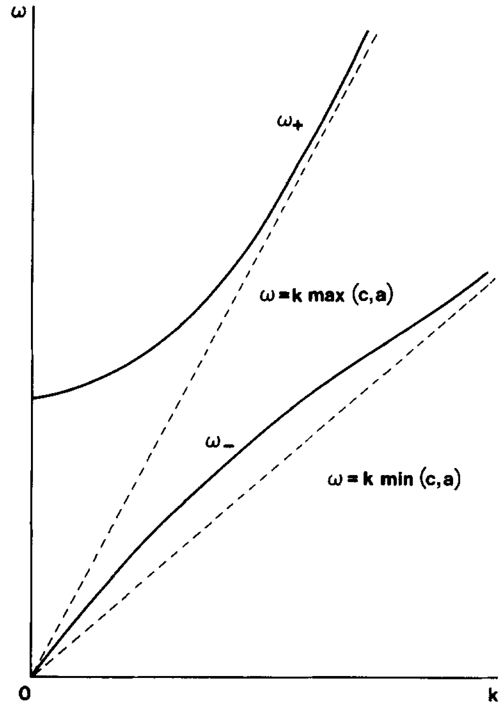

FIG. 3. The propagation–nonpropagation curves for the constant parameter magnetoatmosphere (Sec. VI). The curves are appropriate roots of the quadratic in ω^2 with $k_z = 0$ in Eq. (6.2) [see (6.4)]. Note the asymptotic behavior as illustrated by the broken lines. The ω_- curve may approach the asymptote from below (see Fig. 7).

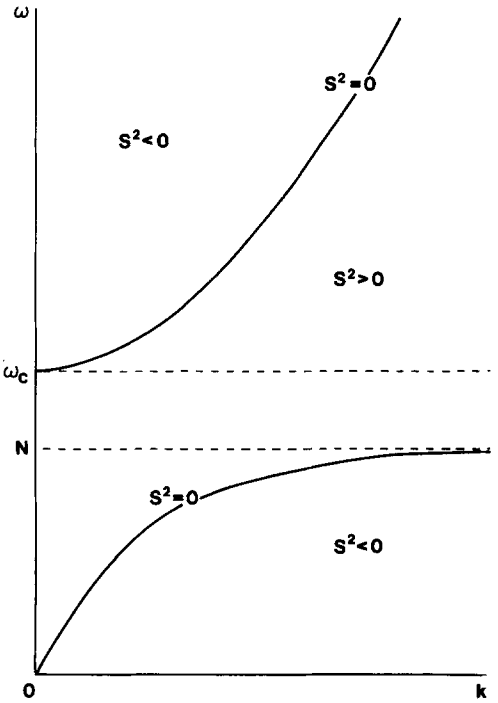

FIG. 2. The locus $S^2 = 0$ in the $\omega - k$ plane delineating regions of vertically propagating acoustic-gravity waves ($S^2 < 0$) from vertically nonpropagating waves ($S^2 > 0$), where S is defined by Eq. (3.12a).

IV. THE SPECTRAL PROBLEM FOR THE EXPONENTIAL MAGNETOATMOSPHERE

A. $S^2>0$

We utilize the notation and results of Van Vleck[44] to investigate the eigenvalue problem corresponding to Eq. (3.8). To this end we define $\lambda_1 = |1-c| = 2K$, $\lambda_2 = |c-a-b| = 0$, and $\lambda_3 = |a-b| = 2S$. Here, $E[q]$ is defined to be the integral part of q if $q>0$, and to be zero if $q\leqslant 0$. We now summarize the pertinent results of Van Vleck without proof in the form of a theorem.

Theorem 4.1: If $1-c<0$ and $S^2>0$, the number of roots of ${}_2F_1\,(a,b,c;w_0)$ in

(a) $(0,1)$ is $E\{\frac{1}{2}(\lambda_3-\lambda_1-\lambda_2+1)\}$;

(b) $(-\infty,0)$ is $E\{\frac{1}{2}(\lambda_2-\lambda_1-\lambda_3+1)\}$;

(c) $(1,\infty)$ is zero, unless $\frac{1}{2}\{\lambda_2+\lambda_3-\lambda_1+1\}$ is a positive integer.

If this is the case, there are four situations to consider: Let $m_i = E(\lambda_i)$, $i=1,2,3$. The number of complex roots of $F(a,b,;w_0)$ within the half-plane $\mathrm{Re}(w_0)>0$ if

(i) $m_1>m_2+m_3$, is zero;

(ii) $m_2>m_1+m_3$, is $E\{U/2\}$,

where

$$U = E\{\tfrac{1}{2}(\lambda_2+\lambda_3-\lambda_1+1)\} - E\{\tfrac{1}{2}(\lambda_2-\lambda_3-\lambda_1+1)\},$$

unless $\lambda_2-\lambda_1-\lambda_3$ is an odd integer, in which case the number of complex roots is zero;

(iii) $m_3>m_1+m_2$

[this is result (ii) with subscripts 2 and 3 interchanged];

(iv) $m_i\leqslant m_j+m_k$, for all $i,j,k=1,2,3$, is $E\{\frac{1}{2}(\lambda_2+\lambda_3-\lambda_1+1)\}$.

We now examine each of (a), (b), and (c) in turn for $R<\frac{1}{4}$ and hence S real.

Lemma 4.1: $F(w_0)$ has no zeros in $(0,1)$.

Proof: The number of roots in $0<w_0<1$ is given by $E[S-K+\frac{1}{2}]$. For exactly n real roots, it is necessary that

$$n+1>S-K+\tfrac{1}{2}\geqslant n\geqslant 1. \tag{4.1}$$

It can be shown that this inequality is inconsistent with the model under consideration for which $N^2>0$ and $K\geqslant 0$. Hence there are no roots of $F(w_0)$ for $w_0\in(0,1)$.

Lemma 4.2: $F(w_0)$ has no zeros in $(-\infty,0)$.

Proof: The number of roots in $-\infty<w_0<0$ is given by $E[\frac{1}{2}-S-K]$. Again, for exactly n real roots, we require $n+1>\frac{1}{2}-S-K\geqslant n$.

This clearly cannot be satisfied for any integer $n\geqslant 1$ since both S and K are non-negative. The result follows.

Lemma 4.3: $F(w_0)$ has no zeros in $(1,\infty)$.

Proof: There are no zeros unless $S-K+\frac{1}{2}=2p+1$, $p=0,1,2,\ldots$, i.e.,

$$S=p+K+\tfrac{1}{2}. \tag{4.2}$$

As noted in Theorem 4.1, there are four cases to consider. Since, however, the operator $\rho_0^{-1}\mathbf{F}(\xi)$ in Eq. (2.7) is formally self-adjoint, we know the eigenvalues ω^2 are real, but we can show that there are no real eigenvalues either in each category. Details can be found elsewhere.[45] Hence in summary, we have the following theorem.

Theorem 4.2: $F(w_0)$ has no zeros in $w_0 \in(0,1)\cup(1,\infty)\cup(-\infty,0)$ for $S^2(\omega,K)>0$.

B. $S^2<0$

Theorem 4.3: Let $R(K)>\frac{1}{4}$, and $S=i\mu$, $\mu=(R-\frac{1}{4})^{1/2}$. Then there exists an infinite number of zeros of (3.18) for $w_0\in(1,\infty)$, and they accumulate at (or diverge to) ∞.

Proof: Since we are interested in $|w_0|>1$ we use the linear transformation formula[46] [valid for $|\arg(-w_0)|<\pi|$], noting that, since w_0 is real, the second term is the complex conjugate of the first, so we may rewrite the eigenvalue equation (3.18) as

$$\mathrm{Re}\{A(-w_0)^{i\mu-(K+1/2)}\,{}_2F_1(K+\tfrac{1}{2}-i\mu,\tfrac{1}{2}-K-i\mu,\; 1-2i\mu;w_0^{-1})\}=0, \tag{4.3}$$

where

$$A=\Gamma(2K+1)\Gamma(2i\mu)/\{\Gamma(K+\tfrac{1}{2}+i\mu)\}^2, \quad \text{for } |w_0|>1. \tag{4.4}$$

It can be shown that, neglecting real multiplicative factors, this result reduces to

$$\mathrm{Re}[\,[1+Bw_0^{-1}] \times\exp i\{\mu\ln w_0-\pi(K+1)+O(\mu^3)\}]=0, \tag{4.5}$$

where $B=B_r+iB_i$ with

$$B_r=(\alpha+2\mu^2)/(1+4\mu^2),$$
$$B_i=\mu(2\alpha-1)/(1+4\mu^2),$$

and

$$\alpha=\tfrac{1}{4}-K^2-\mu^2;$$

thus, on identifying

$$\phi=\mu\ln w_0-\pi(K+1)+O(\mu^3), \tag{4.6}$$

Eq. (4.5) is satisfied when

$$\cot\phi=\frac{B_i}{w_0+B_r}=\frac{\mu(2\alpha-1)}{\alpha+2\mu^2+(1+4\mu^2)w_0}. \tag{4.7}$$

This is a limiting eigenvalue relation for $\mu>0$ ($\mu\ll 1$ for simplicity, but this is not necessary) as $w_0\to\infty$. For given μ and K, the right-hand side is a monotonically increasing function of w_0, tending to 0^- as $w_0\to\infty$ if $\alpha+2\mu^2>0$. Whatever the value of this right-hand side, the left-hand side attains it an infinite number of times in this limit, proving the theorem.

Theorem 4.4: In terms of the parameter $\Lambda=-S^2$, there is a continuous spectrum in $\Lambda\in(0,\infty)$, and a point spectrum (which may be null) in $\Lambda\in(-\infty,0)$.

Proof: We make use of some results of Titchmarsh.[47] Starting with the canonical form (3.8) of the hypergeometric equation, let

$$w=-\sinh^2\tfrac{1}{2}\bar{z} \tag{4.8}$$

and

$$f=r(\bar{z})u, \tag{4.9}$$

where

$$r(\tilde z) = \{(e^{\tilde z} - 1)/(e^{\tilde z} + 1)\}^{1/2 + (1/2)(a+b) - c} \times \sinh^{-(1/2)(a+b)}\tilde z. \tag{4.10}$$

Then Eq. (3.8) reduces to

$$\frac{d^2u}{d\tilde z^2} + \{\Lambda - q(\tilde z)\}u = 0, \tag{4.11}$$

where

$$\Lambda = ab - \tfrac{1}{4}(a+b)^2 = -S^2 \tag{4.12}$$

and

$$q(\tilde z) = [2(a+b-1)(2c-1-a-b)\cosh\tilde z + 2(a+b)^2 - 4c(a+b) + (1-2c)^2]/(4\sinh^2\tilde z) \tag{4.13}$$

(we can simplify these expressions by using $c = a + b$ for this particular problem; but for complete generality at this stage we do not do so). From a theorem of Titchmarsh,[47] since $q(\tilde z)\to 0$ as $|\tilde z|\to\infty$, the result is established.

In fact, for $\Lambda\in(0,\infty)$, the continuous spectrum contributes to the expansion of some $g(\tilde z)$ an amount

$$\frac{1}{2^{a+b+2}\pi}\int_0^\infty \left|\frac{\Gamma(a)\Gamma(c-b)}{\Gamma(c)\Gamma(a-b)}\right|^2 \times \frac{u(\tilde z,\Lambda)d\Lambda}{\sqrt{\Lambda}}\int_0^\infty u(y,\Lambda)g(y)dy, \tag{4.14}$$

for $c>2$ (see Theorem 4.5). There will also be a finite number of simple poles on the negative real axis when $a = -n$ or $c - b = -n$, $n = 0,1,2,\ldots$, with residues $(-1)^n/n!$. Since $a = K - S + \frac{1}{2} = c - b$, this means, for poles to exist, that $S>0$ and hence $S^2>0$. (This also follows from the fact that $\Lambda<0$ implies $S^2>0$.) For the boundary conditions considered in Sec. III, we know from Theorem 4.2 that there are no roots of the dispersion relation in this case, i.e., the point spectrum is null.

Theorem 4.5: For Eq. (4.11) the origin ($\tilde z = 0$; $w = 0$) is of limit circle type if $c<2$ ($K<\frac{1}{2}$), otherwise it is of limit point type. The point $w = 1$ is of limit circle type if $0<a+b-c+1<2$; this is always true for the problem at hand.

Proof: As $\tilde z\to 0$, $r(\tilde z)\sim\tilde z^{1/2-c}$, $w\sim z^2$,

$$f(w)\to 1, \quad \text{so } u(\tilde z)\to\tilde z^{c-1/2}.$$

A second linearly independent solution corresponding to Eq. (3.8) as $w\to 0$ is $f(w)\to w^{1-c}$: thus

$$u(\tilde z)\to\tilde z^{c-1/2}(\tilde z^2)^{1-c}\to\tilde z^{3/2-c}.$$

Examining the square-integrability of these asymptotic solutions, we find

$$\int_0 \tilde z^{2c-1}\,d\tilde z\sim\frac{\tilde z^{2c}}{2c}$$

and

$$\int_0 \tilde z^{3-2c}\,d\tilde z\sim\frac{\tilde z^{4-2c}}{4-2c}.$$

These are both in $\mathscr{L}^2(0,1)$ if (i) $c>0$ and (ii) $4-2c>0$. Hence the origin is limit circle type if $0<c<2$. This corresponds to $K<\frac{1}{2}$, since K is non-negative by definition.

The transformation $\tilde w = 1 - w$ in (3.8) yields the hypergeometric equation in parameters a, b, and $\tilde c$, where $\tilde c = a + b - c + 1$; thus the above analysis holds for $\tilde w\to 0$, i.e., $w\to 1$. Both solutions are in $\mathscr{L}^2$ (0,1) if $0<\tilde c<2$. Since $a + b = c$ for the problem at hand, the point $w = 1$ is of the limit-circle type irrespective of the value of K. The cases in which $a + b = c$ for the hypergeometric equation give rise to logarithmic solution behavior, as is well known. Indeed, the general solution of (3.8) in the neighborhood of $w = 1^-$ ($\tilde\omega = 0^+$) is of the form

$$f(\tilde w)\sim f_1(\tilde w)\ln\tilde w + \sum_{r=1}^\infty c_r\tilde w^r. \tag{4.15}$$

Since $f_1(\tilde w)\to 1$ as $\tilde w\to 0$,

$$\int_0 f^2(\tilde w)d\tilde w\sim\int_0 \ln^2\tilde w\,d\tilde w \sim\{\tilde w[\ln^2\tilde w - 2\ln\tilde w + 2]\}|_0<\infty,$$

thus exhibiting the limit-circle behavior at that point.

Finally, we consider the model equation in the light of some results of Dunford and Schwartz.[48] If we denote the hypergeometric differential operator in (3.8) by $H_w^{a,b,c}$, then the transformations

$$\alpha = \tfrac{1}{2}(c-1), \quad \beta = \tfrac{1}{2}(a+b-c), \quad t = 2w-1, \tag{4.16}$$

and

$$f(w) = (t+1)^{-\alpha}(t-1)^{-\beta}\phi(t), \tag{4.17}$$

give

$$H_w^{a,b,c}\{f\}\to L_t^{\alpha,\beta}\{\phi\}, \tag{4.18}$$

where

$$L_t^{\alpha,\beta}\equiv -\left[\frac{d}{dt}\right](1-t^2)\left[\frac{d}{dt}\right] + \frac{2\alpha^2}{1+t} + \frac{2\beta^2}{1-t}, \tag{4.19}$$

on $(-1,1)$, $\alpha\geqslant 0$, $\beta\geqslant 0$.

Properties of operator $L_t^{\alpha,\beta}$ were discussed by Dunford and Schwartz, and we merely note those that are germane to the analysis presented here. For any λ, $\mathrm{L}-\lambda$ has regular singularities at -1, 1, and ∞ with respective exponents $\{\alpha,-\alpha\}$, $\{\beta,-\beta\}$, and $\{\frac{1}{2}+[\lambda+\frac{1}{4}]^{1/2}, \frac{1}{2}-[\lambda+\frac{1}{4}]^{1/2}\}$. There are three cases to consider.

(i) $\alpha\geqslant\frac{1}{2}$, $\beta\geqslant\frac{1}{2}$. Clearly $\alpha = K$ and $\beta = 0$ for the problem at hand, so this case does not apply. In this case though, the deficiency indices of L are both zero, so L gives to a unique self-adjoint operator in Hilbert space.

(ii) $\alpha\geqslant\frac{1}{2}$, $0\leqslant\beta<\frac{1}{2}$. (The transformation $t\to -t$ interchanges α and β so there is no loss of generality.) In this case ($K\geqslant\frac{1}{2}$, $\beta = 0$) the deficiency indices of L are both unity: one boundary condition is required to define a unique self-adjoint operator in Hilbert space.

(iii) $0\leqslant\alpha<\frac{1}{2}$, $0\leqslant\beta<\frac{1}{2}$. All solutions of $(L-\lambda)\phi = 0$ are square-integrable at both end points. The deficiency indices of L are both 2, so two boundary conditions must be imposed to obtain a self-adjoint operator in Hilbert space.

V. A BOUNDED MAGNETOATMOSPHERE

Consider the domain $0\leqslant z\leqslant d<\infty$ for the system discussed in Sec. III. The boundary conditions

$\phi(z=0)=0=\phi(z=d)$ are chosen, corresponding to $w\in[w_0,\alpha w_0]$, where $\alpha = e^{-d/H}<1$. If $|w_0|<1$, the boundary conditions yield, for constants D and E not both zero,

$$D\,{}_2F_1(a,b;c;w_0) + Ew_0^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;w_0)=0 \tag{5.1}$$

and

$$D_2F_1(a,b;c;\alpha w_0) + E(\alpha w_0)^{1-c}{}_2F_1(a-c+1,b-c+1;2-c;\alpha w_0). \tag{5.2}$$

The necessary and sufficient condition for nontrivial solutions yields the following dispersion relation for the discrete spectrum:

$$w_0^{1-c}\{\alpha^{1-c}F_1(w_0)F_2(\alpha w_0)-F_1(\alpha w_0)F_2(w_0)\}=0, \tag{5.3}$$

where F_1 and F_2 denote the first and second hypergeometric functions in each of the expressions (5.1) and (5.2).

The general solution for $\phi(z)$ valid for $|w|>1$ is the linear combination

$$Gw^{-a}{}_2F_1(a,a-c+1;a-b+1;w^{-1}) + Hw^{-b}{}_2F_1(b,b-c+1;b-a+1;w^{-1}) \tag{5.4}$$

or

$$Gw^{-a}F_1(w^{-1})+Hw^{-b}F_2(w^{-1}). \tag{5.5}$$

If $|\alpha w_0|$ and hence $|w_0|$ is greater than unity, the portions of discrete spectrum residing outside any continua are described in terms of solutions of

$$w_0^{-(a+b)}[\alpha^{-b}F_1(w_0^{-1})F_2(\alpha^{-1}w_0^{-1}) - \alpha^{-a}F_1(\alpha^{-1}w_0^{-1})F_2(w_0^{-1})]=0. \tag{5.6}$$

It is apparent that, in the light of the analysis carried out in Sec. IV B for $S=i\mu$, expressions (5.3) and (5.6) indicate that the singular point $w_0=\infty$ is not a point of accumulation for that part of the spectrum. The case with $|\alpha w_0|<1$ and $|w_0|>1$ is apparently different, containing as it does the major case of interest in Sec. III ($\alpha=0$). Consider Eq. (3.8) for $|w|=|\alpha w_0|<1$, and the appropriate analytic continuation of this equation for $|w|=|w_0|>1$, subject to the boundary conditions $\phi(0)=\phi(d)=0$:

$$DF_1(\alpha w_0)+E(\alpha w_0)^{1-c}F_2(\alpha w_0)=0 \tag{5.7}$$

and

$$D\lambda_1(w_0^{-1})+E(w_0)^{1-c}\lambda_2(w_0^{-1})=0, \tag{5.8}$$

where

$$\lambda_1=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\times(-w_0)^{-a}{}_2F_1(a,1-c+a;1-b+a;w_0^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\times(-w_0)^{-b}{}_2F_1(b,1-c+b;1-a+b;w_0^{-1}), \tag{5.9}$$

with a similar expression for λ_2. From these two equations we have the condition

$$w_0^{1-c}\{F_1(\alpha w_0)\lambda_2-\alpha^{1-c}F_2(\alpha w_0)\lambda_1\}=0. \tag{5.10}$$

The expressions above for λ_1 and λ_2 both contain, in particular, a term $w_0^{i\mu}=\exp(i\mu\ln w_0)$. This implies, as before, the existence of a sequence of eigenvalues accumulating at $w_0=\infty$, *but*, for this representation to be valid, we require that $|\alpha w_0|<1$, which constrains α, and hence the position of the upper boundary, to change accordingly. This rapidly degenerates to the case of the semi-infinite magnetoatmosphere discussed previously.

VI. THE CONSTANT PARAMETER MAGNETOATMOSPHERE

In order for the Alfvén speed to be uniform, it is necessary that $(d/dz)[|\mathbf{B}_0|^2\rho_0^{-1}]=0$. This condition, together with the equation of magnetohydrostatic equilibrium in the form

$$\frac{d}{dz}\left[p_0+\frac{\mathbf{B}_0\cdot\mathbf{B}_0}{2}\right]=-\rho_0 g \tag{6.1}$$

and the perfect gas equation, yield for constant sound speed a density scale height $H=(2c^2+\gamma a^2)/2\gamma g$. For simplicity, we let $B_y=0$, so that the wave vector $\mathbf{k}$ lies in the plane of the magnetic field and gravity vectors. The coefficients A (2.13) and B (2.14) are independent of z, and a Fourier analysis of Eq. (2.12) in the z direction yields the following expression:

$$[\omega^2-a^2k^2][\omega^4-[c^2+a^2]K^2\omega^2 + c^2a^2k^2K^2+N^2c^2k^2]=0. \tag{6.2}$$

The first term represents the decoupled Alfvén mode, while the other term represents the fast and slow magnetoacoustic modes modified by gravity, as is well known.[4] In these expressions,

$$K^2=k^2+k_z^2+1/4H^2, \tag{6.3}$$

k_z being the vertical component of the wavenumber $\mathbf{k}=(k,0,k_z)$, and $N^2=g(1/H-g/c^2)$ being the square of the Brunt–Vaisala frequency, when positive. The fast and slow modes correspond to the plus and minus roots $\omega_\pm^2$ defined by

$$\omega_\pm^2=\tfrac{1}{2}[c^2+a^2]K^2\times\left[1\pm\left\{1-\frac{4c^2k^2(N^2+a^2)K^2}{(c^2+a^2)^2K^4}\right\}^{1/2}\right] \tag{6.4}$$

(see Fig. 3). It is a straightforward matter to show that

$$\omega_2^2=c^2a^2k^2/(c^2+a^2)\leqslant\omega_-^2\leqslant\omega_+^2\leqslant K^2(a^2+c^2). \tag{6.5}$$

In the absence of gravity ($N=0$), it is always the case that $\omega_-^2\leqslant\omega_1^2=a^2k^2\leqslant\omega_+^2$ also. When $N\neq0$, these inequalities still hold provided $k_z^2\geqslant K_z^2=N^2c^2a^{-4}-(4H^2)^{-1}$. For k_z in the range $(0,K_z)$, ω_-^2 is greater than $\omega_1^2=a^2k^2$. However, $\omega_2^2<\max\{\omega_1^2,\omega_-^2\}$ always.

A related matter is that concerning the points of accumulation of the discrete spectra corresponding to the above "+ and −" modes. In a magnetoatmospheric "slab" bounded by the rigid infinite planes $z=0$ and $z=d$, we may identify the wavenumber k_z as $n\pi/d$, $n-1$ being the number of interior nodes of the eigenfunction $\phi(z)$. Thus increasing k_z is heuristically equivalent to increasing the node num-

ber n for the specified value of d. Retaining terms of $O(k_z^{-2})$ as $k_z \to \infty$, k being fixed, it transpires from (6.4) that[11]

$$\omega_+^2 \approx (c^2 + a^2) k_z^2 \to \omega_f^2 = \infty, \tag{6.6}$$

so that ∞ is an accumulation point of the fast mode discrete spectrum, and

$$\omega_-^2 \to c^2 a^2 k^2/(c^2 + a^2) = \omega_2^2. \tag{6.7}$$

The slow mode discrete spectrum accumulates at the point ω_2^2 (see Fig. 4). We may notice the interesting fact that if $N^2 \geqslant 0$, $\omega_-^2 \downarrow$ as $k_z \uparrow$ such that $\omega_-^2 \to \omega_2^{2+}$. This corresponds to an *anti-Sturmian* discrete subspectrum. If, however, $N^2 < 0$, $\omega_-^2 \uparrow$ as $k_z \uparrow$. Thus an unstable gravitational equilibrium gives rise to a *Sturmian* discrete subspectrum. Furthermore, if in this case a^2 is sufficiently small, ω^2 may become negative for some range of finite k_z. This is related to the interchange or hydromagnetic Rayleigh–Taylor instability,[9] stabilized here for sufficiently large k_z.

The behavior of the (point) spectrum for the constant parameter magnetoatmosphere lays the basis for an appreciation of what the spectral theory discussed earlier in Sec. IV means in physical terms. A piecewise uniform model will be used in the next section.

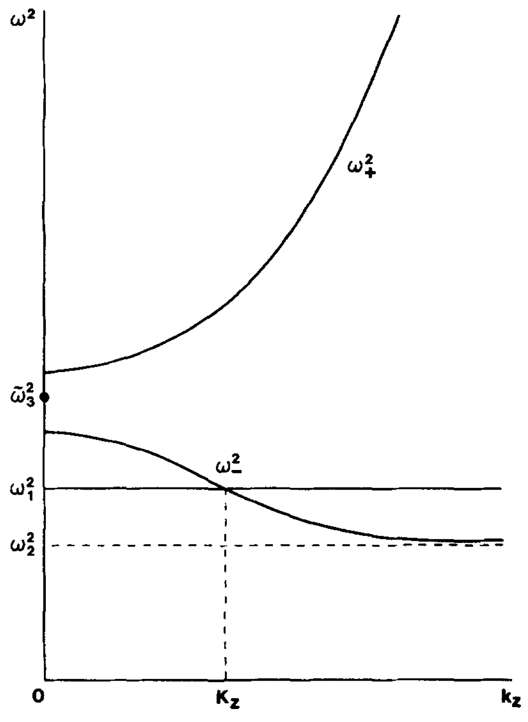

FIG. 4. The $\omega^2 - k_z$ diagram for the constant-parameter magnetoatmosphere (Sec. VI). For the upper branch (the "fast" mode modified by gravity), $\partial\omega^2/\partial k_z > 0$ indicating a Sturmian discrete spectrum accumulating at infinity. For the lower branch (the "slow" mode modified by gravity), $\partial\omega^2/\partial k_z < 0$, indicating an anti-Sturmian discrete spectrum accumulating at $\omega^2 = \omega_2^2$. Note that this point of accumulation $\to 0$ as $a \to 0$, so compressibility-modified gravity waves accumulate at $\omega^2 = 0$. The horizontal line $\omega^2 = \omega_1^2$ represents the infinitely degenerate Alfvén mode (decoupled here for $k_y = 0$).

VII. SPECTRUM OF THE EXPONENTIAL MAGNETOATMOSPHERE: DISCUSSION

It has already been noted that the appropriate dispersion or eigenvalue relation for the discrete spectrum is Eq. (3.18), i.e.,

$${}_2F_1(a,b;c;w_0) = 0,$$

provided $|w_0| < 1$. If $|w_0| > 1$ then the analytic continuation (4.3) is used, being set equal to zero. In what follows, these equations will be referred to as EI and EII, respectively. Since it is precisely these equations that have been used in a model of flare-induced coronal waves, the notation of Ref. 5 will be adopted in the remainder of this section for ease of identification. Thus if

$$\Omega = \omega H/c, \tag{7.1}$$

then

$$w_0 = \Omega^2/\beta^2(K^2 - \Omega^2). \tag{7.2}$$

For EI, the condition $|w_0| < 1$ yields the conditions

$$0 < \Omega^2 < \beta^2 K^2 (1 + \beta^2)^{-1} < K^2 \tag{7.3}$$

and if $\beta^2 > 1$ (which is permitted in this model for which a is not constant)

$$\Omega^2 > \beta^2 K^2 (\beta^2 - 1)^{-1} > K^2. \tag{7.4}$$

For EII, the conditions are

$$K^2 > \Omega^2 > \beta^2 K^2 (1 + \beta^2)^{-1} \tag{7.5}$$

and if $\beta^2 > 1$

$$K^2 < \Omega^2 < \beta^2 K^2 (\beta^2 - 1)^{-1}. \tag{7.6}$$

These regions are illustrated in Fig. 5. In terms of the original variables, the four inequalities (7.3)–(7.6) become, respectively,

$$0 < \omega^2 < \frac{a^2(0) c^2 k^2}{a^2(0) + c^2} = \omega_2^2(0) < c^2 k^2, \tag{7.7}$$

$$a^2(0) > c^2, \quad \omega^2 > \frac{a^2(0) c^2 k^2}{a^2(0) - c^2} > c^2 k^2, \tag{7.8}$$

$$c^2 k^2 > \omega^2 > \frac{a^2(0) c^2 k^2}{a^2(0) + c^2}, \tag{7.9}$$

and if

$$a^2(0) > c^2, \quad c^2 k^2 < \omega^2 < \frac{a^2(0) c^2 k^2}{a^2(0) - c^2}. \tag{7.10}$$

The union of these four regions [or two of them if $a^2(0) < c^2$] represents the domain of the discrete spectrum outside the continuum region. This continuum arises through the term $(\omega^2 - \omega_2^2(z))$ in Eqs. (2.13) and (2.14); in the more general, case of variable sound speed $c(z)$ and $B_y \neq 0$ there will, in general, be four continua, which in weakly inhomogeneous media may be separated from each other by portions of the discrete spectrum and regions of spectral nonmonotonicity,[49] e.g., $\{\omega_3^2(z)\}$ and $\{\omega_4^2(z)\}$.

The expression

$$\omega_2^2(z) = a^2(0) c^2 e^{z/H} k^2 / [a^2(0) e^{z/H} + c^2]$$

is a monotonically increasing function in $z \in [0, \infty)$, $\omega_2^2(0) \leqslant \omega_2^2(z) \leqslant \omega_2^2(\infty) = c^2 k^2$ (see Fig. 6). Hence this may be inverted to give a monotonically increasing profile

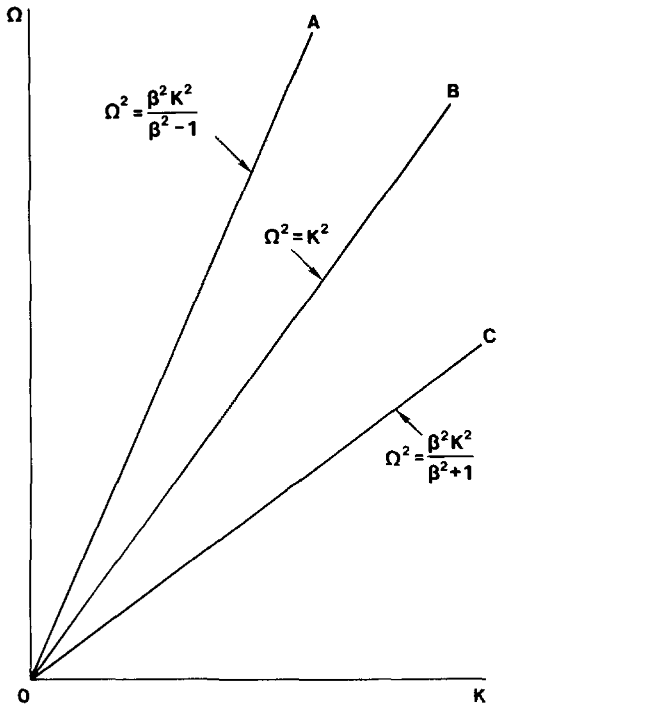

FIG. 5. The Ω-K plane (or dimensionless ω-k plane) illustrating the domains of validity (7.3)–(7.6) of the dispersion relations (3.18) (EI) and (4.3) (EII). For EI and $\beta^2 < 1$, the sector KOC is the appropriate domain; for $\beta^2 > 1$, it is sector ΩOA. For EII and $\beta^2 < 1$, the domain COB is appropriate; for $\beta^2 > 1$ it is sector AOB. Reference to Fig. 9 shows that the domain EII ($\beta^2 < 1$) coincides with the continuous spectral region $\{\omega_2^2\}$: if any discrete spectral values are embedded here, they are in the point-continuous spectrum, and constitute a previously unnoticed set of eigenvalues.

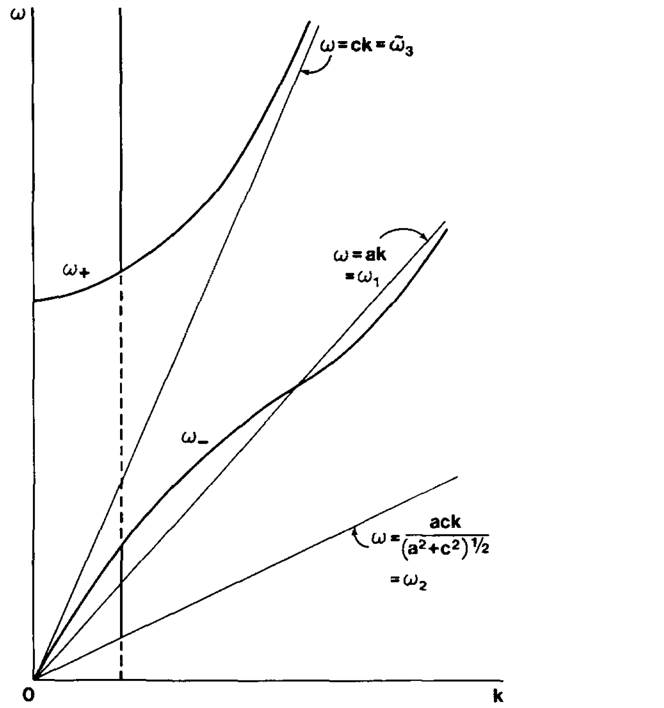

FIG. 7. The propagation curves for the constant parameter magnetoatmosphere (Sec. VI). This is similar to Fig. 3, but with the ω_1, ω_2, and $\tilde{\omega}_3$ lines present, illustrating, respectively, the relative positions of the Alfvén, "limiting" slow, and Lamb modes. The Alfvén mode is the only true mode: the other two are limits related to the accumulation points of the discrete spectra. The solid vertical line segments correspond to branch-line integrals in the related initial-value problem.[54]

$z_2 = z_2(\omega^2)$. The set of frequencies $\omega^2 \in \{\omega_2^2(z) | 0 \leqslant z \leqslant \infty\}$ constitutes the continuous spectrum: the set of improper eigenvalues of the operator $\rho^{-1}\mathbf{F}$ (see also Fig. 7).

It has been shown[49] that although systems of the type

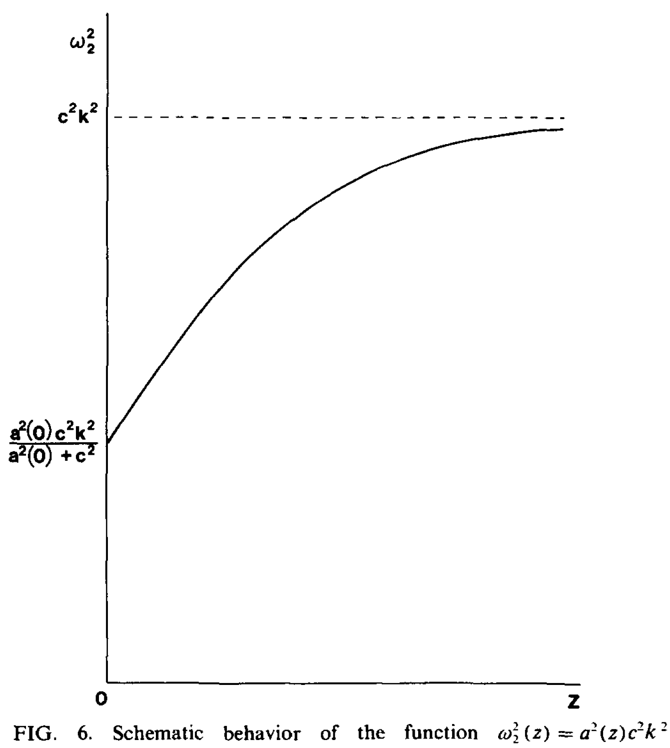

FIG. 6. Schematic behavior of the function $\omega_2^2(z) = a^2(z)c^2k^2 \times (a^2(z) + c^2)^{-1}$ for the exponential magnetoatmosphere (Secs. III and IV).

(2.12)–(2.14) are highly nonlinear and hence non-Sturm–Liouville in the eigenvalue ω^2, partial monotonicity of the discrete spectrum can nevertheless be established outside the regions $\{\omega_i^2(z)\}$, $i = 1,2,3,4$. Specifically, if in Eq. (2.12) the expression for A is positive, then the discrete spectrum is *Sturmian* outside $\cup_i \{\omega_i^2(z)\}$, meaning that the node number (or number of oscillations) of the eigenfunction $\phi(z)$ increases with increasing eigenvalue ω^2. If, on the other hand, $A < 0$, the discrete spectrum is anti-Sturmian, so the node number increases as the eigenvalue decreases (this concept was discussed in terms of the constant parameter atmosphere in Sec. VI). Clearly, if the regions $\{\omega_i^2\}$ are disjoint (by virtue of weak inhomogeneity in the system, for example), then we may have the discrete spectrum changing from Sturmian to anti-Sturmian or vice versa every time one of the factors $[\omega^2 - \omega_i^2(z)]$ changes sign. In the case of the exponential atmosphere of Sec. III, A is defined by

$$A(z,\omega) = \rho_0(a^2 + c^2)(\omega^2 - \omega_2^2(z))/(\omega^2 - c^2k^2). \tag{7.11}$$

For the model at hand,

$$\sup_z \omega_2^2(z) = c^2k^2, \tag{7.12}$$

so $A > 0$ when

$$\text{(i) } \omega^2 < \inf_z \omega_2^2(z) = a^2(0)c^2k^2/[a^2(0) + c^2] \tag{7.13}$$

and when

$$\text{(ii) } \omega^2 > c^2k^2. \tag{7.14}$$

In these regions, the discrete spectrum, if it exists, is Sturmian in nature. There is no region, outside the continuum $\{\omega_2^2(z)\}$, for which the coefficient A is negative, by virtue of (7.12). In the case of a *finite* upper boundary, however,

$$\inf_z \omega_2^2(z) = \frac{a^2(0)c^2k^2}{a^2(0)+c^2} \leqslant \omega_2^2(z) \leqslant \sup_z \omega_2^2(z)$$
$$= \frac{a^2(0)c^2k^2}{a^2(0)+c^2\alpha} < c^2k^2, \qquad (7.15)$$

where $\alpha = e^{-d/H}$ (as in Sec. V) for an upper boundary at $z = d$. Then the spectrum, if it exists, is Sturmian in the regions denoted by (7.13) and (7.14), and anti-Sturmian when

$$a^2(0)c^2k^2/[a^2(0)+c^2\alpha] < \omega^2 < c^2k^2. \qquad (7.16)$$

Returning to the case of a semi-infinite medium ($\alpha = 0$), considerable insight is afforded into the nature of the spectrum by examining a piecewise uniform atmosphere with the same basic features (e.g., increasing Alfvén speed with increasing values of z), To this end, we refer to Eqs. (6.4) in Sec. VI and Fig. 4 [see also Eqs. (6.6) and (6.7)]. First, consider the purely acoustic-gravity case for which $a^2 \equiv 0$. Under these circumstances it is easy to show that the upper branch $\omega_+^2(k_z) \to \infty$ as k_z increases, whereas $\omega_-^2(k_z) \to 0$ as k_z increases. Thus the gravity-modified acoustic spectrum accumulates at ∞ (Sturmian spectrum), whereas the compressibility-modified gravity spectrum accumulates at 0^+; the spectrum is anti-Sturmian[50] (see Fig. 4). As the Alfvén speed is increased from zero, the "line" of accumulation moves upward from $\omega^2 = 0$ to $\omega^2 \approx a^2c^2k^2/(a^2+c^2)$ as the magnetic term gradually dominates the gravitational term. (This can happen for sufficiently large values of k no matter how small a is.)

In Fig. 8 the ω^2-k_z diagram for the $\omega_\pm$ modes is presented schematically: for "weak" stratification the plus and minus modes may emanate, respectively, from above and below their values in the absence of gravity.

Figures 8(a)–8(c) show the progressive behavior of the modes (for a prescribed value of k) as the Alfvén speed increases. This is loosely equivalent to moving upward in the continuously varying exponential atmosphere. As a^2 increases, the line $\omega^2 = (\omega_2^2)_i$ moves progressively closer to $\omega^2 = \tilde{\omega}_3^2 = k^2c^2$. Thus, for the slow mode, the ω^2-k_z curve gradually flattens out and moves slowly upward until it collapses onto the line $\omega^2 = k^2c^2$ (the Lamb mode). It has already been noted that the discrete spectrum for the constant parameter atmosphere is anti-Sturmian for the slow mode, accumulating at $\omega^2 = \omega_2^2$. As a^2 increases, these lines, $\omega^2 = (\omega_2^2)_i$ accumulate from below to $\omega^2 = k^2c^2$, and the spectrum becomes infinitely degenerate in the limit $a^2 \to \infty$. Referring to the exponential model discussed in Secs. III and IV, it can be seen that as $z \to \infty$, $a^2 \to \infty$; thus, for the slow mode, $\omega^2 \to \omega_3^{2-} = (k^2c^2)^-$. This is the result found in Theorem 4.3, namely that $w_0 = \infty$ is a cluster point or accumulation point of the eigenvalues of the system: physically the slow mode accumulates to the Lamb mode eigenvalue ($\omega^2 = c^2k^2$) from below. The fast mode accumulates at $\omega^2 = \infty$ as noted earlier: this corresponds to $w_0 \to (-\beta^{-2})^-$ (see also Fig. 1). Figure 9 illustrates the

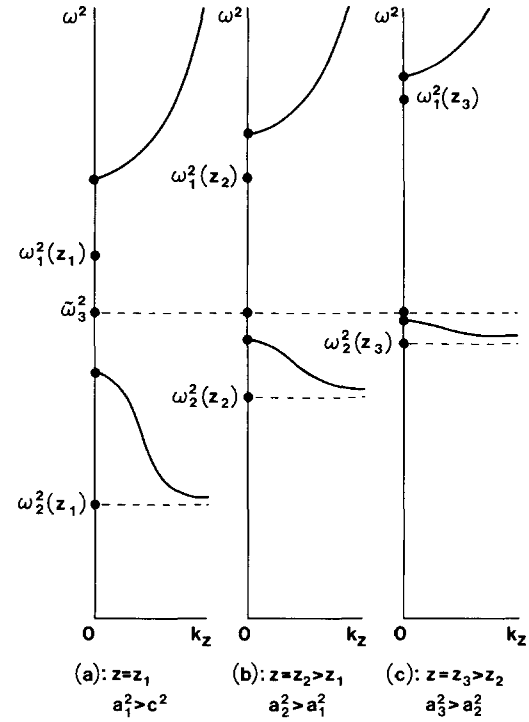

FIG. 8. (a), (b), (c) A schematic sequence of ω^2-k_z diagrams for a constant parameter medium (for simplicity, $g = 0$ here, without much loss of generality). The sequency (a)–(c) may be regarded as mimicking the evolution of a local ω^2-$k_z(z)$ diagram as the Alfvén speed increases with altitude z. The diagrams are drawn for $a > c$ at all locations z; for $c > a$ any *given* diagram is schematically very similar, the only difference being that the positions of ω_1^2 and $\tilde{\omega}_3^2$ are interchanged. Note how the anti-Sturmian slow-mode is collapsing (from below!) onto the Lamb mode $\omega^2 = \tilde{\omega}_3^2$ as a increases.

relationship between real values of ω^2 and w_0, again in a schematic fashion.

It is clear that the restriction of S^2 to negative values for the analysis in Sec. III and parts of Sec. IV will place some

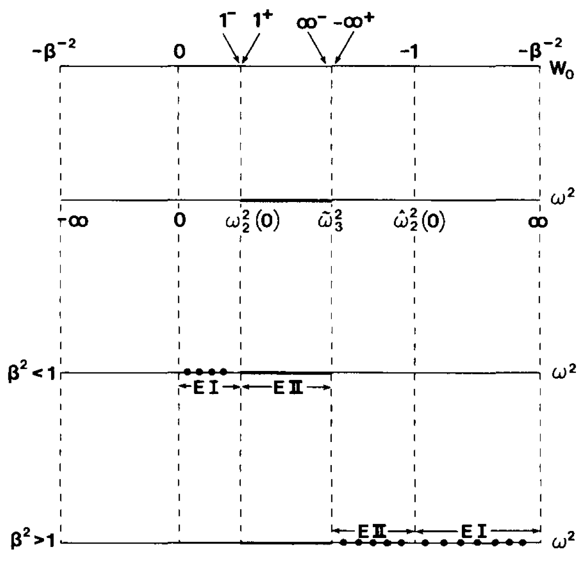

FIG. 9. A schematic representation of the relationship between w_0 and ω^2, as defined by the mapping (3.7b). The quantity $\hat{\omega}_2^2(0) = a^2(0)c^2k^2(a^2(0)-c^2)^{-1}$ exists only for $\beta^2 > 1$. Also illustrated are the subspectra (both discrete and continuous) which are permitted by the various dispersion relations EI (3.18) and EII (4.3).

restriction on w_0 also. To examine the implications of $R > \frac{1}{4}$ (i.e., $S^2 < 0$) for w_0, we have that, from (3.7a),

$$\omega^2 = a^2(0)c^2k^2w_0/[c^2 + a^2(0)w_0] . \qquad (7.17)$$

Note that $\omega^2 < 0$ for $-c^2/a^2(0) < w_0 < 0$, corresponding to convective instability. On substituting for ω^2 in the expression (3.13) for R, the condition $R > \frac{1}{4}$ becomes, after some manipulation,

$$\beta^4\left[\frac{4(\gamma-1)}{\gamma^2} - 1\right]w_0^2 + \beta^2\left[\frac{8(\gamma-1)}{\gamma^2} - (1+4K^2)\right]w_0 + \frac{4(\gamma-1)}{\gamma^2} > 0 . \qquad (7.18)$$

Denoting $1 + 4K^2$ by α, it follows that

$$\beta^2 w_0 > W_+ \quad \text{or} \quad \beta^2 w_0 < W_- , \qquad (7.19)$$

where

$$W_\pm = \frac{-(\alpha\gamma^2 - 8\gamma + 8) \pm \{(\alpha\gamma^2 - 8\gamma + 8)^2 + 16(\gamma-1)(\gamma-2)^2\}^{1/2}}{2(\gamma-2)^2} . \qquad (7.20)$$

Since

$$\beta^2 w_0 = \omega^2(c^2k^2 - \omega^2)^{-1} , \qquad (7.21)$$

these constraints on w_0 correspond to

$$\omega^2 > \frac{c^2k^2W_+}{1+W_+} \quad \text{or} \quad \omega^2 < \frac{c^2k^2W_-}{1+W_-} . \qquad (7.22)$$

For realistic magnetoatmospheres $1 < \gamma < 2$.

In particular, if $\alpha = 2$, i.e., $kH = 1/2$, and $\gamma = 5/3$ then $W_+ = 4$ and $W_- = -6$. Reference to Fig. 1 yields the following inequalities in ω^2:

(i) $0.8c^2k^2 < \omega^2 < c^2k^2$ (subsonic branch)

(ii) $c^2k^2 < \omega^2 < 1.2c^2k^2$ (supersonic branch)

APPENDIX: HYDRODYNAMIC SHEAR FLOW: A COMPARISON

Miles examined the shear flow given by[51]

$$U(z) = V(1 - e^{-z/H}), \quad 0 \leqslant z \leqslant \infty , \qquad (A1)$$

with density profile

$$\ln(\rho(0)/\rho(z)) = \sigma(1 - e^{-z/H}), \quad 0 < \sigma \ll 1 , \qquad (A2)$$

for an inviscid incompressible fluid bounded below by a rigid horizontal plane $z = 0$, filling the half-space $z > 0$. The governing Fourier-transformed ordinary differential equation and boundary conditions are (in terms of the perturbation stream function ϕ)

$$\phi'' + \left\{\frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2\right\}\phi = 0 \qquad (A3)$$

and

$$(U-c)^{-1}\phi = 0 \quad (z = 0) , \qquad (A4)$$

$$\phi = 0 \quad (z = \infty) , \qquad (A5)$$

where $c = \omega/k$ is the (possibly complex) horizontal phase speed. As in Sec. III, a set of transformations of dependent and independent variables yields the hypergeometric differential equation. Thus if

$$\phi(z) = e^{-kz}f(w) , \qquad (A6)$$

$$w = w_0 e^{-z/H} , \qquad (A7)$$

where

$$w_0 = V/(V-c) , \qquad (A8)$$

$$f = P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & K - \sqrt{1+K^2} & \frac{1}{2}(1+\nu) & w \\ -2K & K + \sqrt{1+K^2} & \frac{1}{2}(1-\nu) & \end{Bmatrix} , \qquad (A9)$$

where $K = kH$, $\nu = (1 - 4Jw_0)^{1/2} \equiv i\mu$ if $1 < 4Jw_0$, and $J = \sigma gHV^{-2}$ is the Richardson number for the flow. The above boundary conditions yield the following eigenvalue equation:

$$(1 - w_0)^{1/2(-1+\nu)}\,{}_2F_1(a,b;1+2K;w_0) = 0 , \qquad (A10)$$

where

$$a = \tfrac{1}{2}(1+\nu) + K - (1+K^2)^{1/2} \qquad (A11)$$

and

$$b = \tfrac{1}{2}(1+\nu) + K + (1+K^2)^{1/2} . \qquad (A12)$$

Of physical interest in a system such as this, governed by an equation of the type (A3), are those values of z (if any) for which $U(z) = c$—the so-called critical layer singularities. It is easily seen that this corresponds to the regular singular point $w = 1$. Using techniques similar to those used here, Miles establishes the following results, which we state in the form of a theorem.

Theorem A.1: For $Jw_0 < \frac{1}{4}$, ${}_2F_1(a,b;2K+1;w_0)$ has

(i) one zero in $w_0 \in (0,1)$ iff $0 < \nu < \nu_0(K)$;

(ii) no zeros in $w_0 \in (1,\infty)$;

(iii) n zeros in $w_0 \in (-\infty,0)$ iff $\nu_n < \nu < \nu_{n+1}$,

where $\nu_0(K)$ and $\nu_n(K,n)$ are specified expressions. Miles established that there are no unstable modes for any K and J.

When $Jw_0 > \frac{1}{4}$, the relation (A10) can be written as

$$\mathrm{Re}\{A^*(1-w_0)^{1/2(-1+i\mu)}\,{}_2F_1(a,b;1+i\mu;1-w_0)\} = 0 , \qquad (A13)$$

where

$$A = \Gamma(1+2K)\Gamma(i\mu)/\Gamma(a)\Gamma(b) . \qquad (A14)$$

On examining this as $w_0 \to 1^-$, we obtain the limiting eigenvalue relation

$$-(J - \tfrac{1}{4})^{1/2}\cot\{(J - \tfrac{1}{4})^{1/2}\ln(1 - w_0)\} = \tfrac{1}{2}\nu_0(K) , \qquad (A15)$$

and hence it is proved, as in Sec. IV:

Theorem A.2: There exists an infinite number of zeros in $w_0 \in (0,1)$ for $Jw_0 > \frac{1}{4}$, and they accumulate at $w_0 = 1^-$

$(c=0^-)$. This result corresponds physically to the existence of an infinite number of stable, propagating internal gravity waves with speeds that lie outside the range of flow speeds $(0,V)$.

Recall that in the magnetoatmospheric problem discussed in Sec. VII, the point $w_0=\infty$ $(c^2=\hat{c}^{2-})$ was the corresponding point of accumulation for propagating waves. It is natural to ask, in view of the formal similarity of the singular differential equations (A3) and (3.3), what is the nature of the regular singularity $w_0=1$ in this case? Since the usual transformation formula for ${}_2F_1(a,b;c;w_0)$ has a pole when $c=a+b$, the appropriate formula, exhibiting a logarithmic singularity, is

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}\times[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\ln|1-w_0|]\times(1-w_0)^n=0, \quad \text{(A16)}$$

for $|\arg(1-w_0)|<\pi$, $|1-w_0|<1$, $R>\frac{1}{4}$, where $(a)_n=\Gamma(a+n)/\Gamma(a)$ is the usual Pochammer symbol, and $\psi(\alpha)$ is the digamma function.

A careful examination of expression (A16) precludes the existence of an accumulation point at $w_0=1^-$ [or $\omega^2=(\omega_2^2)^-$]. By contrast, for the hydrodynamic stability problem a standard Frobenius expansion in the neighborhood of the singularity z_c, where $c=U(z_c)$ $(w_0=1)$ yields complex roots of the indicial equation (in fact they are complex conjugates). There is no logarithmic singularity, and the behavior of the solution (and hence wave energy flux) is altogether different (see the detailed analysis of hydrodynamic critical layers by Booker and Bretherton, and the corresponding discussions of the MHD "critical layer").[24,32,52,53]

[1] C. P. Yu, Phys. Fluids **8**, 650 (1965).
[2] J. H. Thomas, Annu. Rev. Fluid Mech. **15**, 321 (1983).
[3] J. A. Adam and J. H. Thomas, Astrophys. Space Sci. **106**, 125 (1984).
[4] A. H. Nye and J. H. Thomas, Solar Phys. **38**, 399 (1974).
[5] A. H. Nye and J. H. Thomas, Astrophys. J. **204**, 573 (1976).
[6] B. Roberts, Solar Phys. **87**, 77 (1983).
[7] H. C. Spruit and B. Roberts, Nature (London) **304**, 401 (1983).
[8] J. M. Davila, Astrophys. J. **317**, 514 (1987).
[9] E. R. Priest, *Solar Magnetohydrodynamics* (Reidel, Dordrecht, 1982).
[10] J. P. Goedbloed, Phys. Fluids **18**, 1258 (1975).
[11] J. P. Goedbloed, "Lecture notes on ideal MHD," Rijnhuizen Report 83-145, Nieuwegein, 1983.
[12] J. Grossmann and W. C. Tataronis, Z. Phys. **261**, 217 (1973).
[13] M. Kruskal and M. Schwarzschild, Proc. R. Soc. London Ser. A **223**, 348 (1954).
[14] R. J. Tayler, J. Nucl. Fusion C **3**, 266 (1961).
[15] L. M. B. C. Campos, Rev. Mod. Phys. **59**, 363 (1987).
[16] Yu. D. Zhugzhda and N. S. Dzhalilov, Geophys. Astrophys. Fluid Dynam. **35**, 131 (1986).
[17] J. A. Adam, Q. J. Mech. Appl. Math. **31**, 77 (1978).
[18] J. A. Adam. J. Phys. A: Math. Gen. **13**, 3325 (1980).
[19] J. A. Adam, Astrophys. Space Sci. **78**, 293 (1981).
[20] J. A. Adam, J. Plasma Phys. **19**, 77 (1978).
[21] P. S. Cally, Geophys. Astrophys. Fluid Dynam. **23**, 43 (1983).
[22] P. S. Cally and J. A. Adam, Geophys. Astrophys. Fluid Dynam. **23** 57 (1983).
[23] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge U.P., Cambridge, 1981).
[24] J. A. Adam, Phys. Rep. **142**, 263 (1986).
[25] K. M. Case, Phys. Fluids **3**, 143 (1960).
[26] K. M. Case, Phys. Fluids **3**, 149 (1960).
[27] E. Balbinski, Mon. Not. R. Astron. Soc. **209**, 145 (1984).
[28] J. A. Adam, Astrophys. Space Sci. **114**, 249 (1985).
[29] H. Ando, Geophys. Astrophys. Fluid Dynam. **35**, 71 (1986).
[30] J. W. Miles, J. Fluid Mech. **10**, 496 (1961).
[31] Z. Sedlacek, J. Plasma Phys. **5**, 239 (1971).
[32] J. A. Adam, Astrophys, Space Sci. **105**, 401 (1984).
[33] J. Boyd, J. Math. Phys. **22**, 1575 (1981).
[34] E. M. Barston, Ann. Phys. (NY) **29**, 282 (1964).
[35] L. N. Howard, J. Fluid Mech. **10**, 509 (1961).
[36] E. Kreyszig, *Introductory Functional Analysis with Applications* (Wiley, New York, 1978).
[37] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Pitman, Boston, 1980).
[38] J. Chaudhuri and W. N. Everitt, Proc. R. Soc. Edinburgh A **68**, 95 (1968).
[39] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London Ser. A **244**, 17 (1958).
[40] J. P. Goedbloed, Physica **53**, 42 (1971).
[41] J. A. Adam, Sol. Phys. **52**, 293 (1977).
[42] J. Tataronis and W. C. Grossmann, Z. Phys. **261**, 203 (1973).
[43] H. Grad, Proc. Natl. Acad. Sci. (USA) **70**, 3277 (1973).
[44] E. R. Van Vleck, Trans. Am. Math. Soc. **3**, 110 (1902).
[45] J. A. Adam, unpublished report, Department of Mathematics and Statistics, Old Dominion University.
[46] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 15.
[47] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Ordinary Differential Equations* (Clarendon, Oxford, 1964), Vol. I.
[48] N. Dunford and J. T. Schwartz, *Linear Operators* (Interscience, New York, 1958), Vol. II.
[49] J. P. Goedbloed and P. H. Sakanaka, Phys. Fluids **70**, 908 (1974).
[50] J. P. Goedbloed, Physica D **12**, 107 (1984).
[51] J. W. Miles, J. Fluid Mech. **28**, 305 (1967).
[52] J. R. Booker and F. P. Bretherton, J. Fluid Mech. **27**, 513 (1967).
[53] J. A. Adam, Phys. Rep. **86**, 217 (1982).
[54] J. A. Adam (to be published).