The Effects of Variable Parameters on the Behavior of Initial Wind Waves

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The EFFECTS OF VARIABLE PARAMETERS ON THE BEHAVIOR OF INITIAL WIND WAVES

by

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A Dissertation Submitted to the Faculty of Old Dominion University in Partial Fulfillment of the Requirements for the Degree of

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Approved by:

Gabriel T. Csanady (Director)
Abstract

THE EFFECTS OF VARIABLE PARAMETERS ON THE BEHAVIOR OF INITIAL WIND WAVES

Glen H. Wheless
Old Dominion University, 1990
Director: Dr. Gabriel T. Csanady

The generation mechanism of short wind waves is generally thought to be a viscous instability at the air-sea interface. The short, regular waves arising from a sudden wind on a still water surface have a dispersion relation which is characteristic of gravity-capillary waves. The effects of variable surface tension, viscosity and shear flow parameters on the behavior of these waves were studied.

A numerical hydrodynamic stability analysis of a coupled laminar shear flow was accomplished by integrating a transformed version of the Orr-Sommerfeld equation, subject to the boundary conditions at a two-fluid interface. Unbounded growth problems usually encountered in a direct numerical integration of the Orr-Sommerfeld equation were avoided by the use of the compound matrix method, an efficient numerical technique based on a Riccati transformation. Phase speeds and growth rates of the waves generated by the instability mechanism were obtained for various surface tension and viscosity values as well as for different shear flow characteristics.

The conjectured maximum growth rate/minimum phase speed relationship is shown to be valid only for specific values of surface tension. Changes in the viscosity of water are shown to have a large effect on the behavior of the waves, while changes in the viscosity of air do not. The role of the air velocity profile characteristics are shown to be subordinate to those of water-side parameters in the generation and subse-
quent growth of the initially appearing short waves. The disturbance is confined to a narrow region on either side of the interface which is of much smaller scale than free wave motion.
To my parents,
Lorraine and Harvey Wheless,
who were always there for me.
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"Oh Joy! Rapture! Now I have a brain!"
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Chapter 1. INTRODUCTION

"Stability can be defined as the quality of being immune to small disturbances."

Betchov and Criminale, 1967

What is hydrodynamic instability and how does it tie into wind-wave generation and growth? Simply put, hydrodynamic instability is the tendency for small perturbations in a flow field to grow without limit. If one finds that an infinitesimal perturbation superimposed on a shear flow grows without limit, one can say that the flow is unstable. In the air-sea interface region, a shear flow is generated by the transfer of momentum from wind to water as the wind moves over the water surface. As this shear flow evolves from rest in time and space, a linear hydrodynamic instability mechanism causes small gravity-capillary waves to form at the interface in the very short time before non-linear effects take over and the surface becomes confused (Valenzuela, 1976; Kawai, 1979). Surface tension and gravity are the restoring forces for these short ($\lambda < 7$ cm) initial wavelets. The dispersion relationship is characterized by a minimum phase speed at a specific wavenumber (Lamb, 1932). Using linear stability theory and experimental data, Kawai (1979) obtained interfacial disturbance phase speeds and growth rates as a function of wavenumber and found that the most unstable disturbance, that with the largest growth rate, had a wavenumber very close to that of minimum phase speed. Additionally, Kawai found that the phase speed of the unstable disturbance was close to the phase speed of free waves.

Although there have been other theoretical investigations of this stability problem (van Gastel et al., 1985; Blennerhassett and Smith, 1985), little has so far been learned
about the physical nature of the unstable disturbances. These disturbances grow on a sheaf flow much as those in a boundary layer on a flat plate, and yet move with the phase speed of free waves. The coincidence of maximum growth rate and minimum celerity is intriguing: are the slowest wavelets in some sense like fixed roughness elements? How important are the two key fluid properties, surface tension and viscosity, in the instability problem?

The present study was undertaken in an attempt to shed light on the above questions. The stability characteristics of a coupled, laminar shear flow at the air-sea interface were explored using different velocity field characteristics. The coincidence or otherwise of the minimum phase speed and the maximum growth rate was checked by examining the effects of variable surface tension. It was found to be true in the Kawai experiments, but not generally so. The mechanism of wavelet growth and the physical factors controlling it were further explored by examining the effects of variable air and water viscosity values on the behavior of the disturbances.

This study builds on earlier work in the field, but differs from previous investigations in that it explores a much larger range of parameters with the aid of an efficient numerical technique. The results of this study show that although there may appear to be a correspondence between minimum disturbance phase speed and maximum growth rate, it is a correspondence which holds true only for certain values of surface tension. It is also shown that the viscosity of water affects the characteristics of a disturbance in a fundamental manner, underscoring the importance of the viscous shear flow on which the waves are generated. Finally, the disturbance velocity profiles reveal the importance of viscous forces in the growth of the disturbance.

This study's practical relevance lies in its connection to the remote sensing of the ocean surface: backscattered microwave energy is highly dependent upon the energy density of ocean waves with wavelengths less than 40 cm. Knowledge of the behavior of small-scale gravity-capillary waves under different physical conditions will help in
the improvement of algorithms used to process satellite scatterometer data.

1.1 Problem Overview

Through the years, fundamental studies of Helmholtz, Kelvin, Reynolds and Rayleigh have provided the underlying framework within which to approach problems of hydrodynamic stability. In the late 1800's, Helmholtz first considered problems of stability in fluid flow, while Lord Kelvin's work produced theorems used today as practical tools of fluid dynamics. Reynolds (1883), through his experiments with pipe flow, demonstrated that laminar flow becomes turbulent above a critical "Reynolds number". Lord Rayleigh, arguably the father of work on hydrodynamic stability, laid the foundation of modern stability theory with his inflection-point theorem and the inviscid form of the governing equations of stability.

The search for a universal critical Reynolds number led to the independent yet almost concurrent derivation in 1907 by Orr and Sommerfeld of the fundamental equation of viscous flow instability involving the second derivative of the mean velocity profile. The Orr-Sommerfeld equation is a fourth-order ordinary differential equation with variable coefficients. The pioneering work of Heisenberg (1924), Schlichting (1979), and Lin (1945) towards the goal of analytically solving this equation yielded values of critical Reynolds numbers for various flows and produced new mathematical techniques of solution. At present, analytical solutions tend to yield more accurate results for higher Reynolds number flows than do numerical methods. The analytic method of Tsuge and Sakai (1985), which matches solutions of high and low Reynolds number flows, allows treatment of a wider range of flows than do other analytical methods and demonstrates rapid convergence to an eigenvalue.

By using numerical computations, Thomas (1953) confirmed that plane Poiseuille flow was inherently unstable and ushered in the era of hydrodynamic stability studies using computers. With the wider use of computers and the advent of more efficient,
accurate numerical methods, the solution of the Orr-Sommerfeld equation has become commonplace for standard boundary conditions corresponding to parallel flow over a rigid surface. Betchov and Criminale (1967) devote much of their book to computational methods to solve the stability problems of parallel flow. In addition to a rigorous treatment of the theory of hydrodynamic stability, Drazin and Reid (1981) discuss several numerical methods of solution, including expansions in orthogonal functions (Grosch and Salwen, 1968), the use of Chebyschev polynomials (Orszag, 1971), finite difference methods (Osborne, 1967), initial value (‘shooting’) methods (Davey, 1973) and the compound matrix method (Ng and Reid, 1978), the last named being the method used in this work. These studies laid the foundation for attacking stability problems with many different boundary conditions, among them those to be satisfied at the air-sea interface.

In explaining wave generation on the sea surface, the initial generation mechanism and the mechanism for continued growth of the disturbance are generally addressed separately. Several theories for both phases of wave growth have been advanced over the years, each with its own shortcomings in light of observational evidence.

Lord Kelvin (1887) postulated a flow consisting of two irrotational parallel fluid streams with differing densities and velocities, where all vorticity was concentrated at the interface of the two fluids in a ‘vortex sheet’. He showed that a small initial disturbance of this interface will grow exponentially given certain velocity and density ratios which lead to an imbalance between inertia and buoyancy forces. This mechanism, termed Kelvin-Helmholtz instability, fails to account for the generation of surface gravity waves because these waves appear at wind speeds much less than the minimum required from the theory.

In 1925, Jeffrey theorized that the boundary layer in the air flow above a wavy surface separates on the leeward side of the wave crest. Low pressure in the air is then
located over an area of upward particle velocity in the water and higher pressure occurs over downward moving water, tending to enhance wave growth. Jeffrey's 'sheltering' theory, meant to be applied to the growth of long and steep waves already present, fell into disfavor after laboratory experiments using solid objects to model a wavy air-sea interface seemed to disprove its applicability. The importance of the separated boundary layer to the growth of long waves became clear only after Banner and Melville's (1976) observations of the large increase in momentum transfer to the sea surface over breaking waves and the realization that flow separation did indeed occur in the breaking wave regime.

Miles (1957, 1959a, 1959b) investigated the laminar shear flow of a logarithmic wind profile over a wavy water surface, but neglected the shear flow in the water. His assumption was that the wavy motion was coupled to perturbations in the airflow. Linearizing the resulting problem, Miles sought solutions to the inviscid form of the Orr-Sommerfeld equation. With turbulence and viscosity neglected, he showed that waves could grow from the momentum transferred via pressure forces in the air in phase with the slope of the wave. According to Miles' theory, energy is transferred at a rate proportional to the curvature of the wind profile at the height where the wave phase speed equals the wind velocity, the so-called critical level. The properties of the wind profile at the critical level determines the rate of energy input to the waves. The physical interpretation by Lighthill (1962) of Miles's theory showed how the airflow just above the critical height was turned back from the trough of the undulating surface due to the higher pressure found there. A similar upward motion was found at the crest, rendering the now familiar picture of the 'cat's eye' streamline pattern located above the surface. The position of the closed 'cat's eye' circulation pattern over the forward wave face causes a net pressure force and a resultant momentum transfer. Miles' 'shear instability' theory predicted growth rates not too far from the laboratory observations of Larson and Wright (1975) yet only explained long wave growth at
wave celerities such that the critical level is well above the sea surface.

In the case of short waves, where the critical level is very close to the surface, Miles (1962) conjectured that laminar dissipation exceeded the energy transfer due to profile curvature effects and that his original shear instability theory was inapplicable. In a further development of this theory, he suggested that an instability process involving the entire viscous sublayer was responsible for the generation and growth of short gravity-capillary waves. The formulation of this theory again relied upon perturbing the shear flow present in the turbulent boundary layer in the air only, with no account for the drift current caused by the wind. Miles extended his previous analysis by using the complete, viscous form of the Orr-Sommerfeld equation, obtaining viscous as well as inviscid solutions to the problem. His results were more or less in agreement with the observations of Larsen and Wright (1975), but there were significant discrepancies in the gravity-capillary wave regime.

Lock (1954) approached the problem of wave generation by examining the stability of small perturbations in a laminar boundary layer flow of wind over a water surface. Unlike Miles, Lock included the motion of the water in his analysis by modeling the mean flow as a solution of the boundary layer equations in two fluids with different densities and viscosities, coupled by the continuity of velocity and stress components (Lock, 1951). He estimated the wavelength of maximum amplification for a given wind speed and produced results in general agreement with observations available at that time. Lock’s results are difficult to compare with recent laboratory data yet the analysis certainly was correct in its concept of including the motion of the water in the analyzed flow.

Valenzuela (1976), in reconsidering the problem, numerically investigated the solution of the Orr-Sommerfeld equation for a coupled, laminar shear flow in the air and in the water. He used a Taylor series expansion to simplify the complex interfacial boundary conditions and was able to obtain growth rates and phase speeds of the
unstable disturbances at the air-water interface. These results were in very close agreement with previously measured values for short waves (Larson and Wright, 1975) and implied that the growth of initial wind wavelets was due to the viscous instability of a coupled, laminar air-water shear flow. Valenzuela further concluded that the motion in the water could not be ignored as the drift current affected the phase speeds of the waves.

In a remarkable study on the development of short wavelets from rest, Kawai (1979) compared time-series of laboratory wind-wave tank data with disturbance phase speeds and growth rates obtained from the numerical solution of the Orr-Sommerfeld equation. As in Valenzuela's calculations, the basic state was the coupled laminar shear flow of the air and the water. However, unlike Valenzuela, Kawai used observed shear flow profiles in his numerical calculations. In every case studied, he found that the calculated phase speeds and wavenumbers of the most unstable waves were almost identical to those initially observed in the wave tank. He also found that the period of exponential wavelet growth was short and that soon other effects took over. Kawai's results verified Lock's and Valenzuela's idea that the total air-water shear flow complete with the four interfacial boundary conditions must be considered in order to account for initial wavelet growth. His data suggested also that maximum growth rate occurred at or near the minimum phase speed of gravity-capillary waves. However, Kawai did not vary surface tension or other parameters in his calculations or in his experiments, so that the coincidence of maximum growth rate and minimum phase speed may have been coincidental, due to the choice of specific flow parameters.

Van Gastel et al. (1985) examined analytically the effects of various flow parameters on the growth of gravity-capillary wavelets. Using a coupled air-water shear flow model and asymptotic methods to solve the Orr-Sommerfeld equation, they found an initial growth rate proportional to $u_*^3$, where $u_*$ is the friction velocity in air, which is a constant factor depending on $u_*$ on account of stress continuity. The growth rate

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was also found to depend strongly on the wind profile, even when the profile change consisted only of a different height of the viscous sublayer. The phase speed of the waves was dependent on the surface current speed, but did not depend on the form of the current profile. Once again, a minimum phase speed/maximum growth rate connection appeared in the results, but for very similar fluid properties and flow parameters as used by Kawai. Neither Valenzuela, Kawai nor van Gastel et al. investigated the effects of variable fluid properties on the behavior of the initial wavelets, although van Gastel et al. remarked that they should be sensitive to changes in the air-sea density and viscosity ratios.

Blennerhassett and Smith (1987) used linear hydrodynamic stability theory to analytically examine the stability of a generic flow comprised of one fluid of specific density and viscosity over another with different properties. Instead of assuming a vanishingly small air-to-water density ratio as Miles (1962) had done, their analysis retained the small fluid-to-fluid density and viscosity ratios in the controlling equations, which were then used as an expansion parameter. The lowest order equation for the disturbance streamfunction was dominated by viscous forces, with inertial forces appearing only at second order. The short waves ($\lambda < 1.7$ cm) resulting from the perturbation analysis, often called 'cat's paws', were shown to travel at or near the surface velocity and to be driven by a balance of surface tension, shear flow vorticity at the interface and viscous diffusion. Although their analysis is applicable only to those waves with a phase speed close to the interfacial surface velocity, their solutions to the Orr-Sommerfeld equation described the wave behavior based on the density and viscosity ratios of the two fluids. Blennerhassett and Smith speculate that gravity and velocity profile characteristics have little effect on the behavior of these very short waves and that the critical level plays no active role in the generation mechanism, contrary to the Miles theory. Their results also seem to indicate that an increase in interfacial shear or a decrease in surface tension would decrease the wavelength of the distur-
Based upon Kawai's (1979) work, the use of linear hydrodynamic stability theory to describe the initial generation of short waves is valid for the short time before non-linear behavior becomes dominant. In the aforementioned studies of wave generation and growth, there were no systematic investigations to quantify the effects of variable fluid properties (surface tension, viscosity) or variable flow parameters (free stream velocity, friction velocity) on the initial behavior of these short waves. It is certainly not clear how changes in these properties and parameters may affect the initial growth rate of these waves or if the wavenumber of the most unstable wave is dependent upon these or other factors.

1.2 Problem Statement

It is evident from the previous discussion that the underlying mechanism by which short wind waves are generated is not well understood. If maximum growth rate and minimum phase speed are somehow related, what is the physics behind such a relationship? Since the Miles theory and its physical explanation by Lighthill (1962) is inappropriate to the short wave generation problem, what is the proximate physical cause of wavelet growth and on what parameters does it depend? What is the role of surface tension and viscosity? The object of this study was to find answers to these questions. The method utilized was a stability analysis of a coupled, laminar shear flow, carried out by a numerical solution of the Orr-Sommerfeld equation subject to the boundary conditions at a two-fluid interface. Turbulent Reynolds stresses were excluded from the analysis. The models of the water and air flow are based on solutions to the equations of motion and are linked at the interface by continuity of velocity and stress. Both the water and air models have smooth curvature, important for both realism and computational concerns. The analysis yielded phase speeds and growth rates as a function of wavenumber for different shear flow characteristics and fluid properties. Also, the eigenfunctions were calculated in an effort to describe the
character of the disturbances.

The formulation of the laminar shear flow model at the air-sea interface, and the equations governing the initial behavior of the disturbance are described in Chapter 2. A brief overview of the methods available to solve the Orr-Sommerfeld equation are covered in Chapter 3 and a description of the compound matrix method used in this study may be found in Chapter 4. Results obtained from the research are presented and discussed in Chapter 5.
Chapter 2. FORMULATION OF THE MODEL

The generation and growth of gravity-capillary wind wavelets is due to the selective amplification, by a linear instability mechanism, of small disturbances occurring at the air-sea interface (Valenzuela, 1976; Kawai, 1979). The initial formation process may be described as follows. Let the air-sea interface be at equilibrium with no motion at time zero. When a wind abruptly begins to blow along the quiescent water surface, a large shear stress appears and the uppermost layer of the water begins to move. A small disturbance imposed at this highly sheared interface will grow exponentially for a short time until more complex wave interactions come into play. The initial behavior of the disturbance in time and space is described by the Orr-Sommerfeld equation, hereafter referred to as the OSE, an equation which is derived by linearization of the Navier-Stokes equations. Although it neglects non-linear effects, the OSE has been shown to adequately describe the initial generation of wind wavelets (Kawai, 1979), as well as serving as a first approximation to the non-linear stability problem.

The OSE must be solved subject to specified boundary conditions and the supposed eigenvalue problem is dependent upon wavenumber and viscosity. Temporal stability characteristics, or how the behavior of the disturbance varies with time, are deduced by considering the wavenumber, k, to be real and the phase speed of the wave, c = c_r + ic_i, to be the complex eigenvalue. The disturbance growth rate is expressed as kc_i. The temporal stability of the disturbance is therefore dependent upon the imaginary portion of the phase speed, c_i, with the disturbance amplified if c_i > 0 and damped if c_i < 0. For spatial stability analysis, a much more difficult
problem, $c$ and $R$ are considered to be real and eigenvalues consisting of complex $k$ values are sought.

Here, the temporal stability problem is considered. The model of the coupled wind-current shear flow is explained in section 2.1, and the OSE is derived in section 2.2. The boundary conditions at the far field and at the interface, taking into account continuity of stresses and the action of gravity and surface tension as restoring forces, completely define the problem and are explained in section 2.3.

2.0 Basic Assumptions

A two-layer model of fluids with different densities, viscosities and velocity profiles is assumed, with the $z$ axis taken to be positive upwards. It is assumed that the flow in each medium is incompressible, that density and temperature are constant and that non-linear effects are small enough to ignore during the initial stages of wavelet generation and growth. A two-dimensional, steady laminar shear flow is considered; the flows in the water and the air are assumed to satisfy the equations of motion and are coupled at the free surface through continuity of velocity and stress components (Batchelor 1974, p. 148-151). The small cross-stream velocity component is ignored for simplicity. Although the fully developed air flow over the sea surface is certainly not laminar, the model may be thought to simulate a sudden gust of wind over a smooth sea surface.

Although the derivation of the OSE normally proceeds from a parallel flow assumption, it is possible to apply the equation to nearly parallel flow, a class of flows defined by the constraints of $W(x,z) \ll U(x,z)$ and $\frac{\partial U}{\partial x} \ll 1$ (Drazin and Reid 1982, p. 154), including shear flows and boundary layer flows. It will be useful to simplify the system of equations governing the three-dimensional stability problem to an equivalent two-dimensional problem by the use of Squire's transformation, which shows that a two-dimensional disturbance will behave identically to a three-dimensional one, but at
a smaller Reynolds number (Drazin and Reid 1982, p. 155). In other words, the disturbance first appearing on an air-water interface due to a shear flow instability will be two-dimensional (Maslowe, 1981). With this simplification, the stability of nearly parallel flow is relatively easily investigated.

2.1 Model of the Shear Flow

Let the velocity distribution in air and water arising from a sudden wind gust be $U(z)$. Computational difficulties may arise if a discontinuous description of $U''(z)$ is used to describe the basic flow. For this reason, an ‘empirical fit’-type profile, such as the standard linear-logarithmic model or van Driest’s mixing length model of the velocity distribution in a turbulent boundary layer (Cebeci and Smith 1984, p. 162), is unsuitable for use in the stability analysis on account of rapid variations of $U''(z)$. In the case of the linear-logarithmic profile, this non-smooth behavior is exhibited at the top of the viscous sublayer where the linear and logarithmic portions are matched. The mixing length profile has a large variation of profile curvature at the interface. Further numerical difficulties are encountered using wind profiles of this type on account of the continual increase of $U_a(z)$ with height and the resulting slow approach of $U_a''(z)$ to zero. This requires either a high upper limit of integration, causing the assumed exponential solutions to blow up at higher Reynolds number, or the imposition of a ‘top’ to the problem, known to produce spurious modes in the eigenfunction spectrum (Lakin and Grosch, 1982). Kawai (1979) overcame this problem by assuming that $U_a(z)$ became constant at an arbitrarily imposed upper limit of integration, then using analytical solutions of the OSE at that upper limit to start the numerical integration. This procedure is somewhat suspect and is, at any rate, inconvenient in the numerical solution.

A wind profile is desired which not only realistically models the flow of air over an aerodynamically smooth interface but which also meets the smoothness and asympt-
totic far field behavior constraints. The wind profile used in this study was modeled by numerically integrating the Blasius equation, with interfacial boundary conditions formulated to ensure continuity of shear stress and surface velocity when coupled with the flow in the water. Developed from the equations of motion, this air flow model is continuous throughout the domain of the problem and describes the flow in the developing laminar boundary layer found over a still water surface upon the application of a wind. By assuming a sufficiently large distance over which the wind blows along the still water surface before wavelets appear, (on the order of a few meters, Kawai, 1979), the wavelengths of the initially appearing waves are assured to be small compared with the distance. Therefore, the changes in the mean wind profile are negligible over a wavelength (Brooke Benjamin, 1959), in effect fixing the basic flow with respect to the disturbance.

The model of the shear flow in the water arising from the sudden application of a constant wind stress will be developed first.

2.1.1 Flow in the Water

The flow in the water has a vertical structure such that the velocity becomes negligible at some finite stress penetration depth. Neglecting horizontal pressure gradients, the momentum balance arising from the applied surface stress is

\[
\frac{\partial U}{\partial T} = \frac{1}{\rho_w} \frac{\partial \tau}{\partial z},
\]

or

\[
\frac{\partial U}{\partial T} = v_w \frac{\partial^2 U}{\partial z^2},
\]

the solution of which is

\[
U(z) = \frac{\mu w^2 L_w}{v_w} \text{ierfc} \left( \frac{z}{L_w} \right)
\]

where \( \xi = \frac{z}{L_w} \) and \( \text{ierfc} \) is the integral of the complementary error function (see...
Carslaw and Jaeger 1959, pp.50 and 483). $u_\ast w$ is the friction velocity in water and $v_w$ is the kinematic viscosity of water. Eqn (2.1.3) is closely approximated by an exponential profile from van Gastel et al. (1985):

$$U_w(z) = U_0 \exp(\lambda \xi) \quad (2.1.4)$$

The length scale in the water is defined as the depth of the boundary layer, or

$$L_w = 2\sqrt{v_w T} \quad (2.1.5)$$

where $T$ is the time after the initial onset of the wind for short wind waves to appear and is on the order of 10 seconds. The factor $\lambda$ is derived from the requirement of continuity of shear stress and connects the current with the wind:

$$\lambda = \frac{\rho_a}{\rho_w} \frac{u_\ast a^2 L_w}{v_w U_0} \quad (2.1.6)$$

Equation (2.1.4) is equivalent to (2.1.3) if $\lambda$ is exactly 2. Values of $\lambda$ using Kawai’s (1979) measured data were approximately 1.75 - 1.8. The profile described by (2.1.4) closely resembles the flow observed by Kawai (1979) at the time of the initial appearance of wind wavelets in a wave tank. The surface velocity, $U_0$, is

$$U_0 = 2\pi^{-\frac{1}{2}} u_\ast w [T/v_w]^{\frac{1}{2}} \quad (2.1.7)$$

arising from $\text{erfc}(0) = \pi^{-\frac{1}{2}}$.

The free stream wind velocity, $U_\infty$, is used as the scale velocity for both wind and current velocity profiles. Reynolds number in the water, $R_w$, is therefore defined as

$$R_w = \frac{U_\infty L_w}{v_w} \quad (2.1.8)$$

Although the use of a time dependent expression for $L_w$ and $U_0$ seems inconsistent with the assumption of non-time dependent flow in the OSE, it can be defended by noting that the temporal rate of change of the basic flow is much smaller than the frequency and the growth rate of the disturbance given by the calculation. The basic
flow is thereby "fixed" flow relative to the time scale of the disturbance.

The current profile was constructed using (2.1.5) and prescribed values of $u_{*w}$ and $T$, taken from Kawai's (1979) observations and shown in Table 2.1.

<table>
<thead>
<tr>
<th>$u_{*w}$</th>
<th>$u_{*a}$</th>
<th>$T$</th>
<th>$U_0$</th>
<th>$U_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cm/sec</td>
<td>cm/sec</td>
<td>sec</td>
<td>cm/sec</td>
<td>cm/sec</td>
</tr>
<tr>
<td>.47</td>
<td>13.6</td>
<td>9.05</td>
<td>7.5</td>
<td>420</td>
</tr>
<tr>
<td>.59</td>
<td>17.0</td>
<td>5.97</td>
<td>9.6</td>
<td>510</td>
</tr>
<tr>
<td>.75</td>
<td>21.4</td>
<td>2.52</td>
<td>9.8</td>
<td>630</td>
</tr>
<tr>
<td>.86</td>
<td>24.8</td>
<td>1.99</td>
<td>10.2</td>
<td>740</td>
</tr>
</tbody>
</table>

Table 2.1: Basic Flow Values (from Kawai, 1979).

2.1.2 Flow in the Air

The air flow over a water surface in the initial stages of wave generation and growth is assumed to be aerodynamically smooth. Kawai's (1979) measurements of the heights of initial wavelets in a wind-wave tank have shown that the initially appearing wavelets are entirely located within the viscous sublayer, thereby causing minimal effects on the flow in the air from form drag. The model of the air flow is of the form

$$U_a(z) = U_\infty f \left(z \over L_a \right).$$

The length scale in air is defined as

$$L_a = \sqrt{\frac{v_a x}{U_\infty}}$$

where $U_\infty$ is the free stream wind velocity, $v_a$ is the kinematic viscosity in air and $x$ is a length over the water analogous to the distance from the edge of a suddenly accelerated flat plate. For convenience, it is assumed that $L_a = L_w$. 

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Let \( f\left[\frac{z}{L_a}\right] \) be the solution of the Blasius equation (Schlichting 1979, p.136),

\[
ff'' + 2f''' = 0 .
\tag{2.1.11}
\]

The boundary conditions at \( z = 0 \) arise from the requirements that \( U_a = U_w \) and that \( \mu_a U'_a = \mu_w U'_w \):

\[
f'(0) = \frac{U_0}{U_\infty} ,
\tag{2.1.12}
\]

and

\[
f''(0) = \frac{u^2_a L_a}{v_a U_\infty} .
\tag{2.1.13}
\]

At \( z = +\infty \),

\[
f'(+\infty) = 1 .
\tag{2.1.14}
\]

Using the assumed air and water length scale equivalency in conjunction with the chosen \( T \) and \( U_\infty \) values prescribes the value of the distance, \( x \), in \( L_a \). If the scale velocity is taken to be \( U_\infty \), the Reynolds number in air is defined as

\[
R_a = \frac{U_\infty L_a}{v_a} .
\tag{2.1.15}
\]

Using a chosen \( u* \) value, \( u*_a \) was calculated according to

\[
\rho_a u*_a^2 = \rho_w u*_w^2 .
\tag{2.1.16}
\]

Prescribing realistic values for \( U_\infty \) and \( T \), a wind profile can be numerically constructed by integrating (2.1.9) via ’shooting’ on \( f(0) \) based on the boundary conditions (2.1.12), (2.1.13) and (2.1.14). Although this process results in a small non-zero value of the non-dimensional streamfunction at \( z = 0 \), the ensuing \( x \) variability is also small and is ignored in the development of the Orr-Sommerfeld equation.
Figure 1. Coupled Wind/Current Velocity Profile

Figure 1(a)

Representative velocity profile consisting of a solution of the Blasius equation to model the wind profile and an exponential model for the drift current. The horizontal axis represents velocity, scaled by the free stream wind velocity, $U_\infty$. The vertical axis represents vertical distance above and below the interface, $z = 0$, scaled by $L_a = L_\nu = 2\sqrt{\nu_\nu T}$.

Figure 1(b)

Velocity profile consisting of an empirical fit model of the turbulent boundary layer in the air, based on the mixing length formulation by van Driest, and an exponential current profile. $L_a = \frac{V_a}{u_{*a}}$. Note the continual increase of $U_a(z)$.
Coupled Wind/Current Velocity Profile

Figure 1(a).

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van Driest (empirical fit) Shear Flow Profile

Figure 1(b).

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2.2 The Orr-Sommerfeld Equation

Scales have now been chosen and explicitly identified in the previous sections. The length and velocity scale variables are written as \( L_s = L_a = L_w \) and \( U_s \), Reynolds numbers are written as \( R_a \) and \( R_w \), with the subscripts standing for air and water, respectively. Nondimensional variables are defined as follows:

\[
\tilde{u} = \frac{u^*}{U_s}, \quad U = \frac{U^*}{U_s}, \quad x = \frac{x^*}{L_s}, \quad t = \frac{t^* U_s}{L_s}, \quad p = \frac{p^*}{\rho U_s^2}
\]

where asterisks denote dimensional quantities and \( t \) is the development time of the disturbance.

The velocity vector \( \tilde{u}(x,z; t) \), is considered to consist of two components, a laminar shear flow and a perturbation,

\[
\tilde{u}(x,z; t) = U(z)t + \tilde{u}''(x,z; t)
\]

where \( U(z) \) is the velocity of the basic laminar shear flow and \( \tilde{u}''(x,z; t) \) is the fluctuating perturbation velocity vector, with components \( u' \) and \( w' \).

The scaled Navier-Stokes equations for an incompressible fluid and the equation of continuity are

\[
\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} = -\nabla p + R^{-1} \nabla^2 \tilde{u}
\]

\[
\nabla \cdot \tilde{u} = 0
\]

where \( R \) is the Reynolds number of the flow. Then, noting that the mean flow satisfies the Navier-Stokes equations, one finds the scaled, linearized equations of motion:

\[
(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x})\tilde{u}'' + w' \frac{dU}{dz} = -\nabla p' + R^{-1} \nabla^2 \tilde{u}''
\]

\[
\nabla \cdot \tilde{u}'' = 0.
\]

The perturbation is assumed to be of a wavelike form, as usual in problems of this kind. The perturbation velocity is then supposed to have the character of a progressive
wave:

\[ \ddot{u}'(x,z; t) = \ddot{u}(z) \exp[ik(x-ct)] \]  

(2.2.7)

where \( k \) is a wavenumber and \( c \) is a wave speed, the latter possibly complex. The same functional dependence is assumed for the pressure:

\[ p(x,z; t) = p + p'(x,z; t), \]  

(2.2.8)

where the perturbation pressure is

\[ p'(x,z; t) = \rho(z) \exp[ik(x-ct)]. \]  

(2.2.9)

The expressions for perturbation velocity (2.2.3) and perturbation pressure (2.2.5) are then substituted into (2.2.8) and (2.2.9) to arrive at a system of ordinary differential equations:

\[ \ddot{u}' - k^2\ddot{u} - ikR(U-c)\dot{u} = R\dot{w}U' + ikR\dot{\rho}. \]  

(2.2.10a)

\[ \ddot{w}' - k^2\ddot{w} - ikR(U-c)\dot{w} = R\frac{d\rho}{dz}. \]  

(2.2.10b)

The equation of continuity becomes

\[ ik\dot{u} + \frac{d\dot{w}}{dz} = 0. \]  

(2.2.10c)

This is satisfied by introducing the disturbance stream function, \( \psi(x,z; t) \): 

\[ \psi(x,z; t) = \phi(z) \exp[ik(x-ct)], \]  

(2.2.11)

where \( \phi(z) \) is a disturbance amplitude. The two disturbance velocity components can then be expressed as

\[ u' = \frac{\partial\psi}{\partial z} \quad \quad \quad w' = -\frac{\partial\psi}{\partial x}. \]  

(2.2.12)

and therefore, their amplitudes are:

\[ \ddot{u} = \phi' \quad \quad \quad \ddot{w} = -ik\phi. \]  

(2.2.13)

Using (2.2.13) and (2.2.10a), an expression for \( \frac{d\rho}{dz} \) in terms of the streamfunction amplitude \( \phi \) may be derived.
Substituting into (2.2.10) the Orr-Sommerfeld equation for the disturbance amplitude, \( \phi(z) \), is obtained. In the air, this is

\[
\phi^{IV}_a - 2k^2\phi''_a + k^4\phi_a = ikR_a[(U_a - c)(\phi''_a - \phi_a k^2) - U''_a \phi_a], \quad (2.2.15)
\]

in the water:

\[
\phi^{IV}_w - 2k^2\phi''_w + k^4\phi_w = ikR_w[(U_w - c)(\phi''_w - \phi_w k^2) - U''_w \phi_w] \quad (2.2.16)
\]

where \( U \) and \( U'' \) describe the flow velocity, the subscripts \( a \) and \( w \) stand for air and water respectively, and all variables have been normalized by the chosen scales.

With a mean velocity distribution prescribed, these equations may be integrated in principle, for a chosen complex value of \( c \). The approach used in this study is to perform a "top-down" numerical integration in the air, and a "bottom-up" integration in the water. The resulting solutions must then be made to satisfy the boundary conditions at the interface by an appropriate choice of the eigenvalue, \( c \).

### 2.3 Boundary Conditions

The eigenvalue problem is completely described by boundary conditions at \( z = \pm \infty \) as well as interfacial boundary conditions at \( z = 0 \). At \( z = \pm \infty \), both disturbance velocity components must vanish, which in terms of the disturbance amplitude, \( \phi(z) \), means that:

\[
\phi(z) = \phi'(z) = 0 \quad z = \pm \infty. \quad (2.3.1)
\]

To these four conditions are added another four for the two fourth-order equations (2.2.15) and (2.2.16). The boundary conditions at the air-sea interface are much more complicated than over a solid surface.

The linearized free surface boundary conditions have been written down by Phillips (1977, p. 35). At \( z = 0 \), he used a Taylor series expansion with the small wave
slope as the expansion parameter. This method is equivalent to Brooke Benjamin's (1959) description of the same conditions in an orthogonal curvilinear coordinate system referenced to the free surface. The conditions are that both components of the velocity, the shear stress and the normal stress must be continuous.

The continuity of the vertical velocity at the free surface is the "kinematic" boundary condition:

\[ w_0(0) = w_w(0) \quad \text{(2.3.2)} \]

This implies

\[ \phi_0 = \phi_w \quad \text{(2.3.3)} \]

The interface displacement may be written as

\[ \zeta(x,z;t) = \eta \exp[ik(x-ct)] \quad \text{(2.3.4)} \]

where \( \eta \) is the displacement amplitude. The vertical velocity is

\[ w = -\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} \zeta = -ik \phi. \quad \text{(2.3.5)} \]

From this, the interface displacement can be expressed in terms of \( \phi \):

\[ \eta = -\frac{\phi}{(U_0-c)} = -\frac{\phi_0}{(U_0-c)} = -\frac{\phi_w}{(U_0-c)} \quad \text{(2.3.6)} \]

The second boundary condition, expressing continuity of horizontal velocity, is to first order

\[ u_0(\eta) = u_w(\eta), \quad \text{(2.3.7)} \]

or

\[ u_0(0) + \eta \frac{\partial U_0(\eta)}{\partial z} = u_w(0) + \eta \frac{\partial U_w(\eta)}{\partial z}. \quad \text{(2.3.8)} \]

With the aid of the kinematic condition this may be rewritten as

\[ \phi'_a(U_0-c) - \phi_a U'_a = \phi'_w(U_0-c) - \phi_w U'_w. \quad \text{(2.3.9)} \]

The non-dimensional shear stress is
\[ \tau(z) = R^{-1} \left[ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right]. \] (2.3.10)

For small \( \eta \) and to first order this is at the interface:

\[ \tau(\eta) = \tau(0) + \epsilon \frac{\partial \tau}{\partial z} \eta = \tau(0) + R^{-1} \left[ \frac{\partial^2 U}{\partial z^2} \right] \eta, \] (2.3.11)

which is also

\[ \tau(\eta) = R^{-1} \left[ \frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z} + \frac{\partial^2 U}{\partial z^2} \eta \right]. \] (2.3.12)

where \( \frac{\partial w'}{\partial x} \) and \( \frac{\partial u'}{\partial z} \) are to be calculated at \( z = 0 \). The continuity of tangential stress is the third interfacial condition:

\[ \tau_a(\eta) = \tau_w(\eta), \] (2.3.13)

from which one finds using the previous results:

\[ \frac{\mu_a}{\mu_w} \left( (U_s - c)(k^2 \phi_a + \phi''_a) - U'' \phi_a \right) = \left( (U_s - c)(k^2 \phi_w + \phi''_w) - U'' \phi_w \right). \] (2.3.14)

The fourth interfacial condition is a balance of vertical forces. The normal pressure is generally discontinuous between the water and air, the difference being balanced by the surface tension force over a curved interface. In terms of dimensional variables the force balance is:

\[ p_w^*(\eta) = p_a^*(\eta) - \sigma^* \left[ \frac{\partial^2 \psi^*}{\partial x^2} \right]. \] (2.3.15)

The scaled form of this equation is

\[ p_w(\eta) + \sigma \frac{\partial^2 \psi}{\partial x^2} = \frac{p_a}{\rho_w} p_a(\eta) \] (2.3.16)

where the scaled surface tension is \( \sigma = \frac{\sigma^*}{\rho_w U_s^2 L_s} \).

This requires \( p_a(\eta) \) and \( p_w(\eta) \) in terms of quantities expressed at \( z = 0 \).
ning with the horizontal component of the linearized equations of motion, (2.2.10a), utilizing the kinematic condition and solving for the pressure, one finds in the air:

\[
p_a(\eta) = \rho_a \left[ (U_0 - c)^2 k \phi_a' - (U_0 - c) k (U_a' \phi_a) + i \nu_a (U_0 - c)(\phi_a''' - k^2 \phi_a) - g k \phi_a \right].
\]  

(2.3.17)

A similar expression can be found for \( p_w(\eta) \),

\[
p_w(\eta) = \rho_w \left[ (U_0 - c)^2 k \phi_w' - (U_0 - c) k (U_w' \phi_w) + i \nu_w (U_0 - c)(\phi_w''' - k^2 \phi_w) - g k \phi_w \right].
\]  

(2.3.18)

Using (2.2.11), it is seen that

\[
\frac{\partial w}{\partial z} = k^2 \phi'
\]  

(2.3.19)

applicable to air and water, and using (2.3.4) and (2.3.5),

\[
\frac{\partial^2 \xi^2}{\partial x^2} = \frac{\phi}{U_s - c} k^2
\]  

(2.3.20)

also applicable to air and water. Therefore, by substituting the last four equations into (2.3.16), the following is found:

\[
\rho_a \left[ (U_0 - c)^2 k \phi_a' - (U_0 - c) k (U_a' \phi_a) + i \nu_a (U_0 - c)(\phi_a''' - k^2 \phi_a) - g k \phi_a \right] =
\rho_w \left[ (U_0 - c)^2 k \phi_w' - (U_0 - c) k (U_w' \phi_w) + i \nu_w (U_0 - c)(\phi_w''' - k^2 \phi_w) - (g k + \sigma k^3) \phi_w \right],
\]  

(2.3.21)

Writing the classical gravity-capillary wave phase speed in a scaled fashion using non-dimensional Froude and Weber numbers as

\[
C_0^2 = (k F_r)^{-1} + k W e^{-1}
\]  

(2.3.22)

where \( F_r \), a Froude number, and \( W e \), a Weber number, are defined by

\[
F_r = \frac{U_s^2}{g L_s}, \quad W e = \frac{U_s L_s (\rho_w - \rho_a)}{\sigma^*}
\]  

(2.3.23)

equation (2.2.21) may be re-written as

\[
\frac{\rho_a}{\rho_w} \left[ (U_a' - \frac{k C_0^2}{c - U_0}) \phi_a + (c - U_0 + 3 \nu a^{-1}) \phi_a' + \frac{\phi_a'''}{i k Re_a} \right] =
(U_w' - \frac{k C_0^2}{c - U_0}) \phi_w + (c - U_0 + 3 \nu w^{-1}) \phi_w' + \frac{\phi_w'''}{i k Re_w}.
\]  

(2.3.24)
Therefore, the scaled interfacial boundary conditions written in terms of $\phi$ and its derivatives are:

$$\phi_a = \phi_w$$

(2.3.25)

$$\frac{U_a}{c-U_0} \phi_a + \phi_a' = \frac{U_w}{c-U_0} \phi_w + \phi_w'$$

(2.3.26)

$$\frac{\mu_a}{\mu_w} \left[ \left( \frac{U''_a}{c-U_0} + k^2 \right) \phi_a + \phi''_a \right] = \left( \frac{U''_w}{c-U_0} + k^2 \right) \phi_w + \phi''_w$$

(2.3.27)

$$\frac{\rho_a}{\rho_w} \left[ \left( U'_a - \frac{kC_0^2}{c-U_0} \right) \phi_a + \left( c-U_0 + 3ikRe_a^{-1} \right) \phi_a' + \frac{\phi_a'''}{ikRe_a} \right]$$

$$= \left( U'_w - \frac{kC_0^2}{c-U_0} \right) \phi_w + \left( c-U_0 + 3ikRe_w^{-1} \right) \phi_w' + \frac{\phi_w'''}{ikRe_w}$$

(2.3.28)

Satisfying these four conditions simultaneously is a major difficulty in finding the eigenvalues. The difficulty and its reduction is discussed in some detail below.
Chapter 3. METHODS OF STABILITY ANALYSIS

The stability analysis of a specified flow may be accomplished by either analytical or numerical means, the choice of which is determined by the problem specifications, the information on the flow in question and the desired results. Analytical methods are useful for examining high Reynolds number flow, but are often difficult to formulate and may not be uniformly valid throughout the entire domain of the problem. Despite these drawbacks, many of the basic principles of hydrodynamic stability where elicited using analytical asymptotic expansion methods applied to simple, bounded flow profiles. Numerical methods are more suitable for low Reynolds number flows and enable the calculation of eigenvalues and eigenfunctions of the OSE to very high levels of accuracy, provided there is suitably accurate information on the curvature of the velocity profile.

Both analytical and numerical methods will be briefly reviewed in this chapter, with emphasis on those methods or results relevant to the wave generation problem. This overview is by no means complete; the interested reader is referred to the texts of Lin (1955), Betchov and Criminale (1967) and Drazin and Reid (1981) for a more comprehensive treatment.

3.1 Analytical Techniques

In the years before high speed computational resources were available, the stability characteristics of a specified flow were sought through analytical means. The earliest attempts, aimed at finding a universal value of the critical Reynolds number, underscored the inadequacy of the analytical techniques of the period in obtaining solutions
of the OSE. The heuristic methods of approximation conceived by Heisenberg (1924), Lin (1955) and Schlichting (1979, p. 466) all centered on expansion analysis and shared a common developmental theme. First, asymptotic approximations for solutions above and below the critical level were derived, usually from the inviscid second-order Rayleigh equation. The singular nature of this equation at the critical level required the matching of these basic approximations to produce a combined solution, which was then used to obtain higher order approximations. The strong viscous effects present at the critical layer and at the end boundary were usually taken into account in the combined solution through the use of a \((kr)^{-1}\) expansion.

These early expansion methods made use of solutions which were valid at the critical level but not valid at the boundaries of the problem, or vice versa. This drawback of non-uniformity of solutions necessitated the development of improved techniques which provided uniformly valid approximations, and which allowed an orderly method for obtaining higher order approximations. The comparison-equation method (Langer, 1957) uses a scheme relying upon the asymptotic representation of the desired solutions in terms of already known solutions of another, simpler equation with properties similar to that of the OSE. The primary drawback to this method lies in the complicated integral representations of the solutions which make it difficult to apply to actual stability calculations. Drazin and Reid (1981, pp. 251-255) discuss other analytic methods providing solutions which are uniform throughout the entire domain of the problem.

The short wave generation problem requires an analysis of the stability of a combined air-water shear flow at a movable interface. The length scales involved are short, and the interfacial boundary conditions far more complex than those of the classical problem of the boundary layer over a flat plate. This scenario presents unique challenges to the formulation of an accurate asymptotic method. Van Gastel et al. (1985) used asymptotic analysis to examine the sensitivity of the growth rate and phase speed...
of short wind waves to changes in the wind and current velocity profiles. By using as an expansion parameter an inverse Reynolds number based on wavelength and friction velocity, they were able to formulate expansions for the streamfunction of the disturbance with the aid of a WKB approximation (see Drazin and Reid 1981, pp. 167-171). Their analysis provided valuable insight into the role of some flow parameters on the behavior of short wind waves. Tsuge and Sakai (1985) formulated an expansion method which yielded a transformed, approximate version of the OSE requiring no information on the curvature of the profile in question. The eigenvalue relation was found to be a simple determinant equation. Blennerhassett and Smith (1987) also used an asymptotic technique to study the instabilities occurring in a laminar shear flow between parallel plates.

3.2 Numerical Methods

Despite being limited to lower Reynolds number flows, the main advantage in using numerical methods over analytical ones to solve the OSE is the ability to conveniently determine eigenvalues, eigenfunctions and curves of marginal stability. The utility of numerical methods is somewhat reduced by difficulties arising from computational problems, usually stemming from the inability of the computer to handle the extremely large or small numbers encountered at singular points or at the boundaries of the flow. Further complications emerge as the critical level approaches the interface, where the complex boundary conditions have to be satisfied. In this case, the determination of a sufficiently accurate eigenvalue requires an adequate number of grid points located between the critical level and the interface. Additionally, the numerical limit of integration on both sides of the interface must be at least as large as half the wavelength in question (Valenzuela, 1976; Kawai, 1979). One is faced with the computational choice between the need to increase the limit of integration as the wavenumber decreases with the requirement of maintaining the necessary number of...
gridpoints between the critical layer and the interface. Valenzuela suggested the use of a variable grid scheme to overcome this problem but did not do so himself. Kawai’s approach was to alter the total number of integration steps to assure that 10 to 40 steps remained below the critical level for all values of wavenumber investigated.

There are several methods available for the numerical solution of the OSE including expansions in orthogonal functions, finite difference techniques and initial value ('shooting') methods. The choice of method is dependent upon the information available on the specified flow and the boundary conditions, and the persistence of the investigator. All of the below referenced calculations were made for a single fluid, solid boundary problem.

Expansions in Orthogonal Functions

The method of expansion in orthogonal functions has been used to accurately analyze the stability of several types of flows, yielding higher-ordered eigenvalues in addition to the most unstable one. The numerical problem may be started with a trial eigenvalue not necessarily close to the real eigenvalue and suffer no ill effects in calculation time (Grosch and Salwen, 1968). Orszag (1971) has shown that the use of Chebyshev polynomials achieves infinite order error‡ and the method, although somewhat involved, is well suited for the analysis of flows with Reynolds numbers above 10,000.

The stability of plane Poiseuille flow was explored by Grosch and Salwen (1968), who used a truncated series of even and odd orthonormal expansion functions to describe the streamfunction of the disturbance. Highly accurate eigenvalues and eigenfunctions for steady as well as for time dependent, modulated flow were obtained by solving the resulting matrix eigenvalue problem. The results indicated that small dis-

‡The term infinite order error implies that the error after \( N \) terms (as \( N \rightarrow \infty \)) is of smaller order than any power of \( N^{-1} \).
turances were stabilized by a pressure distribution which was periodic in time ie; a modulated flow, and that a higher critical Reynolds number existed for modulated flow than for steady flow with the same $kR$ combination. Conversely, large amplitude disturbances in a modulated flow were found to be inherently more unstable. A secondary, perhaps more important, result of their analysis was the finding that there is only one unstable mode for a flow at any given wavenumber and Reynolds number combination.

**Finite Difference Methods**

*Finite difference methods* provide solutions for a set of differential equations on a continuous domain by representing the domain as a grid of points and the equations as pointwise algebraic approximations based on the discrete grid. The algebraic forms used to represent the equations can be written in a number of standard ways, based on the type and order of the equations to be solved. Grid point spacing over the problem domain usually determines how close the finite difference approximation is to the actual solution of the equations. The difference between the approximation and the actual solution is called truncation error, the order of which is minimized as the grid is refined in spacing. Of more serious concern is the problem of *numerical stability*; a method is deemed numerically stable if the total generated error does not grow beyond limit as the method progresses through the grid. A method may have the characteristic of small order truncation error, yet be highly unstable and therefore useless in most problems.

Once the form of the algebraic approximations is decided, they may then be solved using either *marching* or *relaxation* techniques. A marching procedure requires values to be prescribed for the dependent variables at certain initial grid points. The derivatives are then represented by differences in the known values of the dependent variables from one grid point to the next. These algebraic differencing forms are then assembled into a matrix format and solved using matrix inversion.
Marching procedures are not appropriate for equations whose solutions involve two different characteristic scales, as unbounded growth of errors may occur. Relaxation techniques also use differencing representation, but trial solutions are provided at each grid point rather than only at initial points. The true solutions are obtained by adjusting all the values on the grid until they are in agreement with the finite difference approximations as well as the boundary conditions at both initial and final boundaries. Relaxation techniques are known to occasionally produce spurious modes of the OSE, yet have the advantage of finding all of the associated, stable higher modes (Grosch and Salwen, 1968; Gary and Helgason, 1970) while avoiding the unbounded growth problem. Also, if a self-starting numerical method such as the QR algorithm (see Press et al. 1986, p. 342) is used to solve the matrix eigenvalue problem, a good first guess of the eigenvalue is not needed to start the calculation.

Several workers have used finite difference methods to solve the OSE with good results. In examining the temporal stability of Blasius flow over a flat plate, Osborne (1967) reduced truncation error by constructing his differencing equations from a transformed form of the OSE. He then developed a stable method to solve the resulting matrix eigenvalue problem, a scheme subsequently used by Jordinson (1970) to study the spatial stability of the Blasius flow, and by Valenzuela (1976) to examine the growth of initial wind wavelets. Valenzuela's numerical work first demonstrated the importance of linear instability at the interface in the generation and subsequent growth of gravity-capillary wavelets.

Initial Value Methods

A differential equation which has values of the dependent variable prescribed by the boundary conditions at some starting point but not at the final boundary may be solved using initial value methods. Any higher order differential equation may be rewritten as a coupled set of first order equations, whose derivatives may be represented by algebraic functions. The underlying premise of any initial value
method is the numerical integration of the original equation by iteratively "stepping" these algebraic functions from the starting point to the final boundary. The final values of the dependent variable obtained from an initial value method are, in most cases, not subject to end boundary conditions.

The shooting method is a modified initial value method useful for solving certain two-point boundary value problems. Asymptotically valid solutions for all dependent variables are chosen at one boundary to satisfy the conditions there, then the desired equation is directly integrated using a trial eigenvalue. An iterative method is then used to change the trial eigenvalue until the boundary conditions at the final boundary are all satisfied within an acceptable error, a process called 'shooting'. Difficulties in calculating the eigenfunctions may arise because of the necessity of purifying the solutions (Lakin and Grosch, 1982). Certain 'stiff' sets of equations or equations whose solutions consist of exponential terms which grow at different rates usually require mathematical reformulation to allow solution by the shooting method. In the short wind wave generation problem, the OSE on each side of the air-water interface along with the boundary conditions at \(z = \pm \infty\), (2.3.1) and at the interface (2.3.25 - 2.3.28), describe a linear, two-point boundary value problem in each medium. These types of problems may be numerically solved using relaxation techniques, discussed above, or some type of shooting method.

The OSE may be written in the form

\[
\phi'' - C_1 \phi''' - C_2 \phi'' - C_3 \phi' - C_4 \phi = 0 \quad (3.2.1)
\]

where the \(C_i\)'s are functions of \(z\). The total solution for \(\phi(z)\) is usually assumed to consist of two linearly independent, asymptotically valid solutions, a "viscid" and "inviscid" portion. The direct integration of (3.2.1) may encounter 'runaway' growth problems which are due to numerical errors arising from the much larger rate of increase of the viscid portion than the inviscid portion as the critical layer and the interface are approached. The inviscid portion of the total solution becomes
'contaminated' by these errors, thereby rendering the total solution for $\phi(z)$ useless. Kawai (1979) used a purification method which filtered out the viscid mode from the contaminated inviscid mode at every fifth integration step. Purification methods are inherently suspect because there is no guarantee that the contaminated solutions are really being purified. An attempt to carry out a purification scheme similar to Kawai's method in the present project failed.

Another approach designed to fix the runaway growth problem is to solve the OSE from both ends of the domain and match the solutions at the midpoint. Davey (1973) outlined an orthonormalization technique for the flat plate boundary layer which maintained the linearly independent behaviour of the individual portions and removed from the total solution the exponentially growing viscous mode at each integration step. This method is difficult to utilize in the case of more complex boundary conditions and, in any case, the calculated eigenvalue is subject to errors because of the effect of the procedure on the rapidly varying solutions.

In the case of the short wind wave generation problem of this study, the OSE is integrated in each medium from the far field to the interface, where the complex interface boundary conditions must be satisfied. The OSE is a 'stiff' equation and a method must be used which eliminates the associated runaway growth problems as well as lending itself readily to compliance with the four interface boundary conditions. The compound matrix method, based on the use of a Riccati transformation, does this very efficiently. The nature of this technique and its application to the wave generation problem is dealt with in the next chapter.
Chapter 4. METHOD OF COMPOUND MATRICES

A numerical hydrodynamic stability analysis of the shear flow model representing the flow of wind over water may be performed using the compound matrix method, an efficient technique which eliminates the problem of runaway growth encountered when the OSE is integrated by standard shooting methods. Using a trial complex eigenvalue, $c$, a real fixed wavenumber, $k$ and Reynolds number, $R$, a transformed version of the Orr-Sommerfeld equation is integrated from the far field to the interface in each medium, using a fourth-order Runge-Kutta integration method. The set of solutions found from each integration is then substituted into the interfacial boundary conditions to examine the accuracy of the trial eigenvalue. The error is used to adjust the eigenvalue and the procedure repeated until the desired level of accuracy is achieved. This chapter discusses the compound matrix method, its application to the solution of the OSE and other details of the numerical procedure.

4.1 The Compound Matrix Method

The total solution of the OSE, $\phi_T(z)$, where $z$ is positive upwards, is chosen to consist of two linearly independent solutions, an inviscid and a viscid portion:

$$\phi_T(z) = \phi_1(z) + \phi_2(z)$$  \hspace{1cm} (4.1.1)

where

$$\phi_1(z) = \exp(\pm \alpha z) \hspace{1cm} \phi_2(z) = \exp(\pm \beta z).$$  \hspace{1cm} (4.1.2)
The factors $\alpha$ and $\beta$ are specific for each medium and its associated velocity profile; they are valid asymptotically at large $z$, where $U$ and $U''$ vanish. The exponential function satisfies the OSE at large $z$, and the choice of sign ensures that the solutions die out in the far field, in accordance with the boundary condition at $z = \pm\infty$, (2.3.1).

At realistic Reynolds numbers, the viscid portion of the solution changes much more rapidly than the inviscid portion as the interface is approached. This two scale effect causes the OSE to be fairly 'stiff' near the interface and the viscid portion of the total solution rapidly overpowers the inviscid portion and can erroneously dominate the final result. This problem of 'stiffness' may be alleviated by rescaling the response of the dependent variable and ensuring that the total solution does not become 'polluted' by the viscid portion.

A commonly used technique to overcome the 'stiffness' problem is the Riccati method, which utilizes an eigenvalue solution matrix and its inverse to obtain valid solutions (see, for example, Davey, 1979). Inherent to the method, however, is the loss of numerical accuracy arising from singularities encountered on the path of integration. The compound matrix method is based on the Ricatti method, but eliminates these singularities by formulating the solutions as products of the elements of the eigenvalue solution matrix. Descriptions of the method by Davey (1979) and Ng and Reid (1979) treat forward integration of an equation such as the OSE with a range of integration from $z = 0$ to $z = +\infty$. In the present work, a backwards integration from the far field to the interface was required. The notation used in the description of the method below is taken from Ng and Reid (1979).

Letting $\phi = [\phi, \phi', \phi''', \phi''']^T$, where $T$ is the matrix transpose notation, the OSE can be written as

$$\dot{\phi} = C(z)\phi$$  \hspace{1cm} (4.1.3)

where
The coefficients $C(z)$ are

\[
C_1 = 0, \quad C_2 = 2k^2 + ikR(U-c) \\
C_3 = 0, \quad C_4 = -[k^4 + ikR[k^2(U-c) + U'']] \tag{4.1.5}
\]

and $U$ and $U''$ are specified by the shear flow velocity profile. In this manner, the OSE has been rewritten as a system of first-order equations.

$\phi_1$ and $\phi_2$ are taken to be linearly independent solutions of the OSE which satisfy the boundary conditions at $z = \pm \infty$, (2.3.1). The total solution and its derivatives may then be expressed as

\[
\begin{align*}
\phi &= A_1\phi_1 + A_2\phi_2 \\
\phi' &= A_1\phi_1' + A_2\phi_2' \\
\phi'' &= A_1\phi_1'' + A_2\phi_2'' \\
\phi''' &= A_1\phi_1''' + A_2\phi_2'''
\end{align*}
\tag{4.1.6}
\]

where $A_1$ and $A_2$ are constants to be determined in such a way as to satisfy the boundary conditions at the interface. From this system, a $4 \times 2$ solution matrix is constructed,

\[
\Phi(z) = \begin{bmatrix}
\phi_1 & \phi_2 \\
\phi_1' & \phi_2' \\
\phi_1'' & \phi_2''' \\
\phi_1''' & \phi_2'''
\end{bmatrix} \tag{4.1.7}
\]

Next, a new system is defined, $y = [y_1 \cdots y_6]$ , using the $2 \times 2$ minors of (4.1.7),

\[
\begin{align*}
y_1 &= \phi_1\phi_2' - \phi_1'\phi_2, \quad y_4 = \phi_1\phi_2'' - \phi_1''\phi_2' \\
y_2 &= \phi_1\phi_2'' - \phi_1''\phi_2, \quad y_5 = \phi_1\phi_2''' - \phi_1'''\phi_2' \\
y_3 &= \phi_1\phi_2''' - \phi_1'''\phi_2, \quad y_6 = \phi_1\phi_2'''' - \phi_1''''\phi_2'.
\end{align*}
\tag{4.1.8}
\]
The formulation of these $y$ terms as products of the viscid and inviscid portions "mixes" the two, allowing solutions retaining the linear independence of both portions to be found. Calculations show that this process eliminates the unbounded growth arising during the integration of the OSE.

If the $y$ system is differentiated and the OSE is used to eliminate any $\phi^{IV}$ terms, the resulting derivatives are linear (Davey, 1979). The $y_3'$ term is derived below as an example, using the OSE in the form $\phi^{IV} = C_1\phi'' + C_2\phi'' + C_3\phi' + C_4\phi$.

$$
y_3 = \phi_1\phi_2''' - \phi_1''\phi_2
$$

$$
y_3' = \phi_1\phi_2'' - \phi_1''\phi_2 + [\phi_1\phi_2''' - \phi_1''\phi_2']
$$

$$
y_3' = \phi_1\phi_2'' - \phi_1''\phi_2 + y_5
$$

Substitution of the OSE to eliminate the $\phi^{IV}$ terms yields after some manipulation:

$$
y_3' = C_1y_3 + C_2y_2 + C_3y_1 + y_5. \quad (4.1.10)
$$

The complete $y'$ system is:

$$
y_1' = y_2
$$

$$
y_2' = y_3 + y_4
$$

$$
y_3' = C_3y_1 + C_2y_2 + C_1y_3 + y_5
$$

$$
y_4' = y_5
$$

$$
y_5' = -C_4y_1 + C_2y_4 + C_1y_5 + y_6
$$

$$
y_6' = -C_4y_2 - C_2y_4 + C_1y_6. \quad (4.1.11)
$$

In matrix notation, this system is written as

$$
y' = B(z)y \quad (4.1.12)
$$

where

$$
B(z) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
C_3 & C_2 & C_1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-C_4 & 0 & 0 & C_2 & C_1 & 1 \\
0 & -C_4 & 0 & -C_3 & 0 & C_1
\end{bmatrix} \quad (4.1.13)
$$

The $C_n$ are those defined in (4.1.5). Ng and Reid (1979) call the $y$ system the second compound of $\phi$. 

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It is clear that (4.1.12) is a system of linear differential equations with non-constant coefficients equivalent to (4.1.3).

4.2 Initial Conditions

We wish to integrate the OSE in each medium from the far field towards the interface. The amplitude of the disturbance in the air, \( \phi_a(z) \), and its derivatives are asymptotically:

\[
\phi_1a(z) = [1, -\alpha_a, \alpha_a^2, -\alpha_a^3]^T \exp(-\alpha_a z) \\
\phi_2a(z) = [1, -\beta_a, \beta_a^2, -\beta_a^3]^T \exp(-\beta_a z).
\]

The air velocity profile characteristics at a large enough \( z = z_a \) are

\[
U_a(z_a) = 1 \quad U''_a(z_a) = 0.
\]

Substituting these profile characteristics into the OSE, the roots of the resulting algebraic expression are found to be:

\[
\alpha_a = k \quad \beta_a = \left[ k^2 + ikR_a(1-c) \right]^{1/2}.
\]

The initial conditions for the top-down integration are the \( y_a \) terms at \( z = z_a \):

\[
\begin{align*}
y_{1a} &= \phi_1a\phi_2a' - \phi_1a'\phi_2a = (\alpha_a - \beta_a) \exp[(\alpha_a + \beta_a) z] \\
y_{2a} &= \phi_1a\phi_2a'' - \phi_1a''\phi_2a = (\beta_a^2 - \alpha_a^2) \exp[(\alpha_a + \beta_a) z] \\
y_{3a} &= \phi_1a\phi_2a''' - \phi_1a'''\phi_2a = (\alpha_a^3 - \beta_a^3) \exp[(\alpha_a + \beta_a) z] \\
y_{4a} &= \phi_1a\phi_2a' - \phi_1a'\phi_2a = \alpha_a \beta_a (\alpha_a - ba) \exp[(\alpha_a + \beta_a) z] \\
y_{5a} &= \phi_1a\phi_2a'' - \phi_1a''\phi_2a = \alpha_a \beta_a (\beta_a^2 - \alpha_a^2) \exp[(\alpha_a + \beta_a) z] \\
y_{6a} &= \phi_1a\phi_2a''' - \phi_1a'''\phi_2a = (\alpha_a^2\beta_a^2)(\alpha_a - \beta_a) \exp[(\alpha_a + \beta_a) z].
\end{align*}
\]

Discarding the common \( \alpha_a - \beta_a \) term, the system of initial conditions is written in matrix format as

\[
y_a(z_a) = [1, -(\alpha_a + \beta_a), \alpha_a^2 + \alpha_a \beta_a + \beta_a^2, \alpha_a \beta_a, -\alpha_a \beta_a (\alpha_a + \beta_a), \alpha_a^2 \beta_a^2]^T \exp[(\alpha_a + \beta_a) z].
\]

In the water, with \( z \) negative downwards, \( \phi_w(z) \) and its derivatives are written as
\begin{equation}
\phi_{1w}(z) = [1, \alpha_w, \alpha_w^2, \alpha_w^3]^T \exp[(\alpha_w)z]
\end{equation}
\begin{equation}
\phi_{2w}(z) = [1, \beta_w, \beta_w^2, \beta_w^3]^T \exp[(\beta_w)z].
\end{equation}
(4.2.6)

The water velocity profile characteristics at a large depth \( z = z_w \),
\begin{equation}
U_w(z_w) = 0 \quad U''_w(z_w) = 0
\end{equation}
(4.2.7)

are substituted into the OSE, and the roots of the resulting algebraic expression are found to be:
\begin{equation}
\alpha_w = k \quad \beta_w = [k^2 - i \omega R_w]^{1/2}.
\end{equation}
(4.2.8)

The initial conditions for the bottom-up integration are the \( y_w \) terms at \( z = z_w \):
\begin{align*}
y_{1w} &= \phi_{1w} \phi_{2w}' - \phi_{1w}' \phi_{2w} = (\beta_w - \alpha_w) \exp[(\alpha_w + \beta_w) z] \\
y_{2w} &= \phi_{1w} \phi_{2w}'' - \phi_{1w}'' \phi_{2w} = (\beta_w^2 - \alpha_w^2) \exp[(\alpha_w + \beta_w) z] \\
y_{3w} &= \phi_{1w} \phi_{2w}''' - \phi_{1w}''' \phi_{2w} = (\beta_w^3 - \alpha_w^3) \exp[(\alpha_w + \beta_w) z] \\
y_{4w} &= \phi_{1w} \phi_{2w}'''' - \phi_{1w}'''' \phi_{2w} = \alpha_w \beta_w (\beta_w - \alpha_w) \exp[(\alpha_w + \beta_w) z] \\
y_{5w} &= \phi_{1w} \phi_{2w}''''' - \phi_{1w}''''' \phi_{2w} = \alpha_w \beta_w (\beta_w^2 - \alpha_w^2) \exp[(\alpha_w + \beta_w) z] \\
y_{6w} &= \phi_{1w} \phi_{2w}'''''' - \phi_{1w}'''''' \phi_{2w} = (\alpha_w^2 \beta_w^2) \exp[(\alpha_w + \beta_w) z].
\end{align*}
(4.2.9)

After removal of the common factor \((\beta_w - \alpha_w)\), this system of initial conditions may be written as
\begin{equation}
y(z_w) = [1, \alpha_w + \beta_w, \alpha_w^2 + \alpha_w \beta_w + \beta_w^2, \alpha_w \beta_w, \alpha_w \beta_w (\alpha_w + \beta_w), \alpha_w^2 \beta_w^2]^T \exp[(\alpha_w + \beta_w) z].
\end{equation}
(4.2.10)

4.3 Eigenvalue Relation

An eigenvalue relation for the short wave generation problem is sought subject to the initial conditions at \( z_a \), (4.2.4), and at \( z_w \), (4.2.9), and the interfacial boundary conditions, (2.3.25 - 2.3.28). A difficulty arises from the fact that the interfacial boundary conditions are written in terms of \( \phi(z) \), while the solutions obtained from the integration are written in terms of the \( y \) elements. Upon initial inspection, the problem appears almost impossibly complicated. It is certainly different from that of the boun-
boundary layer over a flat plate, where the wall boundary condition is simply \( \phi(0) = \phi'(0) = 0 \) and the eigenvalue relation in terms of the \( y \) elements is \( \gamma_1 = 0 \).

Using the form of (4.1.6), \( \phi_a(z) \) and \( \phi_w(z) \) may be written

\[
\phi_a(z) = K_1 \phi_{1a} + K_2 \phi_{2a} \\
\phi_w(z) = K_3 \phi_{1w} + K_4 \phi_{2w}
\]

These forms are substituted into the interface boundary conditions, which may then be written as

\[
[BC_a - BC_w] [K_n] = 0 ,
\]

where

\[
K_n = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix}
\]

and

\[
BC_a = \begin{bmatrix} \phi_{1a} & \phi_{2a} \\ A_a \phi_{1a} + \phi'_{1a} & A_a \phi_{2a} + \phi'_{2a} \\ m[B_a \phi_{1a} + \phi''_{1a}] & m[B_a \phi_{2a} + \phi''_{2a}] \\ r [D_a \phi_{1a} + E_a \phi'_{1a} + F_a \phi'''_{1a}] & r [D_a \phi_{2a} + E_a \phi'_{2a} + F_a \phi'''_{2a}] \end{bmatrix}
\]

and

\[
BC_w = \begin{bmatrix} \phi_{1w} & \phi_{2w} \\ [A_w \phi_{1w} + \phi'_{1w}] & [A_w \phi_{2w} + \phi'_{2w}] \\ [B_w \phi_{1w} + \phi''_{1w}] & [B_w \phi_{2w} + \phi''_{2w}] \\ [D_w \phi_{1w} + E_w \phi'_{1w} + F_w \phi'''_{1w}] & [D_w \phi_{2w} + E_w \phi'_{2w} + F_w \phi'''_{2w}] \end{bmatrix}
\]

The variables \( A, B, D, E \) and \( F \) are written as

\[
A = \frac{U'(0)}{c - U_0} \\
B = \frac{U''(0)}{c - U_0} + k^2 \\
D = U'(0) - \frac{kC_0^2}{c - U_0} \\
E = c - U_0 + 3ik \text{Re}^{-1} \\
F = (ik \text{Re})^{-1}
\]

As before, all variables have been scaled with respect to the previously identified scale.
variables, the subscripts \( a \) and \( w \) refer to air and water respectively and \( r = \frac{\rho_a}{\rho_w} \) and
\[
m = \frac{\mu_a}{\mu_w}.
\]

The determinant of the matrix \([BC_a - BC_w]\) must be zero for the existence of non-trivial solutions of \( \phi(z) \) which satisfy the interfacial boundary conditions. Expanding this determinant as the sums of the products of the second-order minors, i.e., a Laplace expansion, will produce an expression for the determinant in terms of the \( y \) system, which is the desired eigenvalue relation. For a trial eigenvalue, this determinant will generally not be zero. The non-zero residual value is used as an error term in an iterative method which adjusts the complex trial eigenvalue until the desired accuracy of \( 10^{-8} \) is achieved.

An example of a second-order minor reduced to a linear expression in terms of the \( y \) elements is shown below, in this case the term \([1_a]\) in (4.3.9):
\[
\begin{vmatrix}
\phi_{1a} & \phi_{2a} \\
A_a\phi_{1a} + \phi_{1a}' & A_a\phi_{2a} + \phi_{2a}'
\end{vmatrix}
= \phi_{1a}[A_a\phi_{2a} + \phi_{2a}'] - \phi_{2a}[A_a\phi_{1a} + \phi_{1a}']
\]
\[
(4.3.7)
\]
which, after simplification, yields
\[
\phi_{1a}\phi_{2a}' - \phi_{1a}'\phi_{2a} = y_{1a}.
\]
\[
(4.3.8)
\]
The remaining second-order minors are reduced in similar fashion and the determinant expanded to yield the eigenvalue relation:
\[
[1_a][6_w] - [2_a][5_w] + [3_a][4_w] + [4_a][3_w] - [5_a][2_w] + [6_a][1_w] = 0
\]
\[
(4.3.9)
\]
where
Once again, the subscripts \( a \) and \( w \) stand for air and water respectively.

### 4.4 Calculation of the Eigenfunctions

Once the eigenvalue has been obtained using the procedure described above, the complex eigenfunctions for air and water may be calculated. If the form of the eigenfunction is assumed to be that of (4.1.6), then the eigenfunction system for air is written as:

\[
\phi = K_1 \phi_1 + K_2 \phi_2 \\
\phi' = K_1 \phi_1' + K_2 \phi_2' \\
\phi'' = K_1 \phi_1'' + K_2 \phi_2'' \\
\phi''' = K_1 \phi_1''' + K_2 \phi_2'''.
\] (4.4.1)

A separate eigenfunction system exists for the water. Eliminating the constants \( K_1 \) and \( K_2 \) will yield four equations composed of \( y \) terms and expressions for \( \phi \) from (4.4.1),

\[
y_1 \phi'' - y_2 \phi' + y_4 \phi = 0 \\
y_1 \phi''' - y_3 \phi' + y_5 \phi = 0 \\
y_2 \phi''' - y_3 \phi'' + y_6 \phi = 0 \\
y_4 \phi''' - y_5 \phi'' + y_6 \phi' = 0.
\] (4.4.2a-d)

The above set of equations are in general form; there is a set for the air and one for the water when solving the two-media problem.

In principle, any of the four above equations may be integrated to determine the eigenfunction \( \phi \). Ng and Reid (1979) indicate that a backwards integration of (4.4.2a)
will do the best job of avoiding numerical growth problems.

4.5 Numerical Procedure

Once the mean velocity profile is defined by methods described earlier, the hydrodynamic stability calculations are carried out in the following manner. Initial conditions required for the solution of the initial value problem in the water are calculated according to (4.2.10), using a trial scaled complex eigenvalue, \( c \), a fixed scaled wavenumber, \( k \), and the appropriate Reynolds number. These \( y(z_w) \) terms are then used to initiate an integration of the transformed version of the OSE, (4.1.12), from \( z = -\infty \) up to \( z = 0 \), employing a fourth-order Runge-Kutta scheme. A similar integration of (4.1.12) in the air, from \( z = +\infty \) down to \( z = 0 \), is completed using the initial conditions for the air problem, (4.2.5), and the air velocity profile data. The values of \( y_w(0) \) and \( y_a(0) \) obtained from the integrations are then used in the eigenvalue relation according to (4.3.5) to calculate an error. The bottom-up and top-down integration procedure is repeated twice more with slightly different trial eigenvalues, generating three error values. Muller's method (Gerald and Wheatly 1984, p.35), an iterative method which finds the complex root of a quadratic fitted to the three pairs of trial eigenvalue-error points, is used to adjust the eigenvalue until the error is reduced to an accuracy of at least \( 10^{-8} \). The \( y \) terms for each medium after a successful eigenvalue search are shown in Figure 2. The eigenvalue, a scaled complex wave speed, was used to calculate a scaled disturbance growth rate, \( kc_i \) and a frequency, \( \omega = kc_r \), at a single wavenumber. The eigenvalues for a specific velocity profile were found over a range of wavenumbers, resulting in curves of phase speed and growth rate versus wavenumber, as shown in Figure 3. Stability curves were also generated for velocity profiles based on different values of \( u_ea \) and \( U_\infty \), as well as for different surface tension and viscosity values. These curves allowed analysis of the effects of surface tension, viscosity and variable flow parameters on the growth rate and phase speed of the wave resulting from the viscous instability mechanism.
After obtaining the \( y \) terms and the eigenvalue for a specific wavenumber and Reynolds number, the complex eigenfunctions for air and water were found from a backwards integration of (4.4.2a), as described above.

All calculations were done in complex double precision FORTRAN on a SUN 4/100 workstation. The domain of the problem was divided into 4000 grid points. The stability calculations usually converged to an eigenvalue within four to five iterations, with each iteration taking about two-and-a-half minutes. More iterations were required if the initial trial eigenvalue was far from the actual eigenvalue.
Figure 2. The $y$ terms for air and water.

Real and imaginary portions of the $y$ terms for a typical eigenvalue computation using the compound matrix method. The velocity profile is of the form shown in Figure 1 and was constructed using $U_\infty = 420$ cm/sec, $u_* = 13.6$ cm/sec and $T = 9.05$ secs. In this case, $L_w = L_a = .6$. The fixed, dimensional value of $k$ was 3.0 $cm^{-1}$; non-dimensional $k = 1.8$. The horizontal axis represents scaled distance above or below the interface, $z = 0$. The vertical axis represents the real and imaginary value of the $y$ term. The $a$ and $w$ subscripts refer to air and water, respectively. Figures 2(a) through 2(f) correspond to $y_1$ through $y_6$. Note that the disturbance is effectively confined to within $z/L_z < 0.5$ from the interface, or approximately .3 cm.
Figure 2(a)
Figure 2(b)
Figure 2(c)
Figure 2(d)

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Figure 2(e)

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Y6a [real and imaginary]

![Y6a graph](image)

Y6w [real and imaginary]

![Y6w graph](image)

Figure 2(f)

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Chapter 5. RESULTS

The phase speeds and growth rates obtained for interface conditions similar to those used by Kawai (1979) were very much as those found by Kawai through observation and stability calculations, a result which validates his numerical method. Going further, this study explored the effects of varying certain interfacial characteristics, such as surface tension and viscosity, on the behavior of these short waves. In addition to variable interface parameters, the effects of varying shear flow characteristics were explored, specifically, the free stream air velocity, $U_{\infty}$, and the air friction velocity, $u_{*a}$.

5.1 Effects of Variable Surface Tension

How do changes in surface tension affect the wavenumber and growth rate of the wave first appearing on the interface after a wind begins to blow? Kawai's (1979) plots of numerically generated phase speeds and growth rates suggest a maximum growth rate near the wavenumber of minimum phase speed. Figure 3 illustrates this possible relationship quite clearly for shear flow similar to Kawai's using a standard air-water surface tension value of 75.0 dynes/cm.

Phase speeds and growth rates were also obtained for substantially different surface tension values and are shown plotted against wavenumber in figures 4 and 5. The calculations were made for surface tension values of 75.0 dynes/cm, 45.0 dynes/cm and 1.0 dynes/cm. Values for other parameters were $\rho_a = 1.2 \times 10^{-3}$, $\rho_w = 1.0$, $v_a = 0.15$ and $v_w = 0.01$, all in cgs units.
Figure 3. Phase Speed and Growth Rate vs. Wavenumber

Phase speed ($C_r$) and growth rate ($kC_i$) plotted against wavenumber for different shear flow cases.

Fig. 3a $u_*= 13.6$ cm/sec, $U_* = 420$ cm/sec, $Re_a = 842$.

Fig. 3b $u_*= 17.0$ cm/sec, $U_* = 510$ cm/sec, $Re_a = 830$.

Fig. 3c $u_*= 21.4$ cm/sec, $U_* = 630$ cm/sec, $Re_a = 667$.

Fig. 3d $u_*= 24.8$ cm/sec, $U_* = 740$ cm/sec, $Re_a = 602$.

These plots are similar to Kawai's and show an apparent connection between minimum $c_r$ and maximum $kC_i$. 

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Figure 3

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Figure 4. Effects of Variable Surface Tension on Phase Speed

Phase speed against wavenumber for different values of surface tension. Horizontal axis is dimensional wavenumber, $k$, in $cm^{-1}$. Vertical axis is phase speed in $cm/sec$. Surface tension, $\sigma^*$, values are 1.0, 45.0 and 75.0 dynes/cm. Shear flow parameters are $u_{*w} = .47 \text{ cm/sec}$ and $U_\infty = 550 \text{ cm/sec}$. Minimum phase speed moves to higher wavenumber as surface tension is decreased.
Surface Tension Effects on Phase Speed

- **1.0 dynes/cm**
- **45.0 dynes/cm**
- **75.0 dynes/cm**

\[ u^*a = 13.6 \text{ cm/sec} \quad U_{\text{max}} = 550 \text{ cm/sec} \]

Figure 4.
Growth rate against wavenumber for different values of surface tension using linear hydrodynamic stability theory to calculate growth rate. Horizontal axis is dimensional wavenumber, $k$, in $cm^{-1}$. Vertical axis is growth rate in sec$^{-1}$. Surface tension, $\sigma^*$, values are 1.0, 45.0 and 75.0 dynes/cm. Shear flow parameters are $u_{*w} = .47$ cm/sec and $U_\infty = 550$ cm/sec. Maximum growth rate moves to higher wavenumber as surface tension is decreased, but far less than the minimum phase speed.
Surface Tension Effects on Growth Rate

Figure 5.
As surface tension values were decreased, the phase speed for a given wavenumber decreased as expected according to the theoretical dispersion relation. In addition, the wavenumber corresponding to minimum phase speed was shifted to higher values, or smaller wavelengths. The maximum growth rate wavenumber was also shifted to higher wavenumber with a decrease in surface tension, yet by a much smaller amount than was \( c_{\text{min}} \). Also of interest is the effect on the growth rate magnitude, which increased with a decrease in surface tension.

Discussion of Surface Tension Effects

It can be seen from the phase speed and growth rate curves that the conjectured correlation of minimum phase speed and maximum growth rate wavenumber for initially appearing short waves does not hold true for all surface tension values. In fact, it appears the near-correspondence is true only for specific values of surface tension. As surface tension is decreased, the wavenumber of the most unstable wave shifts to larger wavenumber, but not as much as does the wavenumber of the minimum phase speed wave. The near-correspondence of the two holds only for a range of surface tension values and must be judged fortuitous.

The increase of the wave growth rate with a decrease in surface tension is reasonably attributed to the decrease in the amount of work required to perturb a free surface. Lower surface tension decreases the stiffness of the interface, so that for equal forcing, higher growth rates occur with lower surface tension.

This result may seem contrary to the well-known damping effect of surface films on wavelets. However, the presence of a surface film changes the characteristics of an interface in a much more complex manner than a uniform lowering of the surface tension. As alluded to by Lighthill (1978; p.237) and explained more fully by Alpers and Huhnerfuss (1989), maximum damping occurs when the wavelets and Marangoni waves are in resonance. The Marangoni waves, induced by surface tension gradients.
and highly damped by viscous dissipation, exist only if there is a viscoelastic film present at the interface. The presence of a surface tension gradient on such a film at the air-water interface modifies the tangential stress boundary condition, (2.3.14), so the surface tension gradient becomes part of the stress balance. This causes very strong velocity gradients to be present in a thin layer of water adjacent to the interface, which in turn causes strong viscous dissipation and hence, enhances the damping of small waves.

5.2 Effects of Variable Kinematic Viscosity

The question of how viscosity influences the behavior of short initial wavelets was examined by obtaining the wave phase speeds and growth rates for several different values of the kinematic viscosity of air, \( \nu_a \), and of water, \( \nu_w \). Density was assumed constant in both air and water.

For the short waves examined in this study, there was little effect on the behavior of the free surface due to a change in the kinematic viscosity of air. Stability calculations were made for \( \nu_a \) values of \( .12 \text{ cm}^2/\text{sec} \), \( .15 \text{ cm}^2/\text{sec} \) and \( .18 \text{ cm}^2/\text{sec} \), corresponding to air temperatures of \( 2^\circ \text{C} \), \( 20^\circ \text{C} \) and \( 56^\circ \text{C} \). The kinematic viscosity of water was held at a constant value of \( .01 \text{ cm}^2/\text{sec} \). The results of one such calculation are shown in Figure 6, highlighting the small effects of varying the viscosity of air on the behavior of the disturbance.

In contrast, the effects of varying the viscosity of water on the disturbance are quite substantial. In addition to a standard value of \( .01 \text{ cm}^2/\text{sec} \), corresponding to a water temperature of \( 20^\circ \text{C} \), calculations were also made using \( 1.7 \times 10^{-2} \text{ cm}^2/\text{sec} \) and \( 5.0 \times 10^{-3} \text{ cm}^2/\text{sec} \), corresponding to temperatures of \( 2^\circ \text{C} \) and \( 56^\circ \text{C} \). The results of the calculations are shown in Figure 7. Decreasing the value of the kinematic viscosity of water caused an increase in the phase speed of the wave and shifted \( k_{\text{min}} \) to a slightly lower wavenumber. Growth rates were also increased with a decrease in \( \nu_w \) and the
Figure 6. Effects of Variable Kinematic Viscosity of Air

Phase speed against wavenumber [Fig. 6(a)] and growth rate against wavenumber [Fig. 6(b)] for different values of kinematic viscosity of air, \( \nu_a \). Horizontal axis is dimensional wavenumber, \( k \), in cm\(^{-1} \). Vertical axis is phase speed in cm/sec. Shear flow parameters are \( u_* = .47 \text{ cm/sec} \) and \( U_\infty = 420 \text{ cm/sec} \). Values of \( \nu_a \), are .12 and .15 cm\(^2\)/sec.

Fig. 6(a) -- Vertical axis is phase speed in cm/sec.

Fig. 6(b) -- Vertical axis is growth rate in sec\(^{-1} \). Note the small change in growth rate.
Variable Air Viscosity Effects on Phase Speed

\[ \omega = 0.15 \text{ cm}^2/\text{sec}, \quad \omega = 0.12 \text{ cm}^2/\text{sec} \]

\[ u^*a = 13.6 \text{ cm/sec}, \quad U_{\text{max}} = 420 \text{ cm/sec} \]

Figure 6(a).
Variable Air Viscosity Effects on Growth Rate

Growth Rate (sec\(^{-1}\))

$\kappa \ (cm^{-1})$

$u^*a = 13.6 \ cm/sec$  $U_{max} = 420 \ cm/sec$

Figure 6(b).

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Figure 7. Effects of Variable Kinematic Viscosity of Water

Phase speed against wavenumber [Fig. 7(a)] and growth rate against wavenumber [Fig. 7(b)] for different values of kinematic viscosity of water, $v_w$. Horizontal axis is dimensional wavenumber, $k$, in $cm^{-1}$. Vertical axis is phase speed in $cm/sec$. Shear flow parameters are $u_\star = .47 cm/sec$ and $U_\infty = 420 cm/sec$. Values of $v_w$ are .005, .10 and .017 $cm^2/sec$. Surface velocities, $U_0$, corresponding to the respective values of $v_w$ are 10.6, 7.49 and 5.75 $cm/sec$.

Fig. 7(a) -- Vertical axis is phase speed in $cm/sec$.

Fig. 7(b) -- Vertical axis is growth rate in $sec^{-1}$.
Variable Water Viscosity Effects on Phase Speed

- **.017 cm \(^2\)/sec**
- **.01 cm \(^2\)/sec**
- **.005 cm \(^2\)/sec**

Figure 7(a).

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Variable Water Viscosity Effects on Growth Rate

Figure 7(b).
wavenumber of the most unstable wave was shifted to a higher wavenumber. Thus, the waves which first appear at an air-water interface of low viscosity are of shorter wavelength than those appearing with high viscosity.

Discussion of Viscous Effects

Viscosity is a proportionality factor describing the connection between the applied shear stress, \( \tau \), and the rate of fluid deformation, \( \frac{du}{dy} \). A fluid's viscosity decreases with increasing temperature due to the reduction in cohesive forces arising from an increase in internal fluid energy. The Orr-Sommerfeld equation, (2.2.15-16), which describes the behavior of the amplitude of the disturbance, is essentially a balance between viscid and inertial effects; a change in viscosity will affect the solution in a fundamental way. The eigenvalue problem is affected by viscosity through the boundary conditions as well as the OSE, further complicating the problem.

A simple kinematic effect contained in the OSE is that a wavelet travels at its own celerity relative to the fluid, plus an effective surface current velocity. In the problem as formulated above, the surface current \( U_0 \) increases with decreasing \( \nu_w \). Therefore, the increase in phase speed seen in Figure 7(a) corresponding to a decrease in kinematic viscosity may simply be attributed to the increased surface current speed, \( U_0 \), calculated from (2.1.16).

The reasons for the effects of variable water viscosity on wave growth rate are equally plausible. In general, viscosity tends to dissipate energy and diffuse momentum. The dissipation of energy tends to stabilize a flow and reduce wavelet growth rate. In the short wave generation problem, viscous forces are important primarily in the very narrow boundary layer on the water side of the interface, the depth of which is the length scale used in this study, (2.1.19). Viscous forces are also important in a thin region around the critical level, but due to the critical layer's proximity to the interface in the short wave case, the critical layer and the interface layer are merged.
These viscous forces tend to dissipate the energy of a disturbance. If the dissipation of energy is less than the amount of energy input to the disturbance, whatever the source, growth may occur. A decrease in $\nu_w$ decreases the depth of the boundary layer in the water according to (2.1.19) and decreases the dissipation of energy; accordingly, there are higher growth rates with smaller values of viscosity.

The wavenumber corresponding to the maximum growth rate is shifted to smaller values, or to longer wavelengths with increasing $\nu_w$. Energy loss for deep water waves due to viscous dissipation is proportional to $\nu k^2$ (Lighthill 1978, p.235). Assuming constant energy supply with varying $k$, an increase in $\nu_w$ causes a corresponding decrease in the wavenumber of the most unstable wave.

In the analysis of Mei (1989, p. 407), viscous damping by air is shown to be much smaller than that caused by water. The rate of dissipation in the air boundary layer is $O\left[\frac{\mu_a U_{orb}^2}{\delta_a}\right]$, where $U_{orb}$ is orbital velocity and $\delta_a$ is the thickness of the boundary layer in the air. The corresponding rate of dissipation in the water is $O\left[\frac{\nu_w U_{orb}^2}{\lambda}\right]$, where $\lambda$ is wavelength. In the range of wavelengths and boundary layer thicknesses used in this study, the ratio of dissipation in air to that in water is effectively $O\left[\frac{\mu_a}{\nu_w}\right]$, or $O(10^{-2})$. Therefore, a change in $\nu_a$ would not be expected to substantially affect the behavior of the short initial wavelets, a conclusion which is confirmed by the results of the calculations shown in Figure 6.

### 5.3 Effects of Variable Shear Flow Parameters

The behavior of the short waves was also examined by varying the free stream air flow velocity and the interfacial shear stress. Altering either parameter causes a change in wind profile curvature, with a change in $U_\infty$ having the largest effect. An increase in $U_\infty$ at constant surface stress and water-side development time $T$ increases the
curvature, moves the critical level, $z_c$, closer to the interface and increases the Reynolds number of the flow. Surface current speed, $U_0$, remains constant. Alternatively, increasing $u_\infty^*$ increases the curvature and lowers the height of $z_c$, but only slightly; Reynolds number is decreased. Surface tension and viscosity were held constant for these calculations.

Figure 8 shows phase speed plotted against scaled wavenumber for different $U_\infty$ values with a constant $u_\infty^*$. Phase speeds are portrayed as departures from the celerity, $c_c$, of a free irrotational wave, scaled by the surface velocity:

$$c_\Delta = \frac{(c_r^*-c_c)}{U_0} \quad (5.1)$$

where $c_r^*$ is calculated dimensional phase speed, $c_c$ is the celerity of a wave at a given $k$ according to classical theory, and $U_0$ is the surface speed. The phase speeds were unaffected by a change in $U_\infty$. Growth rates, scaled by water boundary layer development time $T$, exhibit little change with increasing $U_\infty$, as shown in Figure 9. The wavenumber corresponding to maximum growth rate shifts to smaller wavenumbers.

Figure 10 shows scaled phase speed vs. scaled wavenumber for profiles with different $u_\infty^*$ values and constant $U_\infty$. Once again, phase speeds are anomalies, defined by (5.1). The scaled phase speeds are quite similar, diverging slightly as wavenumber decreases. Figure 11 shows scaled growth rate vs. scaled wavenumber curves for the flow data used in Figure 10. The curves are remarkably similar; growth rate magnitudes are almost identical when scaled by $T$, highlighting the importance of water-side parameters to the behavior of the disturbance. Maximum growth rate wavenumber shifts to smaller values, or longer wavelengths, with an increase in $u_\infty^*$.
Figure 8. Effects of Variable Air Profile Curvature on Phase Speed

Scaled phase speed against scaled wavenumber for different shear flow cases. Velocity profile curvature is changed by varying $U_\infty$; $u_* a$, $T$ and $U_0$ are held constant. Horizontal axis is wavenumber scaled by multiplication with the length scale, $L_w$. Vertical axis is scaled phase speed.

Figure 8(a) -- Values of $U_\infty$ are 420, 550 and 650 cm/sec. The fixed value of $u_* a$ is 13.6 cm/sec. Length scale, $L_w = .3008$ cm.

Figure 8(b) -- Values of $U_\infty$ are 510, 550 and 650 cm/sec. The value of $u_* a$ is 17.0 cm/sec. $L_w = .2443$ cm.
Variable Umax Effects on Phase Speed

Scaled Phase Speed \( \frac{\sigma - \sigma_0}{U_0} \)

Scaled Wavenumber \( kL_s \)

\( u^*a = 13.6 \)

Figure 8a.

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Variable $U_{\text{max}}$ Effects on Phase Speed

![Graph showing the effect of $U_{\text{max}}$ on phase speed](image)

Figure 8b.

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Figure 9. Effects of Variable Profile Curvature on Growth Rate

Scaled growth rate against scaled wavenumber for different shear flow cases. Air profile curvature is changed by varying $U_\infty$; $u_{*a}$ is held constant. Horizontal axis is scaled wavenumber, $kL_w$. Vertical axis is growth rate, scaled by waterside development time. Maximum growth rates vary less than 10% even with fairly large changes in $U_\infty$.

Fig. 9a. $u_{*a} = 13.6$ cm/sec. $U_\infty$ values are 420, 550 and 650 cm/sec.

Fig. 9b. $u_{*a} = 17.0$ cm/sec. $U_\infty$ values are 510, 550 and 650 cm/sec.
Variable Umax Effects on Growth Rate

Figure 9a.
Variable Umax Effects on Growth Rate

Figure 9b.
Figure 10. Effects of Variable Friction Velocity on Phase Speed

Scaled phase speed against scaled wavenumber for different $u_{*a}$ values. The $u_{*a}$ and corresponding $T$ values are: 17.0 cm/sec and 5.97 secs.; 21.4 cm/sec and 2.52 secs.; 24.8 cm/sec and 1.49 secs. $U_w$ is held constant at 650 cm/sec. Horizontal axis is scaled wavenumber, $kL_w$, where $L_w = 2\sqrt{V_wT}$. Vertical axis is the difference between calculated phase speed and the phase speed of a classical free wave at a specific wavenumber, scaled by $U_0$. 

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Scaled Phase Speed vs. Wavenumber -- Constant U max

Figure 10.

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Figure 11. Effects of Variable Friction Velocity on Growth Rate

Scaled growth rate against scaled wavenumber for different $u_\ast a$ values. The $u_\ast a$ and corresponding $T$ values are: 17.0 cm/sec and 5.97 secs.; 21.4 cm/sec and 2.52 secs.; 24.8 cm/sec and 1.49 secs. $U_\infty$ is held constant at 650 cm/sec. Horizontal axis is scaled wavenumber, $kL_w$. Vertical axis is scaled growth rate $kC_f/T$. Large changes in friction velocity cause small changes in scaled growth rate, indicating a dependence on water side parameters.
Scaled Growth Rate vs. Wavenumber -- Constant $U_{\text{max}}$

Figure 11.
Discussion of Variable Shear Flow Effects

Although the model of the air flow used in this study is somewhat unrealistic, it is clear from the results of the calculations that the effects on the disturbance behavior from changes in the air flow profile are minimal. Varying $U_\infty$ and therefore, varying the profile curvature, has little effect on either phase speed or growth rate when $u_*$ is held constant. Moreover, when $U_\infty$ is held constant and $u_*$ is varied, the growth rate curves, scaled by development time, may coincide and may be made so with the proper choice of water-side parameters.

Furthermore, the results of Kawai are reproduced by this study; it is quite remarkable that such similar results were obtained from such different air velocity profiles and numerical methods. It is clear that the choice of the wind profile does not affect the behavior of the disturbance to any great degree.

The implications of these results are profound. The influence of the air flow on the behavior of the disturbance is shown to be much less than that of the water-side parameters, especially boundary layer development time. This may be attributed to the importance of the vorticity concentrated in the thin boundary layer below the interface. As the length scale decreases with decreasing boundary layer development time, growth rates increase because the stronger velocity gradients imply faster energy transfer to the disturbance. $\overline{u_w} \frac{\partial U}{\partial z}$ is energy supply to the growing disturbance, a classical result of instability theory.

The instability is generally concentrated very close to the interface itself, evident in the plots of the $y$ elements in Figure 2. The $y$ elements are products of the viscous and inviscid solutions, and describe the basic behavior of the disturbance with height. They reside in a vertical scale much smaller than the scale of the wave motion, $k^{-1}$.
5.4 Future Extensions

There are several possible extensions to this study.

First, the description of the wind profile in the present study is that of flow in a developing laminar boundary layer and is simplistic when compared to the actual flow above the air-sea interface. The use of a turbulent profile, such as that produced by the Cebeci-Smith method (Cebeci and Smith 1974, p.329), would be more realistic physically. Moreover, it would provide a comparison for the present result indicating a much greater influence on the initial disturbance by the water-side parameters than by the air flow.

Secondly, the interface boundary conditions may be re-formulated to include the effects of surface tension gradients. This would allow the damping effect due to the presence of longitudinal Marangoni waves to be examined and compared with the present results regarding surface tension effects.

Finally, a portrayal of the disturbance eigenfunctions in both media for a range of shear flow parameters would show a clear, physical picture of the aforementioned dependence of the disturbance on water-side parameters rather than air profile characteristics. Calculating curves of marginal stability for a wide range of Reynolds numbers may also validate this novel conjecture.

5.5 Summary

The important conclusions drawn from this study are:

1) The apparent correlation between minimum phase speed and maximum growth rate wavenumber holds only for specific surface tension values. As surface tension decreases, the wavenumber of the most unstable wave increases and growth rates increase for all wavenumbers.

2) Varying the viscosity of water has a large effect on the disturbance behavior.
Phase speeds and growth rates increase with a decrease in \( v_w \). The wavelength of the most unstable wave also decreases with reduced \( v_w \).

3) A change in the kinematic viscosity of air has little effect on the behavior of the initial wavelets.

4) For the range of wavenumbers studied, the effects of air profile changes on wavelet behavior are negligible. Wavelet behavior appears dependent upon water-side parameters, such as boundary layer development time or the depth of the developing boundary layer. The form of the laminar wind profile, other than meeting smoothness constraints, does not appear important to the results.
REFERENCES


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Autobiographical Statement

Glen Harvey Wheless was born February 25, 1957 in Brussels, Belgium to Harvey Howell and Lorraine Gersky Wheless.

Mr. Wheless graduated from the U.S. Naval Academy in June, 1978 and was commissioned an Ensign in the United States Navy. He received his Naval Flight Officer wings in August, 1979 and was assigned to Fighter Squadron Fourteen as an F-14 Radar Intercept Officer, operating from the USS John F. Kennedy. He is now medically retired from active duty.

Mr. Wheless received his Masters of Science in Oceanography in August, 1986 from Old Dominion University, Norfolk VA. In 1987, Mr. Wheless was named a Commonwealth Fellow by the Council of Higher Education of Virginia. He was the ODU Oceanography Department’s Outstanding Doctoral Student for 1989. Mr. Wheless is the recipient of a Postdoctoral Fellowship in Ocean Modeling, sponsored by the University Corporation for Atmospheric Research.

Mr. Wheless is a member of the Honor Society of Phi Kappa Phi. He is also a member of the American Geophysical Union and the American Meteorological Society.

In 1978, Mr. Wheless married the former Martha Ann Lingua, of Mountain Lakes, N.J. They have one daughter, Elizabeth Lorraine, born in 1987.