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LIMIT THEOREMS IN THE AREA OF LARGE DEVIATIONS
FOR SOME DEPENDENT RANDOM VARIABLES

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A magnetic body can be considered to consist of \( n \) sites, where \( n \) is large. The magnetic spins at these \( n \) sites, whose sum is the total magnetization present in the body, can be modelled by a triangular array of random variables \((X_1^{(n)}, \ldots, X_{n}^{(n)})\). Standard theory of physics would dictate that the joint distribution of the spins can be modelled by \( dQ_n(x) = z_n^{-1} \exp(-H_n(x)) \prod dP(x_j) \), where \( x = (x_1, \ldots, x_n) \in \mathcal{S}^n \), where \( H_n \) is the Hamiltonian, \( z_n \) is a normalizing constant and \( P \) is a probability measure on \( \mathcal{S} \). For certain forms of the Hamiltonian \( H_n \), Ellis and Newman (1978b) showed that under appropriate conditions on \( P \), there exists an integer \( r \geq 1 \) such that \( S_n/n^{-1/2r} \) converges in distribution to a random variable. This limiting random variable is Gaussian if \( r = 1 \) and non-Gaussian if \( r \geq 2 \). In this article, utilizing the large deviation local limit theorems for arbitrary sequences of random variables of Chaganty and Sethuraman (1985), we obtain similar limit theorems for a wider class of Hamiltonians \( H_n \), which are functions of moment generating functions of suitable random variables. We also present a number of examples to illustrate our theorems.

1. Introduction. In this article we obtain limit theorems for some dependent random variables which are used to describe the distribution of magnetic spins present in a ferromagnetic crystal. A ferromagnetic crystal consists of a large number of sites. The amount of magnetic spin present at site \( i \) will be denoted by \( X_i^{(n)} \), \( i = 1, \ldots, n \), where \( n \) is a positive integer. The magnetic spin present at any site interacts with the magnetic spins at its neighboring sites and hence gives rise to some dependency among the random variables \( X_i^{(n)} \)'s. In the Ising model, the joint distribution, at a fixed temperature \( T > 0 \), of the spin random variables \((X_1^{(n)}, \ldots, X_n^{(n)})\), is given by

\[
dQ_n(x) = z_n^{-1} \exp \left( - \frac{H_n(x)}{T} \right) \prod dP(x_j),
\]

where \( x = (x_1, \ldots, x_n) \in \mathcal{S}^n \) and \( P \) is a probability measure on \( \mathcal{S} \). The function \( H_n(x) \) is known as the Hamiltonian and it represents the energy of the crystal at the configuration \( x \), and \( z_n \) is a normalizing constant which is also known as the partition function. In many cases, an explicit evaluation of \( z_n \) is very difficult and physicists usually try to approximate the limiting free energy per site \( \xi(T) \), at the temperature \( T \), defined as follows:

\[
\xi(T) = - \lim_{n \to \infty} \left( \frac{\log(z_n)}{n} \right).
\]

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628
For some particular types of Hamiltonians, it has been shown by physicists that there exists a temperature level \( T_c \) such that the function \( \xi(T) \) is non-differentiable at \( T = T_c \) [see Kac (1966)]. A phase transition is said to occur at the critical temperature \( T_c \). As pointed out by Ellis and Newman (1978b), the existence of the critical temperature can be demonstrated in yet another way. We may be able to show that for \( T > T_c \), there is a weak dependence among the random variables \( (X_1^{(n)}, \ldots, X_n^{(n)}) \) and a standard central limit theorem is valid for \( S_n/n^{1/2} \) and that for \( T = T_c \), there exists a \( \delta \in (1, 2) \) such that \( S_n/n^{\delta/2} \) converges to a non-Gaussian limit and for \( T < T_c \), due to the strong dependence among the \( X_i^{(n)} \)'s, the random variables tend to cluster in several ergodic components. This shows a marked discontinuity in the asymptotic distribution of \( S_n \) as the temperature \( T \) is allowed to vary and represents our approach to demonstrating a phase transition.

In Section 2, we consider a special case for the Hamiltonian, \( H_n \), by setting it to be equal to \( -(1/2n)\sum x_i x_j \). This is known as the Curie–Weiss model. The asymptotic distribution of \( S_n \) for this model when \( P \) (which appears in Theorem 2.1) is symmetric Bernoulli is obtained by Simon and Griffiths (1973). In a two-paper series, Ellis and Newman (1978a, 1978b) extended Theorem 2.1 of Simon and Griffiths to the class of probability measures \( L \), defined in (2.2) [see also Ellis and Rosen (1979)]. We state their extension precisely in Theorem 2.6. Recently Jeon (1979) in his Ph.D. dissertation gave a simpler and statistically motivated proof of Theorem 2.6. The goal of this article is to extend Theorem 2.6 for a larger class of Hamiltonians \( H_n \) and probability measures \( P \). Our main result, Theorem 3.7 is stated in Section 3. The proof of Theorem 3.7 relies on a recent large deviation local limit theorem of Chaganty and Sethuraman (1985), which is restated in Section 3 as Theorem 3.4.

Let \( T_n, n \geq 1 \), be an arbitrary sequence of random variables with analytic moment generating function \( \phi_n(z) \). We assume that \( T_n \) satisfies the conditions of Theorem 3.4. In our generalized model the Hamiltonian \( H_n(x) \) is taken to be equal to \( -\log[\phi_n(s_n/n)] \), where \( s_n = x_1 + \cdots + x_n \). Thus, the joint distribution of the spin random variables \( (X_1^{(n)}, \ldots, X_n^{(n)}) \) is given by

\[
dQ_n(x) = n^{-1} \phi_n\left(\frac{s_n}{n}\right) \prod dP(x_j),
\]

where \( P \) is an arbitrary probability measure. Let \( S_n = X_1^{(n)} + \cdots + X_n^{(n)} \). Under appropriate conditions on the probability measure \( P \) we show in Theorems 3.7 and 3.18, there exists an integer \( r \geq 1 \) such that \( S_n/n^{1-1/2r} \) converges in distribution to a random variable \( Y_r^* \), which has a nonnormal distribution when \( r \geq 2 \) and normal distribution when \( r = 1 \). The technique of our proof is to introduce a new random variable \( W_n \), conditional on which, \( X_1^{(n)}, \ldots, X_n^{(n)} \) become i.i.d. It is easy to obtain the limiting distribution of \( W_n \) and the conditional asymptotic distribution of \( S_n/n^{1-1/2r} \). Using the results of Sethuraman (1961) we deduce the asymptotic distribution of \( S_n/n^{1-1/2r} \).

We now briefly give our reasons for calling these theorems on the asymptotic distribution of \( S_n \) under \( Q_n \), defined in (1.3), as limit theorems in the area of large deviations. A standard technique to obtain the asymptotic distribution of
$S_n$ under $Q_n$ is to first obtain the asymptotic distribution of $S_n$ under $P_n$, where

$$dP_n(x) = \prod dP(x_j)$$

and then to use contiguity arguments, as in Le Cam (1960). This technique breaks down completely if $r \geq 2$. For the various models considered in physics which are described in greater detail in Sections 2 and 3,

$$|L_n(x)| = \left| \log \frac{dQ_n(x)}{dP_n(x)} \right| = \left| \frac{H_n(x)}{T} + \log z_n \right|$$

converges to $\infty$ in probability under $P_n$ and thus contiguity arguments are not applicable here. Under $P_n$, $S_n/n^{1/2}$ has a limiting normal distribution, and $|L_n(x)|$ is small in the area of ordinary deviations of $S_n$, that is, when $S_n/n^{1/2}$ is finite, while $|L_n(x)|$ is large otherwise. Thus from the point of view of $P_n$, we are looking for the asymptotic distribution of $S_n$, which is substantially different from 1 in the area of large deviations of $S_n$. This view point helps in a statistically motivated proof of the asymptotic distribution under $Q_n$ and describes the background behind the title of this article. One should also note that the normalizing factor on $S_n$ in its asymptotic distribution under $Q_n$ is different from the corresponding factor under $P_n$.

2. A brief summary of the Curie–Weiss model and its extensions. In a ferromagnetic system with only pair interactions and with no external magnetic field, the Hamiltonian $H_n$, may be taken to be $-\frac{1}{2} \sum a_{ij} x_i x_j$, where $a_{ij} \geq 0$. The Curie–Weiss model assumes that $a_{ij} = 1/n$ for all $i$ and $j$, that is to say that each spin interacts equally with every other spin with strength $1/n$, and takes $P$ to be symmetric Bernoulli, i.e., $P(-1) = P(1) = \frac{1}{2}$. Replacing $P$ by $P_\gamma(x) = P(x T^{1/2})$, we get

$$dQ_n(x) = \frac{1}{n^{1/2}} \exp \left( \frac{s_n^2}{2n} \right) \prod dP(x_j),$$

where $s_n = x_1 + \cdots + x_n$. This model has the advantage that the limiting free energy per site can be solved exactly. The existence of the critical temperature and phase transition for this model was demonstrated by Kac (1968). The asymptotic distribution for the total magnetism, $S_n$, for this model was obtained by Simon and Griffiths (1973). This is contained in Theorem 2.1.

**Theorem 2.1 (Simon and Griffiths).** Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables whose joint distribution is given by (2.1) and $P$ be symmetric Bernoulli. Then $S_n/n^{3/4}$ converges in distribution to a random variable whose density function is proportional to $\exp(-y^4/12)$.

Theorem 2.1 was extended to the class of probability measures $L$, which is defined below, by Ellis and Newman (1978b).
DEFINITION 2.2. Let $L$ be the class of probability measures $P$ on $\mathcal{R}$ such that

$$
\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2}\right) dP(x) < \infty.
$$

Fix $P \in L$. It can be shown that condition (2.2) guarantees the existence of the moment generating function (m.g.f.), $m(u)$, of $P$. Let $h(u) = \log m(u)$ be the cumulant generating function (c.g.f.) of $P$. The function $G(u) = u^2/2 - h(u)$ plays an important role in Theorem 2.6 below.

DEFINITION 2.3. A real number $m$ is said to be a global minimum for $G$ if $G(u) \geq G(m)$ for all $u$.

DEFINITION 2.4. A global minimum $m$ for $G$ is said to be of type $r$ if

$$
G(u + m) - G(m) = \frac{c_{2r} u^{2r}}{(2r)!} + O(|u|^{2r+1}), \quad \text{as } u \to 0,
$$

where $c_{2r} = G^{(2r)}(m)$ is strictly positive.

DEFINITION 2.5. A probability measure $P$ is said to be pure if $G$ has a unique global minimum.

Let $Y_r$, $r \geq 1$, be a sequence of random variables with density function $p_r(y)$, where

$$
p_r(y) = \begin{cases}
      d_r \exp\left[-c_{2r} y^{2r}/(2r)!\right], & \text{if } r \geq 2, \\
      N(0, (1 - c_2)/c_2), & \text{if } r = 1,
\end{cases}
$$

and where $d_r$ is the appropriate normalizing constant. With these definitions and notation we are now in a position to state the generalization of Theorem 2.1, due to Ellis and Newman (1978b).

THEOREM 2.6 (Ellis and Newman). Let $P \in L$. Let $P$ be pure, that is, let $m$ be the unique global minimum of type $r$ for $G$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by (2.1). Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. Then

$$
\frac{S_n - nm}{n^{1-1/2r}} \overset{d}{\to} Y_r,
$$

where $Y_r$ is a random variable with density function given by (2.4).

It is easily verified that the symmetric Bernoulli measure is pure and belongs to the class $L$ with the corresponding value of $r$ equal to 2. Thus Theorem 2.6 contains Theorem 2.1.
Note that the moment generating function \( M(s) \) of the standard normal is given by \( \exp(s^2/2) \). Thus we can write (2.1) as

\[
(2.6) \quad dQ_n(x) = z_n^{-1} \left[ M \left( \frac{S_n}{n} \right) \right]^n \prod dP(x_j).
\]

One might ask the question whether it is possible to obtain limit theorems of the type (2.5) when \( [M(s)]^n \) is replaced by the m.g.f. \( \phi_n(s) \) of a random variable \( T_n \), satisfying some conditions. We answer this question in the affirmative in the next section.

3. Further extensions of the Curie–Weiss model. In this section we propose to extend Theorem 2.6 by enlarging the class of Hamiltonians as well as the class of probability measures \( L \). The large deviation local limit theorem for an arbitrary sequence \( T_n, n \geq 1 \), of random variables of Chaganty and Sethuraman (1985) (stated below) plays a key role in this extension. The Hamiltonians, \( H_n \), in our generalized model (3.13) are taken to be the cumulant generating functions of these random variables \( T_n \).

Let \( \{ T_n, n \geq 1 \} \) be a sequence of nonlattice valued random variables with m.g.f.'s \( \phi_n(s), n \geq 1 \), finite for real values of \( s \) such that \( |s| < c \leq \infty \). Assume that \( \phi_n(z), n \geq 1 \), are analytic and nonvanishing for complex \( z \) in \( \Omega = \{ z: |z| < c_1 \} \), where \( 0 < c_1 \leq c \). Let \( I = (-\alpha, \alpha) \) and \( \Omega_\alpha = \{ z: |z| < \alpha \} \), where \( 0 < \alpha < c_1 \). For values of \( z \) such that \( \phi_n(z) \) is nonvanishing we let

\[
(3.1) \quad \psi_n(z) = \frac{1}{n} \log \phi_n(z)
\]

and

\[
(3.2) \quad \gamma_n(u) = \sup_{|s| < c} \left[ us - \psi_n(s) \right], \quad \text{for} \ u \in \mathbb{R}.
\]

Let \( \mathcal{A}_n = \{ \psi_n(s): s \in I \} \). For \( u \in \mathcal{A}_n \), we have \( \gamma_n(u) = [us_n - \psi_n(s_n)] \), where \( s_n \in I \) satisfies \( \psi_n(s_n) = u \). Let \( P \) be a probability measure on \( (-c, c) \) which satisfies the following condition:

\[
(3.3) \quad \int_{-c}^{c} \exp[\psi_n(x)] \, dP(x) < \infty, \quad \text{for all} \ n \geq 1.
\]

Let \( h(u) \) denote the c.g.f. of \( P \). It is easy to check that condition (3.3) implies that \( h(u) \) is finite for \( u \in \mathbb{R}_n \), where

\[
(3.4) \quad \mathbb{R}_n = \{ u: \gamma_n(u) < \infty \}.
\]

Let

\[
(3.5) \quad V_n(u) = \begin{cases} \gamma_n(u) - h(u), & \text{for} \ u \in \mathbb{R}_n, \\ \infty, & \text{for} \ u \notin \mathbb{R}_n. \end{cases}
\]

The function \( V_n \) plays the same role as the function \( G \) of Section 2.
DEFINITION 3.1. Let \( L^* \) be the class of all probability measures \( P \) on \((-c, c)\) satisfying condition (3.3). We assume that there exist \( l, p_1 > 0 \), such that

\[
\int_{\mathbb{R}_n} \exp[-lV_n(u)] \, du = O(n^{-p_1}),
\]

and the \( V_n \)'s have a unique global minimum at some point \( m_n \). Furthermore, there exists \( \eta_1 > 0 \) such that

\[
\inf_{|u| > \delta} \left\{ V_n(m_n + u) - V_n(m_n) \right\} = \min_{s, -1, 1} \left\{ V_n(m_n + s\delta) - V_n(m_n) \right\},
\]

for all \( 0 < \delta < \eta_1 \).

REMARK 3.2. Condition (3.7) is used mainly in inequality (3.27) of Lemma 3.13. An easily verifiable sufficient conditions for (3.7) is

\[
V_n'(u) > 0, \quad \text{for} \quad u > m_n \quad \text{and} \quad V_n'(u) < 0, \quad \text{for} \quad u < m_n.
\]

In all the examples of Section 4 we will be verifying (3.8) instead of (3.7).

REMARK 3.3. Suppose that \( \mathcal{R}_n = (-\infty, \infty) \). If \( \gamma_n(u)/|u| \) converges to \( \infty \) as \( |u| \to \infty \), then condition (3.3) implies (3.7) as seen below:

\[
\exp[-V_n(u)] = \exp[-\gamma_n(u) + h(u)]
= \exp[-\gamma_n(u)] \left[ \int_{|x| \leq A} \exp[ux] \, dP(x) + \int_{|x| > A} \exp[ux] \, dP(x) \right]
\leq \exp[-\gamma_n(u)] + |u|A + \int_{|x| > A} \exp[\psi_n(x)] \, dP(x)
\leq \exp\left[-|u|\left(\frac{\gamma_n(u)}{|u|} - A\right)\right] \int_{|x| > A} \exp[\psi_n(x)] \, dP(x).
\]

The right-hand side can be made close to zero first by choosing \( A \) and then letting \( |u| \to \infty \). This shows that \( V_n(u) \to \infty \) as \( |u| \to \infty \). Since \( m_n \) is the unique global minimum of \( V_n \), this also shows that condition (3.7) holds.

Let \( m_n \in \mathcal{R}_n \). Then there is a \( \tau_n \) in \( I \) such that \( \psi_n'(\tau_n) = m_n \). For \( t \in I \), define

\[
G_n(t) = \psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it).
\]

The following theorem, which provides an asymptotic expansion for the density function \( k_n \) of \( T_n/n \) at \( m_n \), in terms of the large deviation rate \( \gamma_n \), is due to Chaganty and Sethuraman (1985). In fact, in that paper, it was shown that (3.11) holds for any \( m_n \in \mathcal{R}_n \).

THEOREM 3.4. Assume the following conditions for \( T_n \):

(A) There exists \( \beta > 0 \) such that \( |\psi_n(z)| < \beta \) for \( z \in \Omega_a \) and \( n \geq 1 \).

(B) There exists \( \alpha > 0 \) such that \( \psi_n''(\tau) \geq \alpha \) for \( \tau \in I \) and \( n \geq 1 \).

(C) There exists \( \eta > 0 \) such that for any \( 0 < \delta < \eta \),

\[
\inf_{|t| \geq \delta} \text{Real}(G_n(t)) = \min\left\{ \text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta)) \right\}, \quad \text{for} \quad n \geq 1.
\]
(D) There exists \( p > 0 \) such that
\[
\sup_{\tau \in I} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{1/n} \int_{-\infty}^{\infty} dt = O(n^p).
\]

Then
\[
k_n(m_n) = \left[ \frac{n}{2\pi \psi_n''(\xi_n)} \right]^{1/2} \exp\left(-n \gamma_n(m_n) - \frac{1}{n} \right).
\]

**Remark 3.5.** When \( T_n \) is the sum of \( n \) i.i.d. random variables, condition (C) is automatically satisfied and conditions (A), (B) and (D) are easy to verify, since they do not depend on \( n \).

**Remark 3.6.** Suppose that \( m \) is an interior point of \( \cap A_r \). Then there exists \( \xi_n \in I \) such that \( \psi_n(\xi_n) = m \), for \( n \geq 1 \). If conditions (A) and (B) of the above theorem are satisfied, then one can verify that \( \psi_n''(\xi_n) \) is bounded above uniformly in \( n \) and \( (\psi_n''(\xi_n))^{1/2}/(\psi_n''(\xi_n))^{1/2} = [1 + O(m_n - m)] \) [see (2.6) and (2.25) of Chaganty and Sethuraman (1985)]. Thus we can rewrite (3.11) as
\[
k_n(m_n) = \left[ \frac{n}{2\pi \psi_n''(\xi_n)} \right]^{1/2} e^{-n \gamma_n(m_n)} \left[ 1 + O(m_n - m) + O\left(\frac{1}{n}\right) \right].
\]

For each integer \( r \geq 1 \), let \( Y_r^* \) be a random variable with probability density function given by \( d_r \exp[-c_{2r}2^r/[h''(m)][h''(m) + c_r]/c_2] \) if \( r \geq 2 \) and \( N(0, h''(m)h''(m) + c_r)/c_2) \) if \( r = 1 \), where \( c_{2r} \) is the constant that appears in Theorem 3.7 below and \( d_r \) is the normalizing factor. With these assumptions and notation, we are in a position to state the main theorem of this section.

**Theorem 3.7.** Let \( X_j^{(n)}, j = 1, \ldots, n, \) be a triangular array of random variables satisfying \( |X_j^{(n)}| < c \) and having a joint distribution given by
\[
dQ_n(x) = z_n^{-1} \phi_n \left( \frac{s_n}{n} \right) \prod D(x_j),
\]
where \( \phi_n \) is the m.g.f. of \( T_n \) and \( P \in L^* \). Assume that \( V_n \), defined in (3.5), has a unique global minimum of type \( r \) at \( m_n \in A_r \). Let \( m_n \to m \) and \( V_n^{(2r)}(m_n) \to c_{2r} \) as \( n \to \infty \), where \( m \) is an interior point of \( \cap A_r \). Let \( S_n = X_1^{(n)} + \cdots + X_n^{(n)} \). If \( T_n \) satisfies the conditions of Theorem 3.4, then
\[
S_n - n \tau_n \to_d Y_r^*,
\]
where \( \psi_n(\tau_n) = m_n \) and \( Y_r^* \) is as defined above.

The proof of the above theorem is postponed until the end of Lemma 3.13.
REMARK 3.8. The distribution function $Q_n(x)$ is well defined because

$$z_n = \int \exp \left( n \psi_n \left( \frac{x_n}{n} \right) \right) \prod dP(x_j) \leq \left[ \int_{-\epsilon}^{\epsilon} \exp[\psi_n(x)] dP(x) \right]^n < \infty,$$

wherein we have used condition (3.3) and the fact that $\psi_n$ is a convex function.

For $y \in \mathcal{R}$, let

$$g(y) = \exp \left[ -\frac{y^{2r}c_{2r}}{(2r)!} \right]$$

and

$$g_n(y) = \left[ \frac{2\pi\psi''(\xi_n)}{n} \right]^{1/2} k_n(m_n + n^{-1/2}y)$$

$$\times \exp[n \left( h(m_n + n^{-1/2}y) + V_n(m_n) \right)],$$

where the $\xi_n$'s are defined as in Remark 3.6. The functions $g_n$, $n \geq 1$, arise in the proof of Theorem 3.7. Lemma 3.9 shows that $g_n(y)$ converges to $g(y)$ as $n \to \infty$ for each $y$. The next four lemmas, Lemmas 3.10–3.13, show that

$$\int_{-\infty}^{\infty} g_n(y) dy \to \int_{-\infty}^{\infty} g(y) dy, \quad \text{as } n \to \infty.$$

**Lemma 3.9.** Suppose that $V_n$ has a unique global minimum of type $r$ at the point $m_n \in \mathcal{A}_n$. Let $m_n$ converge to $m$, where $m$ is an interior point of $\mathcal{A} \cap \mathcal{A}_n$. Suppose that $V_n^{(2r)}(m_n) = c_{2r, n}$ converges to $c_{2r}$ as $n \to \infty$. Then

$$g_n(y) \to g(y), \quad \text{as } n \to \infty.$$

**Proof.** Fix $y \in \mathcal{R}$. Let $m_{n, r}(y) = m_n + n^{-1/2}y$. Then $m_{n, r}(y)$ converges to $m$ and $m_{n, r}(y) \in \mathcal{A}_n$ for sufficiently large $n$. Applying Theorem 3.4 together with Remark 3.6, with $m_n$ replaced by $m_{n, r}(y)$ we get

$$g_n(y) = \exp[-n \gamma_n(m_{n, r}(y)) + n \left( h(m_{n, r}(y)) + V_n(m_n) \right)]$$

$$\times \left[ 1 + O(|m_{n, r}(y) - m|) + O \left( \frac{1}{n} \right) \right]$$

$$= \exp[-n \left( V_n(m_{n, r}(y)) - V_n(m_n) \right)]$$

$$\times \left[ 1 + O(|m_{n, r}(y) - m|) + O \left( \frac{1}{n} \right) \right]$$

$$= \exp \left[ -\frac{y^{2r}c_{2r, n}}{(2r)!} + n \frac{O(|y|^{2r})}{n} \right] \left[ 1 + O(|m_{n, r}(y) - m|) + O \left( \frac{1}{n} \right) \right]$$

$$\to g(y), \quad \text{as } n \to \infty.$$
LEMMA 3.10. Suppose that the $V_n$'s have a unique global minimum of type $r$ at the point $m_n \in \mathcal{A}_n$. Then there exists an $N$ such that

$$(3.20) \quad n \left[ V_n(m_n + n^{-1/2r} y) - V_n(m_n) \right] \geq \frac{y^{2r} c_{2r}}{2(2r)!},$$

for all $n \geq N$, and $|y| < n^{1/4r}$.

PROOF. Let $0 < \varepsilon < c_{2r}/2$. Since $c_{2r,n}$ converges to $c_{2r}$ we can find $N_1$ such that $c_{2r,n} > c_{2r}/2 + \varepsilon$ for all $n \geq N_1$. Recall that $\gamma_n(m_n) = m_n \tau_n - \psi_n(\tau_n)$, where $\tau_n$ is such that $\psi_n'(\tau_n) = m_n$. It is easy to verify that $\gamma_n'(m_n) = \tau_n$ and $\gamma_n''(m_n) = [\psi_n''(\tau_n)]^{-1}$. Also, $\gamma_n^{(2r+1)}(m_n)$ is the ratio of a polynomial of $(2r + 1)$ derivatives of $\psi_n$ at $\tau_n$ to $[\psi_n''(\tau_n)]^{2r}$. Conditions (A) and (B) of Theorem 3.4 imply that all these derivatives are bounded uniformly in $n$ and that $\psi_n''(\tau_n) \geq \alpha > 0$ [see (2.6) of Chaganty and Sethuraman (1985)]. Hence $\gamma_n^{(2r+1)}(m_n)$ is uniformly bounded in $n$ and consequently $V_n^{(2r+1)}(m_n) = \gamma_n^{(2r+1)}(m_n) - h^{(2r+1)}(m_n)$ is also uniformly bounded in $n$. Therefore

$$V_n(m_n + u) - V_n(m_n) = \frac{u^{2r} c_{2r,n}}{(2r)!} + K_n u^{2r+1},$$

as $u \to 0$, where $|K_n| \leq K < \infty$ for all $n$. Thus

$$n \left[ V_n(m_n + n^{-1/2r} y) - V_n(m_n) \right] = \frac{y^{2r} c_{2r,n}}{(2r)!} + \frac{K_n y^{2r+1}}{n^{1/2r}}$$

$$\geq \frac{y^{2r} c_{2r}}{2(2r)!} + y^{2r} \left[ \frac{\varepsilon}{(2r)!} - \frac{K y}{n^{1/2r}} \right]$$

$$\geq \frac{y^{2r} c_{2r}}{2(2r)!},$$

if $|y| < n^{1/4r}$ and $n \geq N = \max\{N_1, (K(2r)!/\varepsilon)^{4r}\}$. This completes the proof of Lemma 3.10. □

LEMMA 3.11. Let $g$ and $g_n$ be as defined in (3.15) and (3.16). Then under the hypothesis of Lemma 3.9 we have

$$(3.21) \quad \int_{|y| \leq n^{1/4r}} g_n(y) \, dy \to \int_{-\infty}^{\infty} g(y) \, dy, \quad \text{as } n \to \infty.$$
\[ n \geq N_3, \text{ we get} \]
\[
\int_{|y| \leq n^{1/4}} g_n(y) \, dy = \left[ \frac{2\pi \psi''(\xi_n)}{n} \right]^{1/2} \int_{|y| \leq n^{1/4}} \exp\left[ n(h(m_n, r(y)) + V_n(m_n)) \right] \\
\times k_n(m_n, r(y)) \, dy
\]
\[
= \int_{|y| \leq n^{1/4}} \exp\left[ -n(V_n(m_n, r(y)) - V_n(m_n)) \right] \\
\times \left[ 1 + O\left( |m_n, r(y) - m| \right) + O\left( \frac{1}{n} \right) \right] \, dy
\]
\[
= \int_{-\infty}^{\infty} \lambda_n(y) \, dy,
\]
where
\[
\lambda_n(y) = I(|y| \leq n^{1/4}) \exp\left[ -n(V_n(m_n, r(y)) - V_n(m_n)) \right] \\
\times \left[ 1 + O\left( |m_n, r(y) - m| \right) + O\left( \frac{1}{n} \right) \right],
\]
and \( I(\cdot) \) is the indicator function. It follows from Lemma 3.10 that \( |\lambda_n(y)| \) is bounded by an integrable function. We can now conclude from Lemma 3.9 and the Lebesgue dominated convergence theorem that
\[
\int_{-\infty}^{\infty} \lambda_n(y) \, dy \to \int_{-\infty}^{\infty} g(y) \, dy, \quad \text{as } n \to \infty.
\]
The proof Lemma 3.11 is now complete. \( \Box \)

The next Lemma 3.12 is needed in the proof of Lemma 3.13.

**Lemma 3.12.** Let \( T_n, n \geq 1, \) be a sequence of random variables satisfying the conditions of Theorem 3.4. Then
\[
\sup_y \left[ \exp\left( n\gamma_n(m_n + y) \right) k_n(m_n + y) \right] = O(n^{p+1}), \quad \text{as } n \to \infty.
\]

**Proof.** An application of the inversion formula yields [see (2.12) of Chaganty and Sethuraman (1985)],
\[
\left[ \exp\left[ n((m_n + y)s - \psi_n(s)) \right] k_n(m_n + y) \right] \\
= \left| \frac{n}{2\pi} \int_{-\infty}^{\infty} \exp\left[ n(\psi_n(s + it) - \psi_n(s) - it(m_n + y)) \right] \, dt \right|
\]
\[
\leq \frac{n}{2\pi} \int_{-\infty}^{\infty} \left[ \phi_n(s + it) \right]^{1/n} \phi_n(s) \, dt.
\]
Taking the supremum with respect to \( s \in I \) and using condition (D) of Theorem
3.4 we get
\[ \sup_y \exp(n\gamma_n(m_n + y)) k_n(m_n + y) = O(n^{p+1}). \]

**Lemma 3.13.** Suppose that \( V_n \) has a unique global minimum at the point \( m_n \in \mathcal{A}_n \) and let \( g_n \) be as defined in (3.16). Then

\[ \int_{|y| > n^{1/4}} g_n(y) \, dy \to 0, \quad \text{as } n \to \infty. \] (3.25)

**Proof.** Let \( m_{n,r}(y) = m_n + n^{-1/2}r \). By (3.16) we have

\[ \int_{|y| > n^{1/4}} g_n(y) \, dy = \left[ \frac{2\pi \psi'_\infty(\xi_n)}{n} \right]^{1/2} \int_{|y| > n^{1/4}} k_n(m_{n,r}(y)) \times \exp\left[n(h(m_{n,r}(y)) + V_n(m_n))\right] \, dy \]

\[ = \left[ \frac{2\pi \psi'_\infty(\xi_n)}{n} \right]^{1/2} \int_{|y| > n^{1/4}} k_n(m_{n,r}(y)) \times \exp\left[-n(V_n(m_{n,r}(y)) - V_n(m_n))\right] + n\gamma_n(m_{n,r}(y)) \] \hspace{1cm} (3.26)

Substituting \( u = n^{-1/2}r \), we get

\[ \int_{|y| > n^{1/4}} g_n(y) \, dy \]

\[ = \left[ \frac{2\pi \psi'_\infty(\xi_n)}{n} \right] n^{-\left(1-1/r\right)/2} \int_{|u| > n^{-1/4}} \left[ \exp\left[-n(V_n(m_n + u) - V_n(m_n))\right] \right] \times \left[ \exp(n\gamma_n(m_n + u)) k_n(m_n + u) \right] \, du \]

\[ \leq O(n^{p+1/(1+r)/2}) \int_{|u| > n^{-1/4}} \left[ \exp\left[-n(V_n(m_n + u) - V_n(m_n))\right] \right] \, du. \]

The last inequality follows from Lemma 3.12 and the fact that \( \psi'_\infty(\xi_n) \) is uniformly bounded in \( n \) (see Remark 3.6). It is easy to verify that \( V_n(m_n) = m_n r_n - \psi_n(r_n) - h(m_n) \) is uniformly bounded in \( n \) under the conditions of Theorem 3.4 and because \( m_n \to m \). Thus we get

\[ \int_{|y| > n^{1/4}} g_n(y) \, dy \]

\[ \leq O(n^{p+1/(1+r)/2}) \max_{|u| > n^{-1/4}} \exp\left[-(n - l)(V_n(m_n + u) - V_n(m_n))\right] \times \int_{|u| > n^{-1/4}} \exp\left[-l(V_n(m_n + u) - V_n(m_n))\right] \, du \]

\[ \leq O(n^{p+1/(1+r)/2}) \max_{|u| > n^{-1/4}} \exp\left[-(n - l)(V_n(m_n + u) - V_n(m_n))\right] \times \int_{\mathcal{A}_n} \exp\left[-l(V_n(u))\right] \, du. \]
This together with condition (3.6) yields

\[
\left| \int_{|y| > n^{1/4r}} g_n(y) \, dy \right| \\
\leq O(n^q) \max_{|u| > n^{-1/4r}} \exp[-(n-l)(V_n(m_n + u) - V_n(m_n))] \\
= O(n^q) \exp[-(n-l)L_n],
\]

where

\[
q = p_1 + p + \frac{1 + 1/r}{2},
\]

and

\[
L_n = \min_{|u| > n^{-1/4r}} [V_n(m_n + u) - V_n(m_n)].
\]

This minimum is attained at \( z = \pm n^{-1/4r} \) by condition (3.7). Therefore,

\[
L_n = \min \left[ (V_n(m_n + n^{-1/4r}) - V_n(m_n)), (V_n(m_n - n^{-1/4r}) - V_n(m_n)) \right] \\
= \frac{c_{2r,n}}{(2r)!} \frac{1}{n^{1/2}} + K_n n^{-(2r+1)/4r}.
\]

Hence

\[
\left| \int_{|y| > n^{1/4r}} g_n(y) \, dy \right| = O(n^q) \exp \left[ -(n-l) \left[ \frac{c_{2r,n}}{(2r)!} \frac{1}{n^{1/2}} + K_n n^{-(2r+1)/4r} \right] \right],
\]

which goes to zero since \( |K_n| \leq K \) for all \( n \). The proof of Lemma 3.13 is now complete. \( \square \)

**Proof of Theorem 3.7.** We first express \( dQ_n \) defined in (3.13) as follows:

\[
dQ_n(x) = z_n^{-1} \phi_n \left( \frac{s_n}{n} \right) \prod dP(x_j)
\]

(3.28)

\[
= z_n^{-1} \int \exp(ys_n)k_n(y) \, dy \prod dP(x_j).
\]

Substituting \( m_n(r) = m_n + n^{1/2r}y \), we get

\[
dQ_n(x) = z_n^{-1} n^{-1/2r} \int \exp(m_n(r)s_n)k_n(m_n(r)) \, dy \prod dP(x_j)
\]

(3.29)

\[
= z_n^{-1} n^{-1/2r} \int \prod \exp(x_jm_n(r) - h(m_n(r))) \, dP(x_j)
\]

\[
\times k_n(m_n(r)) \exp(nh(m_n(r))) \, dy
\]

\[
= \int \prod dM_{n,y}(x_j) f_n(y) \, dy,
\]
where

\begin{equation}
(3.30)
dM_{n,Y}(x_j) = \exp(x_jm_{n,Y}(y) - h(m_{n,Y}(y)))\,dP(x_j)
\end{equation}

and

\begin{equation}
(3.31)
f_n(y) = z_n^{-1}n^{-1/2}k_n(m_{n,Y}(y))\exp(nh(m_{n,Y}(y))).
\end{equation}

Since \(\int dQ_n(x) = 1\) and \(\int dM_{n,Y}(x_j) = 1\) for each \(y\) and \(j\), we have \(\int f_n(y)\,dy = 1\). Thus we can introduce random variables \(W_n\) with probability density function \(f_n(y)\) and the representation (3.29) of \(dQ_n(x)\) shows that given \(W_n = y\), \(X_j^{(n)}\), \(j = 1, \ldots, n\), are i.i.d. with common distribution \(M_{n,Y}(x)\).

We now proceed to obtain the limiting distribution of \((S_n - n\tau_n)/n^{1-1/2r}\) under \(dM_{n,Y}(x)\).

We first note that

\begin{align*}
\log E_{M_{n,Y}} & \exp \left[ t(S_n - n\tau_n) \right] \\
& = n \left[ -\frac{\tau_n}{n^{1-1/2r}} + h\left( \frac{t}{n^{1-1/2r}} + m_{n,Y}(y) \right) - h(m_{n,Y}(y)) \right] \\
& = n \left[ -\frac{\tau_n}{n^{1-1/2r}} + h'(m_{n,Y}(y)) \frac{t}{n^{1-1/2r}} \\
& \quad + h''(m_{n,Y}(y)) \frac{t^2}{2n^{2-1/r}} + o(n^{-1}) \right] \\
& = h''(m_n)ty + \frac{h''(m_n)t^2}{2n^{1-1/r}} + o(1),
\end{align*}

since \(\tau_n = h'(m_n)\). Thus

\begin{equation}
(3.33) \quad \log E_{M_{n,Y}} \exp \left[ t(S_n - n\tau_n) \right] \to \begin{cases} 
  h''(m)ty, & \text{if } r \geq 2, \\
  h''(m)ty + \frac{h''(m)t^2}{2}, & \text{if } r = 1.
\end{cases}
\end{equation}

This shows that the limiting distribution of \((S_n - n\tau_n)/n^{1-1/2r}\) given \(W_n = y\) is degenerate at \(h''(m)y\) if \(r > 1\) and \(N(h''(m)y, h''(m))\) if \(r = 1\). Next we note that

\begin{align*}
f_n(y) &= z_n^{-1}n^{-1/2}k_n(m_{n,Y}(y))\exp(nh(m_{n,Y}(y))) \\
&= \frac{g_n(y)}{\int g_n(y)\,dy},
\end{align*}

where \(g_n(y)\) is as defined in (3.16). By Lemmas 3.9, 3.11 and 3.13 it follows that

\begin{equation}
(3.34) \quad f_n(y) \to f(y) = \frac{g(y)}{\int g(y)\,dy}, \quad \text{as } n \to \infty,
\end{equation}

\begin{equation}
(3.35) \quad f_n(y) = \frac{g(y)}{\int g(y)\,dy}, \quad \text{as } n \to \infty,
\end{equation}
where \( g(y) = \exp[-y^2 e_{2r}/(2r)!] \). Thus the limiting distribution of \( W_n \) is \( f(y) \).

The unconditional limiting distribution of \((S_n - n \tau_n)/n^{1-1/2r}\) is just the mixture of the limiting conditional distribution and \( f(y) \), by Theorem 3.15 of Sethuraman (1961). This completes the proof of Theorem 3.7. □

**Remark 3.14.** When \( T_n \) is the sum of independent, normally distributed random variables with mean zero and variance one, \( \phi_n(s_n/n) \) becomes \( \exp[s_n^2/2n] \) and the class of probability measures \( L^* \) reduces to the class \( L \). Thus Theorem 3.7 generalizes Theorem 2.6 to a larger class of Hamiltonians and probability measures.

We now state the theorem of Sethuraman (1961) which was crucially used to obtain the limiting marginal distribution of \((S_n - n \tau_n)/n^{1-1/2r}\) in the proof of Theorem 3.7.

**Theorem 3.15** (Sethuraman, 1961). Let \( \Lambda_n \) be a sequence of probability measures on \( V \times W \), where \( V \) and \( W \) are topological spaces. Let \( \nu_n \) be the marginal probability measure of \( \Lambda_n \) and \( V \) and \( \nu_n(v, \cdot) \) be the conditional probability measure on \( W \). Assume that there exists a probability measure \( \mu \) such that \( \mu_n(A) \) converges to \( \mu(A) \) for every measurable set \( A \subset V \). Suppose that for almost all \( v \) with respect to \( \mu \), \( \nu_n(v, \cdot) \) converges weakly to \( \nu(v, \cdot) \). Then \( \Lambda_n \) converges weakly to \( \Lambda \), where

\[
\Lambda(A \times B) = \int_A \nu(v, B) \, d\mu(v),
\]

for every measurable rectangular set \( A \times B \).

We now turn our attention to the case where \( T_n \), \( n \geq 1 \), are lattice valued random variables with spans \( h_n \), \( n \geq 1 \). The following theorem, which is analogous to Theorem 3.4, was proved by Chaganty and Sethuraman (1985).

**Theorem 3.16.** Let \( T_n \), \( n \geq 1 \), be a sequence of lattice valued random variables with spans \( h_n \), \( n \geq 1 \). Let \( m_n \) belong to the range of \( T_n/n \). Assume that conditions (A) and (B) of Theorem 3.4 hold and replace conditions (C) and (D) by the following:

- (C') There exists \( \eta > 0 \) such that for any \( 0 < \delta < \eta \),
  
  \[
  \inf_{\delta \leq |t| \leq \pi/|h_n|} \text{Real}(G_n(t)) = \min\{\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))\}, \quad \text{for } n \geq 1,
  \]
  
  where \( G_n(t) \) is defined by (3.10).

- (D') There exists \( p > 0 \) such that \( |h_n| = O(n^{-p}) \).

Then

\[
\frac{n^{1/2}}{|h_n|} \Pr\left( \frac{T_n}{n} = m_n \right) = \left[ \frac{1}{2\pi \psi''(\tau_n)} \right]^{1/2} \exp(-n\gamma_n(m_n)) \left[ 1 + O\left( \frac{1}{n} \right) \right].
\]
As before for a probability measure $P$ on $\mathcal{A}$, define $V_n(u)$ as in (3.5). The class of probability measures that are of interest is defined below.

**Definition 3.17.** Let $L^*_t$ be the class of probability measures $P$ satisfying conditions (3.3), (3.7) and (3.38) (defined below).

\begin{equation}
\sum_{u \in \mathcal{A}_n} \exp[-lV_n(u)] = O(n^{p_1}), \quad \text{for some } l, p_1 > 0.
\end{equation}

Note that (3.38) is the appropriate replacement of (3.6) for the lattice valued case.

For Hamiltonians which are functions of the moment generating functions of lattice valued random variables we have the following theorem almost identical to Theorem 3.7.

**Theorem 3.18.** Let $P \in L^*_t$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables satisfying $|X_j^{(n)}| < c$ and having a joint distribution given by

\begin{equation}
dQ_n(x) = z_n^{-1} \phi_n \left( \frac{s_n}{n} \right) \prod dP(x_j),
\end{equation}

where $\phi_n$ is the m.g.f. of the lattice valued random variables $T_n$. Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. Let $V_n$ have a unique global minimum of type $r$ at the point $m_n \in \mathcal{A}_n$. Let $m_n$ converge to a point $m$ belonging to the interior of $\cap \mathcal{A}_n$. If $T_n$ satisfies the conditions of Theorem 3.16, then

\begin{equation}
\frac{S_n - n\tau_n}{n^{1-1/2r}} \rightarrow_d Y_r^*,
\end{equation}

where $Y_r^*$ and $\tau_n$ are as defined in Theorem 3.7.

The proof of the above theorem parallels the proof of Theorem 3.7. We therefore outline briefly the modifications that need to be done. Note that $dQ_n$ can be written as

\begin{equation}
dQ_n(x) = \sum_y \prod dM_{n,y}(x_j) f_n^*(y),
\end{equation}

where $f_n^*(y) = z_n^{-1} k_n(m_n + n^{-1/2}y) \exp[nh_m(m_n + n^{-1/2}y)]$ is a probability mass function of a lattice valued distribution with span $h'_n = h_n/n^{1-1/2r}$, and $dM_{n,y}(x_j)$ is as defined in (3.30). We introduce discrete random variables $W_n^*$ with p.m.f. $f_n^*$. It suffices to show that $W_n^*$ converges weakly to a continuous random variable $W$ with probability density function $f$, defined in (3.35). The rest of the proof is identical to the proof of Theorem 3.7. Note that the span, $h'_n$, of $W_n^*$ converges to zero. By a theorem of Okamoto (1959), the sequence of random variables $W_n^*$ will converge in distribution to $W$, once we prove the
following:

**Lemma 3.19.** For \( y \in \mathcal{R} \), define \( y_n = h_n[\gamma / h_n] \). Let the probability mass function \( f_n \) and the probability density function \( f \) be as defined above. Then

\[
\frac{1}{|h'_n|} f_n^*(y_n) \rightarrow f(y), \quad \text{as } n \to \infty,
\]

uniformly on bounded intervals of \( y \).

**Proof (outline).** Note that \( f(y) = g(y) / \|g(y)\) dy, \( g(y) \) is as defined in (3.15). We first write

\[
f_n^*(y_n) = \frac{g_n^*(y_n)}{\Sigma g_n^*(y)},
\]

where

\[
g_n^*(y_n) = \frac{\sqrt{2\pi} \psi_n''(\xi_n)}{n(1-1/r)^{1/2}} k_n(m_n + n^{-1/2r}y_n)
\times \exp \left[ n \left( h(m_n + n^{-1/2r}y_n) + V_n(m_n) \right) \right].
\]

Imitating the proofs of Lemmas 3.9–3.13, one can show the following:

(i) \[
\frac{1}{|h'_n|} g_n^*(y_n) \rightarrow g(y), \quad \text{as } n \to \infty,
\]

uniformly on bounded intervals of \( y \);

(ii) \[
\sum_{|y_n| \leq n^{1/4r}} g_n^*(y_n) \rightarrow \int_{-\infty}^{\infty} g(y) \, dy, \quad \text{as } n \to \infty;
\]

(iii) \[
\sum_{|y_n| > n^{1/4r}} g_n^*(y_n) \rightarrow 0, \quad \text{as } n \to \infty.
\]

The above three steps (i), (ii) and (iii) complete the proof of Lemma 3.19. \( \Box \)

**4. Applications.** In this section we illustrate the main theorems of Section 3 with four applications and demonstrate limit theorems in quite complicated situations of dependent variables. The model (3.13) for the joint distribution of \((X_1^{(n)}, \ldots, X_n^{(n)})\) is completely specified if \( T_n \) and \( P \) that arise in it are specified. To simplify matters, in all the examples of this section we let \( T_n \) be the sum of \( n \) i.i.d. random variables with common distribution function \( F \). The four examples below contain all occurrences of lattice and nonlattice \( T_n \), and continuous and discrete \( P \). The limit distribution of the normalized sum \( S_n/n^{1-1/2r} \) is normal (\( r = 1 \), in the notation of Theorems 3.7 and 3.18) in Example 4.1 and is nonnormal (\( r = 2 \)) in Examples 4.2, 4.3 and 4.4. The results of Ellis and Newman (1978b) show that limit distributions with every possible value of \( r > 2 \) can also arise in suitable models.
In all the examples below we will specify $F$ and $P$ and write down the joint distribution $Q_n$. Since $T_n$ is the sum of independent random variables, the functions $\psi_n = \psi$, $\gamma_n = \gamma$ and $V_n = V$, independent of $n$ and therefore it is straightforward to verify the four conditions of Theorem 3.4 or Theorem 3.16 depending on whether $T_n$ is nonlattice or lattice. We can also show that in all the examples considered, $V$ is symmetric around the origin and $V(u) > 0$ for $u > 0$. Hence $V$ has a unique global minimum at the origin. The verification of conditions (3.3), (3.6) and (3.8) that insure that $P \in L^*$, in the case of continuous $P$ and conditions (3.3), (3.8) and (3.38) that insure that $P \in L^*_1$, in the case of discrete $P$, does not pose any difficulties. The details are left to the reader.

**Example 4.1.** Let the distribution function $F$ and the probability measure $P$ be defined by the probability density functions $\frac{1}{2}\exp(-|x|)$, $-\infty < x < \infty$, and $\frac{3}{4}(1-x^2)$, $|x| < 1$, respectively. Then the joint distribution $Q_n$ given in (3.13) becomes

\begin{equation}
(4.1) \quad dQ_n(x) = z_n^{-1}\left[\frac{3}{4}\right]^n\left(1 - \frac{s_n^2}{n^2}\right)^{-n}\prod (1 - x_j^2)\prod dx_j.
\end{equation}

In this case we can show that for $u \in \mathbb{R}$,

\begin{equation}
(4.2) \quad V(u) = \left[-1 + \sqrt{1 + u^2}\right] + \log|u| + \log\left[-1 + \sqrt{1 + u^2}\right] - |u| - \log(|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})) + \log\left(\frac{4}{3}\right).
\end{equation}

Since $V''(0) = \frac{3}{10} > 0$, the global minimum of $V$ is of type 1. We can therefore conclude in this example that

\begin{equation}
(4.3) \quad S_n/n^{1/2} \to_d N(0, \frac{1}{3}).
\end{equation}

**Example 4.2.** Let $F$ be the triangular distribution function on the interval $(-2b, 2b)$ with $b = 3^{1/2}/2^{1/2}$. Let $P$ be the standard normal probability measure. The joint distribution $Q_n$ is given by

\begin{equation}
(4.4) \quad dQ_n(x) = z_n^{-1}(2\pi)^{-n/2}\left[\frac{n \sinh(bs_n/n)}{(bs_n)}\right]^{2n}\exp\left[-\frac{1}{2}\sum x_j^2\right]\prod dx_j.
\end{equation}

With our choice of $b = 3^{1/2}/2^{1/2}$ one can verify that $V''(0) = 0$ and $V^{(4)}(0) = \frac{3}{2} > 0$. Therefore $V$ has a unique global minimum of order 2 at the origin and hence by the conclusion of Theorem 3.7 we get

\begin{equation}
(4.5) \quad \frac{S_n}{n^{3/4}} \to_d Y_2^*,
\end{equation}

where the p.d.f. of $Y_2^*$ is given by $d_2\exp(-y^4/40)$, $-\infty < y < \infty$.

**Example 4.3.** Let $F$ be as defined in Example 4.2. Let $P$ be symmetric Bernoulli, i.e., $P((-1)) = P((1)) = \frac{1}{2}$. The joint distribution $Q_n$ is given by

\begin{equation}
(4.6) \quad dQ_n(x) = z_n^{-1}\left[\frac{n \sinh(bs_n/n)}{(2^{1/2}bs_n)}\right]^{2n},
\end{equation}

where the p.d.f. of $Y_2^*$ is given by $d_2\exp(-y^4/40)$, $-\infty < y < \infty$. 

where $x_j = \pm 1$ for all $1 \leq j \leq n$, and $b = 3^{1/2}/2^{1/2}$. In this example we can show that $V$ has a unique minimum of order 2 at the origin and $V^{(4)}(0) = 13/5$. Thus by the conclusion of Theorem 3.7 we get

$$
\frac{S_n}{n^{3/4}} \rightarrow_d Y_2^*,
$$

where the p.d.f. of $Y_2^*$ is given by $d_\theta\exp[-13y^4/120], -\infty < y < \infty$.

**Example 4.4.** Let $F$ be symmetric Bernoulli distribution and $P$ be the standard normal probability measure. The joint distribution in this example is given by

$$
dQ_n(x) = z_n^{-1}(2\pi)^{-n/2}\left[\cosh\left(\frac{8n}{n}\right)\right]^n \exp\left[-\frac{1}{2}\sum x_j^2\right] \prod dx_j.
$$

It is easy to check that zero is the unique global minimum of order 2 for the function $V$ and $V^{(4)}(0) = 2$. Note that in this example $T_n$’s are lattice valued random variables. Thus by the conclusion of Theorem 3.18 we get

$$
\frac{S_n}{n^{3/4}} \rightarrow_d Y_2^*,
$$

where $Y_2^*$ is distributed as $d_\theta\exp(-y^4/12), -\infty < y < \infty$.

**REFERENCES**


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