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Analysis of Jump Linear Systems Driven by Lumped Processes

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ABSTRACT

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Safety critical control systems such as flight control systems use *fault-tolerant* technology to minimize the effect of *faults* and increase the *dependability* of the system. In fault-tolerant systems, the *system availability process* indicates the current operational mode of an interconnection of digital logic devices. It is a process that results from the transformation of the stochastic processes characterizing the availability of the devices forming the system. To assess safety critical control systems, the following measures of performance will be considered: the *steady-state mean output power*, the *mean output energy*, the *mean time to failure* and the *mean time to repair*. For this assessment it is important to determine the characteristics of the system availability process since both stability and the aforementioned measure of performance are directly dependent on it. When the system availability process results from a transformation of a homogeneous Markov chain, it is well-known that the resulting process is not necessarily a homogeneous Markov chain. In particular, when the Markov chain characterizing the faults is a zeroth order Markov chain, it is shown that the availability process results in another zeroth order Markov chain. Thus, all the results which are known to analyze closed-loop systems driven by a homogeneous Markov chain can be applied to the zeroth order Markov chain. However, simpler formulas that do not trivially follow from these Markov chain results

can be derived in this case. Part of this dissertation is dedicated to the derivation of these new formulas. On the other hand, when the system availability results in either a non-homogeneous Markov chain or a non-Markov chain, the existing Markov results can not be directly applied. This problem is addressed here. The necessity for better integration of the fault tolerant and the control designs for safety critical systems is also addressed. The dependability of current designs is primarily assessed with measures of the interconnection of fault tolerant devices: the reliability metrics that include the mean time to failure and the mean time to repair. These metrics do not directly take into account the interaction of the fault tolerant components with the dynamics of the system. In this dissertation, a first step to better integrate fault tolerant and the control designs for safety critical systems is made. These are the problems that motivated this work. Therefore, the goals of this dissertation are: to develop a suitable methodology to analyze a jump linear system when the driving process is characterized by a zeroth order Markov chain, a non-homogeneous Markov chain and a non-Markov chain; and to integrate the analysis of jump linear systems with the reliability theory for network architectures.

To my parents: Alejandro Chávez and Juana R. Fuentes

Coplas a la muerte de mi padre:

Recuerde el alma dormida, avive el seso y despierte,

contemplando cómo se passa la vida,

cómo se viene la muerte

tan callando.

Jorge Manrique.

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LIST OF ACRONYMS

MC	Markov Chain
HMC	Homogeneous Markov Chain
NHMC	Non-Homogeneous Markov Chain
NMC	Non-Markov Chain
i.i.d.	Independent and Identically Distributed Process
JLS	Jump Linear System
MSS	Mean Square Stability
MSES	Mean Square Exponential Stability
TTF	Time to Failure
TTR	Time to Repair
MTTF	Mean Time to Failure
MTTR	Mean Time to Repair

CHAPTER I

INTRODUCTION

I.1 MOTIVATION AND GENERAL ASSUMPTIONS

An interconnection of $L \geq 2$ devices that are working together to accomplish a certain function is referred to as a network architecture. This dissertation is focused on the logical rather than the physical layout of a network. The operation of a network in a harsh environment like that caused, for example, by high intensity radiated fields, can result in faults, that is, a deviation from the correct functionality of a device. No fault due to aging or wear of the components forming the system is considered here. In addition, it is assumed that these faults are transient, that is, they only exist for a finite period of time. These faults will be called upsets. Since faults are unavoidable and flight control systems use complex closed-loop digital technology, it is important to consider the construction of dependable control systems with a fault-tolerant communication network architecture capable of recovering after a fault and continuing operation while maintaining the closed-loop system's stability and desired level of performance. Since these fault-tolerant networks are the enabling technology in safety critical distributed control system applications, it is important to analyze the effect of the random jumps of functionality on the performance of the controlled dynamical system caused by an upset.

Assume that the effect of an upset on each device forming a fault-tolerant network architecture for a flight control system is to put it in one of S modes of operation

during a period of time (the faults last one or more control sample periods T_p). Moreover, assume that the i-th device's mode is represented by a state of a zeroth or first order discrete-time homogeneous Markov chain (HMC) (see, e.g., [6], [18]) $\boldsymbol{z}_i(k)$, where $i \in \mathscr{I}_L \triangleq \{1, ..., L\}$ and $k \in \mathbb{Z}^+ \triangleq \{0, 1, ...\}$ denotes the sample period number. If these Markov chains (MCs) are stochastically independent, then the joint process, $\boldsymbol{z}(k) \triangleq (\boldsymbol{z}_1(k), ..., \boldsymbol{z}_L(k))$, is also an HMC [22]. This assumption also implies that the current mode of one device does not depend on the modes of the remaining devices during the same sample period. When the event $\{\boldsymbol{z}_i(k) = 0\}$ occurs, it is said that the i-th device is operating as intended and, in general, the event $\{\boldsymbol{z}_i(k) = s\}$ denotes the s-th mode of operation during the k-th sample period, where $s \in \mathcal{I}_S \triangleq \{0, 1, ..., S-1\}$, $S \geq 2$. A particular case of interest is when $S = 2$, that is, each device only has two modes of operation, 0 and 1. In this case, the event $\{\boldsymbol{z}_i(k) = 1\}$ indicates that the i-th device is not working correctly during the k-th sample period and the probability $\Pr(\boldsymbol{z}_i(k) = 1)$ is called its probability of upset. In general, $\Pr(\boldsymbol{z}_i(k) = s)$ is the state or mode probability for the k-th sample period.

From a control systems point of view, it is important to characterize the modes of operation of the fault-tolerant network as a function of the stochastic processes that characterize the modes of the interconnected components, $\boldsymbol{z}_i(k)$, since they determine the closed-loop system's modes of operation, and the switching between the different modes affects performance. The network's mode during the k-th sample period is assumed to be characterized by the random variable $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, where ϕ is any memoryless and onto transformation of $\boldsymbol{z}(k)$ thereby inducing a well-defined stochastic process $\boldsymbol{\rho}(k)$ with state space $\mathcal{I}_\ell \triangleq \{0, 1, ..., \ell-1\}$ and $1 < \ell < S^L$ (see Section II.2). Since ϕ reduces the number of states of $\boldsymbol{z}(k)$ from S^L to ℓ, it is called

a (MC) *lumping transformation* and $\boldsymbol{\rho}(k)$, a *lumped process.* It is well known (see, e.g., [19], [23]) that $\boldsymbol{\rho}(k)$ can be be either an r-th order HMC, $r \geq 1$, or a lumped non-homogeneous Markov chain (NHMC) or a non-Markov chain (NMC) whenever it is applied to a first order HMC. The expression *lumped NHMC* refers to a lumping transformation that results in an NHMC for some initial state probability vectors of the underlying process. In this dissertation, a new result is given that establishes conditions under which the process $\boldsymbol{\rho}(k)$ is characterized by a zeroth order MC, that is, an independent, identically distributed process (i.i.d.). In addition, by using the concept of a *lumping matrix* (see Definition II.2.5), a test to check when the process $\boldsymbol{\rho}(k)$ results in a first order HMC is provided.

To analyze the effect of $\boldsymbol{\rho}(k)$ in the closed-loop control system, let $\boldsymbol{x}(k) \in \mathbb{R}^n$ represent the state of a system at the k-th sample period and $\boldsymbol{x}(0) = \boldsymbol{x}_0$ be the initial state random vector with finite second moment. Consider now the jump linear system (JLS) driven by the lumped process $\boldsymbol{\rho}(k)$:

$$\boldsymbol{x}(k+1) = A_{\boldsymbol{\rho}(k)}\boldsymbol{x}(k) + B_{\boldsymbol{\rho}(k)}\boldsymbol{w}(k), \quad \boldsymbol{x}(0) = x_0, \tag{I.1.1a}$$

$$\boldsymbol{y}(k) = C_{\boldsymbol{\rho}(k)}\boldsymbol{x}(k), \tag{I.1.1b}$$

where the process $\boldsymbol{w}(k) \in \mathbb{R}^p$ that represents an input disturbance to the system is taken to be a white noise process independent of $\boldsymbol{\rho}(k)$ and $\boldsymbol{x}(0)$, and $\boldsymbol{y}(k) \in \mathbb{R}^q$ is the output of the system. The matrices A, B and C are indexed by the process $\boldsymbol{\rho}(k)$ to represent the switching operational mode of the system. The triple (A_0, B_0, C_0) represents the nominal closed-loop system. The study to be done here includes the analysis of the *mean square stability* (MSS), the *steady-state mean output power*, J_w, and the *mean output energy*, J_0, associated with the JLS (I.1.1) when the lumped

process $\boldsymbol{\rho}(k)$ is not necessarily an HMC. The mean output power and the mean output energy will be referred to as the *output performance metrics* of the system. In addition, the *mean time to failure* (MTTF) and the *mean time to repair* (MTTR) associated with the network architecture will also be analyzed. The metrics MTTF and MTTR will be jointly referred to as the *network performance metrics.* This analysis is done for the different statistical characteristics that the lumped process can take. From a control systems point of view, a particular interest of this work is to find connections between the output and the network performance metrics in order to see how the former are affected by the latter. The following section presents the specific problems that are solved in this dissertation.

I.2 PROBLEM STATEMENT

Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the lumped process that characterizes a network architecture where random upsets switch the modes of the closed-loop system represented by the JLS (I.1.1). To attain the goals of this dissertation, that is, to develop a suitable methodology to analyze a jump linear system driven by a lumped process that is not an HMC, and to integrate the analysis of jump linear systems with the reliability theory of a network architecture, the following problems will be solved.

Problem 1. *When $\boldsymbol{\rho}(k)$ is either a lumped NHMC or an NMC, determine:*

a) *The probability distribution of $\boldsymbol{\rho}(k)$, $p_j(k) \triangleq \Pr(\boldsymbol{\rho}(k) = j)$, $j \in \mathcal{I}_\ell$.*

b) *The availability of the system at steady-state, $\lim_{k\to\infty} \Pr(\boldsymbol{\rho}(k) = 0)$, whenever this limit exists.*

c) *The one-step transition probabilities of* $\boldsymbol{\rho}(k)$,

$$p_{ij}(k) \triangleq \Pr(\boldsymbol{\rho}(k+1) = j | \boldsymbol{\rho}(k) = i), \;\; i, j \in \mathcal{I}_\ell,$$

whenever these transition probabilities are well-defined.

d) *Conditions under which there exists the steady-state values of the transition probabilities* $p_{ij}(k)$, $\lim_{k\to\infty} p_{ij}(k)$, *derive these steady-state values.*

Problem 2. *Assuming that* $\boldsymbol{z}(k)$ *is an i.i.d. process, determine:*

a) *Conditions under which* $\boldsymbol{\rho}(k)$ *is also an i.i.d. process.*

b) *When* $\boldsymbol{\rho}(k)$ *is an i.i.d. process that drives the JLS* (I.1.1), *derive formulas for the output performance metrics,* J_w *and* J_0.

c) *Determine the advantages of using these new formulas versus the formulas that assume an HMC.*

Problem 3. *Develop a methodology to analyze the MSS and the output performance metrics of the JLS* (I.1.1) *when* $\boldsymbol{\rho}(k)$ *is either an NHMC or an NMC.*

Problem 4. *When* $\boldsymbol{\rho}(k)$ *is an i.i.d. process, determine* $\left.\frac{\partial J_w(p)}{\partial p_j}\right|_{p=p^*}$ *and* $\left.\frac{\partial J_0(p)}{\partial p_j}\right|_{p=p^*}$, *where* $j \in \mathcal{I}_\ell$ *and* $p^* \triangleq (p_0^*, ..., p_{\ell-1}^*)$ *is a point in* $[0,1]^L \triangleq \underbrace{[0,1] \times \cdots \times [0,1]}_{L \text{ times}}$ *such that the JLS* (I.1.1) *is MSS.*

Problem 5. *When* $\boldsymbol{\rho}(k)$ *is either an i.i.d. process or an HMC, show that* J_w *and* J_0, *of the JLS* (I.1.1) *driven by* $\boldsymbol{\rho}(k)$ *are explicit functions of the performance metrics of the network architecture represented by* $\boldsymbol{\rho}(k)$.

I.3 ORGANIZATION AND ACHIEVEMENTS

This dissertation has five chapters. The solutions to the problems given in Section I.2 and related results are given in Chapters II, III and IV. Chapter V gives the

conclusions of the dissertation. In addition, there is an appendix at the end of the dissertation to briefly review some concepts about MCs. The organization of this dissertation is as follows.

Problem 1 is entirely solved in Chapter II. A general network architecture and the system availability process, $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, that characterizes it are introduced first. Next, the probability distribution of $\boldsymbol{\rho}(k)$ is derived. Moreover, the availability of the network and the availability at steady-state are defined and calculated. Next, it is shown that $\boldsymbol{\rho}(k)$ has well-defined one-step transition probabilities, which are derived and calculated at steady-state. Most of the reliability analysis has been done in continuous-time, particularly for continuous-time MCs [2]. When the network architecture is characterized by discrete-time MCs, much less literature is available [4]. There is no literature for the case when the network architecture is characterized by a lumped process determined by a lumping transformation. One of the main contributions of this chapter is that the derivations concerning the statistical characterization of the process $\boldsymbol{\rho}(k)$ are completely general results as long as it is a well-defined process (in particular, these results are independent of whether or not $\boldsymbol{\rho}(k)$ is an MC). An application of the results of Chapter II given in Section III.5 is to analyze the exponentially second moment stability of the JLS (I.1.1) when it is driven by a lumped NHMC. The exponentially second moment stability will be referred to here as *mean square exponential stability* (MSES). In this application, a new result is obtained that complements a test for checking MSES given in [11]. Finally, a section is dedicated to characterizing a lumping transformation, ϕ, in a network where the upsets are characterized by i.i.d. processes. The main result in this case is that the zeroth order Markov property is preserved under ϕ. This result is useful in applications (see, e.g.,

Example III.4.1).

Problem 2a is solved in Section II.3 of Chapter II, and **Problems 2b-c** are solved in Chapter III, where a characterization of the JLS (I.1.1) is given when it is driven by the lumped process $\rho(k)$. First, the output performance metrics are derived when $\rho(k)$ is an i.i.d. process. These formulas are simpler than those given in [17], where these metrics were calculated when $\rho(k)$ is a non-lumped HMC. These formulas, which are presented in Section II.3, do not trivially follow from the non-lumped HMC case, and they are derived by using the smaller matrix $\mathcal{A}$ instead of the $\mathcal{A}_2$ matrix used in [17] (see Sections III.2 and III.3). This reduces the dimensionality of the formulas. Control system performance for an i.i.d. JLS has been addressed in [25–27], where the power spectral density is considered as the output performance of the system. The results obtained in this dissertation differ from the formulas given in this literature because the approach followed here is based on [8] and [17], where the definition considered for the output performance induces a norm rather than a semi-norm. In this sense the results obtained here represent, to the best of our knowledge, a new contribution in the theoretical analysis of JLSs.

Problems 3 and 4 are also solved in Chapter III. Sensitivity formulas, which describe how the output performance metrics are affected by a small change in the probabilities of upset, are given in Section III.3. The analysis of MSS and the output performance metrics of the JLS (I.1.1) when $\rho(k)$ is either a lumped NHMC or an NMC, which is Problem 4, have not been addressed before. The last section of the chapter is dedicated to developing a new result to cover this case.

Problem 5 is solved in Chapter IV, where one of the main achievements of this dissertation is given. Specifically, a connection between a fault-tolerant network

architecture, which has been characterized in Chapter II by the system availability process $\boldsymbol{\rho}(k)$, and a closed-loop control system, driven by $\boldsymbol{\rho}(k)$, is established. It is shown that J_0 and J_w are functions of the MTTF and the MTTR. This relationship implies that is not possible to require a certain level of performance for the fault-tolerant network without taking into account the reliability metrics of the system. This connection represents a new contribution in the theory that integrates two fields of study, dynamic system theory and reliability theory, which so far have been addressed separately.

CHAPTER II

THE SYSTEM AVAILABILITY PROCESS OF A NETWORK ARCHITECTURE

II.1 INTRODUCTION

This chapter characterizes the system availability process of a network architecture. As explained in Chapter I, a fault randomly changes the operational mode of the devices forming the network, thereby changing the network's mode. The system availability process indicates at each time instant the operational mode at which the network is performing its intended function [35, 43]. The network's mode is a manifestation of the relationship between the performance of the network and the performance of the devices under the presence of a fault. This relationship is accomplished by a structure function [3, 24] or more general for a lumping transformation, ϕ, which is a function that maps the modes of the devices, modeled here as the states of either a zeroth or a first order HMC, into a finite set resulting in another well-defined stochastic process, the system availability process. To better understand the effect of faults on the performance metrics when the system is operating in a harsh environment, it is important to characterize the system availability process as a function of the processes that represent the modes of the devices.

The term availability is a concept widely used by the computer engineering community. It is defined as the probability of an MC to stay in a set of operational states (Up states) at a given time. Availability has been studied mostly in continuous-time

reliability theory (see, e.g., [1, 16, 38] and the references therein). Results for the discrete-time case are less developed. An account of the state of the art and arguments for the necessity of a discrete-time theory can be found, for example, in [4] and [29]. In [2], continuous and discrete-time reliability models are also presented. For a discrete-time NHMC there are fewer published results. This case has been addressed in [33–35], where the availability and the steady-state availability are defined and computed, and practical applications are given. In Section II.2, the concept of a (discrete-time) system availability process is formally introduced as the transformation ϕ of an HMC $\boldsymbol{z}(k)$ (see Definition II.2.3). The notion of operational and non-operational states (Up and Down states, respectively) that are defined in the literature above are substituted here by a finite set of modes, for example the zero mode represents the Up states. In this chapter, a statistical characterization of the system availability process is done. This analysis, unlike the existing approaches, takes into account the properties of the transformation ϕ, and all the results are given in terms of the statistical characteristics of the underlying process $\boldsymbol{z}(k)$. A transformation of an HMC is not necessarily an MC [19, 37], thus the results derived here are independent of whether or not the system availability is an MC, as long as it is a well-defined process. In this sense, the results presented here cover broader situations than those given in [34, 35].

The rest of this chapter is organized as follows. The definitions of a lumping transformation and the system availability process are introduced in Section II.2. The probability distribution of the system availability process and the availability at steady-state are derived in this section. Furthermore, one-step transition probabilities of the system availability process and the steady-state of these transition

probabilities are derived. A case of interest for applications is when the modes of the devices are represented by a zeroth-order HMC. A new result regarding the preservation of the Markov property when the transformation ϕ is applied to a zeroth-order HMC is given in Section II.3. Section II.4 gives conditions under which the system availability process, which is a transformation of an MC, results in another MC. Finally, Section II.5 gives a summary of the chapter.

II.2 THE SYSTEM AVAILABILITY PROCESS

Consider a particular operation performed by a network of $L \geq 2$ devices, and assume that each device is affected by L independent upset processes. Let the mode of operation at time $k \in \mathbb{Z}^+ \triangleq \{0, 1, \ldots\}$ of the i-th device be modeled by a state of the HMC $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L \triangleq \{1, \ldots, L\}$. For all $i \in \mathscr{I}_L$, the state space of $\boldsymbol{z}_i(k)$ is assumed to be the finite set $\mathcal{I}_S \triangleq \{0, \ldots, S-1\}$, where $S \geq 2$. Let $(\Omega, \mathcal{F}, \Pr)$ be the ambient probability space over which these processes are defined. In this work, an HMC is taken to be a stochastic process satisfying the first-order Markov property (see (A.1.1)). The Markov property is trivially satisfied by a zeroth-order HMC. A zeroth-order HMC is an independent, identically distributed process, and it will be referred to just as an i.i.d. process. A first-order HMC will be referred to here after as just an HMC.

Let $\boldsymbol{z}(k)$ be the joint process of the HMCs $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$. The statistical nature of $\boldsymbol{z}(k)$ is characterized in Lemma II.2.1. Note that the random processes $\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_L(k)$ are independent if the random variables at time k are mutually independent for every $k \in \mathbb{Z}^+$.

Lemma II.2.1. *Let $z_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, initial state probability vector $\pi_{z_i}(0) \triangleq \left[\Pr(z_i(0) = 0) \ldots \Pr(z_i(0) = S-1)\right]$ and transition probability matrix Π_{z_i}. Then the joint process $z(k)$ is an HMC with state space $\mathcal{I}_S^L \triangleq \underbrace{\mathcal{I}_S \times \cdots \times \mathcal{I}_S}_{L\ times}$ initial state probability vector*

$$\pi_z(0) = \left[\prod_{i=1}^{L} \Pr\bigl(\pi_{z_i}(0) = 0\bigr), \ldots, \prod_{i=1}^{L} \Pr\bigl(\pi_{z_i}(0) = S-1\bigr)\right] = \pi_{z_1}(0) \otimes \cdots \otimes \pi_{z_L}(0), \tag{II.2.1}$$

and transition probability matrix

$$\Pi_z = \Pi_{z_1} \otimes \cdots \otimes \Pi_{z_L},$$

where $\otimes$ is the Kronecker product. The joint process $z(k)$ is irreducible and aperiodic if each of the Markov chains $z_i(k)$ satisfies these properties.

Proof: The initial state probability vector $\pi_z(0)$ follows from the independence of the HMCs z_i, $i \in \mathscr{I}_L$. The rest of the theorem is a direct generalization of Lemma 7.19 in [22]. ■

Remark

Theorem A.1.2 shows that for a finite-state HMC $z(k)$, ergodicity is equivalent to the property of being an aperiodic and irreducible MC. Furthermore, ergodicity is equivalent to the transition probability matrix of $z(k)$ being quasi-positive (see Definition A.1.3). Therefore, if either of these conditions is satisfied then the joint process $z(k)$ is ergodic, which implies the existence of a stationary probability vector, π_z.

Let $\mathcal{D}$ and $\mathcal{C}$ be two finite sets such that the cardinality of $\mathcal{C}$ is strictly less than the cardinality of $\mathcal{D}$. The following definition, based on [19], introduces the notion of a lumping transformation.

Definition II.2.1. Any onto function

$$\phi : \mathcal{D} \to \mathcal{C}$$
$$x \mapsto \phi(x) = y \in \mathcal{C}$$

is called a *lumping transformation.*

The lumping transformation ϕ amalgamates elements from $\mathcal{D}$ to associate them with elements in $\mathcal{C}$, thereby reducing the cardinality of the domain $\mathcal{D}$. When applying a lumping transformation to a finite-state MC, Definition II.2.1 becomes

Definition II.2.2. Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC. Let $\mathcal{I}_\ell \triangleq \{0, \ldots, \ell - 1\}$ be a finite set such that $1 < \ell < S^L$. Any onto, memoryless function

$$\phi : \mathcal{I}_S^L \to \mathcal{I}_\ell$$
$$\boldsymbol{z}(k) \mapsto \phi(\boldsymbol{z}(k)) = j \in \mathcal{I}_\ell$$

is called a (MC) *lumping transformation.*

Since ϕ is measurable, observe that $\phi(\boldsymbol{z}(k)$ is a well defined random variable for each $k \in \mathbb{Z}^+$. When $S = \ell = 2$, the mapping ϕ is called a *structure function* [3, p. 2]. A (MC) lumping transformation will be called from now on just a lumping transformation. Lumping transformations have been extensively studied since the 1950's (see [5,19,23,44,45]) with the purpose to establish conditions under which the Markov property of $\boldsymbol{z}(k)$ is preserved after the lumping transformation. Conditions under which the lumping transformation results in an HMC for every initial state probability vector $\pi_{\boldsymbol{z}}(0)$ and a simple test for checking it are presented in Section II.4.

Clearly, a lumping transformation is a measurable mapping. Thus, for each $k \in \mathbb{Z}^+$ the function ϕ induces a random variable defined by $\boldsymbol{\rho}(k) \triangleq \phi(\boldsymbol{z}(k))$, and having range $\mathcal{I}_\ell$. Since the process $\{\boldsymbol{\rho}(k)\ k \in \mathbb{Z}^+\}$ is not necessarily an MC (see, e.g., [19], [23]), in general it is called a lumped process. The process $\boldsymbol{\rho}(k)$ characterizes the network architecture according to the following definition.

Definition II.2.3. Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC. Let $\mathcal{I}_\ell \triangleq \{0, \ldots, \ell - 1\}$ be a finite set such that $1 < \ell < S^L$. The lumped process

$$\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k)) = j \in \mathcal{I}_\ell$$

is called the (induced) *system availability process.*

The system availability process indicates at each time instant the operational mode of the network architecture. For example, the event $\{\boldsymbol{\rho}(k) = 0\}$ is identified with the correct functioning of the network at the k-th sample period.

The onto and lumpability properties of the function ϕ make it possible to partition the state space of $\boldsymbol{z}(k)$ as follows:

$$\mathcal{I}_S^L = \bigcup_{j=0}^{\ell-1} I_j, \tag{II.2.2}$$

where for each $j \in \mathcal{I}_\ell$, $I_j \triangleq \phi^{-1}(j) = \{\zeta \in \mathcal{I}_S^L : \phi(\zeta) = j\}$. This partition is used in this dissertation to derive all the results regarding the system availability process $\boldsymbol{\rho}(k)$. In this section, a statistical characterization of $\boldsymbol{\rho}(k)$ is given. First, the state probability vector and the availability of the system at steady-state are given in Lemma II.2.2 and Theorem II.2.1, respectively. Second, the one-step transition probabilities and their steady-state values are given in Theorems II.2.2 and II.2.3,

respectively. Observe that all these results are general in the sense that they are independent of whether or not the process $\boldsymbol{\rho}(k)$ is an MC.

In [19], the joint distribution $\Pr(\boldsymbol{\rho}(k) = j_k, \cdots, \boldsymbol{\rho}(0) = j_0)$ is given in terms of a matrix called a *lumping projector matrix.* In particular, this result can be used to calculate $\Pr(\boldsymbol{\rho}(k) = j)$. However, is more natural to provide a formula that calculates this probability in terms of what is assumed to be known. The following theorem gives the state probability vector of the system availability process $\boldsymbol{\rho}(k)$ in terms of the transition probability matrix and the initial state probability vector of the underlying HMC $\boldsymbol{z}(k)$.

Lemma II.2.2. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, initial state probability vector $\pi_{z_i}(0)$ and transition probability matrix Π_{z_i}, and let $\boldsymbol{z}(k)$ be the joint HMC. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_\ell$. Then the state probability vector of $\boldsymbol{\rho}(k)$, $\pi_\rho(k) \triangleq [\Pr(\boldsymbol{\rho}(k) = 0) \ \ldots \ \Pr(\boldsymbol{\rho}(k) = \ell - 1)]$, is characterized by*

$$\Pr(\boldsymbol{\rho}(k) = j) = \sum_{\zeta \in I_j} \prod_{i=1}^{L} \pi_{z_i}(0) \Pi_{z_i}^k \begin{bmatrix} 1_{\{\zeta_i = 0\}} \\ \vdots \\ 1_{\{\zeta_i = S-1\}} \end{bmatrix}, \quad j \in \mathcal{I}_\ell, \tag{II.2.3}$$

where $1_{\{\cdot\}}$ is the indicator function of the event $\{\cdot\}$, and ζ_i is the i-th component of the state ζ.

Proof: Since ϕ is a measurable mapping, for each $j \in \mathcal{I}_\ell$ it follows that

$$\Pr(\boldsymbol{\rho}(k) = j) = \sum_{\zeta \in I_j} \Pr(\boldsymbol{z}(k) = \zeta). \tag{II.2.4}$$

From the assumption that the processes $z_i(k)$ are independent HMCs, the following equalities hold

$$\begin{aligned}\Pr(\boldsymbol{\rho}(k)=j) &= \sum_{\zeta\in I_j}\prod_{i=1}^{L}\Pr(z_i(k)=\zeta_i)\\ &= \sum_{\zeta\in I_j}\prod_{i=1}^{L}\pi_{z_i}(k)\begin{bmatrix}1_{\{\zeta_i=0\}}\\ \vdots\\ 1_{\{\zeta_i=S-1\}}\end{bmatrix}.\end{aligned}$$

Since $z_i(k)$, $i\in\mathscr{I}_L$, is an HMC, it follows that

$$\Pr(\boldsymbol{\rho}(k)=j)=\sum_{\zeta\in I_j}\prod_{i=1}^{L}\pi_{z_i}(0)\Pi_{z_i}^{k}\begin{bmatrix}1_{\{\zeta_i=0\}}\\ \vdots\\ 1_{\{\zeta_i=S-1\}}\end{bmatrix}.$$

Finally, the partition in (II.2.2) and (II.2.4) show that $\sum_{j=0}^{\ell-1}\Pr(\boldsymbol{\rho}(k)=j)=1$. ■

The following definition, based on [35], is related with the correct functioning of the network.

Definition II.2.4. The probability $\Pr(\boldsymbol{\rho}(k)=0)$ is called the *(point) availability* of the network, and $\lim_{k\to\infty}\Pr(\boldsymbol{\rho}(k)=0)$ is called the *availability of the system at steady-state.*

According to this definition, availability indicates how likely it is that the network is working correctly at the specific time k. The steady-state availability indicates the same, but considers it in the long term. The availability of an HMC and an NHMC, which are not induced from a lumping transformation, is addressed in [34, 35]. As indicated in Section II.1, this availability is defined as the probability of the MC

to stay in the set of Up states at time k. Lemma II.2.2 above gives a formula for the probability distribution of the system availability process. In particular, this formula gives the availability of the network (when $j = 0$). The availability of the system at steady-state, which in [34] is called the *asymptotic availability*, is derived in Theorem II.2.1 and shown to be constant under the additional assumption that the independent HMCs $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, are ergodic.

Theorem II.2.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent ergodic HMCs with state space $\mathcal{I}_S$ and stationary probability vector π_{z_i}, and let $\boldsymbol{z}(k)$ be the joint HMC. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_\ell$. Then the availability of the system at steady-state is*

$$\lim_{k\to\infty} \Pr(\boldsymbol{\rho}(k) = 0) = \sum_{\zeta \in I_0} \prod_{i=1}^{L} \pi_{z_i} \begin{bmatrix} 1_{\{\zeta_i=0\}} \\ \vdots \\ 1_{\{\zeta_i=S-1\}} \end{bmatrix}. \tag{II.2.5}$$

Proof: Under the given assumptions, the limit exists since $\lim_{k\to\infty} \pi_{z_i}(0)\Pi_{z_i}^k = \pi_{z_i}$. Equation (II.2.5) follows directly from (II.2.3). ■

The following results give one-step transition probabilities of $\boldsymbol{\rho}(k)$ and the steady-state value of these transition probabilities. To simplify the presentation, the S^L possible states of $\boldsymbol{z}(k)$, labeled in their natural last-lexical order [44], are assigned values in $\mathcal{E} = \{1, 2, \ldots, S^L\}$. Let $\xi : \mathcal{I}_S^L \to \mathcal{E}$ denote the bijective function that maps a state to an integer label in $\mathcal{E}$, such as, $\xi((0, 0, \ldots, 0)) = 1$ and $\xi((S-1, S-1, \ldots, S-1)) = S^L$. Thus, ϕ induces through ξ the partition:

$$\mathcal{E} = \bigcup_{j=0}^{\ell-1} \mathcal{E}_j, \tag{II.2.6}$$

where $\mathcal{E}_j = \{l \in \mathcal{E} : l = \xi(\zeta),\ \zeta \in I_j\}$, and I_j belongs to the partition defined in (II.2.2). Observe that there is a one-to-one relationship between the set of labels $\mathcal{E}_j$ and the set of states I_j. The $S^L \times \ell$ matrix M defined below characterizes this partition, and it is useful in the analysis of the lumping operation. (In [45] a similar matrix is defined and it is called a *lumping matrix.*)

Definition II.2.5. Let $M = [m_{ij}]$ be a matrix of dimension $S^L \times \ell$ such that for $j \in \mathcal{I}_\ell$ and $i \in \mathcal{E}$, m_{ij} is defined as follows:

$$m_{ij} = \begin{cases} 1 & : \text{ whenever } \phi(\xi^{-1}(i)) = j, \\ 0 & : \text{ otherwise.} \end{cases}$$

The matrix M will be called *lumping matrix*, and its columns will be denoted sequentially from left to right as $M_0, \ldots, M_{\ell-1}$.

The following lemma gives conditions under which the probability of the system to stay in any mode is positive. Moreover, the lemma gives another formula for calculating the probability distribution of the lumped process $\boldsymbol{\rho}(k)$ in terms of the lumping matrix M and the state probability vector of the joint HMC $\boldsymbol{z}(k)$.

Lemma II.2.3. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$ and state probability vector $\pi_{z_i}(k)$. Let $\boldsymbol{z}(k)$ be the joint HMC with state probability vector $\pi_z(k)$, and let $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ be the system availability process. If for each $i \in \mathscr{I}_L$ and all $k \in \mathbb{Z}^+$, $\pi_{z_i}(k)$ has positive entries, then* $\Pr(\boldsymbol{\rho}(k) = i) > 0$, $i \in \mathcal{I}_\ell$, *and*

$$\Pr(\boldsymbol{\rho}(k) = i) = \pi_z(k) M_i.$$

Proof: Since $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, are independent, Equation (II.2.1) holds for any sample period $k \geq 0$. Therefore, the assumptions of the lemma imply that the state

probability vector $\pi_z(k)$ has positive entries for any k. Now

$$\begin{aligned}\Pr(\boldsymbol{\rho}(k)=i) &= \Pr\big(\boldsymbol{z}(k)\in \cup_{m\in\mathcal{E}_i}\{\xi^{-1}(m)\}\big)\\ &= \Pr\big(\boldsymbol{z}(k)\in \cup_{m\in\mathcal{E}_i}\{\xi^{-1}(m)\}\big)\\ &= \sum_{m\in\mathcal{E}_i}\Pr\big(\boldsymbol{z}(k)=\xi^{-1}(m)\big)\\ &= \sum_{m\in\mathcal{E}_i}\Pr(\boldsymbol{z}(k)=\xi^{-1}(m))\\ &= \pi_z(k)M_i\end{aligned}$$

Since each column of the lumping matrix M has at least one entry different from zero, then $\Pr(\boldsymbol{\rho}(k)=i)=\pi_z(k)M_i>0$. ∎

The following theorem gives the one-step transition probabilities of the process $\boldsymbol{\rho}(k)$.

Theorem II.2.2. *Let $z_i(k)$, $i\in\mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC $\boldsymbol{z}(k)$ with transition probability matrix $\Pi_z=[p^z_{mn}]$, $m,n\in\mathcal{E}$ and initial state probability vector $\pi_z(0)$. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k)=\phi(\boldsymbol{z}(k))$, the system availability process. If for each $i\in\mathscr{I}_L$ and all $k\in\mathbb{Z}^+$, $\pi_{z_i}(k)$ has positive entries, then the one-step transition probabilities of $\boldsymbol{\rho}(k)$, $p_{ij}(k)=\Pr(\boldsymbol{\rho}(k+1)=j|\boldsymbol{\rho}(k)=i)$, are well-defined and given by*

$$p_{ij}(k)=\frac{1}{\pi_z(0)\Pi_z^kM_i}\sum_{n\in\mathcal{E}_j}\sum_{m\in\mathcal{E}_i}p^z_{mn}\pi_z(0)\Pi_z^k e_m,\quad i,j\in\mathcal{I}_\ell, \tag{II.2.7}$$

where $e_m\in\mathbb{R}^{S^L}$ is the vector of zeros with a single 1 in the m-th position.

Proof: By Lemma II.2.3 and since ϕ is a lumping transformation, it follows that

$$
\begin{aligned}
p_{ij}(k) &= \Pr(\boldsymbol{\rho}(k+1) = j \mid \boldsymbol{\rho}(k) = i) \\
&= \Pr\big(\boldsymbol{z}(k+1) \in \cup_{n\in\mathcal{E}_j}\{\xi^{-1}(n)\} \mid \boldsymbol{z}(k) \in \cup_{m\in\mathcal{E}_i}\{\xi^{-1}(m)\}\big) \\
&= \frac{\Pr\big(\boldsymbol{z}(k+1) \in \cup_{n\in\mathcal{E}_j}\{\xi^{-1}(n)\} \cap \boldsymbol{z}(k) \in \cup_{m\in\mathcal{E}_i}\{\xi^{-1}(m)\}\big)}{\Pr\big(\boldsymbol{z}(k) \in \cup_{m\in\mathcal{E}_i}\{\xi^{-1}(m)\}\big)} \\
&= \frac{\left(\displaystyle\sum_{n\in\mathcal{E}_j}\sum_{m\in\mathcal{E}_i} \Pr\Big(\boldsymbol{z}(k+1) = \xi^{-1}(n) \big| \boldsymbol{z}(k) = \xi^{-1}(m)\Big) \Pr\big(\boldsymbol{z}(k) = \xi^{-1}(m)\big)\right)}{\displaystyle\sum_{m\in\mathcal{E}_i} \Pr\big(\boldsymbol{z}(k) = \xi^{-1}(m)\big)} \\
&= \frac{\displaystyle\sum_{n\in\mathcal{E}_j}\sum_{m\in\mathcal{E}_i} p^z_{mn} \Pr(\boldsymbol{z}(k) = \xi^{-1}(m))}{\displaystyle\sum_{m\in\mathcal{E}_i} \Pr(\boldsymbol{z}(k) = \xi^{-1}(m))} \\
&= \frac{\displaystyle\sum_{n\in\mathcal{E}_j}\sum_{m\in\mathcal{E}_i} p^z_{mn}\pi_z(k)e_m}{\pi_z(k)M_i} \\
&= \frac{\displaystyle\sum_{n\in\mathcal{E}_j}\sum_{m\in\mathcal{E}_i} p^z_{mn}\pi_z(0)\Pi^k_z e_m}{\pi_z(0)\Pi^k_z M_i}.
\end{aligned}
$$

■

Observe that the one-step transition probabilities $p_{ij}(k)$ are given in terms of the transition probabilities of the joint process $\boldsymbol{z}(k)$, which are assumed to be known. The steady-state values of these transition probabilities are given in the following theorem.

Theorem II.2.3. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent, ergodic HMCs with state space $\mathcal{I}_S$ and let $\boldsymbol{z}(k)$ be the joint HMC with transition probability matrix $\Pi_z = [p^z_{mn}]$, $m, n \in \mathcal{E}$ and stationary probability vector π_z. Let ϕ be a lumping transformation and*

$\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_\ell$. Then the steady-state values of the transition probabilities $p_{ij}(k)$, $\overline{p}_{ij} = \lim_{k\to\infty} p_{ij}(k)$, are

$$\overline{p}_{ij} = \lim_{k\to\infty} p_{ij}(k) = \frac{1}{\pi_z M_i} \sum_{n\in\mathcal{E}_j} \sum_{m\in\mathcal{E}_i} p^z_{mn} \pi_z e_m, \quad i,j \in \mathcal{I}_\ell. \tag{II.2.8}$$

Proof: Since each HMC $\boldsymbol{z}_i$, $i \in \mathscr{I}_L$, is ergodic, then by Lemma II.2.1 the joint process $\boldsymbol{z}(k)$ is also ergodic and its stationary probability vector, π_z, has positive components. Thus, for k big enough it follows that $\Pi_z^k = \overline{1}\pi_z$, where $\overline{1} \in \mathbb{R}^{S^L}$ is a vector with ones in each entry. Now, for k big enough it follows that $\Pr(\boldsymbol{\rho}(k) = i) = \pi_z(0)\Pi_z^k M_i = \pi_z(0)\overline{1}\pi_z M_i = \pi_z M_i$, which is positive because ϕ is an onto mapping implying that the columns M_i, $i \in \mathcal{I}_\ell$, have at least one entry equal to 1. Then the claim follows directly by taking limits in (II.2.7). ■

From Theorem II.2.2, it is clear that the one-step transition probability matrix $\Pi_\rho(k) \triangleq [p_{ij}(k)]$ is a stochastic matrix. Theorem II.2.3 says that the matrix $\Pi_\rho(k)$ converges point-wise to a constant stochastic matrix $\overline{\Pi} \triangleq [\overline{p}_{ij}]$, where $\overline{p}_{ij}$ is given in (II.2.8).

The 2-state Case

The case when each device and the network only have two operational modes, that is when $S = \ell = 2$, is of particular interest for applications (see, e.g., [18] and Chapter III). In this case the diagonal entries of the 2×2 transition probability matrix $\Pi_\rho(k)$ become

$$p_{00}(k) = \frac{1}{\pi_z(0)\Pi_z^k M_0} \sum_{m,n\in\mathcal{E}_0} p^z_{mn} \pi_z(0)\Pi_z^k e_m \tag{II.2.9}$$

and

$$p_{11}(k) = \frac{1}{\pi_z(0)\Pi_z^k M_1} \sum_{m,n\in\mathcal{E}_1} p^z_{mn} \pi_z(0)\Pi_z^k e_m, \tag{II.2.10}$$

where $p_{ii}(k) = \Pr(\boldsymbol{\rho}(k+1) = i | \boldsymbol{\rho}(k) = i), i \in \mathcal{I}_2$.

For the 2×2 state case, the following steady-state result is obtained.

Corollary II.2.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be independent, ergodic HMCs with state space $\mathcal{I}_2$ and let $\boldsymbol{z}(k)$ be the joint HMC with transition probability matrix $\Pi_z \triangleq [p^z_{mn}]$, $m, n \in \{1, 2\}$, and stationarity probability vector π_z. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ the system availability process with state space $\mathcal{I}_2$. Then the steady-state values of the transition probabilities $p_{00}(k)$ and $p_{11}(k)$ are:*

$$\overline{p}_{00} = \lim_{k\to\infty} p_{11}(k) = \frac{1}{\pi_z M_0} \sum_{m,n \in \mathcal{E}_0} p^z_{mn} \pi_z e_m$$

and

$$\overline{p}_{11} = \lim_{k\to\infty} p_{22}(k) = \frac{1}{\pi_z M_1} \sum_{m,n \in \mathcal{E}_1} p^z_{mn} \pi_z e_m \ .$$

Proof: This follows from Theorem II.2.3 by taking limits in (II.2.9) and (II.2.10), respectively. ■

This result will be used, in particular, in the proof of Theorem III.5.1, where a test for checking MSES is given.

II.3 TRANSFORMATIONS OF I.I.D. PROCESSES

In this section, a useful result for applications is given, where the network components are characterized by i.i.d. processes (see Example III.4.1). The following lemma is a direct consequence of Lemma II.2.1. It is used to prove Theorem II.3.1, which is the main result of this section.

Lemma II.3.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be independent i.i.d. processes with state space $\mathcal{I}_S$ and state probability vector π_{z_i}. Then the joint process $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_L(k))$*

is i.i.d. with state space $\mathcal{I}_S^L \triangleq \underbrace{\mathcal{I}_S \times \cdots \times \mathcal{I}_S}_{L\ times}$ *and its state probability vector is* $\pi_z \triangleq \pi_{z_1} \otimes \cdots \otimes \pi_{z_L}$.

Proof: This is a special case of Lemma II.2.1. ∎

Theorem II.3.1 below shows that a lumping transformation does preserve the zeroth-order Markov property when applied to an i.i.d. process. This theorem also characterizes the distribution of $\boldsymbol{\rho}(k)$.

Theorem II.3.1. *Let* $\boldsymbol{z}_i(k)$ *be a set of independent i.i.d. processes,* $i \in \mathscr{I}_L$, *with state probability vector* π_{z_i} *and common state space* $\mathcal{I}_S$. *Let* ϕ *be a lumping transformation and* $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ *the system availability process. Then* $\boldsymbol{\rho}(k)$ *is an i.i.d. process, and its probability distribution is*

$$\Pr\{\boldsymbol{\rho}(k) = j\} = \sum_{\zeta \in I_j} \prod_{i=1}^{L} \pi_{z_i} \begin{bmatrix} 1_{\{\zeta_i = 0\}} \\ \vdots \\ 1_{\{\zeta_i = S-1\}} \end{bmatrix}, \; j \in \mathcal{I}_\ell. \tag{II.3.1}$$

Proof: Since the joint process $\boldsymbol{z}(k)$ is i.i.d., the sigma algebras generated by the HMC $\boldsymbol{z}(k)$, $\sigma(\{\boldsymbol{z}(k)\})$, and $k \in \mathbb{Z}^+$ are independent. Thus, the claim follows immediately from the fact that ϕ is a memoryless measurable function implying that $\sigma(\{\boldsymbol{\rho}(k)\}) = \sigma(\{\boldsymbol{z}(k)\})$. Equation (II.3.1) follows from Lemma II.2.2 for the i.i.d. case. ∎

II.4 HMC CONDITIONS

Strong lumpability is the name given to the property under which a transformation of a finite-state HMC results in another reduced finite-state HMC for any initial state probability vector of the underlying process (see Appendix A). A result that gives sufficient conditions for a transformation of an HMC to be an HMC was given

by Kemeny and Snell in 1960 [23]. Theorem II.4.1 below reformulates these conditions for the lumped process $\boldsymbol{\rho}(k)$. The statement of the theorem follows the notation given in [37]. Let $\mathcal{P}$ be the partition determined by the lumping transformation ϕ on $\mathcal{I}_S^L$, that is, $\mathcal{P} \triangleq \{I_0, \ldots, I_{\ell-1}\}$ (see (II.2.2)). Denote by $\Pr(m, I_r)$, $r \in \mathcal{I}_\ell$ the probability of moving from the state ζ of $\boldsymbol{z}(k)$, labeled by $m \in \mathcal{E}$, to the set $I_r \in \mathcal{P}$, that is, $\Pr(m, I_r) \triangleq \sum_{n \in I_r} p^z_{mn}$.

Theorem II.4.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$ and let $\boldsymbol{z}(k)$ be the joint HMC. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_\ell$. Then the process $\boldsymbol{\rho}(k)$ is an HMC for every initial state probability vector $\pi_z(0)$ if and only if for every pair of sets I_r and I_t in $\mathcal{P}$, the probability* $\Pr(m, I_t)$ *has the same value for any m in I_r. This common value is the one-step transition probability corresponding the process $\boldsymbol{\rho}(k)$ of moving from the set I_r into the set I_t.*

Proof: It is a direct application of Kemeny-Snell's Theorem 6.3.2 in [23, p. 124]. ∎

The next result shows that $\boldsymbol{\rho}(k)$ can be an NHMC only for some but not all initial state probability vectors.

Lemma II.4.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be independent HMCs with state space $\mathcal{I}_S$ and let $\boldsymbol{z}(k)$ be the joint HMC with initial state probability vector $\pi_z(0)$. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ the system availability process with state space $\mathcal{I}_\ell$. If the process $\boldsymbol{\rho}(k)$ is an MC for all $\pi_z(0)$ then it is an HMC.*

Proof: This follows directly from [19, pp. 105-106]. ∎

By adding the ergodicity property to the HMCs $\boldsymbol{z}_i$, $i \in \mathscr{I}_L$, in the Lemma II.4.1,

one can obtain the following result.

Theorem II.4.2. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent, ergodic HMCs. Let $\boldsymbol{z}(k)$ be the joint HMC with initial state probability vector $\pi_z(0)$. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ the system availability process with state space $\mathcal{I}_\ell$. If the process $\boldsymbol{\rho}(k)$ is an MC for all $\pi_z(0)$ then it is an ergodic HMC.*

Proof: It follows from Lemma II.4.1 that $\boldsymbol{\rho}(k)$ is an HMC. The ergodicity of $\boldsymbol{\rho}(k)$ follows from Lemma II.2.1. ■

Lemma II.4.2 below gives necessary and sufficient conditions under which a 2-state lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ will be an HMC for all initial state probability vectors $\pi_z(0)$. It is a reformulation, in terms of the lumpability matrix given in Definition II.2.5, of Theorem II.4.1. The result is similar, but not exactly equal to Lemma 1 given in [45]. Moreover, it is easier to apply, since it does not require relabeling of the states.

Lemma II.4.2. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_2$ and let $\boldsymbol{z}(k)$ be the joint HMC with transition probability matrix Π_z. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ the system availability process with state space $\mathcal{I}_2$. Then the process $\boldsymbol{\rho}(k)$ is an HMC for every initial state probability vector $\pi_z(0)$ if and only if there exists constants μ_1 and μ_2 in $[0,1]$ satisfying*

$$\Pi_m M_1 = 1 - \mu_0 \;\; \forall\, m \in \mathcal{E}_0 \quad \textit{and} \quad \Pi_m M_0 = 1 - \mu_1 \;\; \forall\, m \in \mathcal{E}_1,$$

where Π_m is the m-th row of Π_z. Furthermore, the transition probability matrix of $\boldsymbol{\rho}(k)$ is $\Pi_\rho = \left[\begin{smallmatrix} \mu_0 & 1-\mu_0 \\ 1-\mu_1 & \mu_1 \end{smallmatrix}\right]$.

Proof: The set of labels $\mathcal{E}_0$ and $\mathcal{E}_1$ correspond to the set of states I_0 and I_1, respectively, in the partition $\mathcal{P} = \{I_0, I_1\}$ induced by the structure function ϕ. The claim follows directly from Theorem II.4.1 by observing that $\Pr(m, I_1) = \Pi_m M_1 = 1 - \mu_0 \ \forall \ m \in \mathcal{E}_0$ and $\Pr(m, I_0) = \Pi_m M_0 = 1 - \mu_1 \ \forall \ m \in \mathcal{E}_1$. ■

The following is an example of a parallel interconnection known as 1-out-of-2, that is, the interconnection is considered to be working correctly if a least 1 of the devices is working.

Example II.4.1. Consider an interconnection of $L = 2$ devices with upset processes given by an HMC with transition probability matrices

$$\Pi_{z_i} \triangleq \begin{bmatrix} p^i_{11} & p^i_{12} \\ p^i_{21} & p^i_{22} \end{bmatrix}$$

and initial state probability vector $\pi_{z_i}(0)$, $i = 1, 2$. If the system availability process is given by the process $\boldsymbol{\rho}(k)$ defined in Table I, then the state space of $\boldsymbol{z}(k)$ is partitioned as $\mathcal{I}_2^2 = I_0 \bigcup I_1$, where $I_0 = \{(0,0), (0,1), (1,0)\}$ and $I_1 = \{(1,1)\}$.

The lumping matrix is

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

TABLE I: Transformation table for Example II.4.1

$z_1(k)$	$z_2(k)$	$z(k)$	$\xi(z(k))$	$\rho(k) = \phi(z(k))$
0	0	$(0,0)$	1	0
0	1	$(0,1)$	2	0
1	0	$(1,0)$	3	0
1	1	$(1,1)$	4	1

By Lemma II.2.2 the probability that the network is working correctly is

$$\begin{aligned}
\Pr(\rho(k) = 0) &= \pi_{z_1}(0)\Pi_{z_1}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pi_{z_2}(0)\Pi_{z_2}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \pi_{z_1}(0)\Pi_{z_1}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{z_2}(0)\Pi_{z_2}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{z_1}(0)\Pi_{z_1}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \pi_{z_1}(0)\Pi_{z_1}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{z_2}(0)\Pi_{z_2}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{z_1}(0)\Pi_{z_1}^k \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{z_2}(0)\Pi_{z_2}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\end{aligned}$$

The stationary probability vector for $\rho(k)$ exists whenever $z_1(k)$ and $z_2(k)$ are ergodic. Let the stationary probability vectors of these processes be $\pi_{z_1} \triangleq \begin{bmatrix} \pi_{z_1}^1 & \pi_{z_1}^2 \end{bmatrix}$ and $\pi_{z_2} \triangleq \begin{bmatrix} \pi_{z_2}^1 & \pi_{z_2}^2 \end{bmatrix}$, respectively. From Theorem II.2.1 it follows that

$$\begin{aligned}
\lim_{k\to\infty} \Pr(\rho(k) = 0) &= \pi_{z_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pi_{z_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \pi_{z_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{z_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{z_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \pi_{z_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pi_{z_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \pi_{z_1}^1 + \pi_{z_1}^2 \pi_{z_2}^1 \\
&= 1 - \pi_{z_1}^2 (1 - \pi_{z_2}^1).
\end{aligned}$$

To calculate the one-step transition probabilities $p_{00}(k)$ and $p_{11}(k)$ given by (II.2.9) and (II.2.10), respectively, observe first that ϕ partitions the set of labels introduced

in (II.2.6) as $\mathcal{E} = \mathcal{E}_0 \bigcup \mathcal{E}_1$, where $\mathcal{E}_0 = \{1, 2, 3\}$ and $\mathcal{E}_1 = \{4\}$. Thus,

$$p_{00}(k) = \frac{1}{\pi_z(0)\Pi_z^k M_0}\Big((p_{11}^z + p_{12}^z + p_{13}^z)e_1 + (p_{21}^z + p_{22}^z + p_{23}^z)e_2 + (p_{31}^z + p_{32}^z + p_{33}^z)e_3 \Big)\pi_z(0)\Pi_z^k$$

and

$$p_{11}(k) = \frac{1}{\pi_z(0)\Pi_z^k M_1} p_{44}^z \pi_z(0)\Pi_z^k,$$

where $M_0 = [1\ 1\ 1\ 0]^T$ and $M_1 = [0\ 0\ 0\ 1]^T$.

Lemma II.4.2 is used to determine the conditions for $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ to be an HMC. The process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ will be an HMC if and only if the following equalities are satisfied:

$$\Pi_1 M_1 = 1 - \mu_0 = p_{12}^1 \times p_{12}^2$$

$$\Pi_2 M_1 = 1 - \mu_0 = p_{12}^1 \times p_{22}^2$$

$$\Pi_3 M_1 = 1 - \mu_0 = p_{22}^1 \times p_{12}^2.$$

Lemma II.4.2 gives a fourth equation, $\Pi_4 M_0 = 1 - \mu_1 = 1 - p_{22}^1 \times p_{22}^2$, which is not needed since it is dependent on the first three equations. These relations imply that

$$p_{12}^1 \times p_{12}^2 = p_{12}^1 \times p_{22}^2 = p_{22}^1 \times p_{12}^2.$$

If these equalities do not hold, then $\boldsymbol{\rho}(k)$ will not be an HMC for all initial state probability vectors $\pi_z(0)$. By Lemma II.4.1, however, $\boldsymbol{\rho}(k)$ could be an NHMC for some but not all $\pi_z(0)$. Whenever the stationary probability vector for $\boldsymbol{z}(k)$ exists then as $k \to \infty$, $\boldsymbol{\rho}(k)$ is characterized by a constant transition probability matrix as shown in Corollary II.2.1. Assume the 2-state HMCs $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ have transition

probability matrices Π_{z_1} and Π_{z_2} with positive entries. Then, the necessary and sufficient conditions for $\boldsymbol{\rho}(k)$ to be an HMC are

$$p^1_{12} = p^1_{22} \text{ and } p^2_{12} = p^2_{22}.$$

In this case, Π_{z_1} and Π_{z_2} have the form

$$\Pi_{z_1} \triangleq \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}, \quad \Pi_{z_2} \triangleq \begin{bmatrix} b & 1-b \\ b & 1-b \end{bmatrix}, \tag{II.4.1}$$

where $a = 1 - p^1_{12}$ and $b = 1 - p^2_{12}$ with $a, b \in]0, 1[$. If the initial state probability vectors are $\pi_{z_1}(0) = [a \;\, 1-a]$ and $\pi_{z_2}(0) = [b \;\, 1-b]$, then the processes $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ with transition probability matrices given in (II.4.1) are i.i.d. processes. Since $\mu_0 = 1 - \mu_1$, then Π_ρ has equal rows, and $\pi_\rho(0) = [\mu_0 \;\, 1-\mu_0]$. Thus, $\boldsymbol{\rho}(k)$ is an i.i.d process for $k \geq 1$.

This example shows, in particular, that for the 2-state MCs $\boldsymbol{z}_1(k)$ and $\boldsymbol{z}_2(k)$ with positive entries in their transition probabilities, the 2-state lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, where ϕ is the 1-out-of-2 structure function, can not be an HMC for all $\pi_{z_1}(0)$ and $\pi_{z_2}(0)$. □

II.5 SUMMARY

In this chapter, the concepts of a lumping transformation, ϕ, and the system availability process induced by ϕ, $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, have been introduced formally. A statistical characterization of this process was given. In particular, its state probability vector was derived and conditions were given under which it is an ergodic HMC. Furthermore, it was established that the process $\boldsymbol{\rho}(k)$ has well-defined one-step transition probabilities. These transition probabilities and their steady-state

values were computed. Conditions under which a transformation of a zeroth-order HMC result in a zeroth order HMC were also given. In addition, a reformulation of Kemeny-Snell's Theorem 6.3.2 in [23], that uses the concept of lumping matrix, was used to check when the system availability process results in an HMC. Finally, an example was presented to demonstrate some of the results obtained in this chapter.

CHAPTER III

DISCRETE-TIME JUMP LINEAR SYSTEMS DRIVEN BY LUMPED PROCESSES

III.1 INTRODUCTION

This chapter analyzes the MSS and the output performance metrics of a JLS driven by a lumped process. The JLS represents the closed-loop control system dynamics and a network architecture comprised of $L \geq 2$ devices. It is assumed that each device forming the system is in one of a finite number of modes of operation. Each operational mode is identified with a state of either an i.i.d. finite state process or an HMC. In particular, suppose that a harsh environment randomly switches each device's mode of operation in the set $\mathcal{I}_S$ such that the mode of operation of the i-th device at time k is represented by a state of the MC process $\boldsymbol{z}_i(k)$. From the point of view of the closed-loop control system, it is important to characterize the modes of operation of the fault-tolerant network since they determine the closed-loop system's modes. The network's modes at time k are characterized by a state of the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, where ϕ is a lumping transformation, and $\boldsymbol{z}(k)$ is the joint MC $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \ldots, \boldsymbol{z}_L(k))$. It is assumed that $\boldsymbol{\rho}(k)$ drives the JLS taking values in the set $\mathcal{I}_\ell$, thereby switching the modes of the closed-loop control system. It is known that the process $\boldsymbol{\rho}(k)$ might not be an MC [19]. In this chapter, a class of networks that result in $\boldsymbol{\rho}(k)$ being either an i.i.d. process or an HMC or a lumped NHMC or an NMC is characterized. New results concerning

the MSS and the performance analysis are given when $\rho(k)$ is not an HMC. Most of the JLS literature has addressed the case where $\rho(k)$ is an HMC that is not the result of a lumping transformation (see, e.g., [8,11,14,46]). Some of these papers and others have presented results for i.i.d. switching processes (see, e.g., [8,10,11,20,27] and their references). Since an i.i.d. process also satisfies the first order Markov property, all the known results would apply in this case. However, simpler formulas can be derived that do not trivially follow from the known Markov results. This has been commented on, e.g., [10,11] regarding stability criterion for an i.i.d. JLS. In particular, the performance of a JLS driven by an i.i.d. process has been defined and addressed in [27]. In Section III.3, the *output performance metrics* based in [8] and [17] are defined and new formulas are derived for these metrics. The analysis of MSS when $\rho(k)$ is either a lumped NHMC or an NMC has not been addressed before. In [11], a test for MSES of a JLS driven by a non-lumped NHMC has been given. A relatively recent publication by Dragan and Morozan, [9], analyzes different types of MSESs of a JLS driven by either an HMC or an NHMC that are not lumped processes. One of the objectives of this chapter is to give analytical tools to analyze the MSS of a JLS driven by either a lumped NHMC or an NMC.

The rest of the chapter is organized as follows. A brief review of the HMC results is done in Section III.2. Next, in Section III.3, a JLS driven by the process $\rho(k)$ when it is i.i.d. is addressed. New analytic expressions for the output performance metrics, including their sensitivity analysis, are also given. In Section III.4, an example is given to demonstrate the results derived in Section III.3. The case when the process $\rho(k)$ is either a lumped NHMC or an NMC is addressed in Section III.5. Finally, a summary of the chapter is given in Section III.6.

III.2 PRELIMINARIES

A brief review, based on [17] and [39], of the MSS and the output performance metrics of a JLS driven by an HMC is given in this section. Let Ξ_z represent the set of all initial state probability vectors $\pi_z(0)$, and Φ_z be a proper subset of Ξ_z. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ a lumped process with state space $\mathcal{I}_\ell$. Consider the JLS driven by $\boldsymbol{\rho}(k)$:

$$\boldsymbol{x}(k+1) = A_{\rho(k)}\boldsymbol{x}(k) + B_{\rho(k)}\boldsymbol{w}(k), \;\; \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{III.2.1a}$$

$$\boldsymbol{y}(k) = C_{\rho(k)}\boldsymbol{x}(k), \tag{III.2.1b}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $\boldsymbol{y}(k) \in \mathbb{R}^p$, $\boldsymbol{x}_0$ is a second-order random vector, and $\boldsymbol{w}(k) \in \mathbb{R}^q$ is a zero mean, second-order, wide sense, stationary process with identity covariance matrix I_q and independent of $\boldsymbol{\rho}(k)$ and $\boldsymbol{x}_0$. Assume that $\boldsymbol{\rho}(k)$ is an ergodic HMC for all $\pi_z(0) \in \Xi_z$ with transition probability matrix Π_ρ and state probability vector $\pi_\rho(k) = [\Pr(\boldsymbol{\rho}(k) = 0) \; \ldots \; \Pr(\boldsymbol{\rho}(k) = \ell - 1)]$. Let Ξ_ρ be the set of all initial state probability vectors of $\boldsymbol{\rho}(k)$. A standard MSS definition for the HMC JLS (III.2.1) follows [39].

Definition III.2.1. The HMC JLS (III.2.1) is MSS if there exists a non-negative constant α such that for any initial state probability vector $\pi_\rho(0) \in \Xi_\rho$ and any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ with finite second moment, it follows that $\lim_{k\to\infty} E\{\|\boldsymbol{x}(k)\|^2\} = \alpha$. If $\boldsymbol{w}(k) = 0$ for $k \in \mathbb{Z}^+$ then $\alpha = 0$.

Remarks

1. It is reported in [39] that according to [8] and [12], the condition of ergodicity for $\boldsymbol{\rho}(k)$ is needed in Definition III.2.1 to ensure the uniqueness of the limit

$\lim_{k\to\infty} E\{\|\boldsymbol{x}(k)\|^2\} = \alpha$ when $\boldsymbol{w}(k) \neq 0$. On the other hand, when $\boldsymbol{w}(k) = 0$, this condition is not needed.

2. For each $k \in \mathbb{Z}^+$ define $Q(k) \triangleq E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\}$. When $\boldsymbol{w}(k) = 0$, it is known that $\lim_{k\to\infty} E\{\|\boldsymbol{x}(k)\|^2\} = 0$ is equivalent to $\lim_{k\to\infty} Q(k) = 0$ [31]. For $\boldsymbol{w}(k) \neq 0$, MSS is defined in [7,8] similarly by requiring the existence of a positive semi-definite matrix Q (independent of $\boldsymbol{x}(0)$ and $\pi_\rho(0)$) such that $\lim_{k\to\infty} Q(k) = Q$. This condition will be used here.

3. In [10,11], MSS and other types of stability are defined with respect to Φ_ρ, that is, a restricted set of initial state probability vectors $\pi_\rho(0)$.

A test for MSS is given next.

Lemma III.2.1. *The HMC JLS (*III.2.1*) is MSS if and only if the spectral radius of $\mathcal{A}_2$ is less than 1, where*

$$\mathcal{A}_2 \triangleq \mathrm{diag}(A_0^T \otimes A_0^T, ..., A_{\ell-1}^T \otimes A_{\ell-1}^T)(\Pi_\rho \otimes I_{n^2}). \qquad \text{(III.2.2)}$$

Proof: See [7]. ∎

The analysis of the output performance metrics of the HMC JLS (III.2.1) summarized below is extensively developed in [17] and [46]. The output performance is defined as follows:

$$J = \begin{cases} J_w \triangleq \lim_{k\to\infty} E\{||\boldsymbol{y}(k)||^2\} & , \boldsymbol{w}(k) \neq 0 \\ J_0 \triangleq E\left\{\sum_{k=0}^{\infty} ||\boldsymbol{y}(k)||^2\right\} & , \boldsymbol{w}(k) = 0, \end{cases}$$

where J_w is called the steady-state mean output power and J_0 the mean output energy. When the system is MSS, analytic expressions for J_w and J_0 exist. These

expressions are given in terms of the following matrix:

$$Q = \varphi^{-1}\Big(\big(I_{\ell n^2} - \mathcal{A}_2\big)\cdot\varphi(\mathcal{C})\Big), \tag{III.2.3}$$

where $\mathcal{A}_2$ is defined in (III.2.2) and $\mathcal{C} \triangleq \left[C_0^T C_0, ..., C_{\ell-1}^T C_{\ell-1}\right]$. The function φ is defined as

$$\varphi(Q) \triangleq [\text{vec}^T(Q_0) \quad \text{vec}^T(Q_1) \cdots \text{vec}^T(Q_{\ell-1})]^T \in \mathbb{R}^{\ell n^2},$$

where "vec" denotes the column stacking operator and, since $\big(I_{\ell n^2} - \mathcal{A}_2\big)\cdot\varphi(\mathcal{C})$ is a square matrix, φ^{-1} yields the contrary effect than φ. The matrix Q_i, $i = 0, ..., \ell - 1$ is comprised of the column vectors $q_{ij} \in \mathbb{R}^n$, $j = 1, ..., n$, that is, $Q_i \triangleq [q_{i1} \;\; q_{i2} \; \dots \; q_{in}]$.

Since the HMC $\boldsymbol{\rho}(k)$ is ergodic, it has a stationary probability vector (see Theorem A.1.3). Denote it by $\pi_\rho \triangleq [\pi_\rho^0, ..., \pi_\rho^{\ell-1}]$. When the JLS (III.2.1) is MSS, it is shown in [17] that

$$J_w = \text{tr}(B^w Q^w), \tag{III.2.4}$$

and

$$J_0 = \text{tr}(X^0 Q^0), \tag{III.2.5}$$

where $B^w \triangleq BB^T$, $Q^w = \sum_{i=0}^{\ell-1} Q_i \pi_\rho^i$, $X^0 = E\{\boldsymbol{x}_0 \boldsymbol{x}_0^T\}$ and $Q^0 = \sum_{i=0}^{\ell-1}(Q_i \Pr(\boldsymbol{\rho}(0) = i))$.

The following section addresses the MSS and the performance analysis of the JLS (III.2.1) when the process $\boldsymbol{\rho}(k)$ is i.i.d. Simpler formulas are derived for J_w and J_0 that do not trivially follow from (III.2.4) and (III.2.5).

III.3 JLS DRIVEN BY I.I.D. PROCESSES

Analysis of MSS

The results derived in this section hold for any i.i.d. process $\boldsymbol{\rho}(k)$ that drives the JLS (III.2.1) including the case when $\boldsymbol{\rho}(k)$ is the result of a lumping transformation.

Recall that in Section II.3, conditions under which the process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ is an i.i.d. process were given. In what follows, the i.i.d. process $\boldsymbol{\rho}(k)$ is assumed to have states in the set $\mathcal{I}_\ell$ such that for $i \in \mathcal{I}_\ell$, $p_i \triangleq \Pr(\boldsymbol{\rho}(k) = i)$. The MSS definition applied to the i.i.d. case is given next.

Definition III.3.1. The i.i.d. JLS (III.2.1) is MSS if there exists a non-negative constant α such that for any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ with finite second moment, it follows that $\lim_{k\to\infty} E\{\|\boldsymbol{x}(k)\|^2\} = \alpha$. If $\boldsymbol{w}(k) = 0$ for $k \in \mathbb{Z}^+$ then $\alpha = 0$.

When $\boldsymbol{\rho}(k)$ is an i.i.d. process, then $\Pr(\boldsymbol{\rho}(0) = i) = \Pr(\boldsymbol{\rho}(k) = i)$ for all $k \geq 1$. Therefore, the expression for "all initial state probability vectors" has been removed from Definition III.2.1. Moreover, due to the remark given after Definition III.2.1, Definition III.3.1 is equivalent to the existence of a positive semi-definite matrix Q (independent of $\boldsymbol{x}(0)$ and $\pi_\rho(0)$) such that $\lim_{k\to\infty} Q(k) = Q$.

A test for MSS is given next.

Lemma III.3.1. *The i.i.d. JLS* (III.2.1) *is MSS if and only if the spectral radius of* $\mathcal{A}$ *is less than 1, where*

$$\mathcal{A} \triangleq \sum_{i=0}^{\ell-1} (A_i \otimes A_i) p_i. \tag{III.3.1}$$

Proof: See [10]. ∎

The matrix $\mathcal{A}$ in (III.3.1) has dimension $n^2 \times n^2$. The i.i.d. process $\boldsymbol{\rho}(k)$ can be represented by the $\ell \times \ell$ transition probability matrix

$$\Pi_\rho \triangleq \begin{bmatrix} p_0 & \cdots & p_{\ell-1} \\ \vdots & \ddots & \vdots \\ p_0 & \cdots & p_{\ell-1} \end{bmatrix}.$$

In this case, the matrix $\mathcal{A}_2$ defined in (III.2.2) has dimension $\ell n^2 \times \ell n^2$. Therefore, the corresponding MSS test for (III.2.1) would require the computation of the spectral radius of a matrix with dimension $\ell n^2 \times \ell n^2$. The lower dimension of $\mathcal{A}$ in (III.3.1) is one benefit of working with an i.i.d. JLS in Lemma III.3.1 as opposed to an MSS stability test for an HMC JLS. An additional benefit is that an equivalent MSS test for an HMC JLS requires solving a set of coupled algebraic generalized Lyapunov equations [8, Theorem 3.9]. For the i.i.d. JLS only one algebraic generalized Lyapunov equation needs to be solved [8, Corollary 3.26], [11].

Two useful properties of the i.i.d. JLS are introduced in Lemma III.3.2. Let $\mathcal{F}_k \triangleq \sigma(\{\boldsymbol{\rho}(k)\})$ denote the σ-algebra generated by $\boldsymbol{\rho}(k),\ k \in \mathbb{Z}^+$.

Lemma III.3.2. *Suppose the JLS* (III.2.1) *is driven by the i.i.d. process $\boldsymbol{\rho}(k)$. Then $\boldsymbol{x}(k)$ and $\mathbf{1}_{\{\boldsymbol{\rho}(k)=i\}}$ are independent for all $i \in \mathcal{I}_\ell$ and $k \geq 1$. In addition, for each $k \in \mathbb{Z}^+$ the random variables $\boldsymbol{x}(k)$ and $\boldsymbol{w}(k)$ are independent.*

Proof: From (III.2.1a) it follows that $\boldsymbol{x}(k)$ is $\mathcal{F}_{k-1}$-measurable for $k \geq 1$. Since $\mathbf{1}_{\{\boldsymbol{\rho}(k)=i\}}$ is $\mathcal{F}_k$-measurable for all $i \in \mathcal{I}_\ell$, the claim follows because $\boldsymbol{\rho}(k)$ is an i.i.d. process implying that the σ-algebras $\mathcal{F}_{k-1}$ and $\mathcal{F}_k$ are independent. The independence between $\boldsymbol{x}(k)$ and $\boldsymbol{w}(k)$ follows from the assumption that $\boldsymbol{w}(k)$ is independent of $\boldsymbol{\rho}(k)$. ∎

Lemma III.3.3. *If the i.i.d. JLS* (III.2.1) *is MSS then Q satisfies*

$$Q = \sum_{i=0}^{\ell-1} A_i Q A_i^T p_i + \sum_{i=0}^{\ell-1} B_i B_i^T p_i. \tag{III.3.2}$$

and

$$Q = \mathrm{vec}^{-1}\left((I_{n^2} - \mathcal{A})^{-1} \mathrm{vec}\left(\mathcal{B}\right) \right), \tag{III.3.3}$$

where $\mathcal{A}$ is defined in (III.3.1) and

$$\mathcal{B} = \sum_{i=0}^{\ell-1} B_i B_i^T p_i. \tag{III.3.4}$$

Proof: Since $\boldsymbol{x}(k)$ and $\boldsymbol{w}(k)$ are independent, and $\boldsymbol{w}(k)$ is zero mean with identity covariance, it follows that

$$\begin{aligned}
E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} &= E\big\{\big(A_{\rho(k-1)}\boldsymbol{x}(k-1) + B_{\rho(k-1)}\boldsymbol{w}(k-1)\big)\cdot \\
&\quad \big(A_{\rho(k-1)}\boldsymbol{x}(k-1) + B_{\rho(k-1)}\boldsymbol{w}(k-1)\big)^T\big\} \\
&= E\big\{A_{\rho(k-1)}\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1)A_{\rho(k-1)}^T\big\}+ \\
&\quad E\big\{B_{\rho(k-1)}\boldsymbol{w}(k-1)\boldsymbol{w}^T(k-1)B_{\rho(k-1)}^T\big\} \\
&= E\bigg\{\sum_{i=0}^{\ell-1} A_i\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1)A_i^T\mathbf{1}_{\{\rho(k-1)=i\}}\bigg\}+ \\
&\quad E\bigg\{\sum_{i=0}^{\ell-1} B_iB_i^T\mathbf{1}_{\{\rho(k-1)=i\}}\bigg\}.
\end{aligned}$$

Using Lemma III.3.2 yields

$$E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} = E\bigg\{\sum_{i=0}^{\ell-1} A_i\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1)A_i^T\bigg\}p_i + \sum_{i=0}^{\ell-1} B_iB_i^Tp_i.$$

MSS of the i.i.d. JLS (III.2.1) makes it possible to take limits as $k \to \infty$ on both sides of this equation resulting in (III.3.2). Finally, (III.3.3) follows from (III.3.2). ∎

Derivation of J_w and J_0

To characterize the output performance metrics of the i.i.d. JLS (III.2.1), analytic expressions are derived. These expressions have been given in (III.2.4) and (III.2.5), based on [17] when the lumped process $\boldsymbol{\rho}(k)$ is an HMC. Since an i.i.d. process is an HMC of order zero, the results in [17] can also be used when $\boldsymbol{\rho}(k)$ is i.i.d.;

however, new simpler and lower dimensional formulas are derived here. The output performance metrics for the JLS (III.2.1) are redefined as follows:

$$J = \begin{cases} J_w & \triangleq \lim_{k\to\infty} E\{||\boldsymbol{y}(k)||^2\} \quad , \ w(k) \neq 0 \\ J_0 & \triangleq \sum_{k=0}^{\infty} E\{||\boldsymbol{y}(k)||^2\} \quad , \ w(k) = 0. \end{cases}$$

Remark

In the definition of J_0, the order of the sum and the expectation has been changed with respect to [17] to match the order given in [8].

Theorem III.3.1. *If the i.i.d. JLS (*III.2.1*) is MSS then $J_w < \infty$, and*

$$J_w = \sum_{i=0}^{\ell-1} \mathrm{tr}(C_i Q C_i^T) p_i, \tag{III.3.5}$$

*where Q is given in (*III.3.3*).*

Proof: From (III.2.1b) it follows that

$$\begin{aligned} E\{\| \boldsymbol{y}(k) \|^2\} &= E\{\boldsymbol{x}^T(k) C_{\rho(k)}^T C_{\rho(k)} \boldsymbol{x}(k)\} \\ &= E\{\mathrm{tr}(C_{\rho(k)}^T C_{\rho(k)} \boldsymbol{x}(k) \boldsymbol{x}^T(k))\} \\ &= E\left\{\mathrm{tr} \sum_{i=0}^{\ell-1} C_i \boldsymbol{x}(k) \boldsymbol{x}^T(k) C_i^T \mathbf{1}_{\{\rho(k)=i\}}\right\}. \end{aligned}$$

By Lemma III.3.2

$$E\{\| \boldsymbol{y}(k) \|^2\} = \mathrm{tr}\left\{\sum_{i=0}^{\ell-1} C_i E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} C_i^T\right\} p_i.$$

Since the JLS (III.2.1) is MSS, taking limits as $k \to \infty$ on both sides of this equation gives (III.3.5). Equation (III.3.3) follows from (III.3.2). ■

For each $k \in \mathbb{Z}^+$ define $M(k) \triangleq \sum_{i=0}^{k} Q(i)$. When $\boldsymbol{w}(k) = 0$, the following lemma gives another equivalent characterization of MSS for the i.i.d. JLS (III.2.1).

Lemma III.3.4. *The i.i.d. JLS (*III.2.1*) with $\boldsymbol{w}(k) = 0$ is MSS if and only if there exists a positive semi-definite matrix $M \in \mathbb{R}^{n \times n}$ such that $M = \sum_{k=0}^{\infty} Q(k)$.*

Proof: By Theorem 2 in [15] the following result holds for each $k \in \mathbb{Z}^+$:

$$\frac{1}{n} E\{\|\boldsymbol{x}(k)\|^2\} \leq \|E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\}\| \leq E\{\|\boldsymbol{x}(k)\|^2\}. \tag{III.3.6}$$

Suppose that the i.i.d. JLS (III.2.1) is MSS. From the second inequality of (III.3.6) it follows that

$$\begin{aligned}
\|M(n) - M(m)\| &= \left\|\sum_{m+1}^{n} Q(i)\right\| \\
&= \left\|\sum_{m+1}^{n} E\{\boldsymbol{x}(i)\boldsymbol{x}^T(i)\}\right\| \\
&\leq \sum_{m+1}^{n} \|E\{\boldsymbol{x}(i)\boldsymbol{x}^T(i)\}\| \\
&\leq \sum_{m+1}^{n} E\{\|\boldsymbol{x}(i)\|^2\}.
\end{aligned}$$

Since MSS is equivalent to stochastic stability, that is, $\sum_{k=0}^{\infty} E\left\{\|\boldsymbol{x}(k)\|^2\right\} < \infty$ [21], then the sequence $\sum_{i=0}^{n} E\{\|\boldsymbol{x}(i)\|^2\}$ is Cauchy which, due to the inequality above, implies that $M(k)$ is also Cauchy. This proves the convergence of the series $\sum_{k=0}^{\infty} Q(k)$ since the normed space of $n \times n$ matrices is complete.

Assume now that the series $\sum_{k=0}^{\infty} Q(k)$ is convergent. Then $\lim_{k \to \infty} Q(k) = 0$. Therefore, the first inequality in (III.3.6) implies that $\lim_{k \to \infty} E\{\|\boldsymbol{x}(k)\|^2\} = 0$, that is, the i.i.d. JLS (III.2.1) is MSS. ■

Let $E(\boldsymbol{x}(0)\boldsymbol{x}^T(0))$ be denoted by X^0. The following lemma gives a formula for the matrix M.

Lemma III.3.5. *If the i.i.d. JLS (*III.2.1*) with $\boldsymbol{w}(k) = 0$ is MSS then*

$$M = \sum_{i=0}^{\ell-1} A_i M A_i^T p_i + X^0 \tag{III.3.7}$$

and

$$M = \text{vec}^{-1}\big((I_{n^2} - \mathcal{A})^{-1}\text{vec}(X^0)\big). \tag{III.3.8}$$

Proof: Equation (III.3.7) is derived as follows

$$\begin{aligned}
M &= X^0 + \sum_{k=1}^{\infty} E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} \\
&= X^0 + \sum_{k=1}^{\infty}\sum_{i=0}^{\ell-1} A_i E\{\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1)\} A_i^T p_i \\
&= X^0 + \sum_{i=0}^{\ell-1} A_i \Big(\sum_{k=1}^{\infty} E\{\boldsymbol{x}(k-1)\boldsymbol{x}^T(k-1)\}\Big) A_i^T p_i \\
&= X^0 + \sum_{i=0}^{\ell-1} A_i \Big(\sum_{k=1}^{\infty} Q(k-1)\Big) A_i^T p_i \\
&= X^0 + \sum_{i=0}^{\ell-1} A_i M A_i^T p_i.
\end{aligned}$$

Finally (III.3.8) follows from (III.3.7). ■

Theorem III.3.2. *If the i.i.d. JLS (*III.2.1*) with $\boldsymbol{w}(k) = 0$ is MSS then $J_0 < \infty$,*

and

$$J_0 = \sum_{i=0}^{\ell-1} \text{tr}(C_i M C_i^T) p_i, \tag{III.3.9}$$

*where M satisfies (*III.3.7*).*

Proof: From (III.2.1b) and Lemmas III.3.2 and III.3.4 it follows that

$$\begin{aligned}
\sum_{k=0}^{\infty} E\{\|\boldsymbol{y}(k)\|^2\} &= \sum_{k=0}^{\infty} \text{tr}\Big(\sum_{i=0}^{\ell-1} C_i^T C_i E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\}\Big) p_i \\
&= \text{tr}\Big(\sum_{i=0}^{\ell-1} C_i^T C_i \sum_{k=0}^{\infty} E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} p_i\Big)
\end{aligned}$$

$$= \operatorname{tr} \sum_{i=0}^{\ell-1} C_i^T C_i M p_i$$
$$= \sum_{i=0}^{\ell-1} \operatorname{tr}(C_i M C_i^T) p_i.$$

■

Sensitivity performance analysis

When the i.i.d. JLS (III.2.1) is MSS, the output performance metrics J_w and J_0 given in (III.3.5) and (III.3.9), respectively, can be seen as the real-valued functions $J_w(p)$ and $J_0(p)$, mapping the mean-square stabilizing subset of $[0,1]^\ell \triangleq \underbrace{[0,1] \times \cdots \times [0,1]}_{\ell \text{ times}}$ into $\mathbb{R}$, where $p \triangleq (p_0, \ldots, p_{\ell-1})$ and $p_j \triangleq \Pr\{\boldsymbol{\rho}(k) = j\}$, $j \in \mathcal{I}_\ell$. In fact, from Theorems III.3.1 and III.3.2 it follows that the performance metrics are rational functions of these mean-square stabilizing probabilities. Moreover, the following lemma makes possible the evaluation of their partial derivatives.

Lemma III.3.6. *Let $p^* \in [0,1]^\ell$ be such that the i.i.d. JLS (*III.2.1*) is MSS. Then there exist a neighborhood of p^* such that for each p in this neighborhood the i.i.d. JLS (*III.2.1*) remains MSS.*

Proof: The result follows because the spectral radius of the matrix $\mathcal{A}$ is a continuous function of p. ■

The sensitivity of J_w and J_0 with respect to p_j are defined next.

Definition III.3.2. Let $p^* \in [0,1]^\ell$ be such that the i.i.d. JLS (III.2.1) is MSS. The sensitivity of J_w and J_0 with respect to p_j are denoted by $S_w(p_j)$ and $S_0(p_j)$,

respectively, and are given by

$$S_w(p_j) = \frac{p_j}{J_w(p)} \left.\frac{\partial J_w(p)}{\partial p_j}\right|_{p=p^*},$$
$$S_0(p_j) = \frac{p_j}{J_0(p)} \left.\frac{\partial J_0(p)}{\partial p_j}\right|_{p=p^*}.$$

Hence, the he sensitivity and the partial derivatives differ by a constant factor. In Theorem III.3.3 below, the partial derivatives of J_w and J_0 with respect to p_j, $j \in \mathcal{I}_\ell$, are evaluated at the mean-square stabilizing probability $p^* \triangleq (p_0^*, \ldots, p_{\ell-1}^*) \in [0,1]^\ell$. A less local result is given in Theorem III.3.4, where the intervals over which the performance metric is monotonic are characterized for a special case.

Theorem III.3.3. *Let $p^* \in [0,1]^\ell$ be such that the i.i.d. JLS (*III.2.1*) is MSS and let $Q^* \triangleq Q(p^*)$ and $M^* \triangleq M(p^*)$ be the values of Q and M at this point, respectively. Then for each $j \in \mathcal{I}_\ell$*

$$\left.\frac{\partial J_w(p)}{\partial p_j}\right|_{p=p^*} = \left(\sum_{i=0}^{\ell-1} \operatorname{tr}\left(C_i \left.\frac{\partial Q(p)}{\partial p_j}\right|_{p=p^*} C_i^T\right) p_i^*\right) + \operatorname{tr}(C_j Q^* C_j^T), \tag{III.3.10}$$

$$\left.\frac{\partial J_0(p)}{\partial p_j}\right|_{p=p^*} = \left(\sum_{i=0}^{\ell-1} \operatorname{tr}\left(C_i \left.\frac{\partial M(p)}{\partial p_j}\right|_{p=p^*} C_i^T\right) p_i^*\right) + \operatorname{tr}(C_j M^* C_j^T), \tag{III.3.11}$$

where

$$\left.\frac{\partial Q(p)}{\partial p_j}\right|_{p=p^*} = \operatorname{vec}^{-1}\Big((I_{n^2} - \mathcal{A})^{-1}\big((A_j \otimes A_j)(I_{n^2} - \mathcal{A})^{-1} \operatorname{vec}(\mathcal{B}) + \operatorname{vec}(B_j B_j^T)\big)\Big),$$
$$\left.\frac{\partial M(p)}{\partial p_j}\right|_{p=p^*} = \operatorname{vec}^{-1}\Big((I_{n^2} - \mathcal{A})^{-1}(A_j \otimes A_j)(I_{n^2} - \mathcal{A})^{-1}\operatorname{vec}(X^0)\Big)$$

*with $\mathcal{A}$ and $\mathcal{B}$ defined in (*III.3.1*) and (*III.3.4*), respectively.*

Proof: The proof is given only for J_w since the other case is similar. Since J_w is a rational function, it is infinitely differentiable at any point where it is well-defined. The partial derivatives of J_w and Q follow by direct application of $\frac{\partial}{\partial p_j}$ and noting

that the trace, vec, and vec^{-1} are linear transformations. Thus, these transformations commute with the partial derivative. ■

To present a less local result, consider the $\ell = 2$ case. Then the i.i.d. JLS (III.2.1) has two modes of operation that are selected by $\boldsymbol{\rho}(k)$. The probability $p_1 = \Pr(\boldsymbol{\rho}(k) = 1)$ can be interpreted as the probability that the closed-loop system is in the upset state, and the performance J_w can be seen as a function of this probability. Let $\mathcal{U}$ denote the union of all the disjoint subintervals of $[0, 1]$ containing the values of p_1 that result in (III.2.1) being MSS. When $\mathcal{U}$ is nonempty, the end points of each open subinterval are consecutive points taken from the sequence $0 \leq \hat{p}_0 < \hat{p}_1 < \cdots < \hat{p}_{r-1} \leq 1$, where $\hat{p}_i$, $i = 0, \ldots, r-1$, satisfy one or more of the following conditions: $\hat{p}_0 = 0$ ($\hat{p}_{r-1} = 1$) when A_0 (A_1) is Hurwitz; $\hat{p}_i$ are the values of p_1 that result in a unit spectral radius for $\mathcal{A}$; and $\hat{p}_i$ can also be the distinct real roots of $\frac{dJ_w(p_1)}{dp_1}$. If $\hat{p}_0 = 0$ ($\hat{p}_{r-1} = 1$), then its subinterval is closed on the left (right).

Theorem III.3.4. *When the i.i.d. JLS (*III.2.1*) is MSS, the sign $\frac{dJ_w(p_1)}{dp_1}$ is constant over each subinterval in $\mathcal{U}$, that is, $J_w(p_1)$ is monotonic on these subintervals.*

Proof: Since J_w and $\frac{dJ_w(p_1)}{dp_1}$ are rational functions of p_1, the only possible endpoints for the subintervals are those in $\mathcal{U}$. ■

Mean square stability for the JLS (III.2.1) driven by the HMC $\boldsymbol{\rho}(k)$ requires one to take into account all initial state probability vectors $\boldsymbol{\rho}(0)$. If the HMC JLS (III.2.1) is MSS, from (III.2.5) other partial derivatives can be derived by observing that J_0 can be seen as a function of the initial state probability vector (observe from (III.2.4) that this is no the case for J_w). To do this, define $\rho_i \triangleq \Pr(\boldsymbol{\rho}(0) = i)$, $i \in \mathcal{I}_\ell$,

and $\rho \triangleq (\rho_0, ..., \rho_{\ell-1})$. Then $\frac{\partial J_0(\rho)}{\partial \rho_i}$ evaluated at the specific point $\rho^* = (\rho_0^*, ..., \rho_{\ell-1}^*)$ is

$$\left.\frac{\partial J_0(\rho)}{\partial \rho_i}\right|_{\rho^*} = \mathrm{tr}(X^0 Q_i), \tag{III.3.12}$$

where X^0 and Q_i were defined in Section III.2. Equation (III.3.12) says that a change in the initial state probability vector affects at a constant rate of change the value of the performance J_0. Actually, this conclusion can be drawn directly from (III.2.5) by noting that J_0 is linear with respect to ρ_i.

III.4 AN APPLICATION IN DISTRIBUTED CONTROL

An application of the results of Section III.3 to a distributed control system is presented in the following example.

Example III.4.1. Consider the following discretized state space realization of a plant:

$$\begin{aligned} \boldsymbol{x}_{\mathrm{p}}(k+1) &= A_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}}(k) + B_{\mathrm{p}}\boldsymbol{u}(k) \\ \boldsymbol{y}_{\mathrm{p}}(k) &= C_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}}(k), \end{aligned} \tag{III.4.1}$$

where $\boldsymbol{x}_{\mathrm{p}}(k) \in \mathbb{R}^{n_{\mathrm{p}}}$ is the plant's state vector, $\boldsymbol{y}_{\mathrm{p}}(k) \in \mathbb{R}^m$ is the plant's output, and $\boldsymbol{u}(k) \in \mathbb{R}^m$ is the plant's input. The nominal control law used to close the loop to attain a desired level of regulation performance is $\boldsymbol{u}(k) = \boldsymbol{w}(k) - \boldsymbol{y}_{\mathrm{c}}(k)$, where $\boldsymbol{w}(k) \in \mathbb{R}^q$ is a zero mean, second-order, wide sense stationary process with identity covariance matrix I_q and independent of $\boldsymbol{x}_{\mathrm{p}}(0)$, and $\boldsymbol{y}_{\mathrm{c}}(k) \in \mathbb{R}^m$ is the controller's output. The designed observer-based controller's state space representation is

$$\begin{aligned} \boldsymbol{x}_{\mathrm{c}}(k+1) &= A_{\mathrm{p}}\boldsymbol{x}_{\mathrm{c}}(k) + B_{\mathrm{p}}\boldsymbol{u}(k) + L_{\mathrm{p}}\left(\boldsymbol{y}_{\mathrm{p}}(k) - C_{\mathrm{p}}\boldsymbol{x}_{\mathrm{c}}(k)\right) \\ \boldsymbol{y}_{\mathrm{c}}(k) &= K\boldsymbol{x}_{\mathrm{c}}(k), \end{aligned} \tag{III.4.2}$$

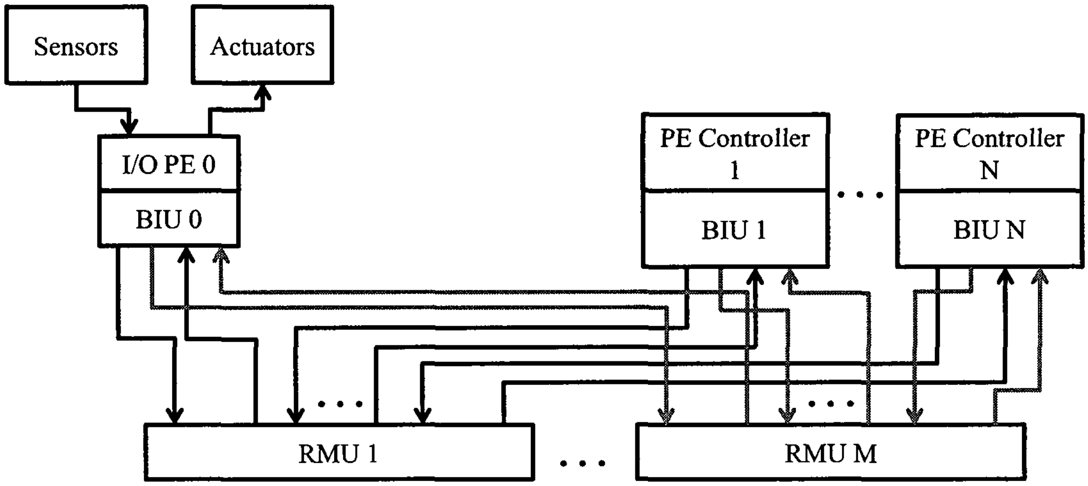

Fig. 1: Schematic of a distributed closed-loop system implemented with a ROBUS-2 fault tolerant communication system.

where $\boldsymbol{x}_{\mathrm{c}}(k) \in \mathbb{R}^{n_{\mathrm{p}}}$ is the controller's state vector, K and L_{p} are the pole placement and observer matrices, respectively. The nominal closed-loop system is obtained when the nominal control law is applied. It results in a nominal regulation level of closed-loop performance given by $J_w = \lim_{k\to\infty} E\{||\boldsymbol{y}_{\mathrm{p}}(k)||^2\}$. The results in this section make it possible to determine the performance degradation when an update to the control law is not received by the actuators at each control cycle due to random events caused by a harsh environment acting on a distributed control system as shown in Fig. 1. It consists of redundant and equivalent implementations of the controller dynamics in N Processing Elements (PEs). Each of the PEs connects to a fault tolerant communication network with a Bus Interface Unit (BIU) and each BIU is connected to M Redundancy Management Units (RMUs). For simplicity, all the sensors and actuators are connected using a single I/O PE and BIU. This PE-BIU node is assumed not to fail. This network is based on NASA's SPIDER (Scalable Processor-Independent Design for Enhanced Reliability) architecture, which uses the

ROBUS-2 communication system [28,41,42]. The network shown in Fig. 1 is referred to as an N PE $\times$ M RMU distributed control system, where the N PE-BIU nodes and M RMUs will be assumed to be the only components that can randomly fail silently, i.e., the devices produce no output during an upset control cycle but can recover and restart operation by the next control cycle.

To analyze this distributed control system, suppose that for each control cycle $k \in \mathbb{Z}^+$ the modes of operation of the i-th PE and the j-th RMU are denoted by the indicator random variables $\boldsymbol{z}_i(k)$ and $\tilde{\boldsymbol{z}}_j(k)$, respectively. The convention for all the indicator random variables is that a value of '0' denotes that the device is available and that a value of '1' denotes that the device has failed silently. Assume that a valid controller output is delivered to the actuators if at least one PE and one RMU are available; otherwise, no controller output is delivered to the actuators. This event is denoted with the indicator random variable $\boldsymbol{z}_v(k)$ that uses the same convention assumed for the components. An application of the results in this section leads to the following statistical characterization of $\boldsymbol{z}_v(k)$.

Lemma III.4.1. *Consider an N PE $\times$ M RMU distributed control system as shown in Fig. 1. Assume that all the availability processes $\{\boldsymbol{z}_i(k),\ i = 1,\dots,N\}$ and $\{\tilde{\boldsymbol{z}}_j(k),\ j = 1,\dots,M\}$ are i.i.d. and mutually independent. Let $p_{\theta_i} \triangleq \Pr\{\boldsymbol{z}_i(k) = 1\}$ and $p_{\nu_j} \triangleq \Pr\{\tilde{\boldsymbol{z}}_j(k) = 1\}$ then $\boldsymbol{z}_v(k)$ is an i.i.d. process with distribution characterized by*

$$p_1 \triangleq \Pr\{\boldsymbol{z}_v(k) = 1\} = 1 - \left(1 - \prod_{i=1}^{N} p_{\theta_i}\right)\left(1 - \prod_{j=1}^{M} p_{\nu_j}\right).$$

Proof: The proof follows by repeated application of Theorem II.3.1 since

$$\boldsymbol{z}_v(k) = \phi_{2|2}\Big(\phi_{1|N}\big(\boldsymbol{z}_1(k), ..., \boldsymbol{z}_N(k)\big), \phi_{1|M}\big(\tilde{\boldsymbol{z}}_1(k), ..., \tilde{\boldsymbol{z}}_M(k)\big)\Big),$$

where the mappings $\phi_{1|N}$ (1-out-of-N) and $\phi_{1|M}$ (1-out-of-M) are parallel structure functions, and $\phi_{2|2}$ (2-out-of-2) is a series structure function. ∎

The effect of the random upsets acting on the N PEs and M RMUs on the closed-loop system can be characterized as follows. When $\boldsymbol{z}_v(k) = 1$, no control input is delivered to the plant's actuators and the communication system restarts the N PEs resulting in the controllers' state vectors getting reset to zero. When $\boldsymbol{z}_v(k) = 0$, the closed-loop system behaves as the nominal one. Thus, the random upsets result in a switched control system indexed by $\boldsymbol{z}_v(k)$. In particular, the control law is also switched, i.e., $\boldsymbol{u}(k) \triangleq \boldsymbol{u}_{\boldsymbol{z}_v(k)}(k)$. The value of $\boldsymbol{u}_{\boldsymbol{z}_v(k)}(k)$ depends on the type of actuators, which can be memoryless or have memory. Memoryless actuators assume a zero command when no data is received. The effective control input is then

$$\boldsymbol{u}_{\boldsymbol{z}_v(k)}(k) = \boldsymbol{w}(k) - (1 - \boldsymbol{z}_v(k))\, \boldsymbol{y}_{\mathrm{c}}(k), \tag{III.4.3}$$

where the process $\boldsymbol{w}(k)$ is assumed to be independent of $\boldsymbol{z}_v(k)$. Actuators with memory belong to a class of smart actuators. When no data is received, these actuators use the previous control command. The effective control input is

$$\boldsymbol{u}_{\boldsymbol{z}_v(k)}(k) = \boldsymbol{w}(k) - (1 - \boldsymbol{z}_v(k))\, \boldsymbol{y}_{\mathrm{c}}(k) - \boldsymbol{z}_v(k)\boldsymbol{y}_{\mathrm{c}}(k-1). \tag{III.4.4}$$

A realization of the switched closed-loop system follows from (III.4.1), (III.4.2) and either (III.4.3) or (III.4.4) to be

$$\begin{aligned} \boldsymbol{x}_{\mathrm{CL}}(k+1) &= \bar{A}_{\boldsymbol{z}_v(k)}\boldsymbol{x}_{\mathrm{CL}}(k) + \bar{B}_{\boldsymbol{z}_v(k)}\boldsymbol{w}(k) \\ \boldsymbol{y}_{\mathrm{CL}}(k) &= \bar{C}_{\boldsymbol{z}_v(k)}\boldsymbol{x}_{\mathrm{CL}}(k), \end{aligned} \tag{III.4.5}$$

where $\boldsymbol{y}_{\mathrm{CL}}(k) = \boldsymbol{y}_{\mathrm{p}}(k)$. For memoryless actuators the state vector is $\boldsymbol{x}_{\mathrm{CL}}(k) = [\boldsymbol{x}_{\mathrm{p}}^T(k)\ \boldsymbol{x}_{\mathrm{c}}^T(k)]^T \in \mathbb{R}^{2n_{\mathrm{p}}}$. The state space realizations $\left(\bar{A}_{\boldsymbol{z}_v(k)}, \bar{B}_{\boldsymbol{z}_v(k)}\right)$ for $\boldsymbol{z}_v(k) \in$

$\{0, 1\}$ are

$$\bar{A}_0 = \begin{bmatrix} A_{\rm p} & -B_{\rm p}K \\ L_{\rm p}C_{\rm p} & A_{\rm c} \end{bmatrix}, \qquad \bar{B}_0 = \begin{bmatrix} B_{\rm p} \\ B_{\rm p} \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_{\rm p} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $A_{\rm c} = A_{\rm p} - B_{\rm p}K - L_{\rm p}C_{\rm p}$. The output equation is given by $C_{\rm CL} = \bar{C}_0 = \bar{C}_1 = [C_{\rm p}\ 0]$. When the actuators have memory, the closed-loop system is augmented with an additional state vector that remembers the previous value of the controller's state vector. So the state vector in (III.4.5) is $\boldsymbol{x}_{\rm CL}(k) = [\boldsymbol{x}_{\rm p}^T(k)\ \boldsymbol{x}_{\rm c}^T(k)\ \boldsymbol{x}_a^T(k)]^T \in \mathbb{R}^{3n_{\rm p}}$, $\boldsymbol{x}_a(k) = \boldsymbol{x}_{\rm c}(k-1)$. The state equation realizations in this case are

$$\bar{A}_0 = \begin{bmatrix} A_{\rm p} & -B_{\rm p}K & 0 \\ L_{\rm p}C_{\rm p} & A_{\rm c} & 0 \\ 0 & I & 0 \end{bmatrix}, \qquad \bar{B}_0 = \begin{bmatrix} B_{\rm p} \\ B_{\rm p} \\ 0 \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} A_{\rm p} & 0 & -B_{\rm p}K \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \qquad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The output equation is not switched. It is given by $C_{\rm CL} = \bar{C}_0 = \bar{C}_1 = [C_{\rm p}\ 0\ 0]$.

The degradation in regulation performance can now be characterized. The case of memoryless actuators and actuators with memory are considered in parallel. First, the nominal closed-loop realization for $\boldsymbol{z}_v(k) = 0$, $k \in \mathbb{Z}^+$ follows from (III.4.5) to be

$$\begin{aligned} \boldsymbol{x}_{\rm n}(k+1) &= \bar{A}_0(k)\boldsymbol{x}_{\rm n}(k) + \bar{B}_0(k)\boldsymbol{w}(k) \\ \boldsymbol{y}_{\rm n}(k) &= C_{\rm CL}\boldsymbol{x}_{\rm n}(k), \end{aligned} \tag{III.4.6}$$

where $\boldsymbol{x}_{\mathrm{n}}(k) = \boldsymbol{x}_{\mathrm{CL}}(k)$ for $k > 0$ is the nominal closed-loop state vector. The regulation error caused by the random upsets is $\boldsymbol{y}_{\mathrm{e}}(k) \triangleq \boldsymbol{y}_{\mathrm{CL}}(k) - \boldsymbol{y}_{\mathrm{n}}(k)$ when (III.4.5) and (III.4.6) have the same disturbance input $\boldsymbol{w}(k)$. A realization of this error system is

$$\begin{bmatrix} \boldsymbol{x}_{\mathrm{CL}}(k+1) \\ \boldsymbol{x}_{\mathrm{n}}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{\boldsymbol{z}_v(k)} & 0 \\ 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\mathrm{CL}}(k) \\ \boldsymbol{x}_{\mathrm{n}}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_{z_v} \\ \bar{B}_0 \end{bmatrix} \boldsymbol{w}(k), \tag{III.4.7a}$$

$$\begin{bmatrix} \boldsymbol{x}_{\mathrm{CL}}(0) \\ \boldsymbol{x}_{\mathrm{n}}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{x}_{n,0} \end{bmatrix},$$

$$\boldsymbol{y}_e(k) = \begin{bmatrix} C_{\mathrm{CL}} & -C_{\mathrm{CL}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\mathrm{CL}}(k) \\ \boldsymbol{x}_{\mathrm{n}}(k) \end{bmatrix}. \tag{III.4.7b}$$

The error system in (III.4.7) is an i.i.d. JLS switched by $\boldsymbol{z}_v(k)$. Let its realization be denoted by $\left(\tilde{A}_{\boldsymbol{z}_v(k)}, \tilde{B}_{\boldsymbol{z}_v(k)}, \tilde{C}\right)$ and the state vector be $\tilde{\boldsymbol{x}}(k) \triangleq \left[\boldsymbol{x}_{\mathrm{CL}}^T(k), \boldsymbol{x}_{\mathrm{n}}^T(k)\right]^T$. The performance metrics for (III.4.7) have been derived in Section III.3. In particular, the steady-state mean error power is $J_{w,e} \triangleq \lim_{k\to\infty} E\{||\boldsymbol{y}_e(k)||^2\}$. When $\boldsymbol{w}(k)$ is applied to (III.4.7), and if it is MSS, then Theorem III.3.1 gives the closed form expression for $J_{w,e}$. The partial derivatives of this metric with respect to $p_1 = \Pr\{\boldsymbol{z}_v(k) = 1\}$ follow from Theorem III.3.3. For the distributed closed-loop system in Fig. 1, the partial derivatives with respect to the upset probabilities of the PEs and RMUs can also be derived. A special case is considered next. □

Lemma III.4.2. *Consider an N PE $\times$ N RMU distributed control system as in Fig.* 1. *Assume that all the availability processes $\{\boldsymbol{z}_i(k),\ i = 1\ldots,N\}$ and $\{\tilde{\boldsymbol{z}}_j(k),\ j = 1,\ldots,N\}$ are i.i.d. and mutually independent. Let $p_\theta \triangleq \Pr\{\boldsymbol{z}_i(k) = 1\} = p_\nu = \Pr\{\tilde{\boldsymbol{z}}_j(k) = 1\}$. Let p_θ^* be such that* (III.4.7*) is MSS and $Q^* = Q(p_\theta^*)$.*

Then

$$\left.\frac{dJ_{w,e}(p_\theta)}{dp_\theta}\right|_{p_\theta^*} = \left.\left[\frac{\partial J_w(p_0,p_1)}{\partial p_1} - \frac{\partial J_w(p_0,p_1)}{\partial p_0}\right]\right|_{(p_0^*,p_1^*)} \left(2N(1-p_\theta^*)(p_\theta^*)^{N-1}\right),$$

where $p_0(p_\theta) = 1 - p_1(p_\theta)$ *and* $p_1(p_\theta) = 1-(1-(p_\theta)^N)^2$.

Proof: Apply Theorem III.3.3 and Lemma III.3.6. ∎

Example III.4.2. Consider the simplified longitudinal dynamics of the AFTI-F16 aircraft given in [13], where the aircraft model has four states (change in speed, angle of attack, pitch rate, and pitch angle) and the output of interest is the pitch rate. The sampled-data closed-loop system has sampling period $T = 0.004$ sec., the pole placement controller places the nominal continuous-time closed-loop poles at $\{-0.2 \pm j0.9798, -0.01 \pm j0.0995\}$, and the observer's discrete-time poles were chosen to be five times faster than the plant's closed-loop poles. The distributed control system consists of 2 PEs and 2 RMUs. When these four devices are allowed to randomly fail independently then $\mathcal{U}$ consists of one nonempty interval and (III.4.5) is MSS for $p_\theta^* \in [0, 0.0174[$ when memoryless actuators are used and $p_\theta^* \in [0, 0.2461[$ when actuators with memory are used. Figure 2 shows the analytically computed steady-state mean error power for both actuator cases. Assuming zero initial conditions for the closed-loop and nominal state vectors in (III.4.7), $J_{w,e}$ starts at zero and is finite only for each value p_θ^* that results in MSS. By Theorem III.3.3 this error metric is known to be monotonically increasing since the nominal closed-loop system (III.4.6) is asymptotically stable. Finally, the partial derivatives of the error metric with respect to p_θ is shown in Figure 3. Observe that by using actuators with memory, the closed-loop is MSS over a larger interval, and the error metric is smaller and has less sensitivity.

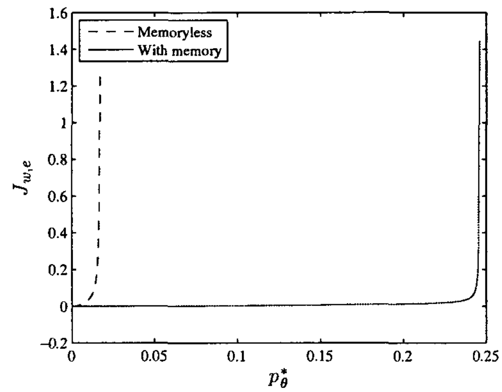

Fig. 2: $J_{w,e}$ for the pitch rate output versus p_θ^* when $p_\theta = p_\nu$ for a 2 PE $\times$ 2 RMU distributed control system.

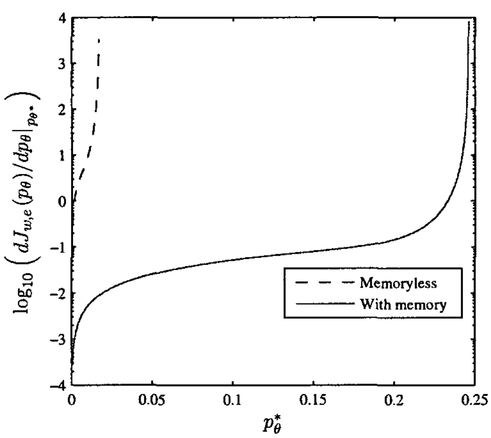

Fig. 3: The sensitivity with respect to p_θ is shown on a log scale when $p_\theta = p_\nu$ for a 2 PE $\times$ 2 RMU distributed control system.

III.5 JLS DRIVEN BY AN NHMC OR AN NMC LUMPED PROCESS

In this section, the case when the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ is either an NHMC or an NMC is addressed. As stated in [19], it is rare for a lumping transformation of an HMC to result in an HMC (see also Example II.4.1). Thus, a suitable tool is needed to perform the system analysis.

The case when $\boldsymbol{\rho}(k)$ results in a lumped NHMC is addressed first. Necessary and sufficient conditions for $\boldsymbol{\rho}(k)$ to be an NHMC for some initial state probability vectors, that is for $\pi_z(0) \in \Phi_z$, can be found for instance in [19, Theorem 22]. Theorem III.5.1 below gives an important application of the result obtained in Corollary II.2.1. In order to present this result, MSES of a dynamical system driven by a NHMC is introduced first in Definition III.5.1. Assume that $\boldsymbol{\rho}(k)$ is an NHMC (not necessarily a lumped process) with state space $\mathcal{I}_\ell$ and transition probability matrix $\Pi_\rho(k)$, and let Φ_ρ be a set of initial state probability vectors of $\boldsymbol{\rho}(k)$. Now consider the following JLS

$$\boldsymbol{x}(k+1) = A_{\rho(k)}\boldsymbol{x}(k),\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{III.5.1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I}_\ell$; and $\boldsymbol{x}_0$ is a random vector with finite second moment that is independent of $\boldsymbol{\rho}(k)$ for $k \geq 0$. Exponential second moment stability (or mean square exponential stability, MSES) is defined next [11].

Definition III.5.1. The equilibrium point at 0 of system (III.5.1) is called MSES with respect to Φ_ρ if for every value of the initial condition $\boldsymbol{x}_0$ and every initial state probability vector $\pi_\rho(0) \in \Phi_\rho$ there exists α and β, both positive and independent of $\boldsymbol{x}_0$ and $\pi_\rho(0)$ such that $E\{\|\boldsymbol{x}(k)\|^2\} \leq \alpha\|\boldsymbol{x}_0\|^2 e^{-\beta k},\ \forall\ k \geq 0$.

MSES and MSS of the JLS (III.2.1) are equivalent [30]. A MSES test for (III.5.1) follows.

Theorem III.5.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent, ergodic HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, a lumped process with state space $\mathcal{I}_\ell$. For $\pi_z(0) \in \Phi_z$ assume that $\boldsymbol{\rho}(k)$ is an NHMC with transition probability matrix $\Pi_\rho(k)$. If $\lim_{k\to\infty} \Pi_\rho(k) = \Pi$,*

where Π *is a stochastic matrix, then the system (*III.5.1*) is exponentially second moment stable if the spectral radius of* $\mathcal{A}_2$ *is less than one, where*

$$\mathcal{A}_2 = \mathrm{diag}(A_0^T \otimes A_0^T, ..., A_{\ell-1}^T \otimes A_{\ell-1}^T)(\Pi \otimes I_{n^2}).$$

Proof: When $\boldsymbol{\rho}(k)$ is an NHMC for $\pi_z(0) \in \Phi_z$, Theorem II.2.3 gives conditions that lead to a constant matrix approximation of the transition probability matrix $\Pi_\rho(k)$. In this case, the result follows from Corollary 2.6 in [11]. ∎

Now the case where the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ results in an NMC is considered. Observe that for each $k \in \mathbb{Z}^+$ the function

$$\psi : \mathcal{I}_S^L \to \mathcal{I}_S^L \times \mathcal{I}_\ell$$

$$\boldsymbol{z}(k) \mapsto \psi(\boldsymbol{z}(k)) \triangleq (\boldsymbol{z}(k), \boldsymbol{\rho}(k))$$

defines a two dimensional random variable denoted by $\boldsymbol{\theta}(k)$. Since $\boldsymbol{\rho}(k)$ is a function of $\boldsymbol{z}(k)$, the only possible values that $\boldsymbol{\theta}(k)$ can take are determined by the state space of $\boldsymbol{z}(k)$ and the lumping transformation ϕ. For instance consider the joint HMC and the structure function given in Example II.4.1. In this case $\boldsymbol{\theta}(k)$ can take the values $\{((0,0),0),((0,1),0),((1,0),0),((1,1),1)\}$ and no other element in $\mathcal{I}_2^2 \times \mathcal{I}_2$ is possible. Thus, the function ψ induces the (well) defined finite-state stochastic process given in the following lemma.

Lemma III.5.1. *Let* $\boldsymbol{z}_i(k)$*,* $i \in \mathscr{I}_L$*, be a set of independent HMCs with state space* $\mathcal{I}_S$*, and let* $\boldsymbol{z}(k)$ *be the joint HMC. Assume* ϕ *is a lumping transformation and* $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$*, a lumped process with state space* $\mathcal{I}_\ell$*. Then the family of random variables* $\{\boldsymbol{\theta}(k) : k \in \mathbb{Z}^+\}$ *is a well-defined stochastic process with range* $\mathcal{I}_\theta \triangleq \{(\zeta, \phi(\zeta)) : \zeta \in \mathcal{I}_S^L\}$*, which is a proper subset of* $\mathcal{I}_S^L \times \mathcal{I}_\ell$*.*

Proof: As explained above, the claim follows because ψ and ϕ are measurable functions of $\boldsymbol{z}(k)$. ■

Since there exists a one-to one relationship between the states of $\boldsymbol{z}(k)$ and the values that $\boldsymbol{\theta}(k)$ can take, it is natural to identify with the same labels in $\mathcal{E}$ the states ζ of $\boldsymbol{z}(k)$ with the states $(\zeta, \phi(\zeta))$ of $\boldsymbol{\theta}(k)$. The following theorem shows that the process $\boldsymbol{\theta}(k)$ is an HMC.

Theorem III.5.2. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC with state space $\mathcal{I}_S^L$, transition probability matrix Π_z and initial state probability vector $\pi_z(0)$. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, a lumped process with state space $\mathcal{I}_\ell$. Then $\boldsymbol{\theta}(k)$ is an HMC with transition probability matrix $\Pi_\theta = \Pi_z$ and initial state probability vector $\pi_\theta(0) = \pi_z(0)$. Moreover, $\boldsymbol{\theta}(k)$ is ergodic if $\boldsymbol{z}(k)$ satisfies this property.*

Proof: By Theorem 5 in [40] the following σ-algebra relationship holds $\sigma(\boldsymbol{\theta}(k), \ldots, \boldsymbol{\theta}(0)) = \sigma(\boldsymbol{z}(k), \ldots, \boldsymbol{z}(0))$. To simplify the notation, for any $k \in \mathbb{Z}^+$, denote the events $\{\boldsymbol{\theta}(k+1) = \theta(k+1)\}$, $\{\boldsymbol{\theta}(k) = \theta(k), \ldots, \boldsymbol{\theta}(0) = \theta(0)\}$, $\big\{\boldsymbol{z}(k+1) = z(k+1)\big\}$ and $\{\boldsymbol{z}(k) = z(k), \ldots, \boldsymbol{z}(0) = z(0)\}$ by $\{\theta(k+1)\}$, $\{\theta(k), \ldots, \theta(0)\}$, $\big\{z(k+1)\big\}$ and $\{z(k), \ldots, z(0)\}$, respectively. Thus,

$$\Pr\big\{\theta(k), \ldots, \theta(0)\big\} = \Pr\big\{z(k), \ldots, z(0)\big\} \tag{III.5.2}$$

Now, since $\boldsymbol{z}(k)$ is Markov

$$\begin{aligned}
&\Pr\big\{\theta(k+1)|\{\theta(k), \ldots, \theta(0)\}\big\} \\
&\quad = \Pr\big\{(z(k+1), \phi(z(k+1)))|\{(z(k), \phi(z(k))), \ldots, (z(0), \phi(z(0)))\}\big\} \\
&\quad = \Pr\big\{z(k+1)|\{z(k), \ldots, z(0)\}\big\}
\end{aligned}$$

$$= \Pr\{z(k+1)|z(k)\}$$
$$= \Pr\{(z(k+1), \phi(z(k)))|\{(z(k), \phi(z(k)))\}\}$$
$$= \Pr\{\theta(k+1)|\theta(k)\}.$$

Therefore, $\boldsymbol{\theta}(k)$ is Markov. Moreover, since the states of $\boldsymbol{z}(k)$ and $\boldsymbol{\theta}(k)$ are identified with the same labels, $\boldsymbol{\theta}(k)$ has the same transition probability matrix as $\boldsymbol{z}(k)$. Furthermore, by (III.5.2) it follows that

$$\Pr(\boldsymbol{\theta}(0) = j) = \Pr(\boldsymbol{z}(0) = j), \ j \in \mathcal{E},$$

that is, $\boldsymbol{\theta}(k)$ has the same initial state probability vector as $\boldsymbol{z}(k)$. Finally, since $\boldsymbol{\theta}(k)$ is completely characterized by $\boldsymbol{z}(k)$, Lemma II.2.1 also determines whether it is ergodic or not. ∎

Remarks

1. Theorem III.5.2 is particularly useful when the process $\boldsymbol{\rho}(k)$ is either a lumped NHMC or an NMC.

2. A similar result is presented in [32], where the Markovian nature of the joint process formed by the input and the output of a finite-state machine is used. However, note that in the case of Theorem III.5.2 there is no penalty to considering the joint process $\boldsymbol{\theta}(k) = (\boldsymbol{z}(k), \boldsymbol{\rho}(k))$ in the sense that the transition probability matrix is of the same dimension as that of the joint HMC $\boldsymbol{z}(k)$. As explained before, this is a consequence of $\boldsymbol{\rho}(k)$ being a function of $\boldsymbol{z}(k)$. In the finite-state machine case commented above, the input and the output are independent processes.

The Markov chain $\boldsymbol{\theta}(k)$ can be used to define the following HMC JLS

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k) + B_{\boldsymbol{\theta}(k)}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= C_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k), \end{aligned} \qquad \text{(III.5.3)}$$

which is selected to be *model equivalent* to the randomly switched system in (III.2.1), that is, for each $k \in \mathbb{Z}^+$ $A_{\theta(k)} \equiv A_{\rho(k)}$, $B_{\theta(k)} \equiv B_{\rho(k)}$, and $C_{\theta(k)} \equiv C_{\rho(k)}$ [46]. Therefore, if (III.2.1) and (III.5.3) have the same initial conditions and input processes, their state and output processes will be the same. Consequently, when the process $\boldsymbol{\rho}(k)$ is either an NHMC or a NMC, the JLS (III.2.1) driven by $\boldsymbol{\rho}(k)$ can be analyzed with regards to its stability and performance by means of the equivalent JLS (III.5.3), where $\boldsymbol{\theta}(k)$ is an HMC. An application of Theorem III.5.2 follows.

Corollary III.5.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC with state space $\mathcal{I}_S^L$ and transition probability matrix Π_z. Let ϕ be a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, a lumped process with state space $\mathcal{I}_\ell$. Then the JLS (*III.2.1*) is MSS if and only if the spectral radius of $\mathcal{A}_3$ is less than 1, where*

$$\mathcal{A}_3 \triangleq \operatorname{diag}(A_0^T \otimes A_0^T, \ldots, A_{S^L-1}^T \otimes A_{S^L-1}^T)(\Pi_z \otimes I_{n^2}). \qquad \text{(III.5.4)}$$

Proof: The claim follows from Lemma III.2.1 and Theorem III.5.2. ■

Remark

Note that if the spectral radius of $\mathcal{A}_3$ is less than 1, that is, if the JLS (III.2.1) is MSS then (III.2.4) and (III.2.5) can be used to calculate J_w and J_0, respectively. In this case, the matrix $\mathcal{A}_2$ of (III.2.3) must be substituted by the matrix $\mathcal{A}_3$ to calculate Q.

Example III.5.1. Let $z_1(k)$ and $z_2(k)$ be two independent HMCs with state space $\mathcal{I}_2$ and transition probability matrices

$$\Pi_{z_1} \triangleq \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}, \quad \Pi_{z_2} \triangleq \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}.$$

By Lemma II.2.1, the transition probability matrix of the joint HMC $\boldsymbol{z}(k) = (\boldsymbol{z}_1(k), \boldsymbol{z}_2(k))$ is

$$\Pi_z = \Pi_{z_1} \otimes \Pi_{z_2} = \begin{bmatrix} 0.18 & 0.02 & 0.72 & 0.08 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.54 & 0.06 & 0.36 & 0.04 \\ 0.3 & 0.3 & 0.2 & 0.2 \end{bmatrix}$$

Observe that $\boldsymbol{z}(k)$ is an ergodic HMC. Define ϕ as a 1-out-of-2 structure function. From Example II.4.1 it is known that the lumped process $\boldsymbol{\rho}(k)$, given in Table I, is not an HMC for all $\pi_z(0) \in \Xi_z$. Therefore, Corollary III.5.1 is the only mathematical tool that can be used to analyze the JLS (III.2.1) driven by $\boldsymbol{\rho}(k)$. To be more specific, take $\ell = 2$, that is, the JLS has two modes: 0 and 1 (the state space of $\boldsymbol{\rho}(k)$ is $\mathcal{I}_2$), and suppose that

$$A_{\{\rho(k)=0\}} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad A_{\{\rho(k)=1\}} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Observe that in this case $S^L = 2^2 = 4$. According to Theorem III.5.2, the process $\boldsymbol{\theta}(k) = (\boldsymbol{z}(k), \boldsymbol{\rho}(k))$ is an ergodic HMC with transition probability matrix $\Pi_\theta = \Pi_z$, and the matrix $\mathcal{A}_3$ becomes

$$\mathcal{A}_3 = \mathrm{diag}(A_0^T \otimes A_0^T, A_1^T \otimes A_1^T, A_2^T \otimes A_2^T, A_3^T \otimes A_3^T,)(\Pi_z \otimes I_4),$$

where, by the model identification explained above, $A_0 = A_1 = A_2 = A_{\{\rho(k)=0\}}$ and $A_3 = A_{\{\rho(k)=1\}}$. Since the spectral radius of $\mathcal{A}_3$ is 3.7637 then, according to Corollary III.5.1, the JLS (III.2.1) is not MSS.

III.6 SUMMARY

New analytical formulas for the output performance metrics, J_w and J_0, of the JLS III.2.1 driven by an i.i.d. process $\boldsymbol{\rho}(k)$ (not necessarily a lumped process) have been derived. These new formulas do not follow trivially from the ones known when the JLS is driven by an HMC. Sensitivity formulas for these output performance metrics with respect to the probabilities $p_i = \Pr(\boldsymbol{\rho}(k) = i)$, $i \in \mathcal{I}_\ell$ were also derived. An example based on NASA's ROBUS-2 communication system was presented. Finally, the case where the JLS III.2.1 is driven by the process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, when it is either a lumped NHMC or an NMC was addressed. First, a new result for analyzing MSES when $\boldsymbol{\rho}(k)$ is a NHMC for $\pi_z(0) \in \Phi$ was given. Next, for the general case where $\boldsymbol{\rho}(k)$ is simply a lumped process, it was shown that in this case the process $\boldsymbol{\theta}(k) = (\boldsymbol{z}(k), \boldsymbol{\rho}(k))$ becomes an HMC making it possible to apply the known results for the stability and performance analysis of the system through the concept of model equivalence.

CHAPTER IV

PERFORMABILITY ANALYSIS OF A JLS DRIVEN BY A LUMPED PROCESS

IV.1 INTRODUCTION

In Chapter I, the term *output performance metrics* was introduced to refer to the steady-state mean output power, J_w, and the mean output energy, J_0. Likewise, the term *network performance metrics* was introduced to refer to the mean time to failure, MTTF, and the mean time to repair, MTTR. A unified framework for the output and network performance metrics is what is called here *performability analysis*. In order to attain this goal, Problem 5 is entirely solved in this chapter. It is shown in Section IV.3 that the output performance metrics of the closed-loop control system driven by the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ are explicit functions of the network performance metrics of the network architecture characterized by the system availability process $\boldsymbol{\rho}(k)$. This connection implies that it is not possible to require a certain level of performance for the closed-loop control system without explicitly taking into account the performance of the network architecture. In effect, the sensitivity formulas given in Section IV.3 show how a small change in the network performance affects the output performances. This unified framework represents, to the best of our knowledge, a new contribution in the theory that integrates two fields of study, (discrete-time) dynamic system theory and (discrete-time) reliability theory, that so far have been addressed separately.

The chapter is organized as follows. In Section IV.2, new sufficient conditions for the existence of the MTTF and the MTTR are given when the system availability process $\boldsymbol{\rho}(k)$ is a 2-state lumped NHMC. Sufficient conditions for the existence of these network metrics have been given in [36] for an NHMC $\boldsymbol{\xi}(k)$, which is not the result of a lumping transformation. The conditions given in Section IV.2 are simpler and easier to test compared with those given in [36]. Indeed, the results obtained here take into account that $\boldsymbol{\rho}(k)$ is a lumped process. This facilitates the analysis because the derivations can be done in terms of the underlying process of the lumping transformation ϕ and the joint process $\boldsymbol{z}(k)$. Some examples are given to show how these new conditions work. In Section IV.3, the derivation of a functional relationship between the output performance metrics of the JLS (III.2.1) and the network performance metrics is done. Finally, a summary of the results obtained in this chapter is given in Section IV.4.

IV.2 NETWORK PERFORMANCE METRICS

In this section, a brief review of the network performance metrics, MTTF and MTTR, is presented. Let $\boldsymbol{\rho}(k)$ be a 2-state HMC, not necessarily a lumped process. The time to failure (TTF) and the time to repair (TTR) are defined next.

Definition IV.2.1. Let $k_0 \in \mathbb{Z}^+$ and assume that at this time instant the network is working correctly, that is, $\boldsymbol{\rho}(k_0) = 0$. The random variable

$$\boldsymbol{\tau}^{k_0} = \inf\{k > k_0 : \boldsymbol{\rho}(k) = 1\}$$

is called the *time to failure* (of the network architecture). The expectation of the TTF, $E(\boldsymbol{\tau}^{k_0})$, is called the MTTF.

Definition IV.2.2. Let $k_1 \in \mathbb{Z}^+$ and assume that at this time instant the network is not working correctly, that is, $\boldsymbol{\rho}(k_1) = 1$. The random variable

$$\boldsymbol{\gamma}^{k_1} = \inf\{k > k_1 : \boldsymbol{\rho}(k) = 0\}$$

is called the *time to repair* (of the network architecture). The expectation of the TTR, $E(\boldsymbol{\gamma}^{k_0})$, is called the MTTR.

Remarks

As usual, the infimum of the empty set is taken to be ∞. The TTF and the TTR, as defined above, are special cases of a more general concept called *hitting times* [36].

Let the transition probability matrix of $\boldsymbol{\rho}(k)$ be:

$$\Pi_\rho \triangleq \begin{bmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{bmatrix},$$

where $p_{00} < 1$ and $p_{11} < 1$. Then it is known (see, e.g., [2, 43]) that the MTTF, $E(\boldsymbol{\tau}^{k_0})$, and the MTTR, $E(\boldsymbol{\gamma}^{k_1})$, are given by

$$E(\boldsymbol{\tau}^{k_0}) = \frac{1}{1 - p_{00}} \tag{IV.2.1}$$

and

$$E(\boldsymbol{\gamma}^{k_1}) = \frac{1}{1 - p_{11}}, \tag{IV.2.2}$$

respectively.

Remarks

1. Since the network performance metrics given in (IV.2.1) and (IV.2.2) do not really depend on the specific time where they are calculated, the upper indexes k_0 and k_1 can be removed.

2. To simplify the notation, the MTTF and the MTTR will be denoted by α and β respectively:

$$\alpha \triangleq MTTF = \frac{1}{1 - p_{00}} \tag{IV.2.3a}$$

$$\beta \triangleq MTTR = \frac{1}{1 - p_{11}}. \tag{IV.2.3b}$$

The NHMC Case

The formulas given above are widely known in the literature. However, the case when the process $\boldsymbol{\rho}(k)$ is an NHMC (not necessarily a lumped process) is less known. This case has been addressed, for example, by Platis et al. in [36], where sufficient conditions are given for the existence of the MTTF and the MTTR, and explicit values of these metrics are given for specific examples. When the process $\boldsymbol{\rho}(k)$ is a 2-state lumped NHMC, simpler sufficient conditions can be derived in terms of the transition probabilities of the joint process, $\boldsymbol{z}(k)$. Moreover, a general formula to approximate the value of the MTTF and the MTTR can also be derived.

For all the following results in this section concerning the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, it is assumed that $\boldsymbol{\rho}(k)$ is an NHMC for $\pi_z(0) \in \Phi$ with transition probability matrix

$$\Pi_\rho(k) \triangleq \begin{bmatrix} p_{00}(k) & p_{01}(k) \\ p_{10}(k) & p_{11}(k) \end{bmatrix}.$$

In Lemma IV.2.1, the distribution probabilities of the random variables $\boldsymbol{\tau}$ and $\boldsymbol{\gamma}$ are given.

Lemma IV.2.1. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs with state space $\mathcal{I}_2$, and let $\boldsymbol{z}(k)$ be the joint HMC with state space $\mathcal{I}_2^L$. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space*

$\mathcal{I}_2$. *Assume that* $\boldsymbol{\rho}(k)$ *is an NHMC with transition probability matrix* $\Pi_\rho(k)$. *Let* $k_0, k_1 \in \mathbb{Z}^+$ *such that* $\boldsymbol{\rho}(k_0) = \phi(\boldsymbol{z}(k_0)) = 0$ *and* $\boldsymbol{\rho}(k_1) = \phi(\boldsymbol{z}(k_1)) = 1$. *Then* $\Pr(\boldsymbol{\tau}^{k_0} = 1) = p_{01}(k_0)$, *and*

$$\Pr(\boldsymbol{\tau}^{k_0} = t) = p_{01}(k_0 + t - 1) \prod_{k=0}^{t-2} p_{00}(k_0 + k), \quad t \geq 2, \ t \in \mathbb{Z}^+. \tag{IV.2.4}$$

Likewise, $\Pr(\boldsymbol{\gamma}^{k_1} = 1) = p_{10}(k_1)$ *and*

$$\Pr(\boldsymbol{\gamma}^{k_1} = t) = p_{10}(k_1 + t - 1) \prod_{k=0}^{t-2} p_{11}(k_1 + k), \quad t \geq 2, \ t \in \mathbb{Z}^+. \tag{IV.2.5}$$

Proof: For $t = 1$ it follows that

$$\Pr(\boldsymbol{\tau}^{k_0} = 1) = \Pr(\boldsymbol{\rho}(k_0 + 1) = 1 | \boldsymbol{\rho}(k_0) = 0) = p_{01}(k_0).$$

Similarly,

$$\Pr(\boldsymbol{\gamma}^{k_1} = 1) = \Pr(\boldsymbol{\rho}(k_1 + 1) = 0 | \boldsymbol{\rho}(k_1) = 1) = p_{10}(k_1).$$

Equations (IV.2.4) and (IV.2.5) follow by induction and the Markov property of $\boldsymbol{\rho}(k)$.

■

Therefore, whenever the series (IV.2.6) and (IV.2.7) below converge, the MTTF and the MTTR are:

$$E(\boldsymbol{\tau}^{k_0}) = p_{01}(k_0) + \sum_{t=2}^{\infty} t p_{01}(k_0 + t - 1) \prod_{k=0}^{t-2} p_{00}(k_0 + k) \tag{IV.2.6}$$

and

$$E(\boldsymbol{\gamma}^{k_1}) = p_{10}(k_1) + \sum_{t=2}^{\infty} t p_{10}(k_1 + t - 1) \prod_{k=0}^{t-2} p_{11}(k_1 + k), \tag{IV.2.7}$$

respectively.

Theorem IV.2.1 gives sufficient conditions for the convergence of these series and, thereby, for the existence of the MTTF and MTTR.

Theorem IV.2.1. *Let $z_i(k)$, $i \in \mathscr{I}_L$, be a set of independent ergodic HMCs with state space $\mathcal{I}_2$, and let $z(k)$ be the joint HMC. Assume ϕ is a lumping transformation and $\rho(k) = \phi(z(k))$, the system availability process with state space $\mathcal{I}_2$. Assume that $\rho(k)$ is an NHMC with transition probability matrix $\Pi_\rho(k)$. Then the limits $\overline{p}_{00} = \lim_{k\to\infty} p_{00}(k)$ and $\overline{p}_{11} = \lim_{k\to\infty} p_{11}(k)$ exist and if $\overline{p}_{00} < 1$ then the series (*IV.2.6*) converges. Likewise, if $\overline{p}_{11} < 1$ then the series (*IV.2.7*) converges.*

Proof: The existence of the limits $\overline{p}_{00}$ and $\overline{p}_{11}$ is guaranteed by Corollary II.2.1. The sufficiency part of the theorem is only proved for the first case since the other one is similar. Observe that

$$\sum_{t=2}^{\infty} t p_{01}(k_0+t-1)\prod_{k=0}^{t-2} p_{00}(k_0+k) \le \sum_{t=2}^{\infty} t \prod_{k=0}^{t-2} p_{00}(k_0+k).$$

Let $R(t) \triangleq t\prod_{k=0}^{t-2} p_{00}(k_0+k)$. Then

$$\frac{R(t+1)}{R(t)} = \left(\frac{t+1}{t}\right)\frac{\prod_{k=0}^{t-1} p_{00}(k_0+k)}{\prod_{k=0}^{t-2} p_{00}(k_0+k)} = (1+\frac{1}{t})p_{00}(k_0+t-1). \qquad \text{(IV.2.8)}$$

Taking limits on both sides of this equality gives $\lim_{t\to\infty}\frac{R(t+1)}{R(t)} = \overline{p}_{00} < 1$. Therefore, by the ratio test for convergence, the claim follows. ∎

By observing that

$$\overline{p}_{00} = \frac{1}{\pi_z M_0}\sum_{m,n\in\mathcal{E}_0} p^z_{mn}\pi_z e_m \le \frac{1}{\pi_z M_0}\sum_{m,n\in\mathcal{E}_0} p^z_{mn}, \qquad \text{(IV.2.9)}$$

a variation of Theorem IV.2.1 can be given.

Theorem IV.2.2. *Let $z_i(k)$, $i \in \mathscr{I}_L$, be a set of independent ergodic HMCs with state space $\mathcal{I}_2$, and let $z(k)$ be the joint HMC with state space $\mathcal{I}_2^L$ and transition*

probability matrix $\Pi_z = \left[p^z_{mn}\right]$, $m, n \in \mathcal{E}$. *Assume* ϕ *is a lumping transformation and* $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, *the system availability process with state space* $\mathcal{I}_2$. *Assume that* $\boldsymbol{\rho}(k)$ *is an NHMC. If*

$$\frac{1}{\pi_z M_0} \sum_{m,n \in \mathcal{E}_0} p^z_{mn} < 1 \tag{IV.2.10}$$

*then the series in (*IV.2.6*) converges. If*

$$\frac{1}{\pi_z M_1} \sum_{m,n \in \mathcal{E}_1} p^z_{mn} < 1 \tag{IV.2.11}$$

*then the series in (*IV.2.7*) converges.*

Proof: The proof follows directly from (IV.2.9) and Theorem IV.2.1 ■

Remarks

1. Notice that if (IV.2.10) and (IV.2.11) are satisfied with the inequality taken in the other direction, then the series do not converge, hence, the MTTF and the MTTR are not defined.

2. To obtain the results given in Theorem IV.2.2, what is actually needed is that the inequalities $\pi_z M_0 > 0$ and $\pi_z M_1 > 0$ hold. These inequalities might be satisfied without some of the HMCs $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, being ergodic.

The following example shows how conditions (IV.2.10) and (IV.2.11) work.

Example IV.2.1. Consider the transformation of $L = 3$ HMCs with transition probability matrices

$$\Pi_{z_i} \triangleq \begin{bmatrix} p^i & 1 - p^i \\ 1 - q^i & q^i \end{bmatrix}, \; i = 1, 2, 3,$$

where $p^1 = 0, q^1 = 0.3, p^2 = 0, q^2 = 0.5$ and $p^3 = 0, q^3 = 1$. Observe that the HMC $\boldsymbol{z}_3$ is not ergodic since its transition probability matrix, Π_{z_3}, is not quasi-positive

TABLE II: Transformation table for Example IV.2.1

$z_1(k)$	$z_2(k)$	$z_3(k)$	$z(k)$	$\xi(z(k))$	$\rho(k)$
0	0	0	$(0,0,0)$	1	0
0	0	1	$(0,0,1)$	2	0
0	1	0	$(0,1,0)$	3	0
0	1	1	$(0,1,1)$	4	1
1	0	0	$(1,0,0)$	5	0
1	0	1	$(1,0,1)$	6	1
1	1	0	$(1,1,0)$	7	1
1	1	1	$(1,1,1)$	8	1

(see Theorem A.1.2). Since the transformation in Table II is a 2-out-of-3 system, $\mathcal{E}_0 = \{1,2,3,5\}$ and $\mathcal{E}_1 = \{4,6,7,8\}$. By taking $\pi_{z_i}(0) = [1 \;\; 0]$, $i = 1,2,3$, one can show that the criterion of Theorem 22 in [19] is satisfied. Therefore, the system availability process $\rho(k) = \phi(z(k))$ is an NHMC (for the specific assumed initial state probability vectors $\pi_{z_i}(0)$).

Now observe that

$$\Pi_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0.35 & 0 & 0.35 & 0 & 0.15 & 0 & 0.15 \\ 0 & 0.35 & 0 & 0.35 & 0 & 0.15 & 0 & 0.15 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and the stationary probability is $\pi_z = [0 \;\; 0.1373 \;\; 0 \;\; 0.2745 \;\; 0 \;\; 0.1961 \;\; 0 \;\; 0.3921]$. Therefore, the condition (IV.2.10), $\sum_{m,n\in\mathcal{E}_0} p^z_{mn}/\pi_z M_0 = 0 < 1$, is satisfied, which ensures the existence of the MTTF. However, the MTTR does not exist since the

condition (IV.2.11), $\sum_{m,n\in\mathcal{E}_1} p^z_{mn}/\pi_z M_1 = 3.83 < 1$, is not satisfied. $\square$

Theorem IV.2.3 below gives general formulas that approximate the values of the MTTF and the MTTR when $\boldsymbol{\rho}(k)$ is an NHMC. First, to simplify the notation write

$$1 - h(k) = p_{00}(k) = \frac{\sum\limits_{m,n\in\mathcal{E}_0} p^z_{mn}\pi_z(0)\Pi^k_z e_m}{\pi_z(0)\Pi^k_z M_0}, \tag{IV.2.12}$$

$$1 - g(k) = p_{11}(k) = \frac{\sum\limits_{m,n\in\mathcal{E}_1} p^z_{mn}\pi_z(0)\Pi^k_z e_m}{\pi_z(0)\Pi^k_z M_1}.$$

Then by substituting $h(k)$ into (IV.2.6) and $g(k)$ into (IV.2.7), it follows that

$$E(\boldsymbol{\tau}^{k_0}) = h(k_0) + \sum_{t=2}^{\infty} t(h(k_0 + t - 1)) \prod_{k=0}^{t-2}(1 - h(k_0 + k)), \tag{IV.2.13}$$

and

$$E(\boldsymbol{\gamma}^{k_1}) = g(k_1) + \sum_{t=2}^{\infty} t(g(k_1 + t - 1)) \prod_{k=0}^{t-2}(1 - g(k_1 + k)),$$

respectively. In addition, associate with the stochastic matrix $\overline{\Pi}$, introduced in Chapter II (see Corollary II.2.1), the HMC $\overline{\boldsymbol{\rho}}$ with state space $\mathcal{I}_2$.

Let $\overline{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\gamma}}$ be the TTF and TTR corresponding to the HMC $\overline{\boldsymbol{\rho}}$, and let $E(\overline{\boldsymbol{\tau}})$ and $E(\overline{\boldsymbol{\gamma}})$ be the MTTF and the MTTR, respectively.

Theorem IV.2.3. *Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent ergodic HMCs with state space $\mathcal{I}_2$, and let $\boldsymbol{z}(k)$ be the joint HMC with state space $\mathcal{I}_2^L$ and transition probability matrix $\Pi_z = \left[p^z_{mn}\right], m, n \in \mathcal{E}$. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_2$. Assume that $\boldsymbol{\rho}(k)$ is an NHMC. Then there exist $t_0 \in \mathbb{Z}^+$ large enough such that $E(\boldsymbol{\tau}^{k_0})$ and $E(\boldsymbol{\gamma}^{k_0})$ can be approximated, respectively, by $E(\overline{\boldsymbol{\tau}})$ and $E(\overline{\boldsymbol{\gamma}})$ as follows:*

$$E(\boldsymbol{\tau}^{k_0}) \cong h(k_0) + S^{k_0}_h - H(\overline{p}_{00}) + E(\overline{\boldsymbol{\tau}}), \tag{IV.2.14}$$

$$E(\gamma^{k_0}) \cong g(k_1) + S_g^{k_1} - G(\overline{p}_{00}) + E(\overline{\gamma}),$$

where

$$S_h^{k_0} = \sum_{t=2}^{t_0} t(h(k_0+t-1)) \prod_{k=0}^{t-2} (1 - h(k_0+k)), \quad t_0 > 2, \tag{IV.2.15}$$

$$S_g^{k_1} = \sum_{t=2}^{t_0} t(g(k_1+t-1)) \prod_{k=0}^{t-2} (1 - g(k_1+k)), \quad t_0 > 2,$$

$$H(\overline{p}_{00}) = (1-\overline{p}_{00}) \sum_{t=1}^{t_0} t\overline{p}_{00}^{t-1}, \tag{IV.2.16}$$

$$G(\overline{p}_{11}) = (1-\overline{p}_{11}) \sum_{t=1}^{t_0} t\overline{p}_{11}^{t-1}.$$

Proof: Since each HMC $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, is ergodic, the joint process $\boldsymbol{z}(k)$ is also ergodic according to Lemma II.2.1. Let π_z be the stationary probability vector of $\boldsymbol{z}(k)$. Thus, for any $\varepsilon > 0$ it is possible to find a value $t_0(\varepsilon) \in \mathbb{Z}^+$ large enough such that

$$\|\Pi_z^{t_0} - \overline{1}\pi_z\| < \varepsilon < 1.$$

By (IV.2.15) and the inequality above it follows that

$$\begin{aligned}
E(\boldsymbol{\tau}^{k_0}) &= h(k_0) + S_h^{k_0} + \sum_{t=t_0+1}^{\infty} th(k_0+t-1) \prod_{k=0}^{t-2} (1 - h(k_0+k)) \\
&\cong h(k_0) + S_h^{k_0} + \sum_{t=t_0+1}^{\infty} t \left(1 - \frac{\sum_{m,n \in \mathcal{E}_0} p_{mn}^{z} \pi_z(0) \overline{1} \pi_z e_m}{\pi_z(0)\overline{1}\pi_z M_0} \right) \prod_{k=0}^{t-2} \frac{\sum_{m,n \in \mathcal{E}_0} p_{mn}^{z} \pi_z(0) \overline{1} \pi_z e_i}{\pi_z(0)\overline{1}\pi_z M_0} \\
&= h(k_0) + S_h^{k_0} + \sum_{t=t_0+1}^{\infty} t \left(1 - \frac{\sum_{m,n \in \mathcal{E}_0} p_{mn}^{z} \pi_z e_m}{\pi_z M_0} \right) \prod_{k=0}^{t-2} \frac{\sum_{m,n \in \mathcal{E}_0} p_{mn}^{z} \pi_z e_m}{\pi_z M_0} \\
&= h(k_0) + S_h^{k_0} + \sum_{t=t_0+1}^{\infty} t(1-\overline{p}_{00}) \prod_{k=0}^{t-2} \overline{p}_{00} \\
&= h(k_0) + S_h^{k_0} + \sum_{t=t_0+1}^{\infty} t(1-\overline{p}_{00})\overline{p}_{00}^{t-1}
\end{aligned}$$

$$= h(k_0) + S_h^{k_0} + (1-\overline{p}_{00})\left(\frac{1}{(1-\overline{p}_{00})^2} - \sum_{t=1}^{t_0} t\overline{p}_{00}^{t-1}\right)$$

$$= h(k_0) + S_h^{k_0} + E(\overline{\boldsymbol{\tau}}) - (1-\overline{p}_{00})\sum_{t=1}^{t_0} t\overline{p}_{00}^{t-1}$$

$$= h(k_0) + S_h^{k_0} - H(\overline{p}_{00}) + E(\overline{\boldsymbol{\tau}}).$$

Similar arguments prove the approximate formula for $E(\gamma^{k_0})$. ■

IV.3 PERFORMABILITY ANALYSIS

In this section, it is shown that the output performance metrics of the closed-loop control system driven by the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ are explicit functions of the network performance metrics of the network architecture characterized by the system availability process $\boldsymbol{\rho}(k)$. This *performability analysis* is first done for the i.i.d. case, that is, when $\boldsymbol{\rho}(k)$ is an i.i.d. process. Next, it is generalized for the HMC case, that is, when the lumped process $\boldsymbol{\rho}(k)$ is in an HMC.

The i.i.d. Case

Let $\boldsymbol{z}_i(k)$, $i \in \mathscr{I}_L$, be a set of independent i.i.d. processes with state space $\mathcal{I}_S$, and let $\boldsymbol{\rho}(k)$ be the system availability process with state space $\mathcal{I}_2$. By Theorem II.3.1, it is known that $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ is also an i.i.d. process for any lumping transformation. In Section III.3, the probabilities $\Pr(\boldsymbol{\rho}(k) = i)$, $i \in \mathcal{I}_2$, have been denoted by p_i. Thus if $0 < p_1 < 1$, the MTTF, α, and the MTTR, β, can be expressed in terms of the probabilities p_0 and p_1 (see (IV.2.3)) as follows

$$\alpha = \frac{1}{1-p_0}, \qquad \text{(IV.3.1a)}$$

$$\beta = \frac{1}{1-p_1}. \qquad \text{(IV.3.1b)}$$

From these equations it follows that

$$p_0 = 1 - \frac{1}{\alpha}, \tag{IV.3.2a}$$

$$p_1 = 1 - \frac{1}{\beta}. \tag{IV.3.2b}$$

Equations (IV.3.2a) and (IV.3.2b) are used in Theorem IV.3.1 below to express the output performance metrics J_w and J_0 as explicit functions of the network performance metrics (see **Problem 5** in Chapter I).

Theorem IV.3.1. *Let $z_i(k)$, $i \in \mathscr{I}_L$, be a set of independent i.i.d. processes with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint i.i.d. process. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process with state space $\mathcal{I}_2$, that drives the JLS (*III.2.1*). Then the output performance metrics J_w and J_0 are functions of the network performance metrics α and β given, respectively, by*

$$J_w(\alpha,\beta) = \mathrm{tr}\Big(C_0 Q(\alpha,\beta) C_0^T\Big)\Big(1-\frac{1}{\alpha}\Big) + \mathrm{tr}\Big(C_1 Q(\alpha,\beta) C_1^T\Big)\Big(1-\frac{1}{\beta}\Big), \tag{IV.3.3}$$

$$J_0(\alpha,\beta) = \mathrm{tr}\Big(C_0 M(\alpha,\beta) C_0^T\Big)\Big(1-\frac{1}{\alpha}\Big) + \mathrm{tr}\Big(C_1 M(\alpha,\beta) C_1^T\Big)\Big(1-\frac{1}{\beta}\Big), \tag{IV.3.4}$$

where

$$Q(\alpha,\beta) = \mathrm{vec}^{-1}\Big(\big(I_{n^2} - \mathcal{A}(\alpha,\beta)\big)^{-1}\mathrm{vec}\big(\mathcal{B}(\alpha,\beta)\big)\Big),$$

$$\mathcal{A}(\alpha,\beta) = A_0 \otimes A_0\Big(1-\frac{1}{\alpha}\Big) + A_1 \otimes A_1\Big(1-\frac{1}{\beta}\Big), \tag{IV.3.5}$$

$$\mathcal{B}(\alpha,\beta) = B_0 B_0^T\Big(1-\frac{1}{\alpha}\Big) + B_1 B_1^T\Big(1-\frac{1}{\beta}\Big), \tag{IV.3.6}$$

$$M(\alpha,\beta) = \mathrm{vec}^{-1}\Big(\big(I_{n^2} - \mathcal{A}(\alpha,\beta)\big)^{-1}\mathrm{vec}(X^0)\Big).$$

Proof: It follows directly from Theorems III.3.1 and III.3.2 by taking into account (IV.3.2a) and (IV.3.2b). ∎

The sensitivity of J_w and J_0 with respect to α and β are defined next.

Definition IV.3.1. Let $\delta^* \triangleq (\alpha^*, \beta^*)$ be such that the i.i.d. JLS (III.2.1) is MSS. The sensitivity of J_w and J_0 with respect to α and β are denoted by $S_w(\alpha)$, $S_w(\beta)$ and $S_0(\alpha)$, $S_0(\beta)$, respectively, and are given by

$$
\begin{aligned}
S_w(\alpha) &= \frac{\alpha}{J_w(\alpha,\beta)} \left. \frac{\partial J_w(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*}, \quad S_w(\beta) = \frac{\beta}{J_w(\alpha,\beta)} \left. \frac{\partial J_w(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} \\
S_0(\alpha) &= \frac{\alpha}{J_0(\alpha,\beta)} \left. \frac{\partial J_0(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*}, \quad S_0(\beta) = \frac{\beta}{J_0(\alpha,\beta)} \left. \frac{\partial J_0(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*}
\end{aligned}
$$

The partial derivatives of J_w and J_0 with respect to α and β are given next. The result can be derived directly from Theorem IV.3.1

Theorem IV.3.2. *Let $\delta^* \triangleq (\alpha^*, \beta^*)$ be such that the i.i.d. JLS (*III.2.1*) is MSS and let $Q^* \triangleq Q(\delta^*)$, $M^* \triangleq M(\delta^*)$, $\mathcal{A}^* \triangleq \mathcal{A}(\delta^*)$ and $\mathcal{B}^* \triangleq \mathcal{B}(\delta^*)$ be the values of Q, M, $\mathcal{A}$ and $\mathcal{B}$ at this point, respectively. Then*

$$
\begin{aligned}
\left. \frac{\partial J_w(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} &= \operatorname{tr}\left(C_0 \left. \frac{\partial Q(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} C_0^T \right) \left(1 - \frac{1}{\alpha^*}\right) + \operatorname{tr}\left(C_0 Q^* C_0^T \right) \left(\frac{1}{\alpha^*}\right)^2 + \\
&\quad \operatorname{tr}\left(C_1 \left. \frac{\partial Q(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} C_1^T \right) \left(1 - \frac{1}{\beta^*}\right), \\
\left. \frac{\partial J_w(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} &= \operatorname{tr}\left(C_1 \left. \frac{\partial Q(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} C_1^T \right) \left(1 - \frac{1}{\beta^*}\right) + \operatorname{tr}\left(C_1 Q^* C_1^T \right) \left(\frac{1}{\beta^*}\right)^2 + \\
&\quad \operatorname{tr}\left(C_0 \left. \frac{\partial Q(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} C_0^T \right) \left(1 - \frac{1}{\alpha^*}\right), \\
\left. \frac{\partial J_0(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} &= \operatorname{tr}\left(C_0 \left. \frac{\partial M(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} C_0^T \right) \left(1 - \frac{1}{\alpha^*}\right) + \operatorname{tr}\left(C_0 M^* C_0^T \right) \left(\frac{1}{\alpha^*}\right)^2 + \\
&\quad \operatorname{tr}\left(C_1 \left. \frac{\partial M(\alpha,\beta)}{\partial \alpha} \right|_{\delta=\delta^*} C_1^T \right) \left(1 - \frac{1}{\beta^*}\right), \\
\left. \frac{\partial J_0(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} &= \operatorname{tr}\left(C_1 \left. \frac{\partial M(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} C_1^T \right) \left(1 - \frac{1}{\beta^*}\right) + \operatorname{tr}\left(C_1 M^* C_1^T \right) \left(\frac{1}{\beta^*}\right)^2 + \\
&\quad \operatorname{tr}\left(C_0 \left. \frac{\partial M(\alpha,\beta)}{\partial \beta} \right|_{\delta=\delta^*} C_0^T \right) \left(1 - \frac{1}{\alpha^*}\right),
\end{aligned}
$$

where

$$\left.\frac{\partial Q(\alpha,\beta)}{\partial \alpha}\right|_{\delta=\delta^*} = \mathrm{vec}^{-1}\Big(\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\big((A_0\otimes A_0)\Big(\frac{1}{\alpha^*}\Big)^2\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\mathrm{vec}(\mathcal{B}^*)+$$
$$\mathrm{vec}(B_0B_0^T)\Big(\frac{1}{\alpha^*}\Big)^2\big)\Big),$$
$$\left.\frac{\partial Q(\alpha,\beta)}{\partial \beta}\right|_{\delta=\delta^*} = \mathrm{vec}^{-1}\Big(\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\big((A_1\otimes A_1)\Big(\frac{1}{\beta^*}\Big)^2\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\mathrm{vec}(\mathcal{B}^*)+$$
$$\mathrm{vec}(B_1B_1^T)\Big(\frac{1}{\beta^*}\Big)^2\big)\Big),$$
$$\left.\frac{\partial M(\alpha,\beta)}{\partial \alpha}\right|_{\delta=\delta^*} = \mathrm{vec}^{-1}\Big(\left(I_{n^2}-\mathcal{A}^*\right)^{-1}(A_0\otimes A_0)\Big(\frac{1}{\alpha^*}\Big)^2\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\mathrm{vec}(X^0)\Big),$$
$$\left.\frac{\partial M(\alpha,\beta)}{\partial \beta}\right|_{\delta=\delta^*} = \mathrm{vec}^{-1}\Big(\left(I_{n^2}-\mathcal{A}^*\right)^{-1}(A_1\otimes A_1)\Big(\frac{1}{\beta^*}\Big)^2\left(I_{n^2}-\mathcal{A}^*\right)^{-1}\mathrm{vec}(X^0)\Big).$$

Proof: These identities follow directly from taking partial derivatives in (IV.3.3), (IV.3.4), (IV.3.5) and (IV.3.6). ∎

Therefore, a change in the value of J_w at the specific point $\delta^* = (\alpha^*, \beta^*)$ caused by a small change in δ, $d\delta \triangleq (d\alpha, d\beta)$, is given by

$$dJ_w(\alpha,\beta)|_{\delta=\delta^*} = \left.\begin{bmatrix}\frac{\partial J_w(\alpha,\beta)}{\partial\alpha} & \frac{\partial J_w(\alpha,\beta)}{\partial\beta}\end{bmatrix}\right|_{\delta=\delta^*}\begin{bmatrix}d\alpha\\ d\beta\end{bmatrix}. \tag{IV.3.7}$$

Similarly for J_0,

$$dJ_0(\alpha,\beta)|_{\delta=\delta^*} = \left.\begin{bmatrix}\frac{\partial J_0(\alpha,\beta)}{\partial\alpha} & \frac{\partial J_0(\alpha,\beta)}{\partial\beta}\end{bmatrix}\right|_{\delta=\delta^*}\begin{bmatrix}d\alpha\\ d\beta\end{bmatrix}. \tag{IV.3.8}$$

Since $p_0 + p_1 = 1$, from (IV.3.2) it follows

$$\alpha = \frac{\beta}{\beta-1}, \quad \beta = \frac{\alpha}{\alpha-1}.$$

Hence, α and β can not change arbitrarily. If we consider α as a function of β then $d\alpha = -\frac{1}{(\beta-1)^2}d\beta$. Likewise, if one considers β as a function of α then $d\beta = -\frac{1}{(\alpha-1)^2}d\alpha$.

In this case, (IV.3.7) and (IV.3.8) take the scalar form

$$dJ_w(\beta)|_{\beta=\beta^*} = \left[-\frac{\partial J_w(\alpha,\beta)}{\partial \alpha}\frac{1}{(\beta^*-1)^2} + \frac{\partial J_w(\alpha,\beta)}{\partial \beta}\right]\Bigg|_{\delta=\delta^*} d\beta, \tag{IV.3.9a}$$

$$dJ_w(\alpha)|_{\alpha=\alpha^*} = \left[\frac{\partial J_w(\alpha,\beta)}{\partial \alpha} - \frac{\partial J_w(\alpha,\beta)}{\partial \beta}\frac{1}{(\alpha^*-1)^2}\right]\Bigg|_{\delta=\delta^*} d\alpha, \tag{IV.3.9b}$$

and

$$dJ_0(\beta)|_{\beta=\beta^*} = \left[-\frac{\partial J_0(\alpha,\beta)}{\partial \alpha}\frac{1}{(\beta^*-1)^2} + \frac{\partial J_0(\alpha,\beta)}{\partial \beta}\right]\Bigg|_{\delta=\delta^*} d\beta,$$

$$dJ_0(\alpha)|_{\alpha=\alpha^*} = \left[\frac{\partial J_0(\alpha,\beta)}{\partial \alpha} - \frac{\partial J_0(\alpha,\beta)}{\partial \beta}\frac{1}{(\alpha^*-1)^2}\right]\Bigg|_{\delta=\delta^*} d\alpha,$$

respectively.

The following example computes the sensitivity of the steady-state mean output power, J_w, with respect to the MTTF and the MTTR.

Example IV.3.1. Let $\boldsymbol{z}_i$, $i \in \mathscr{I}_L$, be a set of independent i.i.d. processes with state space $\mathcal{I}_2 = \{0,1\}$. Let $\boldsymbol{z}(k)$ be the joint i.i.d. process and $\boldsymbol{\rho} = \phi(\boldsymbol{z}(k))$, the system availability driving the i.i.d. JLS (III.2.1). Let $p_0^* = \Pr(\boldsymbol{\rho}(k) = 0) = 0.8$ and $p_1^* = \Pr(\boldsymbol{\rho}(k) = 1) = 0.2$ be the probability distribution of $\boldsymbol{\rho}(k)$. Consider the following matrices

$$A_0 = \begin{bmatrix} 0.5 & -1 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0 \\ -2 & 0.8 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.2 & 0.5 \\ 1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.8 & 1 \\ 0 & -1 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 2 & 0.3 \\ 0.4 & 0.8 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -2 & 0.5 \\ 1 & 0 \end{bmatrix}.$$

Since the spectral radius of $\mathcal{A}$ is 0.128, then by Lemma III.3.1 the JLS (III.2.1) is MSS.

From (IV.3.1a) and (IV.3.1b), the specific value of δ is $\delta^* = (\alpha^*, \beta^*) = (5, 1.25)$. Following Theorem IV.3.1, from the matrices above the specific values of $\mathcal{A}$, $\mathcal{B}$ and Q are determined to be

$$\mathcal{A}^* = \mathcal{A}(\delta^*) = \begin{bmatrix} 0.218 & -0.4 & -0.4 & 0.8 \\ -0.12 & -0.352 & 0 & 0.8 \\ -0.12 & 0 & -0.352 & 0.8 \\ 0.8 & -0.32 & -0.32 & 0.928 \end{bmatrix}$$

$$\mathcal{B}^* = \mathcal{B}(\delta^*) = \begin{bmatrix} 0.56 & -0.04 \\ -0.4 & 1 \end{bmatrix}, \quad Q^* = Q(\delta^*) = \begin{bmatrix} 14.6809 & 16.5195 \\ 16.5195 & 30.1701 \end{bmatrix}$$

From Theorem IV.3.2, it follows that

$$\left.\frac{\partial Q(\alpha,\beta)}{\partial \alpha}\right|_{\delta=\delta^*} = \begin{bmatrix} 0.7044 & 0.8844 \\ 0.8844 & 1.2468 \end{bmatrix},$$

$$\left.\frac{\partial Q(\alpha,\beta)}{\partial \beta}\right|_{\delta=\delta^*} = \begin{bmatrix} 1.8952 & -3.7401 \\ -3.7401 & 16.7489 \end{bmatrix}.$$

Then, Equation (IV.3.7) becomes

$$dJ_w(\alpha,\beta)|_{\delta=\delta^*} = \begin{bmatrix} 9.3271 & 45.4732 \end{bmatrix} \begin{bmatrix} d\alpha \\ d\beta \end{bmatrix}.$$

When J_w is only taken as a function of α, (IV.3.9b) yields

$$dJ_w(\alpha)|_{\alpha=\alpha^*} = 6.485\ d\alpha. \qquad \text{(IV.3.11)}$$

Likewise, when J_w is only taken as a function of β, (IV.3.9a) yields

$$dJ_w(\beta)|_{\beta=\beta^*} = -103.7604\ d\beta. \qquad \text{(IV.3.12)}$$

From (IV.3.11) and (IV.3.12), one can conclude that the steady-state mean output power is more sensitive with respect to the MTTR than with respect to the MTTF. However, it is observed that a positive change in the MTTF increases the value of J_w and, on the other hand, a positive change in the MTTR significantly decreases the value of J_w.

The HMC Case

The performability analysis when the lumped process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$ is i.i.d. can also be done when $\boldsymbol{\rho}(k)$ results in an HMC for all initial state probability vectors of $\boldsymbol{z}(k)$, that is, when $\pi_z(0) \in \Xi_z$. From (IV.2.3) it follows

$$p_{00} = 1 - \frac{1}{\alpha}, \tag{IV.3.13a}$$

$$p_{11} = 1 - \frac{1}{\beta}. \tag{IV.3.13b}$$

Thus,

$$\Pi_\rho \triangleq \begin{bmatrix} 1 - 1/\alpha & 1/\alpha \\ 1/\beta & 1 - 1/\beta \end{bmatrix}.$$

The sensitivity of J_w and J_0 with respect to α and β are defined similarly as for the i.i.d. case. By taking into account (III.2.3), (III.2.4) and (III.2.5), the following theorem relates the output performance metrics with the MTTF and the MTTR.

Theorem IV.3.3. *Let $z_i(k)$, $i \in \mathscr{I}_L$, be a set of independent HMCs processes with state space $\mathcal{I}_S$, and let $\boldsymbol{z}(k)$ be the joint HMC process. Assume ϕ is a lumping transformation and $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$, the system availability process that drives the JLS (*III.2.1*). Further assume that $\boldsymbol{\rho}(k)$ is an HMC for all $\pi_z(0) \in \Xi_z$ and has state space $\mathcal{I}_2$. Let $\delta^* \triangleq (\alpha^*, \beta^*)$ be such that the JLS (*III.2.1*) is MSS. Then the output*

performances metrics, J_w and J_0 are functions of the network performance metrics α and β as given below:

$$J_w(\alpha,\beta) = \mathrm{tr}\Big[B^w\Big(Q_0(\alpha,\beta)\pi_\rho^0 + Q_1(\alpha,\beta)\pi_\rho^1\Big)\Big], \tag{IV.3.14}$$

$$J_0(\alpha,\beta) = \mathrm{tr}\Big[X^0\Big(Q_0(\alpha,\beta)\Pr(\boldsymbol{\rho}(0)=0) + Q_1(\alpha,\beta)\Pr(\boldsymbol{\rho}(0)=1)\Big)\Big], \tag{IV.3.15}$$

where $Q \triangleq (Q_0, Q_1)$. The partial derivatives are given by

$$\begin{aligned}
\frac{\partial J_w(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*} &= \mathrm{tr}\Bigg(B^w\bigg[\frac{\partial Q_0(\alpha,\beta)}{\partial\alpha}\pi_\rho^0 + \frac{\partial Q_1(\alpha,\beta)}{\partial\alpha}\pi_\rho^1\bigg]\bigg|_{\delta=\delta^*}\Bigg),\\
\frac{\partial J_w(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*} &= \mathrm{tr}\Bigg(B^w\bigg[\frac{\partial Q_0(\alpha,\beta)}{\partial\beta}\pi_\rho^0 + \frac{\partial Q_1(\alpha,\beta)}{\partial\beta}\pi_\rho^1\bigg]\bigg|_{\delta=\delta^*}\Bigg),\\
\frac{\partial J_0(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*} &= \mathrm{tr}\Bigg(X^0\bigg[\frac{\partial Q_0(\alpha,\beta)}{\partial\alpha}\Pr(\boldsymbol{\rho}(0)=0) + \frac{\partial Q_1(\alpha,\beta)}{\partial\alpha}\Pr(\boldsymbol{\rho}(0)=1)\bigg]\bigg|_{\delta=\delta^*}\Bigg),\\
\frac{\partial J_0(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*} &= \mathrm{tr}\Bigg(X^0\bigg[\frac{\partial Q_0(\alpha,\beta)}{\partial\beta}\Pr(\boldsymbol{\rho}(0)=0) + \frac{\partial Q_1(\alpha,\beta)}{\partial\beta}\Pr(\boldsymbol{\rho}(0)=1)\bigg]\bigg|_{\delta=\delta^*}\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial Q(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*} &= \varphi^{-1}\Bigg(\big(I_{n^2}-\mathcal{A}^*\big)^{-1}\bigg(\frac{\partial\mathcal{A}(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*}\bigg)\big(I_{n^2}-\mathcal{A}^*\big)^{-1}\varphi(\mathcal{C})\Bigg),\\
\frac{\partial Q(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*} &= \varphi^{-1}\Bigg(\big(I_{n^2}-\mathcal{A}^*\big)^{-1}\bigg(\frac{\partial\mathcal{A}(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*}\bigg)\big(I_{n^2}-\mathcal{A}^*\big)^{-1}\varphi(\mathcal{C})\Bigg),\\
\frac{\partial\mathcal{A}(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*} &= \mathrm{diag}\big(A_0\otimes A_0, A_1\otimes A_1\big)\bigg(\frac{\partial\Pi_\rho(\alpha,\beta)}{\partial\alpha}\bigg|_{\delta=\delta^*}\otimes I_{n^2}\bigg),\\
\frac{\partial\mathcal{A}(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*} &= \mathrm{diag}\big(A_0\otimes A_0, A_1\otimes A_1\big)\bigg(\frac{\partial\Pi_\rho(\alpha,\beta)}{\partial\beta}\bigg|_{\delta=\delta^*}\otimes I_{n^2}\bigg).
\end{aligned}$$

Proof: Equations (IV.3.14) and (IV.3.15) follow from (III.2.4) and (III.2.5), respectively. The partial derivatives follow directly from (IV.3.14) and (IV.3.15) and by taking into account (III.2.3). ∎

Since in this case α and β are not related, these parameters can change arbitrarily. Therefore, a change in the value of J_w at the specific point $\delta^* = (\alpha^*, \beta^*)$ caused by

a small change in δ, $d\delta = (d\alpha, d\beta)$, is given by

$$dJ_w(\alpha,\beta)|_{\delta=\delta^*} = \left[\frac{\partial J_w(\alpha,\beta)}{\partial\alpha} \quad \frac{\partial J_w(\alpha,\beta)}{\partial\beta}\right]\bigg|_{\delta=\delta^*} \begin{bmatrix} d\alpha \\ d\beta \end{bmatrix}.$$

Similarly for J_0:

$$dJ_0(\alpha,\beta)|_{\delta=\delta^*} = \left[\frac{\partial J_0(\alpha,\beta)}{\partial\alpha} \quad \frac{\partial J_0(\alpha,\beta)}{\partial\beta}\right]\bigg|_{\delta=\delta^*} \begin{bmatrix} d\alpha \\ d\beta \end{bmatrix}.$$

IV.4 SUMMARY

In this chapter, new sufficient conditions have been given to guarantee the existence of the MTTF and the MTTR when a network architecture is characterized by a 2-state lumped NHMC system availability process $\boldsymbol{\rho}(k) = \phi(\boldsymbol{z}(k))$. Since these conditions were given in terms of the transition probabilities of the underlying process, $\boldsymbol{z}(k)$, the criterion is easy to check. In addition, general formulas to approximate the values of the MTTF and the MTTR were given in terms of the steady-state probabilities $\overline{p}_{00}$ and $\overline{p}_{11}$ introduced in Corollary II.2.1. A new unified framework between closed-loop control system theory and fault-tolerant network architecture has been given in Section IV.3 when the lumped process $\boldsymbol{\rho}(k)$ is an i.i.d. process or an HMC. The output performance metrics J_w and J_0 have been expressed as a function of the MTTF and the MTTR, and sensitivity formulas were given to see how a small change in these network performance metrics affect the output performance metrics.

CHAPTER V

CONCLUSIONS AND FUTURE RESEARCH

In this chapter, the main conclusions of the dissertation are given. The objectives established in Problem 1 through Problem 5 in Chapter I have been successfully reached as explained below.

Problem 1

a) The probability distribution of $\boldsymbol{\rho}(k)$, $p_j(k) = \Pr(\boldsymbol{\rho}(k) = j)$, $j \in \mathcal{I}_\ell$, was given in Lemma II.2.2. This result only assumes that $\boldsymbol{\rho}(k)$ is a well-defined stochastic process, which is the case since the lumping transformation, ϕ, is a measurable function. Therefore, the probability distribution of $\boldsymbol{\rho}(k)$, given in II.2.3, is valid in particular when the system availability process results in either an NHMC or an NMC. These probabilities are easy to calculate as they are given in terms of the initial state probability vectors $\pi_{z_i}(0)$, $i \in \mathscr{I}_L$, and the transition probability matrix of the joint process $\boldsymbol{z}(k)$, Π_z, that are assumed to be known.

b) The availability of the system at steady-state, $\lim_{k\to\infty} \Pr(\boldsymbol{\rho}(k) = 0)$, was derived directly from Lemma II.2.2, and the result is presented in Theorem II.2.1.

c) The one-step transition probabilities of $\boldsymbol{\rho}(k)$, $p_{ij}(k) \triangleq \Pr(\boldsymbol{\rho}(k+1) = j|\boldsymbol{\rho}(k) = i)$, $i, j \in \mathcal{I}_\ell$, were derived in Theorem II.2.2. It was shown that they are well-defined probabilities if the probabilities of the system to stay in each mode satisfies $\Pr(\boldsymbol{\rho}(k) = i) > 0$, $i \in \mathcal{I}_\ell$. The one-step transition probabilities given in (II.2.7) result in the well defined time-varying stochastic matrix $\Pi_\rho(k)$ for the particular case when the system availability process, $\boldsymbol{\rho}(k)$, has only two states.

d) The steady-state value of the one-step transition probabilities $p_{ij}(k)$, $\lim_{k\to\infty} p_{ij}(k)$, were derived in Theorem II.2.3 assuming that the HMCs $z_i(k)$, $i \in \mathscr{I}_L$ are ergodic. With this result, the matrix $\Pi_\rho(k)$ becomes the stochastic matrix $\overline{\Pi}$ at steady-state. This matrix was used in Theorem III.5.1 to get a new result regarding the MSES of a JLS driven by an NHMC.

Problem 2

a) Under the hypothesis that the i.i.d. processes $z_i(k)$, $i \in \mathscr{I}_L$, are mutually independent, it was established that the lumped process $\boldsymbol{\rho}(k)$ is also an i.i.d. process. The result is given in Theorem II.3.1.

b) The output performance metrics J_w and J_0 for the i.i.d. JLS (III.2.1) were presented in Theorems III.3.1 and III.3.2, respectively.

c) The benefit of using these new formulas for J_w and J_0, instead of the known ones for the HMC case, was explained in the same section where the formulas were derived. Essentially, this benefit is based on computational issues related to the lower dimension of the matrix $\mathcal{A}$ in comparison to the matrix $\mathcal{A}_2$.

Problem 3

To analyze the MSS and the output performance metrics of the JLS (III.2.1) driven by $\boldsymbol{\rho}(k)$, when it is an NHMC or an NMC, a new result, Theorem III.5.2, is presented. Specifically, it was proved that the joint process $\boldsymbol{\theta}(k) = (\boldsymbol{z}(k), \boldsymbol{\rho}(k))$ becomes an HMC with the same transition probability matrix as the joint HMC $\boldsymbol{z}(k)$. Therefore, by introducing a new JLS, driven by the process $\boldsymbol{\theta}(k)$, and taking into account the notion of model equivalence, it is possible to analyze the MSS and the output performance metrics of the JLS (III.2.1) driven by $\boldsymbol{\rho}(k)$.

Problem 4

Sensitivity formulas to analyze the effect of a small change in the probability of upset on J_w and J_0 have been given in Theorem III.3.3. These results directly follow from the ones given for J_w and J_0 in Theorems III.3.1 and III.3.2, respectively.

Problem 5

When the lumped process $\boldsymbol{\rho}(k)$ is either an i.i.d. process or an HMC, it was shown that the performance metrics J_w and J_0 of the JLS (III.2.1) are explicit functions of the MTTF and MTTR for the network architecture represented by $\boldsymbol{\rho}(k)$. These results, which are given in Theorems IV.3.1 and IV.3.3, are one of the main contributions of this dissertation. They represent a new theoretical approach to better integrating system theory with the reliability theory.

Future Research

The following problems need further work.

1. In Theorem II.2.3 it has been shown that the one-step transition probability matrix $\Pi(k)$ of the lumped process $\boldsymbol{\rho}(k)$ converges at steady-state to the constant stochastic matrix $\overline{\Pi}$. It is not clear if there exist a stochastic process, related with the matrix $\overline{\Pi}$, such that $\boldsymbol{\rho}(k)$ converges in some sense to this process.

2. Even though Theorem III.5.2 provides analytical tools for analyzing the MSS of the JLS III.2.1 driven by the lumped process $\boldsymbol{\rho}(k)$ when it is not an MC, there is still a need to solve the computational problem regarding the dimensionality of the matrix $\mathcal{A}_2$ when one wants to check MSS.

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APPENDIX A

MARKOV CHAINS

A.1 BASIC CONCEPTS

Let $(\Omega, \mathcal{F}, \Pr)$ be a probability space over which all the stochastic processes considered in this work will be defined. Let $\mathcal{I}_S \triangleq \{0, ..., S-1\}$, $S \geq 2$, be a finite set. In this appendix, a brief review is given about MCs that take values in $\mathcal{I}_S$. The set $\mathcal{I}_S$ is called the state space of the MC.

Definition A.1.1. Let $A \triangleq [a_{ij}]$, $i, j \in \mathcal{I}_S$, be a square matrix with components from $\mathbb{R}$. It is said that A is a *stochastic matrix* (by rows) if

1. For all $i, j \in \mathcal{I}_S$: $a_{ij} \geq 0$.
2. For all $i \in \mathcal{I}_S$: $\sum_{j=0}^{S-1} a_{ij} = 1$.

All the stochastic matrices considered in this dissertation are taken to be stochastic by rows.

Definition A.1.2. Let $\{z(k) : k \in \mathbb{Z}^+\}$ be a *stochastic process* with state space $\mathcal{I}_S$, and let

$$p_{ij}(k) \triangleq \Pr(z(k+1) = j \mid z(k) = i)$$

be the one-step transition probability from the state i at time k to the state j at time $k+1$ such that $\Pi(k) \triangleq [p_{ij}(k)]$ $i, j \in \mathcal{I}_S$ is a stochastic matrix. Let $\pi(0) \triangleq (p_0, ..., p_{S-1})$ with $p_i \triangleq \Pr(z(0) = i)$, $i \in \mathcal{I}_S$, be a vector called the *initial state probability vector* of $z(k)$. It is said that the process $z(k)$ is an MC with transition

probability matrix $\Pi(k)$ and initial state probability $\pi(0)$ if the following *Markov property* is satisfied:

$$\Pr\bigl(z(k+1)=\zeta(k+1)\mid z(k)=\zeta(k),\ldots, \qquad \text{(A.1.1)}$$
$$z(0)=\zeta(0)\bigr)=\Pr\bigl(z(k+1)=\zeta(k+1)\mid z(k)=\zeta(k)\bigr),$$

where $\Pr(z(k)=\zeta(k),\ldots,z(0)=\zeta(0))>0$, and $\zeta(k)$ is a state of $z(k)$ in $\mathcal{I}_S$ at time k.

Remarks

1. When the one-step probabilities $p_{ij}(k),\ i,j\in\mathcal{I}_S$, do not depend on time k, the MC is said to be an HMC. Otherwise, it is called an NHMC.

2. Let $z(k)$ be an HMC. The expression $p_{ij}^{(k)}$ is used to denote the k-step transition probability from the state i to the state j, that is, $p_{ij}^{(k)}\triangleq\Pr(z(k)=j\mid z(0)=i)$. Correspondingly, the stochastic matrix $\Pi^{(k)}\triangleq[p_{ij}^{(k)}]$ is called the k-step transition probability matrix of the HMC $z(k)$. It is known that $\Pi^{(k)}=\Pi^k\triangleq\underbrace{\Pi\times\cdots\times\Pi}_{k\text{ times}}$.

Let $z(k)$ be an HMC with state space $\mathcal{I}_S$. The vector $\pi(k)\triangleq\bigl[(\Pr z(k)=0),\ldots,(\Pr z(k)=S-1)\bigr]$ is called the state probability vector of $z(k)$ at time k. The following theorem will be used throughout this work.

Theorem A.1.1. *Let $z(k)$ be an HMC with transition probability matrix Π and initial state probability vector $\pi(0)$. Then*

$$\pi(k)=\pi(0)\Pi^k,\quad k\in\mathbb{Z}^+,$$

where Π^0 is identified with the identity matrix $I_{S\times S}$.

Definition A.1.3. Let $z(k)$ be an HMC with state space $\mathcal{I}_S$, one-step transition probability matrix $\Pi = [p_{ij}]$ and k-step transition probability $\Pi^{(k)} = [p_{ij}^{(k)}]$. It is said that $z(k)$ is *ergodic* if the limits

$$\pi_j = \lim_{k \to \infty} \mathrm{p}_{ij}^{(k)}$$

1. exist for all $j \in \mathcal{I}_S$,
2. are independent of $i \in \mathcal{I}_S$, and
3. for all $j \in \mathcal{I}_S$, $\pi_j > 0$ such that $\sum_{j=0}^{S-1} \pi_j = 1$.

Remarks

1. The vector $\pi \triangleq \left[\pi_1, \ldots, \pi_{S-1}\right]$ is called the stationary probability vector of $z(k)$ and can be found by solving the left eigenvector equation:

$$\pi = \pi\Pi.$$

2. Since the limits $\pi_j = \lim_{k \to \infty} \mathrm{p}_{ij}^{(k)}$ are independent of i, then $\lim_{k \to \infty} \pi(k) = \pi$.

Definition A.1.4. Let $z(k)$ be an HMC with state space $\mathcal{I}_S$ and transition probability matrix $\Pi = [p_{ij}]$. If all entries of Π^k are positive for some $k \in \{2, 3, ...\}$, it is said that Π is *quasi-positive*. If for each pair of indexes $i, j \in \mathcal{I}_S$ there exists an $n \in \mathbb{Z}^+$ such that $p_{ij}^{(n)} > 0$, it is said that the MC is *irreducible*. If $1 = \text{gcd}\{n \geq 1 : p_{ii}^{(n)} > 0 \ \forall \ i \in \mathcal{I}_S\}$, where "*gcd*" denotes the greatest common divisor, it is said that the MC is *aperiodic*.

Theorem A.1.2. *Let $z(k)$ be an HMC with state space $\mathcal{I}_S$ and transition probability matrix Π. Then the following statements are equivalent.*

1. The HMC $z(k)$ is ergodic.

2. The transition probability matrix Π *is quasi-positive.*

3. The HMC $z(k)$ *is aperiodic and irreducible.*

When $z(k)$ is an ergodic HMC with transition probability matrix Π, the sequence of matrices $\{\Pi^k : k \in \mathbb{Z}^+\}$ converges to a stochastic matrix, $\overline{\Pi}$, whose rows are precisely equal to the stationary probability vector π.

A.2 A NOTE ABOUT LUMPABILITY

Let $z(k)$ be an HMC with state space $\mathcal{I}_S$, and let Ξ be the set of all initial state probability vectors, $\pi(0)$. Let ξ be any function that lumps or aggregates the states of $z(k)$. The function ξ is called a lumping transformation, and lumpability is the theory that determines conditions under which the lumped process, $\xi(z(k))$, results in a MC. When the lumped process is an HMC for all $\pi(0) \in \Xi$, it is said that the lumpability is strong. On the other hand, when this lumping transformation results in an HMC for $\pi(0) \in \Phi$, where Φ is a proper subset of Ξ the lumpability is said to be weak ([23, p. 134]). Conditions under which a lumping transformation results in an NHMC have been established (see, e.g., [19]). These conditions also depend on the initial state probability vector $\pi(0)$ of the HMC $z(k)$. Therefore, MCs that result from a lumping transformation can be called *lumped MCs* to distinguish them from the MCs described in Definition A.1.2, as they depend on the initial distribution of the underlying HMC $z(k)$. Not all lumping transformations result in an MC. In this case, the resulting process is simply called a lumped process. This dissertation considers the effect of a lumping transformation on the MSS and performance of a closed-loop control system when it is driven by a lumped process.

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