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POLYNOMIAL CALCULUS: RETHINKING THE ROLE OF ARCHITECTURE AND ACCESS TO ADVANCED STUDY

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Access to advanced study in mathematics, in general, and to Calculus, in particular, depends in part on the conceptual architecture of these knowledge domains, and in this paper we outline an alternative architecture. Our general strategy is to separate advanced concepts from the particular advanced techniques used in their definition and exposition. This alternative architecture, thus, affords access to advanced concepts from an elementary standpoint to a larger group of learners than is presently accomplished. In the case of the Calculus we develop the beginning concepts of the Differential and Integral Calculus using only concepts and skills found in secondary algebra and geometry. The purpose of this reconstruction is not to alter the teaching of limit-based Calculus but rather to affect the content and pedagogy of the secondary mathematics courses which precede it while developing student understanding of key foundational concepts that may enhance the potential for success in post-secondary Calculus for more students.

Key words: calculus, transition line, conceptual architecture

INTRODUCTION

The Calculus is often defined by its central and predominate procedures, by a set of defining techniques. Under this perspective Calculus is synonymous with analysis – the study and application of limits. An alternative is to define a discipline by its constitutive or defining problems. Under this alternative perspective the Calculus is defined by two basic problems – the Tangent Problem and the Area Problem. Such a definition of a knowledge domain makes a clear separation between the problems which the domain addresses and "the technical procedures … invented [for] their exact presentation" (Whitehead, 1911, p.1). In turn, the conceptual architecture of a knowledge domain can either facilitate access to the domain or act as a barrier to entry. A re-conceptualization of the Elementary Calculus that is dependent upon only the basics of high school algebra and geometry holds potential for future access with greater understanding to the more traditional Calculus with limits and the quantitative sciences which depend upon it.

Historically, there have been a number of alternative architectures for the Elementary Calculus. In the 19th century Lagrange launched an ultimately failed attempt to base the Calculus on Algebra (Grabiner, 2010). During the 20th century there have been a number of formulations of the Elementary Calculus without limits (e.g., van der Waerden, 1953; School Mathematics Study Group, 1960; Levi, 1968; Saletan, 1973; Marsden & Weinstein, 1981).

In this paper we present a conceptual architecture for the Elementary Calculus designed to lower the bar of access to underserved students who have had limited or no opportunity for success with advanced studies in mathematics. Our purpose is not to teach Calculus with limits at the secondary level, but to develop a secure and coherent foundation across the secondary curriculum in preparation for successful learning and understanding of post-secondary Calculus. Our general strategy for reconstructing advanced topics in more elementary settings is by separating its foundational concepts from the advanced procedures utilized in their definition.

AN ALTERNATIVE ARCHITECTURE FOR THE ELEMENTARY CALCULUS

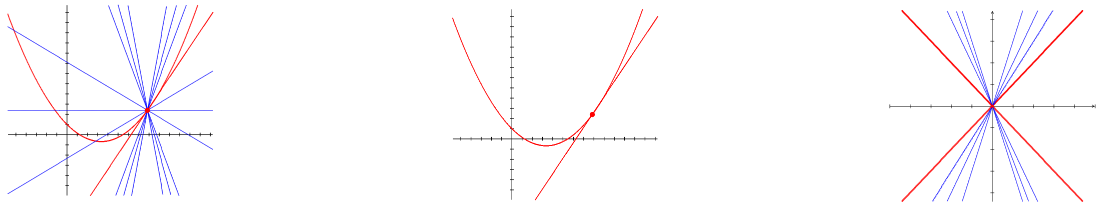
The approach to the Calculus typically begins with a discussion of the need for approximation. The standard argument states that while problems of slopes (rates of change) and areas (net accumulations) are easily obtained for recti-linear figures the same cannot be said for curvi-linear figures and it is necessary to first resort to approximations. Exact results are obtained by recourse to procedures involving limits. In the case of the Differential Calculus the slopes of secant lines are the first approximation to the desired slopes of tangent lines to curves. In the case of the Integral Calculus, Riemann Sums are the first approximation to the desired area under a curve.

In this alternative architecture we capture the basic geometric objects that define the derivative and the integral, but these objects are conceptualized in terms of transition lines rather than approximations and limits. In each case, for the Differential and the Integral Calculus, we consider a pencil of lines, i.e. a family of lines sharing some common property, which is separated into two ordered sets by a transition line. The properties of these transition lines are determined by the application of concepts at the level of secondary algebra and geometry.

Straight Lines as Transition Lines in the Differential Calculus

The Differential Calculus begins with the problem of determining the slope or rate of change of a smooth graph at a given point. We begin by considering the collection of non-vertical straight lines passing through a point on a smooth graph as shown in Figure 1(*a*).

Some of these lines pass from above to below the graph. Some of these lines pass from below to above the graph. We will focus on a particular type of line that we call a transition line. A transition line is a line that, at the given point, separates the lines that pass above the graph from the lines that pass below the graph. The transition line is the basis for our reconceptualization of the notion of the tangent line. The following three theorems characterize the properties of straight lines as transition lines that are relevant to the Differential Calculus.



(*a*) Transition Line (*b*) Closest Line to the Graph (*c*) Transition Region

Figure 1. Transition Lines

Theorem 1: The transition line is unique. Suppose there are two distinct transition lines. If the transition lines are distinct and they each pass through the given point on the graph they must form a non-zero angle between them. Consider a third line that passes through the angle formed by the two transition lines. By definition, the transition line separates the lines that pass above the graph from those that pass below the graph at a given point. From the transition line above the third line, the third line passes below the graph, but from the lower transition line the third line passes above the graph. However, at a given point the same line cannot be both above and below the graph, the two transition lines must coincide.

Theorem 2: The slope of the graph at a given point is the slope of the transition line. At the given point consider the lines passing above the graph, the lines passing below the graph, and the transition line. The lines passing above the graph are changing faster than the graph. The lines passing below the graph are changing slower than the graph. Therefore at the given point the rate of change of the graph is equal to the rate of change of the transition line.

This theorem provides a characterization of the idea of instantaneous rate of change without recourse to arguments based on limits of average rates of change. It uses the notion of trichotomy – the instantaneous rate of change of the graph at the given point is either greater than, equal to, or less than the rate of change of the transition line.

Theorem 3: The transition line is the closest line to the graph. Suppose there is a distinct line passing between the graph and the transition line at the given point. Then this line either passes from one side of the graph to the other or it passes through the point but remains on the same side of the graph. If the line passes from one side of the graph to the other then the transition line would not separate the lines passing above the graph from the lines passing below the graph in contradiction to the definition of the transition line. If the line did not pass from one side of the graph to the other side of the graph at the given point then it is separates the lines passing from one side to the other side of the graph – namely it is the transition line. But Theorem 2 tells us there can be only one transition line. Therefore there can be no line passing between the graph and the transition line.

The transition line, the closest line to the graph, is historically referred to as the tangent line Figure 1(*b*). This is the characterization which Euclid gives for the tangent line in Book III

Proposition 13 of the Elements – the straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed (Heath, 1956, p.37). This last theorem gives a geometric characterization of the transition line which is intuitively and perceptually accessible to secondary school students.

An Absolute Exception

Of course there are cases where transition lines (i.e., tangent lines) do not exist. Consider the notorious case of the absolute value function Figure 1(*c*). At the origin, the tangent line does not exist. Similarly, at the origin there is no transition line which separates the lines passing from above to below the absolute value graph from the lines passing from below to above. There is however a transition region which separates these two collections of lines. The transition region is the double cone, centered at the origin, with boundaries given by $y = \pm x$, and spanning the region between the boundary lines which includes the *x*-axis.

An Application of the Closest Line to a Polynomial Graph

This section describes the conceptual approach to the notion of the tangent line. For a description of how these ideas are applied in the classroom see (Grant, 2012) in this volume.

(*a*) Tangent Line at the *y*-intercept (*b*) Tangent Line at an arbitrary point

Figure 2. Tangent lines at different points

We first consider the tangent line at the *y*-intercept (refer to Figure 2(a)). That is, in general terms, we will find the closest line to a parabola given by an equation of the form $y = ax^2 + bx + c$ at its *y*-intercept (0, *c*). This geometric condition places an arithmetic condition on the coordinates of the polynomials. The closest line to the parabola requires the closest numerical agreement between the coordinates of the parabola and the coordinates of the line. In turn, this arithmetic condition requires a consequent condition on the polynomials for both the parabola and the closest line (i.e., tangent line). For the coordinates of the parabola and the tangent line to have the closest numerical agreement the polynomials that define the two graphs must maximally agree, which occurs when their coefficients are the same to first order. Specifically, the polynomials agree to first order for the parabola (Eq. 1) and the tangent line (Eq. 2) at $x = 0$ are given by the following equations:

$$y = ax^2 + bx + c \quad (1)$$

$$y_t = bx + c \quad (2)$$

Next we consider the tangent line to a parabola given by $y = ax^2 + bx + c$ at an arbitrary point, $x = x_0$ (refer to Figure 2 (*b*)). We will use the same approach used for finding the closest line at the *y*-intercept by applying a basic transformation. That is, move the coordinate axes so that the point of interest is at the *y*-intercept of the transformed axes. The transformation in *x* will have new coordinates, $x = x_0 + x'$ and the tangent line at $x = x_0$ becomes the tangent line at $x' = 0$ (i.e., the transformed *y*-intercept). We can rewrite the equation for the parabola using this transformation (Eq. 3) and simplify (Eq. 4) the transformed polynomial.

$$y = a(x_0 + x')^2 + b(x_0 + x') + c \quad (3)$$

$$y = a(x')^2 + (2ax_0 + b)x' + (ax_0^2 + bx_0 + c) \quad (4)$$

Then, the previous approach allows us to write the equation for the tangent line (Eq. 5) at the *y*-intercept as the linear equation that agrees to first order to the polynomial for the parabola (Eq. 4). Now, rewriting the equation of the tangent line at $x = x_0$ in terms of the original coordinates, the closest line (i.e., tangent line) is given by Eq. 6.

$$y_t = (2ax_0 + b)x' + (ax_0^2 + bx_0 + c) \quad (5)$$

$$y_t = (2ax_0 + b)(x - x_0) + (ax_0^2 + bx_0 + c) \quad (6)$$

Summarizing, under this construction, the equation of the tangent line to a polynomial graph agrees with the equation of the polynomial graph to first order. This concept of agreement is an equivalence relation and replaces the order relation characterized by limits in determining the tangent line to a smooth graph. This equivalence relation may also be applied to polynomials in two variables, covering both rational and algebraic functions.

Horizontal Lines as Transition Lines in the Integral Calculus

The Integral Calculus begins with the problem of determining the area between a continuous boundary graph and a given interval on the *x*-axis. In this case we begin by considering the collection of all horizontal lines on a given interval. Each horizontal line forms a rectangle bounded by the horizontal line and the interval on the *x*-axis. The *y*-intercept of the horizontal line gives the height of the rectangle and the base is the length of the interval on the *x*-axis.

(*a*) Horizontal Transition Line (*b*) Center of the Graph on an Interval

Figure 3. Horizontal transition lines on graphs

The area for each rectangle is the product of the base multiplied by the height of the horizontal line. Some of the rectangles formed by the horizontal lines have an area greater than the area between the graph and the *x*-axis and other rectangles have an area less than the area between the graph and the *x*-axis. We will focus on a particular horizontal line, another type of transition line. This transition line is the horizontal line that forms a rectangle across the interval that separates the horizontal lines that correspond to rectangles with areas greater than the area of the graph from horizontal lines that correspond to rectangles with areas less than the area of the graph (Figure 3 (*a*)). This transition line is the basis for our reconceptualization of the notion of the mean value line. The following three theorems characterize the properties of horizontal lines as transition lines that are relevant to the Integral Calculus.

Theorem 4: The transition line is unique. Suppose there are two distinct horizontal transition lines. Ceteris paribus the argument is the same as for the straight line as transition line above in Theorem 1.

Theorem 5: The area under the transition line is equal to the area under the graph. The argument for the equality of area under the graph and the area under the horizontal transition line, again, is, ceteris paribus, the same as in Theorem 2 above for the straight line as transition line.

In the previous section Theorem 3 gave a geometric characterization of the straight line as transition line which is intuitively and perceptually accessible to secondary school students. Theorem 6 has a similar purpose. For this theorem we define the notion of the center of a graph on a given interval, a generalization from the discrete case follows.

We define the center of a finite collection of points distributed along the x-axis as the location where the total displacement from the location of the center to the points is zero and can be represented symbolically as (Eq. 7). This definition of a center is also designated as the mean of a distribution. A short derivation from the defining equation (Eq. 7) below establishes that the center is determined by a rate (Eq. 9), the rate that we call the arithmetic average of the distribution.

$$\sum (x_i - M) = 0 \quad (7)$$

$$\sum x_i - NM = 0 \quad (8)$$

$$M = (\sum x_i) / N \quad (9)$$

It is important to note that this derivation marks an important conceptual distinction between two ideas that are often conflated – the mean, as a position in a distribution of values and the arithmetic average, as a constant rate associated with the distribution of values.

In a similar fashion we define the center of a graph across an interval, say (*a*, *b*), as the vertical location where the total signed area between the graph and the horizontal line at that location is zero. Symbolically, rather than as a discrete sum, we can represent the defining condition as an area function from $x = a$ to $x = b$ of the difference given in (Eq. 10).

$$A_a^b (y - M) = 0 \quad (10)$$

Theorem 6: The transition line passes through the center of the graph across the given interval. Because the area under the graph is equal to the area under the horizontal transition line the geometric area above the transition line and below the graph must be equal to the geometric area below the transition line and above the graph across the given interval. Consequently the signed area between the graph and the Transition Line across the given interval must be zero (Figure 3(*b*)), and can be represented symbolically (Eq. 11 & 12). We believe these transition line characteristics will be accessible to secondary mathematics students with a basic background in algebra and geometry.

$$A_a^b(y) = A_a^b(M) \quad (11)$$

$$A_a^b(y-M) = A_a^b(y) - A_a^b(M) = 0 \quad (12)$$

In the continuous case, the equality of the mean and the arithmetic average, i.e., the equality of the mean of the slope of the tangent line across an interval and the slope of the secant line across the same interval is an equivalent form of the Fundamental Theorem of Calculus.

An Application to a New Quadrature of the Parabola

Secondary geometry is viewed from a few definite perspectives – congruence, similarity, and scissor congruence or what Hilbert (1971) defined as equi-decomposibility. These perspectives correspond to equivalence relations on size and shape, shape, and size respectively. Scissor congruence (size equivalence) is usually applied to polygonal figures. This application of secondary geometry to the Elementary Calculus focuses on equi-area geometry utilizing notions of symmetry and transformations. The sequence of graphs depicted in Figure 4 and Figure 5 were captured from a program that runs on the TI-84 graphing calculator. The calculator shades the region of interest. The grid is on for these graphs so students can generate reasonable conjectures about the areas of the shaded regions. The calculator's trace function is used to enable students to access equations for the regions' boundary graphs. Students are tasked with working through the sequence of pictures in order to determine the reasons which account for this proof without words.

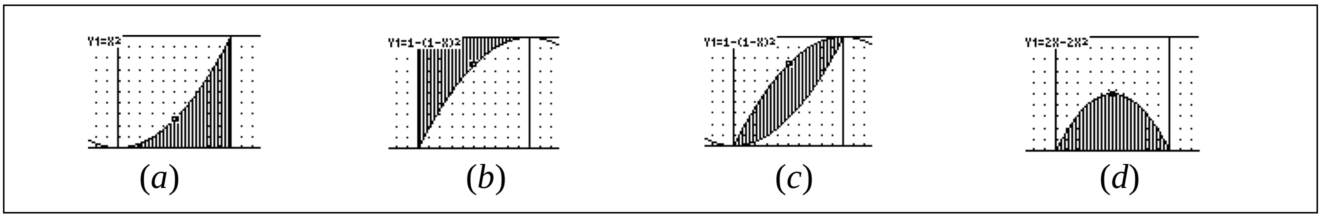


Figure 4. TI-84 graphs depicting areas of interest

In this example the task is to determine the area of the region between the parabola ($y = x^2$) and the positive *x*-axis within the enclosing unit square. We symbolize the area of the shaded region as *P* and refer to the region as a parabolic "triangle" (Figure 4(*a*)). A double reflection (or a rotation) of *P* was used to create the region shown in Figure 4(*b*). The area of the unit square, *S*, in Figure 4(*c*) is comprised of the area of the two parabolic triangles, $2P$, and the shaded region within S, the gap, G; thus, $S = 2P + G$. The strategy used in this proof is to first determine the relationship between the area of the gap and the parabolic triangle and then the relationship between the parabolic triangle and enclosing square.

The relationship between the areas in Figure 4(c) and (d) raise the toughest questions for students. An algebraic approach states that the area of the gap is determined solely by the distance between its upper and lower boundaries. So the area under the difference between the upper and lower boundaries in Figure 4(c), given by the equation $y = 2x - 2x^2$ is equal to the area of the gap, G. Specifically, the gap shown in Figure 4(c) and the hill shown in Figure 4(d) have equal areas.

(a) (b) (c) (d) (e)

Figure 5. TI-84 graphs depicting areas of interest

The area of the translated region shown in Figure 5(a) is half the area of the gap, ½ G. Horizontally scaling the shaded region in Figure 5(a) by a factor of 2 results in Figure 5(b) with a shaded region of area equal to the gap, 2(½ G).Vertically scaling the region in Figure 5(b) by a factor of 2 results in a figure that has twice the area of the gap, $2G$. The shaded region in Figure 5(c) is the same size and shape as the original gap and parabolic triangle, $G+P$, which can be derived from the original square containing two parabolic triangles and the gap by a reflection about a vertical line located at the midpoint of the base. Since the area of the shaded region in Figure 5(c) was equal to twice the area of the gap, the area of the lower parabolic triangle must also be equal to the area of the gap, $G = P$.

Consequently, the area of the enclosing square is three times the area of the parabolic triangle or $P = \frac{1}{3} S$. The rectangle with area equal to the area under the parabola is the rectangle of height 1/3 on the same base. A similar geometric analysis yields the area under the cubic. And with the help of two combinatoric identities and a proof by induction the area of all figures with polynomial boundaries can be determined algebraically and exactly without Limits (Crombie, 2001).

Conventional wisdom suggests the use of rectangles as approximating figures in determining the area of figures that are difficult to deal with because they contain curved boundaries. From the time of Archimedes it appeared that figures with curvilinear boundaries – including those we now define as polynomial boundaries – fell into this “difficult” class. Using the afore-described approach, we now know that curves with polynomial boundaries no longer fall into the difficult category. Figures with polynomial boundaries fall into the class of approximating figures. Further, rectangles can be categorized as the first in the series of polynomial figures whose area can be determined exactly using only algebraic and geometric means.

SUMMARY: CALCULUS REFORM WITHOUT REFORMING CALCULUS

As a pedagogical tool, Strang (1990) describes the basic architecture of the Elementary Calculus before limits. He restricts his examination of derivatives and integrals to piece-wise linear and piece-wise constant functions defined on the integers. And consequently, derivatives and integrals are described by differences and sums, respectively. Strang's presentation is meant to support student understanding when they move on to limit-based Calculus. This paper describes a Polynomial Calculus architecture that is not before limits, but is independent of limits. This type of calculus suggests a large body of the Elementary Calculus, defined by foundational problems and not technique, is accessible with only secondary algebra and geometry. Under the constructions presented here the properties of derivatives and integrals are consequences of the properties of transition lines defined on pencils of straight line and pencils of horizontal lines, respectively.

Kaput (1997) rethinks Calculus as a system of knowledge and technique (C-KNOWL) and as an institution (C-INST). C-KNOWL is an intrinsically difficult subject and C-INST has acted as a filter allowing only a limited few access to move on to higher mathematics and the sciences. A fundamental change in calculus as a system of knowledge and technique can radically alter the role of calculus as an institution and potentially support a redistribution of opportunity for all or at least most students completing a standard secondary mathematics sequence.

Both supporters and detractors of calculus reform have operated under a tacit assumption that the historically derived form of the calculus was for all intents and purposes immutable – close to a law of nature. We have focused not on reforming the institution of the calculus but rather the calculus as a system of knowledge and technique. As a consequence of our analysis we are now able to make a critical distinction between the conceptual foundations of the Elementary Calculus and its logical foundations. There are different conceptual foundations each with its unique logical foundations. In this reformulation the algebraic notion of equivalence replaces the analytic notion of limit as a conceptual foundation of the calculus.

We recognize that the conceptual architecture of a knowledge domain can act as a passive barrier to entry as surely as the architecture of a building determines the presence or absence of physical barriers to entry. Architecture, whether conceptual or physical, is intentional. The conceptual architecture of mathematics in general and of the calculus in particular is neither predetermined nor immutable. There is a conceptual architecture of the Elementary Calculus which is potentially accessible to students at the level of secondary algebra and geometry and provides a gateway to the calculus with limits and advanced mathematics. This work is part of an on-going effort to develop understanding of key elementary calculus concepts, at the secondary level, to increase access to and a success oriented pathway toward advanced mathematics.

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