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Nonlinear Flutter of Curved Panels Under Yawed Supersonic Flow Using Finite Elements

Mohamed Salim Azzouz
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Nonlinear Flutter of Curved Panels Under Yawed Supersonic Flow Using Finite Elements

by

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M.Sc. January 1990, Swiss Federal Institute of Technology, Lausanne, Switzerland

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ABSTRACT

In the extensive published literature on panel flutter, a large number of papers are dedicated to investigation of flat plates in the supersonic flow regime. Very few authors have extended their work to flutter of curved panels. The curved geometry generates a pre-flutter behavior, triggering a static deflection due to a static aerodynamic load (SAL) over the panel as well as dynamic characteristics unique to this geometry. The purpose of this dissertation is to provide new insights in the subject of flutter of curved panels. Finite element frequency and time domain methods are developed to predict the pre/post flutter responses and the flutter onset of curved panels under a yaw flow angle. The first-order shear deformation theory, the Marguerre plate theory, the von Karman large deflection theory, and the quasi-steady first-order piston theory appended with SAL are used in the formulation. The principle of virtual work is applied to develop the equations of motion of the fluttering system in structural node degrees of freedom. In the frequency domain method, the Newton-Raphson method is used to determine the panel static deflection

under the SAL, and an eigen-value solution is employed for the determination of the stability boundary margins at different panel height-rises and yaw flow angles. Pre-flutter static deflection shape, flutter coalescence frequency, and damping rate of various cylindrical panels are thoroughly investigated. The main results revealed that the pre-flutter static response of cylindrical panels is fundamentally different from the one associated with flat plates. It is shown that curvature has a detrimental effect for 2-dimensional (2-D) curved panels, and is beneficial for 3-D components at an optimum height-rise. In the time domain method, the system equations of motion are transformed into modal coordinates, and solved by a fourth-order Runge-Kutta numerical scheme. Time history responses, phase plots, power spectrum density plots, and bifurcation diagrams uncovered the pre/post flutter responses of cylindrical panels. The computed stability boundary margins and onset frequencies matched very well with the ones computed by the frequency domain method. Bifurcation diagrams revealed limit-cycles oscillations (LCO) and chaotic motion. It was found that 2-D cylindrical panels settle in a multiplicity of LCO as the height-rise of the panel increases, whereas chaotic motion characterize the dynamic behavior of 3-D cylindrical panels at high height-rises.

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TABLE OF CONTENTS

	<u>Page</u>
ACKNOWLEDGEMENTS.....	ii
LIST OF TABLES.....	vii
LIST OF FIGURES.....	viii
LIST OF SYMBOLS.....	xii

Chapter

1	INTRODUCTION AND LITERATURE REVIEW	1
1.1	Flutter History.....	1
1.2	Nature of the Flutter Phenomenon.....	2
1.3	Flutter Flow Configuration	4
1.4	Flutter Theoretical Background	4
1.5	Literature Survey	6
1.5.1	Cross-Stream Curvature Flow.....	7
1.5.2	Stream-Wise Curvature Flow	9
1.5.3	Arbitrary Yaw Flow Angle	11
1.5.4	Static Load Effect	14
1.5.5	Aerodynamic Load Effect.....	15
1.5.6	Chaotic Motion	16
1.6	Scope.....	20
2	FINITE ELEMENT FORMULATION	28
2.1	Introduction.....	28
2.2	Element Displacement Vectors.....	28

2.3	Element Displacement Functions	29
2.4	Non-linear Total Strain Deformation Vector	32
2.5	Total Transverse Shear Strain Deformation Vector.....	34
2.6	Constitutive Relations	36
2.7	Resultant Laminates Forces and Moments	38
2.8	Curved System Element Matrices.....	39
2.8.1	Development of the Virtual Work Done by the Internal Forces.....	40
2.8.2	Element Linear Stiffness Matrices	44
2.8.3	Element Linear Shear Stiffness Matrices.....	45
2.8.4	Element Linear Stiffness Matrices Due to $w_o(x,y)$	45
2.8.5	Expansion of the Element First Order Non-Linear Stiffness Matrices.....	46
2.8.6	Element First Order Non-linear Matrices	50
2.8.7	Element First-Order Non-Linear Stiffness Matrices Due to $w_o(x,y)$	51
2.8.8	Element First-Order Non-Linear Stiffness Matrices Due to $[N_b]$	52
2.8.9	Element First-Order Non-Linear Stiffness Matrices Due to $[N_m]$	52
2.8.10	Element First-Order Non-Linear Stiffness Matrices Due to $[N_{\theta_o}^{wb}]$	52
2.8.11	Element First-Order Non-Linear Stiffness Matrices Due to $[N_{\theta_o}^{wv}]$	53
2.8.12	Element Second-Order Non-Linear Stiffness Matrices	53
2.8.13	Virtual Work Done by External Forces	54
2.8.14	Element Mass Matrices.....	54
2.8.15	Quasi-Steady First-Order Aerodynamic Piston Theory.....	56
2.8.16	Element Aerodynamic Stiffness Matrices	60
2.8.17	Element Aerodynamic Damping Matrices.....	60

2.8.18	Element Static Aerodynamic Load Vector	61
2.8.19	Element External Load Vectors	62
2.9	Element Equations of Motion of the Curved Panel	62
2.10	Global Equations of Motion.....	67
3	SOLUTIONS PROCEDURES	75
3.1	Introduction.....	75
3.2	Preliminary Assumptions.....	76
3.3	Separation of the EOM into Static and Dynamic Equations.....	79
3.4	Solution Procedure for the Static Aerodynamic Equation.....	80
3.4.1	Solution Procedure for the Static Aerodynamic Deflection $\{W\}_s$	85
3.4.2	Solution Procedure for the Dynamic Small Deflection $\{W\}_t$	86
3.5	Preliminary Process for Eigen-Solutions	91
3.5.1	Equations of Motion Expressed in $\{W^b\}$	91
3.5.2	Eigen-Analysis.....	96
3.6	Non-Linear Post-Flutter Panel Response.....	97
3.6.1	Neglecting the In-plane Inertia Term in the EOM.....	97
3.6.2	Equations in Modal Coordinates.....	101
3.7	Modal Participation Definition	107
3.8	Multimode Fourth-Order Runge-Kutta Method	107
3.8.1	Runge-Kutta State -Space Scheme	109
3.8.2	Considerations when Applying Runge-Kutta Scheme	110
3.9	Motion Categories Investigation.....	111
3.9.1	Time History Responses	112

3.9.2	Phase Plots	112
3.9.3	Power Spectral Density	113
3.9.4	Bifurcations Diagrams	114
3.10	Strain and Stress Computation.....	115
4	RESULTS AND DISCUSSIONS: FREQUENCY DOMAIN METHOD	120
4.1	Introduction.....	120
4.2	2-D Cylindrical Curved Geometry.....	121
4.3	3-D Doubly Curved Geometry.....	122
4.4	Derivative of the Cylindrical and Spherical Curved Geometry	123
4.5	Static Aerodynamic Load (SAL)	123
4.6	Frequency Domain Solutions for Flutter Stability Boundaries.....	124
4.6.1	Aerostatic Deflection of 2-D Cylindrical Panels	125
4.6.2	Aerostatic Deflection of 3-D Cylindrical Panels	127
4.6.3	Flutter Investigation of 2-D Cylindrical Panels	128
4.6.4	Flutter Investigation of 3-D Cylindrical Panels	131
4.7	Critical Dynamic Pressure Analysis for 2-D and 3-D Panels	132
5	MODE SHAPE CATEGORIES	169
5.1	First Category: Sinusoidal Mode Shapes.....	170
5.2	Second Category: Natural Mode Shapes	170
5.2.1	2-D Natural Mode Shapes.....	171
5.2.2	3-D Natural Mode Shapes.....	173
5.3	Third Category: Aerostatic Mode Shapes.....	174
5.3.1	2-D Aerostatic Mode Shapes	174

5.3.2	3-D Aerostatic Mode Shapes at $A = 0^\circ$	176
6	TIME DOMAIN ANALYSIS.....	191
6.1	Convergence Study	191
6.1.1	Modal participation.....	191
6.1.2	Time Integration Step	193
6.2	2-D Panel Time Domain Analysis	193
6.2.1	Bifurcation Diagrams for 2-D Panels with $H/h = 1$	194
6.2.2	Flutter Time History Analysis for 2-D Panels with $H/h = 1$	195
6.2.3	Phase Plots for 2-D Panels with $H/h = 1$	196
6.2.4	Power Spectral Density for 2-D Panels with $H/h = 1$	197
6.2.5	Bifurcation Diagrams for 2-D Panels with $1 < H/h \leq 1.625$	198
6.2.6	Bifurcation Diagrams for 2-D Panels with $H/h > 1.625$	200
6.3	3-D Panel Time Domain Analysis	201
6.3.1	Bifurcation Diagrams for 3-D Panel with $H/h = 1$ and $A = 0^\circ$	203
6.3.2	Modal Participation for 3-D Panel with $H/h = 1$ and $A = 0^\circ$	204
6.3.3	Time History Analysis for 3-D Panel with $H/h = 1$ and $A = 0^\circ$	205
6.3.4	Phase Plots for 3-D Panel with $H/h = 1$ and $A = 0^\circ$	205
6.3.5	Power Spectral Density for 3-D Panel with $H/h = 1$ and $A = 0^\circ$	206
6.3.6	General Study of 3-D Panel with $H/h = 1$ and $A = 45^\circ$	207
6.3.7	General Study of 3-D Panels with $H/h > 1$ and $A = 0^\circ$	208
6.4	Stress Analysis: Case Study of 2-D Panels.....	211
7	SUMMARY AND CONCLUSIONS	248
8	REFERENCES	253

9	APPENDICES	263
	A. SYSTEM SPLIT MATRICES	263
	B. MODE SHAPES FOR HEIGHT-RISE $H/h = 1$	265
	C. 2-D AND 3-D SINUSOIDAL FREQUENCIES FOR CURVED PLATES	269
1	Sinusoidal Frequency Comparison for 2D-Curved Panels	269
2	Sinusoidal Frequency Comparison for 3D-Curved Panels	270

LIST OF TABLES

<u>Table</u>	<u>Page</u>
Table 4.1 Critical dynamic pressure convergence study for a 2-D simply supported cylindrical panel of height-rise $H/h = 1$	167
Table 4.2 Critical dynamic pressure convergence study for a 3-D simply supported cylindrical panel of height-rise $H/h = 1$	168
Table 5.1 Curved panels natural frequencies computation with Dowell's Galerkin Method and Finite Element Method	189
Table 5.2 Natural frequencies computation with Dowell's Galerkin Method and Finite Element Method for a 3-D doubly curved panel	190
Table 6.1 Modal participation in % at a specified non-dimensional dynamic pressure $\lambda = 800$ for a curved with height-rise $H/h = 5$	246
Table 6.2 Modal participation values at various dynamic pressures for simply supported square isotropic cylindrical panel at $H/h = 1$, flow angle $\Lambda = 0^\circ$ and $Ca = 0.01$	247

LIST OF FIGURES

<u>Figure</u>	<u>Page</u>
Fig. 1.1 Handley Page O/400 twin-engine biplane	23
Fig. 1.2 Lockheed P-80.....	24
Fig. 1.3 Cylindrical panel with cross-stream curvature flow	25
Fig. 1.4 Cylindrical panel with stream-wise curvature flow.....	26
Fig. 1.5 Cylindrical panel with yawing flow angle.....	27
Fig. 2.1 MIN 3 element geometry including area coordinates.....	68
Fig. 2.2 MIN3 element nodal displacements and element dimensions.....	69
Fig. 2.3 Coordinates details of a point belonging to the curved panel.....	70
Fig. 2.4 Curved panel geometry characterized by $w_o(x, y)$, the height-rise H/h and MIN3 element	71
Fig. 2.5 Cylindrical panel geometry and dimensions.....	72
Fig. 2.6 Spherical panel geometry and dimensions.....	73
Fig. 2.7 First-order piston theory pressure modeling	74
Fig. 3.1 Four point slope estimates of the Runge – Kutta method	118
Fig. 3.2 Relative maximums and minimums projection for a given λ	119
Fig. 4.1 Cylindrical panel curvature R_x and height-rise H/h	134
Fig. 4.2 Cylindrical panel underneath the $z = 0$ plan	135
Fig. 4.3 Cylindrical panel above the $z = 0$ plan	136
Fig. 4.4 Spherical panel with the $z = 0$ plan.....	137
Fig. 4.5 Static aerodynamic load along x axis at various dynamic pressures for 3-D cylindrical panel of height-rise $H/h = 1$	138

Fig. 4.6	Static aerodynamic load along x axis at dynamic pressure $\lambda = 700$ and various height-rises for 3-D cylindrical panel	139
Fig. 4.7	2-D curved panel discretized with a mesh size of 79×1 encompassing 158 MIN3 elements.....	140
Fig. 4.8	Aerodynamic static equilibrium deflection $\{W_s\}$ of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1$	141
Fig. 4.9	Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1$	142
Fig. 4.10	Aerodynamic static equilibrium deflection $\{W_s\}$ of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1.625$	143
Fig. 4.11	Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1.625$	144
Fig. 4.12	Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 3$	145
Fig. 4.13	Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 5$	146
Fig. 4.14	3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel with a height-rise of $H/h = 1$ and yaw flow angle $\Lambda = 0^\circ$	147
Fig. 4.15	3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel at yaw flow angle $\Lambda = 0^\circ$ and dynamic pressure $\lambda = \lambda_{cr}$.	148
Fig. 4.16	3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel with a height-rise $H/h = 1$ and dynamic pressure $\lambda = 350$	149

Fig. 4.17	3-D aerodynamic static equilibrium deflection $\{Ws\}/h$ of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$	150
Fig. 4.18	3-D aerostatic deflection shape of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$	151
Fig. 4.19	3-D aerodynamic static equilibrium deflection $\{Ws\}/h$ of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$	152
Fig. 4.20	3-D aerostatic deflection shape of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$	153
Fig. 4.21	Flutter coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 1$ with damping parameter $Ca = 0.01$	154
Fig. 4.22	Damping rate versus dynamic pressure for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$	155
Fig. 4.23	Aerostatic mode softening/hardening for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$	156
Fig. 4.24	Flutter coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 1.625$ with damping parameter $Ca = 0.01$	157
Fig. 4.25	Damping rate versus dynamic pressure for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1.625$ and damping parameter $Ca = 0.01$	158

Fig. 4.26	Aerostatic mode softening/hardening for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1.625$ and damping parameter $Ca = 0.01$	159
Fig. 4.27	Flutter non-coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 3$ with damping parameter $Ca = 0.01$	160
Fig. 4.28	Aerostatic mode softening/hardening for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$	161
Fig. 4.29	Aerostatic mode softening/hardening for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 3.5$ and damping parameter $Ca = 0.01$	162
Fig. 4.30	Damping rate versus non- D dynamic pressure for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 5$ and damping parameter $Ca = 0.01$	163
Fig. 4.31	Aerostatic mode softening/hardening for a 3-D isotropic simply supported curved panel of height-rise $H/h = 5$ and damping parameter $Ca = 0.01$	164
Fig. 4.32	Critical dynamic pressure versus panel height-rise for a 2-D isotropic simply supported cylindrical panel with damping parameter $Ca = 0.1$	165
Fig. 4.33	Critical dynamic pressure versus panel height-rise for a 3-D isotropic simply supported cylindrical panel with damping parameter $Ca = 0.01$ and yaw flow angle $A = 0^\circ$	166
Fig. 5.1	Lowest four natural mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rises of $H/h = 0$ and $H/h = 2$	179

Fig. 5.2	Lowest four natural mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rises of $H/h = 0$ and $H/h = 5$	180
Fig. 5.3	Lowest four natural mode shapes of a 3-D isotropic simply supported cylindrical panel of height-rise of $H/h = 0$	181
Fig. 5.4	Lowest four natural mode shapes of a 3-D isotropic simply supported cylindrical panel of height-rise of $H/h = 4$	182
Fig. 5.5	Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 1$	183
Fig. 5.6	Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 3$	184
Fig. 5.7	Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 5$	185
Fig. 5.8	First and second aerostatic mode shape progression toward flutter onset of ... 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and yaw flow angle $\Lambda = 0^\circ$	186
Fig. 5.9	First and second aerostatic mode shape evolution toward flutter onset of a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 3$ and yaw flow angle $\Lambda = 0^\circ$	187
Fig. 5.10	Sixth and seventh aerostatic mode shape evolution toward flutter onset of a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 5$ and yaw flow angle $\Lambda = 0^\circ$	188
Fig. 6.1	Time integration step convergence for 2-D simply supported cylindrical panel with height-rise $H/h = 1$	213

Fig. 6.2	Bifurcation diagram for simply supported 2-D cylindrical panels of height-rise $H/h = 1$	214
Fig. 6.3	Flutter time history responses of a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25$ % location with LCO patterns	215
Fig. 6.4	Flutter time history responses of a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at 81.25 % location with random ocillation patterns...	216
Fig. 6.5	Phase plots for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25$ % location with LCO patterns	217
Fig. 6.6	Phase plots for simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25$ % location with random patterns	218
Fig. 6.7	Power spectral density for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25$ % location	219
Fig. 6.8	Bifurcation diagram for simply supported 2-D cylindrical panels of height-rise $H/h = 1.625$	220
Fig. 6.9	Static portion of bifurcation diagram illutrating the snap-through phenomenon for simply supported 2-D cylindrical panels of height-rise $H/h = 1.625$	221
Fig. 6.10	Flutter time history responses of a simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$ at 81.25 % location	222
Fig. 6.11	Flutter phase plots of a simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$ at 81.25 % location	223
Fig. 6.12	Power spectral density for a simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$ at $x/a = 81.25$ % location	224

Fig. 6.13	Bifurcation diagrams for simply supported 2-D cylindrical panels of height-rise $H/h = 2.4, 3$, and 5	225
Fig. 6.14	Flutter time history responses, phases plots, and power spectrum density of a simply supported 2-D cylindrical panel of height-rise $H/h = 3$, and 5 at $x/a = 81.25\%$ location	226
Fig. 6.15	Flutter time history responses, and power spectrum density illustrating alternating periodic LCO's of a simply supported 2-D cylindrical panel of height-rise $H/h = 5$ at $x/a = 81.25\%$ location	227
Fig. 6.16	Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 0^\circ$ at $x/a = 75\%$ location involving the 8 lowest stream-wise modes	228
Fig. 6.17	Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 0^\circ$ at $x/a = 75\%$ location involving the 16 lowest stream-wise modes	229
Fig. 6.18	Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 0^\circ$ at $x/a = 75\%$ location.....	230
Fig. 6.19	Phase plots for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 0^\circ$ at $x/a = 75\%$ location.....	231
Fig. 6.20	Power spectral plot for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 0^\circ$ at $x/a = 75\%$ location.....	232
Fig. 6.21	Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $A = 45^\circ$ at $x/a = 75\%$ location involving the 16 lowest stream-wise modes	233

Fig. 6.22	Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location.....	234
Fig. 6.23	Phase plots for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location.....	235
Fig. 6.24	Power spectral plot for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location.....	236
Fig. 6.25	Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 3$ and $\Lambda = 0^\circ$ at $x/a = 75\%$ location	237
Fig. 6.26	Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 3$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location.....	238
Fig. 6.27	Time history response and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 3$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location.....	238
Fig. 6.28	Bifurcation diagrams of simply supported 3-D cylindrical panels of height-rise $H/h = 5$ and $\Lambda = 0^\circ$ at $x/a = 75\%$ location	239
Fig. 6.29	Time history responses, phase plots, and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 5$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location	240
Fig. 6.30	Time history responses, phase plots, and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 5$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location	241
Fig. 6.31	LCO deflection for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$	242

Fig. 6.32	Stress distribution of a simply supported 2-D cylindrical panel of height-rise $H/h = 1$	243
Fig. 6.33	LCO deflection for a simply supported 2-D cylindrical panel of height-rise $H/h = 5$	244
Fig. 6.34	Stress distribution of a simply supported 2-D cylindrical panel of height-rise $H/h = 5$	245

LIST OF SYMBOLS

English Symbols

a	panel length
$A, [A]$	area of element, extension laminate stiffness matrix
$[a], [A_a]$	element and system aerodynamic stiffness matrices
$[A_x], [A_y]$	aerodynamic stiffness matrices in x, y direction
$[A_s]$	shear laminate stiffness matrix
b	panel width
$[B]$	coupling laminate stiffness matrix
$[C]$	matrix relating strains to displacements
C_a	damping parameter
$[D]$	bending laminate stiffness matrix
D_{110}	first entry in laminate bending stiffness $[D]$ with all fibers in x direction
E	Young's moduli
F_d	local external force
G	shear moduli
$\{G\}$	slope vector
$[g], [G]$	element and system damping matrices
g_a	non-dimensional aerodynamic damping
h	panel thickness, small time step interval
H	panel maximum height-rise
$[H]$	interpolation functions matrix

$[k], [K]$	element and system linear stiffness matrices
$[K_{tan}]$	tangent stiffness matrix
L, M	interpolation function
$[m], [M]$	element and system mass matrices
$\{M\}$	bending moment vector
M_∞	Mach number
N	interpolation function
$\{N\}$	in-plane force vector
$[N]$	in-plane force matrix
$[n1], [N1]$	element and system 1 st order nonlinear system stiffness matrices
$[n2], [N2]$	element and system 2 nd order nonlinear system stiffness matrices
p	unsteady pressure
$\{p(t)\}$	external forces: mechanical, acoustic, etc.
p_a	aerodynamic pressure
p_∞	far field pressure
$\{P_{sal}\}$	static aerodynamic load
$\{q\}$	generalized modal coordinate vector
q_a	dynamic pressure
$[\bar{Q}]$	transformed reduced stiffness matrix
$[\bar{Q}_s]$	transformed reduced shear stiffness matrix
R	curvature radius

$\{R\}$	shear force vector
$[T]$	coordinate transformation matrix
u, v	in-plane displacements
V_∞	far field fluid velocity
$\{w\}, \{W\}$	element, system nodal out-of-plane displacements
w_o	geometry of curved panel
x, y, z	cartesian coordinates

Greek Symbols

α	damping rate, angle
α_d	non-dimensional damping rate
α_s	shear correction factor
β	Prandtl-Glauert factor
$\{\gamma\}$	shear strain vector
Γ	non-dimensional curvature
$\{\varepsilon\}$	strain vector
ζ	time variable
$\theta, [\theta]$	fiber laminate angle, slope matrix
$[\theta_o]$	slope curvature matrix
$\kappa, \{\kappa\}$	eigen-value, curvature vector
λ	non-dimensional dynamic pressure
Λ	yaw flow angle
μ	air-panel mass ratio
ν	Poisson's ratio

ξ	area coordinate
ρ	panel mass density
ρ_a	air mass density
$\{\sigma\}$	local stress
$\{\tau\}$	local shear stress
$\{\phi\}$	eigen-vector
$\Phi, [\Phi]$	static deflection function, eigenvector matrix
ψ_x, ψ_y	rotations of the normal about x and y axes
ω	frequency
ω_o	reference frequency
Ω	complex frequency

Subscripts

a	air, aerodynamic
b	bending and rotational component
B	coupling bending/rotational and in-plane components
cr	critical
ext	external
int	internal
k	lamina layer
L	linear
m	in-plane component
max	maximum
min	minimum

o	curvature, initial conditions, reference frequency
q	quadratic non-linearity
qq	cubic non-linearity
r	mode number
s	mode number, shear, static component
sal	static aerodynamic load
st	static and time dependent
t	time dependent, time derivative
tan	tangent
u, v	in-plane components
u_o, v_o	in-plane components due to curved panel geometry
w	out of plane component
w_o	out of plane component due to curved panel geometry
x, y	x and y direction, derivative with respect to x and y
$X, X1, X2$	any b , ψ and m indices
γ	referring to shear
ψ	rotational component
∞	far field component
<u><i>Superscript</i></u>	
b	bending, rotational component
i	iterative process
m	in-plane component
N_b	referring to $\{N_b\} = [C_b]\{w_b\}$

N_m	referring to $\{N_m\} = [C_m]\{w_m\}$
N_{θ_o}	referring to $\{N_{\theta_o}\}$ including $\{N_{\theta_o}^{\psi b}\} = [C_{\psi b}]\{w_b\}$ and $\{N_{\theta_o}^{\psi \psi}\} = [C_{\psi \psi}]\{w_{\psi}\}$
o	curved geometry
r	mode number
s	mode number, shear, static component
sal	static aerodynamic load
T	transpose
u, v	in-plane components
θ	contain the slope matrix
θ_o	curved geometry

Chapter 1

1 INTRODUCTION AND LITERATURE REVIEW

In the extensive published literature on panel flutter, large numbers of papers were dedicated to investigate flat plates in the supersonic flow regime. Very few authors have extended their work to flutter of curved panels. The curved geometry generates a pre-flutter behavior due to a static aerodynamic load (SAL) over the panel, resulting in the emergence of a static deflection. Only three papers dealing with the SAL were found in the literature after an extensive survey. The objective of this dissertation is to provide new insights in the subject of flutter of curved panels. Particular attentions are dedicated to the influence of parameters such as dynamic pressures, flow angles, and panel curvatures on the flutter behavior. Additionally, investigation of the pre/post-flutter static/dynamic behaviors of cylindrical panels under yawed supersonic flow is conducted herein. Consistent non-linear finite element formulation and efficient solution procedures are developed and presented. The von Karman large deflection theory [1], the Marguerre shallow shell theory [2], and the quasi-steady piston theory [3] are the basic pillars of the non-linear finite element formulation.

1.1 Flutter History

The first ever-recorded flutter incident was on a Handley Page O/400 [4] twin-engine biplane British heavy bomber in 1916, Figure 1.1. The flutter phenomenon consisted of a coupling between the fuselage torsion-mode and an antisymmetric elevator rotation mode [5]. During World War I, wing-aileron flutter was widely encountered [6] and was generally eliminated by the use of a mass balance about the control surface hinge line. As

aircraft became more sophisticated and flew at higher airspeed, new flutter phenomenon configurations appeared. Wing flutter and servo tab flutter came into sight and were often reported during the 1920's and 1930's flight tests. Moreover, additional aeroelastic problems emerged as aircraft could fly at transonic speeds. The first transonic-recorded incident was an aileron buzz on a Lockheed P-80 airplane [7], Figure 1.2. Between 1947 and 1956, 21 transonic surface buzz incidents were reported. After the achievement of the first supersonic flight by Chuck Yeager in 1947, a new type of flutter known as panel flutter appeared. This phenomenon involves constant amplitude standing or traveling waves in aircraft skin covering layers. This type of flutter instability could lead to a sudden fatigue failure. In the 1950's a fighter airplane was lost because of a failed hydraulic line attached to a panel that had experienced panel flutter [8]. It is interesting to mention that in the early years of aviation, no formal flutter testing of full-scale aircraft was carried out. The aircraft was simply flown to its maximum speed to demonstrate the aeroelastic stability of the vehicle. The first series and formal flutter test was carried out by Von Schlippe in 1935 in Germany [9].

1.2 Nature of the Flutter Phenomenon

Although no specific speed regime is actually immune from flutter, high-speed aircraft flying at supersonic or hypersonic speeds are the most inclined to the flutter phenomena. This aeroelastically induced, self-excited dynamic instability phenomenon involves an odd interaction between aerodynamic, elastic, and inertia forces on structures to produce unwanted oscillations that often results in structural failure. With regards to aircraft skin panels, the oscillating phenomenon is called panel flutter. Linear flutter

analysis is meant to discover the flutter stability boundaries, in other terms the critical speed at the flutter onset. Investigation of amplitudes, non-linear frequencies, and maximum stresses due to flutter motion beyond flutter boundaries predicts lifetime expectancy of curved panels. The flutter phenomena itself can be described through the dynamic behavior of either flat or curved panels subjected to an arbitrary yaw flow angle at fixed Mach number, dynamic pressure, and temperature. Increasing the dynamic pressure over the panel leads to the determination of the flutter boundary. Before the flutter onset, random oscillations with small amplitudes with respect to the panel thickness are observed. The panel is responding to the fluctuations of the dynamic pressure within the turbulent boundary layer separating the panel and the flow stream. At some critical dynamic pressure, i.e. flutter onset, the panel experiences oscillations of the order of the thickness. Experimentally the flutter onset can be determined within a 10% of critical dynamic pressure range. The flutter motion beyond the flutter onset is essentially dominated by the non-linear structural coupling between bending and stretching. The coupling induces tension within the panel as it bends/stretch. During this phase, the in-plane stretching strains on the surface panel become significant, and generate internal forces that tend to contain the panel vibrations, so that delimited limit-cycle, periodic, aperiodic or chaos motion oscillations are observed. If the experimenter keeps increasing the dynamic pressure the stress amplitudes due to the flutter phenomena exceeds the yielding stress of the panel leading to a rapid failure. On the other hand, if the stress is relatively small with respect to the yielding stress, a long time material fatigue failure can occur. In the present introduction, a literature survey depicting the state of the

art in the flutter field of curved panels will be presented. Dominant aspects intervening and shaping the flutter behavior of the mentioned panels will be thoroughly reviewed.

1.3 Flutter Flow Configuration

Historically, researchers have subdivided their work into two major configurations dealing with the direction of the flow field and the curved panel geometry. They considered curved panels with cross-stream curvature, Figure 1.3 and curved panels with stream-wise curvature, Figure 1.4. The third configuration dealing with yawed flow angle did not receive any attention until recently, Figure 1.5. The term curved panels as used in the present work, encompass all curved panels in general, the curvature could be in any direction, and it can range from a curvature due to imperfections, or, a curvature of shallow shell panels. In the forthcoming chapters, the geometry of the curved panels will be considered within the framework of the Marguerre shallow shell theory. The discussion will be restricted to curved panels with a constant curvature in x and y direction.

1.4 Flutter Theoretical Background

Critical dynamic pressure and flutter frequency for curved panels can be predicted by linear flutter theory. If the panel amplitude of vibration grows up to the order of the thickness when flutter occurs, the non-linear flutter theory has to be applied. Theoretical assumptions accompanying the aforementioned configurations can be classified in one of the following four categories [10]:

1. Linear structural theory appended with quasi-steady aerodynamic theory (type 1),

2. Linear structural theory appended with fully linearized (inviscid potential) aerodynamic theory (type 2),
3. Non-linear structural theory append with quasi-steady aerodynamic theory (type 3), and
4. Non-linear structural theory appended with fully linearized (inviscid potential) aerodynamic theory (type 4).

Gary and Mei [11] added a fifth category for hypersonic panel flutter. Traditionally, a Partial Differential Equation (PDE) combined with the Galerkin modal approach method has been popular and widely used by many investigators. The core of the technique itself consists in assuming the structural response as a weighted linear combination of a sine/cosine truncated series supposedly representing the mode shapes of a simply supported flat plate. Replacing the sought transverse displacement by the assumed series expression in the non-linear equations of motion appended with the appropriate aerodynamic forces, and applying the Galerkin method, a set of ordinary, non-linear, integral-differential equations in time for the modal amplitudes are obtained. These equations are known as Duffing's equations. They can be easily solved by a numerical integration scheme. Many alternative methods were also reported in the literature, among them, the finite element method, the finite difference method, and the so-called exact method, which is the separation of variables method. The latter method was occasionally used by a certain number of researchers just for linear analysis. Olson [12], conducted a type 1 analysis using the finite element method, he successfully reproduced critical dynamic pressure results obtained by PDE/normal mode method. The finite element method requires a small number of elements, though; particular care has to

be taken in choosing the element type to insure the satisfaction of the convergence criteria. The flexibility and the simple feasibility of the finite element method made this approach popular and attractive particularly in dealing with complex structures as larger and faster computing machine becomes more and more available. The finite difference was proven to be very efficient in computing the flow field above the panel; it requires in general a grid generator code conjugated with Euler (inviscid) or Navier-Stokes (viscous) solvers.

1.5 Literature Survey

In the following paragraphs, a literature survey, investigating different parameters shaping the non-linear flutter behavior of curved panel systems is presented for the aforementioned flow configurations: cross-stream curvature, stream-wise curvature and arbitrary flow direction, called yawing flows. The non-linear flutter of stream-wise curvature flows involves the Static Aerodynamic Load (SAL) generated by the inherent geometry of the curved panel. The association of the SAL and the load generated by the pressure differential modeled by the traditional quasi-steady first-order piston theory brings additional difficulties; the SAL is intimately related to panel geometry. The open literature accounts only for three papers considering the effect of the SAL load conjugated with the non-linear von Karman large deflection effects [13, 14], and [15]. The present non-linear finite element flutter investigation stresses particularly the influence of stream-wise flows on the dynamic behavior of curved panels. On the other hand, every yawing flow over a curved panel implies the presence of a SAL. Therefore,

the determination of the dynamic influence of yawing flow over curved panel is intimately related to the study of the SAL.

1.5.1 Cross-Stream Curvature Flow

In this configuration, the theoretical problem is similar to the flutter of a section along the generator of a circular cylinder see Fig. 1.3. Voss [16] carried out a type 1 analysis on flutter of a thin cylindrical shallow shell. He investigated first the panel curvature effect via a corrective approach for the effect of slight curvature on the flutter critical dynamic pressure of flat plates using Reissner's shallow shell theory. Then, he presented an exact formulation using the Gold' denveizer equilibrium equation. A simply supported thin shallow shell was studied with a two-mode PDE/normal mode approach conjugated with a simple form of the piston theory. Assuming free in-plane stress on the curved panels boundary, Voss has found that a cross-stream curvature creates a large stiffening effect and consequently is stabilizing with respect to flutter. For structural systems classified as type 3 and type 4, Bolotin [17] has formulated the non-linear structural flutter problem for general curved panels appended with the quasi-steady aerodynamic theory. Dowell [13, 14] carried out type 3 analyses on 2-dimensional (2-D) and 3-D simply supported and clamped curved panels with SAL. The SAL for the cross-stream curvature is null. Assuming non-zero in-plane stresses in the boundaries, he found that increasing the cross-stream curvature decreases the flutter boundaries limits for a 3-D simply supported curved plate. Comparing his results with Voss, he demonstrated that the in-plane boundary conditions are a major parameter in shaping the flutter boundaries margins. He postulated that higher cross-stream modes could be more critical with

respect to flutter. Matsuzaki [18] investigated the influence of a broadband set of in-plane boundary conditions for simply supported cylindrical panels using Reissner shallow shell theory with a PDE/Galerkin procedure. 2-D quasi-steady first-order piston theory was used to approximate the aerodynamic forces. Twelve assumed modes, specifically, a combination of the first six cross-stream modes and the first two symmetrical stream-wise modes were used in a spanwise symmetrical solution in the form of a double Fourier series. His work also emphasized the critical influence of the in-plane boundary conditions on the flutter boundary margins. For flutter prevention, he recommended that panels with aspect ratio of one should be able to move freely along the spanwise direction and restrained tangentially at the curved edges.

Very few investigators applied the finite element method to curved panels. Bismark-Nasr [19] conducted a type 1 finite element analysis of square cylindrical curved panels based on Reissner's two-field variables principle with transverse displacement appended with the quasi-steady first-order piston theory. He concluded that a curvature less than or equivalent to two panel thickness has a stabilizing effect. For a larger curvature, the panel's critical dynamic pressure passes through a transition region characterized by dips, knees and cups. He demonstrated again the great influence of the in-plane edge restraints on the flutter boundaries. Ganapathi and Varadan [20] extended the flutter boundary analysis to laminated composite curved panels using a type 1 analysis. They used a doubly curved, quadrilateral, shear flexible, shell element based on field-consistency approach. The formulation included transverse shear deformation, in-plane and rotary inertias. The aerodynamic forces were evaluated using quasi-steady first-order piston theory without damping term. They investigated the influence of number of layers, ply

angles, aspect ratios, radius-to-side ratios, side-to-thickness ratios, and boundary conditions. They showed that flutter boundaries increase with high aspect and thickness ratios independently of boundary conditions and ply orientations. For deep shells they found that transverse boundary conditions and number of layers affect the flutter behavior.

1.5.2 Stream-Wise Curvature Flow

In this configuration, the flow espouses the curvature streamline, Fig. 1.4. Changes of the flow field velocity direction create an additional pressure over the curved panel called the SAL. Fung [21] considered flutter of buckled plates in the framework of the effect of imperfections. Using a type 1 analysis, Yates and Zeijel [22] showed that curvature has a destabilizing effect. Performing a type 3 analysis, Dowell [13, 14] showed that the stream-wise curvature reduces the flutter critical dynamic pressure, and increases the flutter amplitude. He found that 3-D curved panels are more affected by the SAL than the 2-D panels. He emphasized the need to employ non-linear structural theory accounting for the deflection of the curved panels under the SAL, which depends on the panel geometry and the dynamic pressure. This need was explicitly implemented in the dissertation finite element formulation, which yields an exact set of governing equations for the determination of critical dynamic pressure for the first time in the literature. Oppositely to flat plates, no critical dynamic pressure minima were found for curved panels. Dowell concluded that curved panels with stream-wise curvature are also sensitive to in-plane boundary conditions.

Recently, Krause and Dinkler [15] investigated flutter boundaries, post-flutter non-linear large amplitudes and frequencies for 2-D and 3-D curved panels using finite elements. A Timoshenko beam theory using two-node beam elements with linear shape functions was adopted for the 2-D curved panel, whereas the Reissner-Mindlin plate theory using four/nine-node plate elements with bilinear shape functions was adopted for 3-D curved panels. The panel stream-wise curvature was described by imperfections and the aerodynamic loads were described by the non-linear 3rd order piston aerodynamic theory appended with a SAL resulting from the imperfections of the panel. Their work showed particularly that, for slightly curved 2-D panels the flutter behavior is similar to that of the flat panel, while for higher curvature the flutter boundaries as a function of the plate height-rise drops significantly. They demonstrated that, the critical flutter boundary could decrease significantly by about 50% for 2-D panels at imperfection amplitude to thickness ratio H/h equal to 1.65. For square curved 3-D panels, the flutter boundaries were building up of several independent curves, each one of them is related to a specified pair of modes. They showed particularly that the main part of the flutter boundary versus panel height-rise is generated by modes one and two, but higher modes like five and six can lower the flutter boundary for a certain range of the panel height-rise. Their work also showed for 3-D curved panels that very small flutter amplitude in the range of 0.001 to 0.01 panel height-rise H/h has the characteristic of acoustical vibrations rather than a classical flutter motion. At this point, it is important to remind the reader that there are only three non-linear curved panel flutter papers in the literature [13-15] dealing with stream-wise curvature and the SAL effect.

1.5.3 Arbitrary Yaw Flow Angle

In the sixties and early seventies researchers investigated the effect of yawing flows on the flutter stability boundaries of isotropic and orthotropic flat rectangular panels at supersonic speeds. Several excellent review articles devoted sections to the influence of yaw flow angle [23, 24] on panel flutter. Little literature has been dedicated to their effects on curved panels. A brief and quick review of yawing flow effects on the flutter of flat isotropic and composite plates will be instructive and inspiring for the present work. Kordes and Noll [25], and Bohon [26] studied analytically the influence of yawing flow angles on flutter of isotropic and composite rectangular panels with simply supported boundary conditions. Using the Raleigh-Ritz method appended with a 16-term trigonometric beam function, Dursasula [27] studied the plate obliquity effect on an isotropic rectangular plate subjected to a flow yawing for simply supported and clamped boundary conditions. Kari-Appa et al. [28], and Sander et al. [29] used the finite element method to study the effect of flow yawing of isotropic parallelogram panels. Shyprykevich and Sawyer [30], and Sawyer [31] have shown experimentally and theoretically that critical dynamic pressure is intimately related to the nature of the boundary conditions and the yaw flow angle. They demonstrated that orthotropic panels mounted on flexible support experienced large reduction in critical dynamic pressure for only small changes of flow angles. Additional developments on the linear finite element method applied to the aeroelastic stability of plates and shells under supersonic flow were reported by Bismarck-Nasr [23].

An extensive search of the open literature reveals that few investigations on non-linear panel flutter have considered the effects of flow yawing. Friedmann and Hanin [32]

used first order piston theory and PDE/Galerkin method to investigate non-linear flutter under yawed supersonic flow. They solved the reduced coupled non-linear ordinary differential modal equations with numerical integration using a four by two (4x2) mode model in vacuo, four natural modes in the x direction, and two modes in the y direction. They obtained limit cycles for simply supported isotropic and orthotropic rectangular panels. Chandiramani et al, [33] used third order piston theory in conjunction with PDE/Galerkin method. They solved the reduced coupled non-linear ordinary differential modal equations using a predictor and a Newton-Raphson type corrector technique for limit-cycle periodic solutions. They employed direct numerical integration for non-periodic and chaotic solutions and used a two by two (2x2) mode model, two natural modes in the x direction and two natural modes in the y direction for simply supported rectangular laminated panels. Recently Abdel-Motagaly et al. [34] presented a finite element formulation with an efficient solution procedure for analysis of supersonic non-linear flutter of composite panels with arbitrary flow direction. The finite element non-linear panel flutter equations were first formulated in the structural-node degrees of freedom (DOF). The number of equations was reduced using a modal transformation. The minimum number of linear natural modes needed for an accurate and convergent limit cycle flutter response was accurately determined. The reduced non-linear modal equations were solved using the linearized update mode with non-linear time function (NTF/LUM). Isotropic and composite panels at yawed supersonic flow were treated. They showed that yaw flow angle significantly affects significantly the critical dynamic pressure and the limit cycle deflection.

The flutter of curved panels under arbitrary yaw flow angle received little attention until recently. Pidaparti and Yang [35] was the only team who investigated a type 1 flutter analysis of laminated composite plates and shells using finite elements. Their aerodynamic load did not account for the SAL. They demonstrated that the critical dynamic pressure versus yaw flow angle has a maximum for cylindrical panels with cross-stream curvature; whereas for spherical panels the relationship is increasing monotonically. Recently, a type 3 flutter analyses of curved panels at an arbitrary yaw flow angle was investigated [36, 37]. A non-linear finite element formulation using the extended triangular Mindlin (MIN3) plate element was first used to analyze the effects of arbitrary yaw flow angle on large-amplitude flutter response of isotropic and composite shallow shell panels. First-order shear deformation theory and quasi-steady piston theory appended with SAL were used in the formulation. A modal reduction procedure combined with a numerical integration scheme was applied to solve the multi-modes reduced non-linear panel flutter modal equations. Results showed that isotropic shallow shell panels exhibited Limit-Cycle Oscillations (LCO), whereas shallow shell composite laminate panels exhibited periodic and chaotic motions. The second publication crafted a frequency/time domain method to investigate pre/post flutter behavior of isotropic shallow shell panels. The frequency domain method has shown capability to predict pre-flutter panel deflection shape, aerostatic modes shapes, and critical dynamic pressure at a prescribed dynamic pressure, whereas time domain method predicted post-flutter behavior through bifurcation diagrams. The conjugation of the aerodynamic forces with the SAL demonstrated the softening of the panel stiffness and the decrease of the flutter

critical dynamic pressure. Cross-stream and stream-wise configurations were simply treated as special cases of the general arbitrary yaw flow angle configuration.

1.5.4 Static Load Effect

Experiments performed by Presnell and McKinney [38], Hess and Gibson [39], and Anderson [40] have brought out the SAL effect due to a pressure loading over the curved panel that tends to flatten the panel. This phenomenon can lead to the possibility of snap-through behavior of the curved panel. Dowell [13, 14] underlined the necessity to involve the pre-flutter deformation due to a static pressure load, and has shown that the static aerodynamic load generated by a stream-wise curvature can significantly affect the flutter boundary. He demonstrated quantitatively that a 3-D curved panel with stream-wise curvature will be more subjected to pre-flutter static deformation than 2-D curved panels, and identified the pre-flutter phenomenon to a buckling behavior. He also found that a static pressure differential applied to a stream-wise curved panel changes the flutter motion in a manner that is qualitatively similar to the one found experimentally by Stearman et al. [41] for a cylindrical shell. The predictions of his analysis are in qualitative agreement with the experimental data given by Anderson for stream-wise curved panels. Recent study on 2-D curved panel by Krause and Dinkler [15] pointed out that for certain Mach numbers, different static equilibrium positions may coexist, particularly for medium and large curvatures. Five solution branches may occur and they range from un-deformed panels (symmetric, stable) to snapped-through panels (symmetric, stable) as well as asymmetric and unstable positions.

1.5.5 Aerodynamic Load Effect

The fluid-structure interaction is described by the fluctuations of the pressure flow field over the exposed side of the curved panel. Several aerodynamic theories were proposed in the literature. Historically, the linear Ackeret linearized 2-D supersonic theory has been proven useful in predicting the first cylinder flutter experiments [41, 42]. Ashley and Zartarian [3] proposed the quasi-steady aerodynamic piston theory. The theory was widely used by researchers to model supersonic flow with high Mach numbers ($M_\infty > \sqrt{2}$). Dowell did a comprehensive review of the theory [43]. He pointed out that the use of the quasi-steady aerodynamics theory neglects the three-dimensionality effect and the unsteadiness or memory of the flow. The theory cannot be used too close to a Mach number of one, where memory or unsteadiness of the flow field becomes a significant phenomenon (transonic flow). He also pointed out that most of the experimental work on flutter showed that flutter would likely occur in the vicinity of Mach number equal to one. Additionally, he postulated that for sufficiently large curvature, the non-linear aerodynamic effects become significant and the quasi-steady piston theory might not be applicable. Generally, it has been accepted that using the local flow pressure, density and Mach number to model the aerodynamic forces (piston theory) in place of the free stream values was a satisfactory approximation for the fluid forces [44]. A study by Baillie and McFeely [45] has shown the validity of the piston theory model for panels mounted on a wedge. Using a full unsteady hypersonic theory that account for the change in the steady flow field, their results agreed very closely with the simpler piston theory. Dowell [43] discussed two possible improvements concerning the boundary layer effects and the local flow effects. If the boundary layer thickness is of the

order of the flutter wavelength, viscous effect will be important. To simplify flutter calculations while accounting for viscous effects most investigators have examined infinitely long plates. These studies pointed out ways in which more realistic investigations may proceed. The first proposed practical improvement is to consider a multi-layer potential shear model suggested by Zeijdel [46] following earlier work by Anderson and Fung [47]. The second improvement deals with local flow effects simulation. Dowell suggested that the finite difference method could be a good candidate for capturing such phenomena. Recently, Kiiko [48] showed the example of a plate located on one of the sides of a wedge, complemented the piston theory formula with an additional term, which has the meaning of a compressive force in the plane of the plate. He showed particularly that when this additional term is accounted for, a decrease in the critical aerodynamic pressure is noted. Recently, studies carried out on non-linear flutter of 3-D flat plate at supersonic flow using Euler equations [49] have shown that the flutter dynamic pressure for LCO is only 5% difference between the Euler and piston theories. Coupling the full Navier-Stokes equations with a finite-difference scheme for the von-Karman plate equations at low supersonic flow [50, 51], the LCO results showed good agreement with Dowell's PDE/Galerkin analysis in conjunction with the piston theory.

1.5.6 Chaotic Motion

Chaotic motion [52] can be defined as a bounded steady state behavior that is not an equilibrium solution, neither a periodic solution, or a quasi-periodic solution. More precisely, a chaotic motion can be defined as the superposition of a very large number of unstable periodic motions. There is no precise definition for chaotic solution because it

cannot be represented with conventional mathematical tools. The frequency spectrum of a chaotic signal has a continuous broadband character, and contains spikes that often indicate the predominant frequencies of the signal. Chaotic systems are also characterized by sensitivity to initial conditions; a tiny perturbation in the input can be quickly and overwhelmingly amplified in the output. Nayfeh and Balachandran [52] described this phenomenon named the butterfly effect in these terms, that is a small perturbation created by the wings of a butterfly today in Beijing, China can produce a torrential rain storm next month in California. Historically in 1963, meteorologist Ed Lorenz [53] derived a 3-D differential equation from a significantly simplified model of atmosphere dynamics. He made a stunning discovery that this simple-looking deterministic system could have exceptionally erratic dynamics behavior featuring a chaotic motion. His work particularly showed that the solution oscillates erratically, never exactly repeating but constantly remaining in a bounded region in the phase space plot. When he plotted the trajectories in three dimensions, he discovered that they settled onto a complicated set, now called a strange attractor. These attractors are not a simple geometrical object like a finite number of points, neither a closed curve, nor a torus. In fact, it is not even a smooth surface, the strange attractor are complicated objects that possess fractal characteristics, with a fractional dimension secluded between 2 and 3. Comprehensive fundamental theories and physical understanding of chaos can be found in the books of Strogatz [54] and Alligood et al. [55]. Gottwald et al. [56] designed an experiment to mimic the non-linear free and forced Duffing equation. Several specific non-linear dynamic behaviors were experimentally established. Competing steady state attractors, jump phenomenon, sensitivity to initial conditions, sub-harmonic oscillations and chaotic motion were

demonstrated by a set of experiments and assessed by numerical simulations. Many structural dynamic systems, particularly, beams, plates and curved panels exhibit a chaotic behavior under some specific loads. A review of theoretical and experimental studies of chaos and strange attractors for non-linear mechanical systems up to 1983 was given by Holmes and Moon [57]. Dowell [58, 59], and Virgin and Dowell [60] scrutinized the motion evolution of a buckled plate from the flutter onset to the established chaotic motion. Time histories, phase plots, power spectrum density plots and Poincare maps were given for the most significant phases of the flutter motion. The conclusion was that chaotic motion seems to arise as a consequence of the presence of two parameters, the flow velocity and the mechanical in-plane load. These two types of loads govern two distinct types of instability, the flutter instability and the Euler buckling instability. Chaos emerges as the flutter and buckling stability boundaries merge. Coker and Johnson [61] studied the chaotic motion of simply supported plates under thermal load and sinusoidal excitation forces. The 2-D largest Lyapunov exponent plane was plotted as a function of temperature and amplitude of the excitation force. The intermittence between chaotic motion and regular motion was shown in the plots. Critical temperature and force amplitude parameters for chaos occurrence were evaluated, respectively. Recently Hsin et al. [62] characterized the conditions that can possibly lead to a chaotic motion for simply supported large deflection rectangular plates by utilizing the fractal criteria dimension and the maximum Lyapunov exponent. The governing plate equations were simplified to a set of two ordinary differential equations by the Galerkin method. Numerical results indicated that large deflection motion of a rectangular plate exhibits many bifurcation points. The numerical simulation showed that the computed

bifurcation point could lead either to a trans-critical bifurcation or a pitchfork bifurcation. The latter bifurcation can migrate gradually to chaotic motion under some specific loading conditions. Some jump phenomena were also observed under various lateral loading.

In the experimental field, Moon and Holmes [63] performed the first structural set of experiments to demonstrate the existence of chaotic motions in structural mechanics systems by investigating the nonlinear vibrations of a magnetically buckled beam under sinusoidal force excitation driven by a shaker. Moon [64] in a new study performed a series of experiments aiming to characterize the chaotic motion. Several critical parameters triggering chaotic motion were identified. He particularly showed that by fixing the shaker driving frequency and varying gradually its amplitude, chaotic motion occurs at sufficiently high amplitude. Bolotin et al. [65], studied non-linear panel flutter in the remote post-critical domains by investigating the dynamic behavior of elastic infinitely long-span plate in cylindrical bending subjected to supersonic cross-curvature flow, and initial compression in the middle surface. The cross-curvature flow was modeled by the simplest form of the piston theory. Two control parameters were considered, forward and backward variations of the compressive in-plane forces and the dynamic pressure, respectively. A number of pertinent flutter patterns and bifurcations were found. Symmetric flutter, asymmetric flutter, chaotic, as well as hysteric motion were observed. The most striking result was the existence of temporary exits from chaos motion to regular motion behavior. The adopted numerical experimentation showed that some of the cases exhibited a strong dependence on the initial conditions due to the proximity of different attractors in the multi-dimensional phase-space.

1.6 Scope

The previous literature survey showed that only three non-linear based papers dealt with the streamwise configuration under the SAL. Two of them are based on the PDE/Galerkin method with the utilization of sinusoidal modes shapes, and one of them is based on the finite element method with the Equation Of Motion (EOM) derived in the structural Degrees Of Freedom (DOF). The three papers studied essentially the influence of boundary conditions and panel curvature on the flutter stability margins under a zero yaw flow angle. The present dissertation has the scope to build a better understanding of the static/dynamic processes behind the flutter of curved skin panels in yawed supersonic flow environment. To that end, a Finite Element (FE) frequency and time domain methods were developed to predict the flutter stability boundaries for arbitrary yawed flow configurations. Moreover, the curved panel pre-flutter static deflection and stiffness are accurately determined under the SAL for a specific dynamic pressure. The mode shapes of the deflected panel, called herein aerostatic modes are precisely computed. Those aerostatic mode shapes are fundamentally different from the sine-function mode shapes used in the classic analytical PDE/Galerkin method. In the frequency domain procedure an eigen-solution derived from the system EOM in structural node DOF is developed to investigate flutter mode coalescence mechanisms and damping-rates. Stability boundary margins, and degree of material hardening/softening are then determined. In the time domain procedure system EOM in structural DOF are transformed into modal coordinates. The non-linear modal equations of motion are then solved for flutter response by a Runge-Kutta numerical scheme. Time histories, phase

plots, power spectrum density, and bifurcation diagrams for 2-D and 3-D isotropic cylindrical panels with yaw flow angle are thoroughly investigated. The following contents are included in the present work.

- **CHAPTER 2 – FINITE ELEMENT FORMULATION**

The main assumptions for the non-linear finite element formulation are outlined. The element displacement vectors and the element displacement functions featuring the MIN3 element are presented. The constitutive relations are derived for curved panels and used to finalize the finite element formulation. The Marguerre theory is incorporated in the formulation to describe the curved panel geometry. The principle of virtual work is applied to derive the non-linear finite element flutter governing equations of motion. Expressions of element mass matrices, aerodynamic damping matrices, linear and non-linear stiffness matrices, aerodynamic stiffness matrices, and external loads are given.

- **CHAPTER 3 – SOLUTION PROCEDURES**

In this chapter frequency and time domain solution procedures are presented. In the frequency domain procedure, the Newton-Raphson method is used to determine the panel deflection under the SAL, and an eigen-value solution is employed for the determination of the flutter stability boundaries and the aerostatic mode shapes. In the time domain procedure system equations of motion in structural node degrees-of-freedom are transformed into the modal coordinates. A Runge-Kutta numerical scheme is then employed to solve the aforementioned equations.

- **CHAPTER 4 – NUMERICAL RESULTS AND DISCUSSION**

Flutter coalescence and damping rate diagrams are obtained for 2-D and 3-D curved panels to define the stability boundary margins using the frequency domain method. The communality and differences of the sinusoidal, natural, and aerostatic modes shapes are thoroughly investigated. Time histories, phase plots, power spectrum, and bifurcation diagrams for different height-rise, flow angle are fully investigated for 2-D and 3-D curved panels.

- **CHAPTER 5 - CONCLUDING REMARKS**

Major conclusions drawn from the study are outlined with recommended future work.

Figure 1.1 Handley Page O/400 twin-engine biplane

Figure 1.2 Lockheed P-80

Figure 1.3 Cylindrical panel with cross-stream curvature flow

Figure 1.4 Cylindrical panel with stream-wise curvature flow

Figure 1.5 Cylindrical panel with yawing flow angle

Chapter 2

2 FINITE ELEMENT FORMULATION

2.1 Introduction

In this chapter the element and system non-linear governing equations of motion for a curved panel subjected to a yawed flow angle, and a static aerodynamic load will be developed. The following assumptions were adopted in the forthcoming formulation:

- The material could be either isotropic or composite laminate; Hook's law and composite laminate theory are valid for the cited materials, respectively.
- The first order shear deformation theory is considered for the present formulation. The curved panel could be either thin or thick. The in-plane inertia term is neglected.
- The effect of large deflection is included in the formulation through the von Karman non-linear strain-displacement relations.
- The flow field over the curved panel is supersonic ($1.6 < M_\infty < 5$), and is modeled by the quasi-steady aerodynamic first-order piston theory.
- The Marguerre curved plate theory is used to develop the curved panel element.

2.2 Element Displacement Vectors

In order to develop the curved panel governing equations of motion, a distinctive three-nodes (MIN3) finite element is extended to discretize the panel system into many finite triangular elements. Originally the three-node MIN3 laminate element illustrated in Fig. 2.1 and Fig. 2.2 was developed by Tessler and Hughes [66] to address the problem of shear locking frequently encountered when using the standard iso-parametric interpolation approach. The problem was resolved through the implementation of special

shape functions, the anisoparametric interpolations. Extensive numerical tests by Tessler and Hughes have shown that the MIN3 element is well suited for linear problems. Chen [67] extended and demonstrated the effectiveness of the element in solving non-linear problems such as the dynamic response of flat panels under combined acoustic and thermal loads. The element nodal displacement vector $\{w\}$, Fig. 2.2 and Fig. 2.3, includes components for each of the three triangular nodes. Each node is characterized by three displacements and two rotations (5 DOF). The DOF are composed of the bending displacements w_i , the normal rotations ψ_{xi} with respect to x and ψ_{yi} with respect to y axes, and the in-plane displacements u_i, v_i , respectively. The aforementioned nodal displacement vectors are defined as

$$\{w_b\}^T = [w_1 \quad w_2 \quad w_3] \quad (2.1)$$

$$\{w_\psi\}^T = [\psi_{x1} \quad \psi_{x2} \quad \psi_{x3} \quad \psi_{y1} \quad \psi_{y2} \quad \psi_{y3}] \quad (2.2)$$

$$\{w_m\}^T = [u_1 \quad u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3] \quad (2.3)$$

where subscripts 1, 2, and 3 represent the displacements at each node. The build up element displacement vector is defined as

$$\{w\}^T = \begin{bmatrix} [w_b] & [w_\psi] & [w_m] \end{bmatrix} \quad (2.4)$$

In an advanced step in this chapter the element displacements $\{w\}$ for various elements are ultimately assembled to construct a system nodal displacement vector $\{W\}$.

2.3 Element Displacement Functions

The geometry of the arbitrary curved panel is described by the function $w_o(x, y)$ shown in Fig. 2.3 to Fig. 2.6. In the present work, the curved geometry is characterized

by H/h , where $H = \max[w_o(x, y)]$ and h is the thickness of the curved panel. The element displacement functions used in the derivation of the equations of motions are defined as

$$u_x = u(x, y, t) + \bar{z} \psi_y(x, y, t) \quad (2.5)$$

$$u_y = v(x, y, t) + \bar{z} \psi_x(x, y, t) \quad (2.6)$$

$$u_z = w(x, y, t) \quad (2.7)$$

where u_x , u_y and u_z are the three displacement components at any point within the element, u, v and w are the displacements of the plate mid-plane, and ψ_x , ψ_y are the rotations of the plate mid-plane normals about the x and y axes. The local coordinate $\bar{z}$ is defined as $\bar{z} = z - w_o(x, y)$.

The interpolation functions for the MIN3 element, Fig. 2.4, are according to [66]:

$$\begin{aligned} w(x, y, t) &= [H_w] \{w_b\} + [H_{w\psi}] \{w_\psi\} \\ &= [\xi_1 \quad \xi_2 \quad \xi_3] \{w_b\} + [L_1 \quad L_2 \quad L_3 \quad M_1 \quad M_2 \quad M_3] \{w_\psi\} \end{aligned} \quad (2.8)$$

$$\begin{aligned} \psi_x(x, y, t) &= [H_{\psi\alpha}] \{w_\psi\} \\ &= [\xi_1 \quad \xi_2 \quad \xi_3 \quad 0 \quad 0 \quad 0] \{w_\psi\} \end{aligned} \quad (2.9)$$

$$\begin{aligned} \psi_y(x, y, t) &= [H_{\psi\gamma}] \{w_\psi\} \\ &= [0 \quad 0 \quad 0 \quad \xi_1 \quad \xi_2 \quad \xi_3] \{w_\psi\} \end{aligned} \quad (2.10)$$

$$\begin{aligned} u(x, y, t) &= [H_u] \{w_m\} \\ &= [\xi_1 \quad \xi_2 \quad \xi_3 \quad 0 \quad 0 \quad 0] \{w_m\} \end{aligned} \quad (2.11)$$

$$\begin{aligned} v(x, y, t) &= [H_v] \{w_m\} \\ &= [0 \quad 0 \quad 0 \quad \xi_1 \quad \xi_2 \quad \xi_3] \{w_m\} \end{aligned} \quad (2.12)$$

where ξ_1 , ξ_2 and ξ_3 are the area coordinates. Parameters L_1 , L_2 , L_3 , M_1 , M_2 and M_3 are defined as

$$L_1 = \frac{1}{8}(b_3N_4 - b_2N_6), \quad L_2 = \frac{1}{8}(b_1N_5 - b_3N_4), \quad L_3 = \frac{1}{8}(b_2N_6 - b_1N_5) \quad (2.13)$$

$$M_1 = \frac{1}{8}(a_2N_6 - a_3N_4), \quad M_2 = \frac{1}{8}(a_3N_4 - a_1N_5), \quad M_3 = \frac{1}{8}(a_1N_5 - a_2N_6) \quad (2.14)$$

where N_4 , N_5 and N_6 are defined as

$$N_4 = 4\xi_1\xi_2, \quad N_5 = 4\xi_2\xi_3, \quad N_6 = 4\xi_3\xi_1 \quad (2.15)$$

and a_1 , a_2 , a_3 , b_1 , b_2 and b_3 are defined as

$$\begin{aligned} a_1 &= x_{32}, & a_2 &= x_{13}, & a_3 &= x_{21} \\ b_1 &= y_{23}, & b_2 &= y_{31}, & b_3 &= y_{12} \end{aligned} \quad (2.16)$$

The transformation between x , y and ξ_i is given through the matrix relation

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} \quad (2.17)$$

where the area coordinates are defined as

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad (2.18)$$

The element area is defined as

$$A = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \quad (2.19)$$

and the element coordinates are defined as

$$x_{ij} = x_i - x_j, \quad y_{ij} = y_i - y_j \quad (2.20)$$

The integration of the area coordinates ξ_1 , ξ_2 and ξ_3 over the triangular element area yields

$$\int_A \xi_1^k \xi_2^l \xi_3^m dA = 2A \frac{k!l!m!}{(2+k+l+m)!} \quad (2.21)$$

where k, l, m are integer numbers.

2.4 Non-linear Total Strain Deformation Vector

Assuming small in-plane strains and moderately large transverse displacement, the total strain-displacement vector is expressed by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \{\varepsilon^o\} + \bar{z}\{\kappa\} \quad (2.22)$$

where $\{\varepsilon^o\}$ is the in-plane strain vector and $\{\kappa\}$ is the curvature vector. Developing the in-plane strains along the x and y axes give

$$\varepsilon_x^o = \left[\frac{\partial(u_o + u)}{\partial x} + \frac{1}{2} \left(\frac{\partial(w_o + w)}{\partial x} \right)^2 \right] - \left[\frac{\partial u_o}{\partial x} + \frac{1}{2} \left(\frac{\partial w_o}{\partial x} \right)^2 \right] \quad (2.23)$$

$$\varepsilon_y^o = \left[\frac{\partial(v_o + v)}{\partial y} + \frac{1}{2} \left(\frac{\partial(w_o + w)}{\partial y} \right)^2 \right] - \left[\frac{\partial v_o}{\partial y} + \frac{1}{2} \left(\frac{\partial w_o}{\partial y} \right)^2 \right] \quad (2.24)$$

$$\begin{aligned} \gamma_{xy}^o = & \left[\frac{\partial(u_o + u)}{\partial y} + \frac{\partial(v_o + v)}{\partial x} + \frac{1}{2} \left(2 \frac{\partial(w + w_o)}{\partial x} \frac{\partial(w + w_o)}{\partial y} \right) \right] \\ & - \left[\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + \frac{1}{2} \left(2 \frac{\partial w_o}{\partial x} \frac{\partial w_o}{\partial y} \right) \right] \end{aligned} \quad (2.25)$$

The in-plane strains along x and y axes are then

$$\varepsilon_x^o = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w_o}{\partial x} \frac{\partial w}{\partial x} \quad (2.26)$$

$$\varepsilon_y^o = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w_o}{\partial y} \frac{\partial w}{\partial y} \quad (2.27)$$

$$\gamma_{xy}^o = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w_o}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial w_o}{\partial x} \quad (2.28)$$

The in-plane strain vector consists of three strain components:

- The linear membrane strain vector $\{\varepsilon_m^o\}$ is related to the in-plane displacements as

$$\{\varepsilon_m^o\} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} \quad (2.29)$$

- The non-linear stretching strain vector $\{\varepsilon_b^o\}$ due to moderately large deflection is related to the transverse displacement [1] as

$$\{\varepsilon_b^o\} = \frac{1}{2} \begin{Bmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,x}w_{,y} \end{Bmatrix} \quad (2.30)$$

- The curvature strain vector $\{\varepsilon_{w_o}^o\}$ due to the curved panel geometry $w_o = w_o(x, y)$ is related to the first derivative of the transverse displacement and the first derivative of the curvature geometry [2] as

$$\{\varepsilon_{w_o}^o\} = \begin{Bmatrix} w_{,x} w_{o,x} \\ w_{,y} w_{o,y} \\ w_{,x} w_{o,y} + w_{,y} w_{o,x} \end{Bmatrix} \quad (2.31)$$

The bending curvature vector $\{\kappa\}$ is expressed as

$$\{\kappa\} = \begin{Bmatrix} \psi_{y,x} \\ \psi_{x,y} \\ \psi_{y,y} + \psi_{x,x} \end{Bmatrix} \quad (2.32)$$

therefore the expression for the total strain becomes

$$\{\mathcal{E}\} = \{\mathcal{E}_m^o\} + \{\mathcal{E}_b^o\} + \{\mathcal{E}_{w_o}^o\} + \bar{z}\{\kappa\} \quad (2.33)$$

$$\{\mathcal{E}\} = \begin{Bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,x}w_{,y} \end{Bmatrix} + \begin{Bmatrix} w_{,x}w_{o,x} \\ w_{,y}w_{o,y} \\ w_{,x}w_{o,y} + w_{,y}w_{o,x} \end{Bmatrix} + \bar{z} \begin{Bmatrix} \psi_{y,x} \\ \psi_{x,y} \\ \psi_{y,y} + \psi_{x,x} \end{Bmatrix} \quad (2.34)$$

2.5 Total Transverse Shear Strain Deformation Vector

The total transverse shear strain consists of two shear vectors

- The shear angle vector due to transverse displacement, and is related to the transverse displacement as

$$\{\gamma_b^o\} = \begin{Bmatrix} w_{,y} \\ w_{,x} \end{Bmatrix} \quad (2.35)$$

- The additional shear angles introduced by the shear deformation theory is defined as

$$\{\gamma_\psi^o\} = \begin{Bmatrix} \psi_x \\ \psi_y \end{Bmatrix} \quad (2.36)$$

Therefore the expression for the total transverse shear strain becomes

$$\{\gamma\} = \{\gamma_b^o\} + \{\gamma_\psi^o\} \quad (2.37)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} w_{,y} \\ w_{,x} \end{Bmatrix} + \begin{Bmatrix} \psi_x \\ \psi_y \end{Bmatrix} \quad (2.38)$$

Substituting u , v , w , and ψ_x , ψ_y with their interpolation functions stated in Eqs. (2.8) to

(2.12) and expressing the components of the strain displacement vector in functions of

the nodal displacement vectors, the membrane strain vector is expressed as

$$\{\mathcal{E}_m^o\} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} = \begin{bmatrix} [H_u]_{,x} \\ [H_v]_{,y} \\ [H_u]_{,y} + [H_v]_{,x} \end{bmatrix} \{w_m\} = [C_m] \{w_m\} \quad (2.39)$$

The non-linear stretching strain vector is expressed as

$$\{\mathcal{E}_b^o\} = \frac{1}{2} \begin{Bmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2w_{,x}w_{,y} \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} w_{,x} & 0 \\ 0 & w_{,y} \\ w_{,y} & w_{,x} \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} = \frac{1}{2} [\theta] \{G\} \quad (2.40)$$

where the derivative matrix of the transverse displacements, denoted as slope matrix $[\theta]$, is defined as

$$[\theta] = \begin{bmatrix} w_{,x} & 0 \\ 0 & w_{,y} \\ w_{,y} & w_{,x} \end{bmatrix} \quad (2.41)$$

and the derivative vector of the transverse displacements, denoted as slope vector $\{G\}$, is defined as

$$\begin{aligned} \{G\} = \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} &= \begin{bmatrix} [H_w]_{,x} \\ [H_w]_{,y} \end{bmatrix} \{w_b\} + \begin{bmatrix} [H_{w\psi}]_{,x} \\ [H_{w\psi}]_{,y} \end{bmatrix} \{w_\psi\} \\ &= [C_{wb}] \{w_b\} + [C_{w\psi}] \{w_\psi\} \end{aligned} \quad (2.42)$$

Finally the non-linear stretching strain vector can be written as

$$\{\mathcal{E}_b^o\} = \frac{1}{2} [\theta] ([C_{wb}] \{w_b\} + [C_{w\psi}] \{w_\psi\}) \quad (2.43)$$

The in-plane strain vector due to the curved geometry is expressed as

$$\{\mathcal{E}_{w_o}^o\} = \begin{Bmatrix} w_{,x} w_{o,x} \\ w_{,y} w_{o,y} \\ w_{,x} w_{o,y} + w_{,y} w_{o,x} \end{Bmatrix} = \begin{bmatrix} w_{o,x} & 0 \\ 0 & w_{o,y} \\ w_{o,y} & w_{o,x} \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} = [\theta_o] \{G\} \quad (2.44)$$

$$\{\varepsilon_h^o\} = [\theta_o] \left([C_{\psi b}] \{w_b\} + [C_{\psi\psi}] \{w_\psi\} \right) \quad (2.45)$$

The in-plane strain vector due to bending is expressed as

$$\{\kappa\} = \begin{Bmatrix} \psi_{y,x} \\ \psi_{x,y} \\ \psi_{y,y} + \psi_{x,x} \end{Bmatrix} = \begin{bmatrix} [H_{\psi y}]_{,x} \\ [H_{\psi x}]_{,y} \\ [H_{\psi y}]_{,y} + [H_{\psi x}]_{,x} \end{bmatrix} \{w_\psi\} = [C_b] \{w_\psi\} \quad (2.46)$$

The total strains vectors can be expressed in reduced form in functions of the nodal displacements as

$$\{\varepsilon^o\} = [C_m] \{w_m\} + \frac{1}{2} [\theta] \left([C_{\psi b}] \{w_b\} + [C_{\psi\psi}] \{w_\psi\} \right) + [\theta_o] \left([C_{\psi b}] \{w_b\} + [C_{\psi\psi}] \{w_\psi\} \right) \quad (2.47)$$

$$\{\kappa\} = [C_b] \{w_\psi\} \quad (2.48)$$

The total shear strain in functions of the nodal displacement vectors is expressed as

$$\begin{aligned} \{\gamma\} = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} &= \begin{Bmatrix} w_{,y} \\ w_{,x} \end{Bmatrix} + \begin{Bmatrix} \psi_x \\ \psi_y \end{Bmatrix} = \begin{bmatrix} [H_w]_{,y} \\ [H_w]_{,x} \end{bmatrix} \{w_b\} \\ &+ \begin{bmatrix} [H_{w\psi}]_{,y} + [H_{\psi x}] \\ [H_{w\psi}]_{,x} + [H_{\psi y}] \end{bmatrix} \{w_\psi\} \end{aligned} \quad (2.49)$$

In reduced form the total shear strain can be written in function of the nodal displacements as

$$\{\gamma\} = [C_{\gamma b}] \{w_b\} + [C_{\gamma\psi}] \{w_\psi\} \quad (2.50)$$

2.6 Constitutive Relations

Considering the general case of a composite lamina, the stress-strain relations for the k^{th} layer are given by

$$\begin{aligned}\{\sigma\}_k &= \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \\ &= [\bar{Q}]_k \{\varepsilon\}\end{aligned}\quad (2.51)$$

and

$$\{\tau\}_k = \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = [\bar{Q}_s]_k \{\gamma\} \quad (2.52)$$

where the reduced lamina stiffness matrix is

$$[\bar{Q}]_k = [T_\sigma]^{-1} [Q]_k [T_\varepsilon] \quad (2.53)$$

and, the reduced shear lamina stiffness matrix is

$$[\bar{Q}_s]_k = [T_{\sigma s}]^{-1} [Q_s]_k [T_{\varepsilon s}] \quad (2.54)$$

The transformation matrices $[T_\sigma]$, $[T_\varepsilon]$ and $[T_{\sigma s}]$ are given by

$$[T_\sigma] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \quad (2.55)$$

$$[T_\varepsilon] = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix} \quad (2.56)$$

$$[T_{\sigma s}] = [T_{\varepsilon s}] = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \quad (2.57)$$

with $c = \cos \theta$ and $s = \sin \theta$, where θ is the composite material fibers orientation.

2.7 Resultant Laminates Forces and Moments

The resultant forces, moments and shear forces per unit length, acting on the composite laminated curved panel element are obtained by integration of the stresses through each layer along the laminate thickness. The in-plane forces vector is given by the relation

$$\{N\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma\}_k dz \quad (2.58)$$

The bending moments vector is given by the relation

$$\{M\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma\}_k z dz \quad (2.59)$$

and the shear forces vector is given by the relation

$$\{R\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\tau\}_k dz \quad (2.60)$$

Introducing the laminate extensional, coupling, bending and shear stiffness matrices, $[A]$, $[B]$, $[D]$ and $[A_s]$ respectively, the in-plane forces and bending moments can be expressed as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon^o \\ \kappa \end{Bmatrix} \quad (2.61)$$

and, the shear forces can be expressed as

$$\{R\} = [A_s] \{\gamma\} \quad (2.62)$$

where the laminate extensional matrix is defined as

$$[A] = \sum_{k=1}^n [\bar{Q}]_k (\bar{z}_{k+1} - \bar{z}_k) \quad (2.63)$$

the laminate coupling matrix is defined as

$$[B] = \frac{1}{2} \sum_{k=1}^n [\bar{Q}]_k (\bar{z}_{k+1}^2 - \bar{z}_k^2) \quad (2.64)$$

the laminate bending matrix is defined as

$$[D] = \frac{1}{3} \sum_{k=1}^n [\bar{Q}]_k (\bar{z}_{k+1}^3 - \bar{z}_k^3) \quad (2.65)$$

and, the shear matrix is defined as

$$[A_s] = \sum_{k=1}^n [\bar{Q}_s]_k (\bar{z}_{k+1} - \bar{z}_k) \quad (2.66)$$

In Eqs. (2.63) – (2.66), n is the total number of layers.

2.8 Curved System Element Matrices

The governing equations of motion for an isotropic or laminated composite curved panel element subjected to time varying aerodynamic loads, and static aerodynamic loads, are derived using the principle of virtual work with the d'Alembert's principle. The principle states that for a structure in equilibrium, the total work done by internal and external forces for an infinitesimal virtual displacement is zero

$$\delta W = \delta W_{\text{int}} - \delta W_{\text{ext}} = 0 \quad (2.67)$$

The virtual work of the internal forces over a curved panel element is given by

$$\delta W_{\text{int}} = \int_A \left(\{\delta \varepsilon^o\}^T \{N\} + \{\delta \kappa\}^T \{M\} + \alpha_s \{\delta \gamma\}^T \{R\} \right) dA \quad (2.68)$$

where α_s is the shear correction factor [66] for the laminated composite element defined as

$$\alpha_s = \left(1 + \frac{1}{2} \frac{\text{tr}([k]_s)}{\text{tr}([k]_w)} \right)^{-1} \quad (2.69)$$

and the virtual work of the external forces over a curved panel element is given by

$$\delta W_{ext} = \int_A \left(\{\delta w\}^T (-\rho h \{w_{,tt}\} + \{Ap\}) + \{\delta u\}(-\rho h \{u_{,tt}\}) + \{\delta v\}(-\rho h \{v_{,tt}\}) \right) dA \quad (2.70)$$

2.8.1 Development of the Virtual Work Done by the Internal Forces

Development of the virtual work done by the internal forces (2.68) gives the following relations:

$$\delta W_{int} = \int_A \{\delta \varepsilon^o\}^T \{N\} dA \quad (2.71-1)$$

$$+ \int_A \{\delta \kappa\}^T \{M\} dA \quad (2.71-2)$$

$$+ \int_A \alpha \{\delta \gamma\}^T \{R\} dA \quad (2.71-3)$$

From Eq. (2.47) the variation of the in-plane strain can be expanded as

$$\{\delta \varepsilon^o\}^T = \delta([C_m]\{w_m\})^T + \delta\left(\frac{1}{2}[\theta][C_{\psi b}]\{w_b\}\right)^T + \delta\left(\frac{1}{2}[\theta][C_{\psi\psi}]\{w_\psi\}\right)^T + \delta([C_o][C_{\psi b}]\{w_b\})^T + \delta([C_o][C_{\psi\psi}]\{w_\psi\})^T \quad (2.72)$$

$$= \{\delta w_m\}^T [C_m]^T + \quad (2.73-1)$$

$$\frac{1}{2}\{\delta w_b\}^T [C_{\psi b}]^T [\theta]^T + \frac{1}{2}\{w_b\}^T [C_{\psi b}]^T [\delta \theta]^T + \quad (2.73-2)$$

$$\frac{1}{2}\{\delta w_\psi\}^T [C_{\psi\psi}]^T [\theta]^T + \frac{1}{2}\{w_\psi\}^T [C_{\psi\psi}]^T [\delta \theta]^T + \quad (2.73-3)$$

$$\{\delta w_b\}^T [C_{\psi b}]^T [\theta_o]^T + \{\delta w_\psi\}^T [C_{\psi\psi}]^T [\theta_o]^T \quad (2.73-4)$$

After performing all operations, Eq. (2.73) can be written as

$$\begin{aligned} \{\delta \varepsilon^o\}^T &= \{\delta w_m\}^T [C_m]^T + \{\delta w_b\}^T [C_{\psi b}]^T [\theta]^T + \{\delta w_\psi\}^T [C_{\psi\psi}]^T [\theta]^T \\ &+ \{\delta w_b\}^T [C_{\psi b}]^T [\theta_o]^T + \{\delta w_\psi\}^T [C_{\psi\psi}]^T [\theta_o]^T \end{aligned} \quad (2.74)$$

The curved panel in-plane internal stresses per unit-length are related to the strain displacements as

$$\{N\} = [A]\{\varepsilon^o\} + [B]\{\kappa\} \quad (2.75)$$

Substituting Eq. (2.74) and Eq. (2.75) in Eq. (2.71-1) and expanding the latter expression gives

$$\int_A \{\delta \varepsilon^o\}^T \{N\} dA = \{\delta w_m\}^T \int_A [C_m]^T [A] [C_m] dA \{w_m\} \quad (2.76-1)$$

$$+ \{\delta w_m\}^T \int_A [C_m]^T [B] [C_b] dA \{w_b\} \quad (2.76-2)$$

$$+ \{\delta w_m\}^T \frac{1}{2} \int_A [C_m]^T [A] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.76-3)$$

$$+ \{\delta w_m\}^T \frac{1}{2} \int_A [C_m]^T [A] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-4)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [C_m] dA \{w_m\} \quad (2.76-5)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [C_m] dA \{w_m\} \quad (2.76-6)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} \quad (2.76-7)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} \quad (2.76-8)$$

$$+ \{\delta w_b\}^T \frac{1}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.76-9)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.76-10)$$

$$+ \{\delta w_b\}^T \frac{1}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-11)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-12)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.76-13)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-14)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.76-15)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-16)$$

$$+ \{\delta w_b\}^T \frac{1}{2} \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.76-17)$$

$$+ \{\delta w_b\}^T \frac{1}{2} \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-18)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.76-19)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-20)$$

$$+ \{\delta w_m\}^T \int_A [C_m]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.76-21)$$

$$+ \{\delta w_m\}^T \int_A [C_m]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.76-22)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta_o]^T [A] [C_m] dA \{w_m\} \quad (2.76-23)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [C_m] dA \{w_m\} \quad (2.76-24)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [C_m] dA \{w_m\} \quad (2.76-25)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta_o]^T [B] [C_b] dA \{w_\psi\}$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta_o]^T [B] [C_b] dA \{w_\psi\} \quad (2-76-26)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2-76-27)$$

$$+ \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2-76-28)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2-76-29)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2-76-30)$$

The variation of bending strain Eq. (2.48), can be expanded as

$$\{\delta \kappa\}^T = \{\delta w_\psi\}^T [C_b]^T \quad (2.77)$$

The internal moments per unit-length are related to the strain displacements as

$$\{M\} = [B] \{\varepsilon^o\} + [D] \{\kappa\} \quad (2.78)$$

Substituting Eq. (2.77) and Eq. (2.78) in Eq. (2.71-2) and expanding the latter expression gives

$$\int_A \{\delta \kappa\}^T \{M\} dA = \{\delta w_\psi\}^T \int_A [C_b]^T [B] [C_m] dA \{w_m\} \quad (2.79-1)$$

$$+ \{\delta w_\psi\}^T \int_A [C_b]^T [D] [C_b] dA \{w_\psi\} \quad (2.79-2)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_b]^T [B] [\theta] [C_{\psi b}] dA \{w_b\} \quad (2.79-3)$$

$$+ \{\delta w_\psi\}^T \frac{1}{2} \int_A [C_b]^T [B] [\theta] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.79-4)$$

$$+ \{\delta w_\psi\}^T \int_A [C_b]^T [B] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.79-5)$$

$$+ \{\delta w_\psi\}^T \int_A [C_b]^T [B] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.79-6)$$

The variation of the transverse shear strain Eq. (2.50) can be expanded as

$$\{\delta\gamma\}^T = \{\delta w_b\}^T [C_{\gamma b}]^T + \{\delta w_\psi\}^T [C_{\gamma\psi}]^T \quad (2.80)$$

The internal shear forces per unit-length are related to the transverse shear strain angle as

$$\{R\} = [A_s] \{\gamma\} \quad (2.81)$$

Substituting Eq. (2.80) and Eq. (2.81) in Eq. (2.71-3) and expanding the latter expression gives

$$\int_A \{\delta\gamma\}^T \{R\} dA = \{\delta w_b\}^T \int_A [C_{\gamma b}]^T [A_s] [C_{\gamma b}] dA \{w_b\} \quad (2.82-1)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\gamma\psi}]^T [A_s] [C_{\gamma b}] dA \{w_b\} \quad (2.82-2)$$

$$+ \{\delta w_b\}^T \int_A [C_{\gamma b}]^T [A_s] [C_{\gamma\psi}] dA \{w_\psi\} \quad (2.82-3)$$

$$+ \{\delta w_\psi\}^T \int_A [C_{\gamma\psi}]^T [A_s] [C_{\gamma\psi}] dA \{w_\psi\} \quad (2.82-4)$$

2.8.2 Element Linear Stiffness Matrices

From Eqs. (2.76) and Eqs. (2.79), the linear stiffness matrices can be expressed as

$$[k]_{mm} = \int_A [C_m]^T [A] [C_m] dA \quad (2.83)$$

$$[k]_{m\psi} = \int_A [C_m]^T [B] [C_b] dA \quad (2.84)$$

$$[k]_{\psi m} = \int_A [C_b]^T [B] [C_m] dA \quad (2.85)$$

$$[k]_{\psi\psi} = \int_A [C_b]^T [D] [C_b] dA \quad (2.86)$$

2.8.3 Element Linear Shear Stiffness Matrices

From Eq. (2.82), the linear shear stiffness matrices can be expressed as

$$[k]_{bb}^s = \int_A [C_{\gamma b}]^T [A_s] [C_{\gamma b}] dA \quad (2.87)$$

$$[k]_{b\psi}^s = \int_A [C_{\gamma b}]^T [A_s] [C_{\gamma\psi}] dA \quad (2.88)$$

$$[k]_{\psi b}^s = \int_A [C_{\gamma\psi}]^T [A_s] [C_{\gamma b}] dA \quad (2.89)$$

$$[k]_{\psi\psi}^s = \int_A [C_{\gamma\psi}]^T [A_s] [C_{\gamma\psi}] dA \quad (2.90)$$

2.8.4 Element Linear Stiffness Matrices Due to $w_o(x,y)$

From Eqs. (2.72) and Eqs. (2.75), the linear stiffness matrices due to curved panel geometry including a single matrix $[\theta_o]$ can be expressed as

$$[k_o]_{mb}^1 = \int_A [C_m]^T [A] [\theta_o] [C_{\psi b}] dA \quad (2.91)$$

$$[k_o]_{bm}^1 = \int_A [C_{\psi b}]^T [\theta_o]^T [A] [C_m] dA \quad (2.92)$$

$$[k_o]_{m\psi}^1 = \int_A [C_m]^T [A] [\theta_o] [C_{\psi\psi}] dA \quad (2.93)$$

$$[k_o]_{\psi m}^1 = \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [C_m] dA \quad (2.94)$$

$$[k_o]_{b\psi}^1 = \int_A [C_{\psi b}]^T [\theta_o]^T [B] [C_b] dA \quad (2.95)$$

$$[k_o]_{\psi b}^1 = \int_A [C_b]^T [B] [\theta_o] [C_{\psi b}] dA \quad (2.96)$$

$$[k_o]_{\psi\psi}^1 = \int_A [C_{\psi\psi}]^T [\theta_o]^T [B] [C_b] dA + \int_A [C_b]^T [B] [\theta_o] [C_{\psi\psi}] dA \quad (2.97)$$

From Eqs. (2.76) and Eqs. (279), the linear stiffness matrices due to curved geometry including a double matrix $[\theta_o]$ can be expressed as

$$[k_o]_{bb}^2 = \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi b}] dA \quad (2.98)$$

$$[k_o]_{b\psi}^2 = \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi\psi}] dA \quad (2.99)$$

$$[k_o]_{\psi b}^2 = \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi b}] dA \quad (2.100)$$

$$[k_o]_{\psi\psi}^2 = \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta_o] [C_{\psi\psi}] dA \quad (2.101)$$

2.8.5 Expansion of the Element First Order Non-Linear Stiffness Matrices

In order to express the first order non-linear matrices, the following transformations have to be done. Consider first the expressions containing the slope matrix $[\theta]$ or its transpose $[\theta]^T$.

Taking the last four terms of expressions (2.76-5) and (2.76-6), they can be expanded according to the following scheme

$$[\theta]^T [A] [C_m] \{w_m\} = [\theta]^T \{N_m\} = [N_m] \{G\} = [N_m] [C_{\psi b}] \{w_b\} + [N_m] [C_{\psi\psi}] \{w_\psi\} \quad (2.102)$$

where

$$\{N_m\} = \{N_{mx} \quad N_{my} \quad N_{mxy}\}^T \quad (2.103)$$

and

$$[N_m] = \begin{bmatrix} N_{mx} & N_{mxy} \\ N_{mxy} & N_{my} \end{bmatrix} \quad (2.104)$$

Expression (2.76-5) can be written as

$$\{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [C_m] dA \{w_m\} = \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [C_m] dA \{w_m\} \quad (2.105)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_m] [C_{\psi b}] dA \{w_b\} \quad (2.106)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_m] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.107)$$

and expression (2.76-6) can be written as

$$\{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [C_m] dA \{w_m\} = \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [C_m] dA \{w_m\} \quad (2.108)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_m] [C_{\psi b}] dA \{w_b\} \quad (2.109)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_m] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.110)$$

The last four terms of expression (2.76-7) and (2.76-8) can be expanded according to the following scheme

$$[\theta]^T [B] [C_b] \{w_\psi\} = [\theta]^T \{N_b\} = [N_b] \{G\} = [N_b] [C_{\psi b}] \{w_b\} + [N_b] [C_{\psi\psi}] \{w_\psi\} \quad (2.111)$$

where

$$\{N_b\} = \{N_{bx} \quad N_{by} \quad N_{bxy}\}^T \quad (2.112)$$

and

$$[N_b] = \begin{bmatrix} N_{bx} & N_{bxy} \\ N_{bxy} & N_{by} \end{bmatrix} \quad (2.113)$$

Expression (2.76-7) can be written as

$$\{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} = \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} \quad (2.114)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_b] [C_{\psi b}] dA \{w_b\} \quad (2.115)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_b] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.116)$$

and expression (2.76-8) can be written as

$$\{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} = \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [B] [C_b] dA \{w_\psi\} \quad (2.117)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_b] [C_{\psi b}] dA \{w_b\} \quad (2.118)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_b] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.119)$$

The last four terms of expression (2.76-13), and (2.76-15) can be expanded according to the following scheme

$$[\theta]^T [A] [\theta_o] [C_{\psi b}] \{w_b\} = [\theta]^T \{N_{\theta_o}^{\psi b}\} = [N_{\theta_o}^{\psi b}] \{G\} = [N_{\theta_o}^{\psi b}] [C_{\psi b}] \{w_b\} + [N_{\theta_o}^{\psi b}] [C_{\psi\psi}] \{w_\psi\} \quad (2.120)$$

where

$$\{N_{\theta_o}^{\psi b}\} = \{N_{\theta_o, x}^{\psi b} \quad N_{\theta_o, y}^{\psi b} \quad N_{\theta_o, xy}^{\psi b}\}^T \quad (2.121)$$

and

$$[N_{\theta_o}^{\psi b}] = \begin{bmatrix} N_{\theta_o, x}^{\psi b} & N_{\theta_o, xy}^{\psi b} \\ N_{\theta_o, xy}^{\psi b} & N_{\theta_o, y}^{\psi b} \end{bmatrix} \quad (2.122)$$

Expression (2.76-13) can be written as

$$\{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} = \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.123)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi b}] [C_{\psi b}] dA \{w_b\} \quad (2.124)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi b}] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.125)$$

and expression (2.76-15) can be written as

$$\{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A][\theta_o] [C_{\psi b}] dA \{w_b\} = \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A][\theta_o] [C_{\psi b}] dA \{w_b\} \quad (2.126)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi b}] [C_{\psi b}] dA \{w_b\} \quad (2.127)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi b}] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.128)$$

The last four terms of expression (2.76-14) and (2.76-16) can be expanded according to the following scheme

$$[\theta]^T [A][\theta_o] [C_{\psi\psi}] \{w_\psi\} = [\theta]^T \{N_{\theta_o}^{\psi\psi}\} = [N_{\theta_o}^{\psi\psi}] \{G\} = [N_{\theta_o}^{\psi\psi}] [C_{\psi b}] \{w_b\} + [N_{\theta_o}^{\psi\psi}] [C_{\psi\psi}] \{w_\psi\} \quad (2.129)$$

where

$$\{N_{\theta_o}^{\psi\psi}\} = \{N_{\theta_o x}^{\psi\psi} \quad N_{\theta_o y}^{\psi\psi} \quad N_{\theta_o xy}^{\psi\psi}\}^T \quad (2.130)$$

and

$$[N_{\theta_o}^{\psi\psi}] = \begin{bmatrix} N_{\theta_o x}^{\psi\psi} & N_{\theta_o xy}^{\psi\psi} \\ N_{\theta_o xy}^{\psi\psi} & N_{\theta_o y}^{\psi\psi} \end{bmatrix} \quad (2.131)$$

Expression (2.76-14) can be written as

$$\{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A][\theta_o] [C_{\psi\psi}] dA \{w_\psi\} = \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [\theta]^T [A][\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.132)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi b}] dA \{w_b\} \quad (2.133)$$

$$+ \frac{1}{2} \{\delta w_b\}^T \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.134)$$

and expression (2.76-16) can be written as

$$\{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} = \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.135)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi b}] dA \{w_b\} \quad (2.136)$$

$$+ \frac{1}{2} \{\delta w_\psi\}^T \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi\psi}] dA \{w_\psi\} \quad (2.137)$$

2.8.6 Element First Order Non-linear Matrices

$$[n1]_{\psi b} = \int_A [C_b]^T [B] [\theta] [C_{\psi b}] dA \quad (2.138)$$

$$[n1]_{b\psi} = \int_A [C_{\psi b}]^T [\theta]^T [B] [C_b] dA \quad (2.139)$$

$$[n1]_{mb} = \int_A [C_m]^T [A] [\theta] [C_{\psi b}] dA \quad (2.140)$$

$$[n1]_{bm} = \int_A [C_{\psi b}]^T [\theta]^T [A] [C_m] dA \quad (2.141)$$

$$[n1]_{\psi\psi}^1 = \int_A [C_b]^T [B] [\theta] [C_{\psi\psi}] dA \quad (2.142)$$

$$[n1]_{\psi\psi}^2 = \int_A [C_{\psi\psi}]^T [\theta]^T [B] [C_b] dA \quad (2.143)$$

$$[n1]_{m\psi} = \int_A [C_m]^T [A] [\theta] [C_{\psi\psi}] dA \quad (2.144)$$

$$[n1]_{\psi m} = \int_A [C_{\psi\psi}]^T [\theta]^T [A] [C_m] dA \quad (2.145)$$

For subsequent development, denote

$$[n1]_{\psi\psi} = [n1]_{\psi\psi}^1 + [n1]_{\psi\psi}^2 \quad (2.146)$$

2.8.7 Element First-Order Non-Linear Stiffness Matrices Due to $w_o(x,y)$

$$[n1]_{bb}^o = \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \quad (2.147)$$

$$[n1]_{b\psi}^o = \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \quad (2.148)$$

$$[n1]_{\psi b}^o = \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi b}] dA \quad (2.149)$$

$$[n1]_{\psi\psi}^o = \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta_o] [C_{\psi\psi}] dA \quad (2.150)$$

$$[n1]_{bb}^{o^T} = \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta] [C_{\psi b}] dA \quad (2.151)$$

$$[n1]_{\psi b}^{o^T} = \int_A [C_{\psi b}]^T [\theta_o]^T [A] [\theta] [C_{\psi\psi}] dA \quad (2.152)$$

$$[n1]_{b\psi}^{o^T} = \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta] [C_{\psi b}] dA \quad (2.153)$$

$$[n1]_{\psi\psi}^{o^T} = \int_A [C_{\psi\psi}]^T [\theta_o]^T [A] [\theta] [C_{\psi\psi}] dA \quad (2.154)$$

For subsequent development, denote

$$[n1]_{bb}^{\theta_o} = [n1]_{bb}^o + [n1]_{bb}^{o^T} \quad (2.155)$$

$$[n1]_{b\psi}^{\theta_o} = [n1]_{b\psi}^o + [n1]_{b\psi}^{o^T} \quad (2.156)$$

$$[n1]_{\psi b}^{\theta_o} = [n1]_{\psi b}^o + [n1]_{\psi b}^{o^T} \quad (2.157)$$

$$[n1]_{\psi\psi}^{\theta_o} = [n1]_{\psi\psi}^o + [n1]_{\psi\psi}^{o^T} \quad (2.158)$$

2.8.8 Element First-Order Non-Linear Stiffness Matrices Due to $[N_b]$

$$[n1]_{bb}^{N_b} = \int_A [C_{\psi b}]^T [N_b] [C_{\psi b}] dA \quad (2.159)$$

$$[n1]_{b\psi}^{N_b} = \int_A [C_{\psi b}]^T [N_b] [C_{\psi\psi}] dA \quad (2.160)$$

$$[n1]_{\psi b}^{N_b} = \int_A [C_{\psi\psi}]^T [N_b] [C_{\psi b}] dA \quad (2.161)$$

$$[n1]_{\psi\psi}^{N_b} = \int_A [C_{\psi\psi}]^T [N_b] [C_{\psi\psi}] dA \quad (2.162)$$

2.8.9 Element First-Order Non-Linear Stiffness Matrices Due to $[N_m]$

$$[n1]_{bb}^{N_m} = \int_A [C_{\psi b}]^T [N_m] [C_{\psi b}] dA \quad (2.163)$$

$$[n1]_{b\psi}^{N_m} = \int_A [C_{\psi b}]^T [N_m] [C_{\psi\psi}] dA \quad (2.164)$$

$$[n1]_{\psi b}^{N_m} = \int_A [C_{\psi\psi}]^T [N_m] [C_{\psi b}] dA \quad (2.165)$$

$$[n1]_{\psi\psi}^{N_m} = \int_A [C_{\psi\psi}]^T [N_m] [C_{\psi\psi}] dA \quad (2.166)$$

2.8.10 Element First-Order Non-Linear Stiffness Matrices Due to $[N_{\theta_o}^{\psi b}]$

$$[n1]_{bb}^{N_{\theta_o}^{\psi b}} = \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi b}] [C_{\psi b}] dA \quad (2.167)$$

$$[n1]_{b\psi}^{N_{\theta_o}^{\psi b}} = \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi b}] [C_{\psi\psi}] dA \quad (2.168)$$

$$[n1]_{\psi b}^{N_{\theta_o}^{\psi b}} = \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi b}] [C_{\psi b}] dA \quad (2.169)$$

$$[n1]_{\psi\psi}^{N_{\theta_o}^{\psi b}} = \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi b}] [C_{\psi\psi}] dA \quad (2.170)$$

2.8.11 Element First-Order Non-Linear Stiffness Matrices Due to $[N_{\theta_o}^{\psi\psi}]$

$$[n1]_{bb}^{N_{\theta_o}^{\psi\psi}} = \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi b}] dA \quad (2.171)$$

$$[n1]_{b\psi}^{N_{\theta_o}^{\psi\psi}} = \int_A [C_{\psi b}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi\psi}] dA \quad (2.172)$$

$$[n1]_{\psi b}^{N_{\theta_o}^{\psi\psi}} = \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi b}] dA \quad (2.173)$$

$$[n1]_{\psi\psi}^{N_{\theta_o}^{\psi\psi}} = \int_A [C_{\psi\psi}]^T [N_{\theta_o}^{\psi\psi}] [C_{\psi\psi}] dA \quad (2.174)$$

For subsequent development, denote

$$[n1]_{bb}^{N_{\theta_o}} = [n1]_{bb}^{N_{\theta_o}^{\psi b}} + [n1]_{bb}^{N_{\theta_o}^{\psi\psi}} \quad (2.175)$$

$$[n1]_{b\psi}^{N_{\theta_o}} = [n1]_{b\psi}^{N_{\theta_o}^{\psi b}} + [n1]_{b\psi}^{N_{\theta_o}^{\psi\psi}} \quad (2.176)$$

$$[n1]_{\psi b}^{N_{\theta_o}} = [n1]_{\psi b}^{N_{\theta_o}^{\psi b}} + [n1]_{\psi b}^{N_{\theta_o}^{\psi\psi}} \quad (2.177)$$

$$[n1]_{\psi\psi}^{N_{\theta_o}} = [n1]_{\psi\psi}^{N_{\theta_o}^{\psi b}} + [n1]_{\psi\psi}^{N_{\theta_o}^{\psi\psi}} \quad (2.178)$$

2.8.12 Element Second-Order Non-Linear Stiffness Matrices

$$[n2]_{bb} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \quad (2.179)$$

$$[n2]_{\psi b} = \frac{3}{2} \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \quad (2.180)$$

$$[n2]_{b\psi} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi\psi}] dA \quad (2.181)$$

$$[n2]_{\psi\psi} = \frac{3}{2} \int_A [C_{\psi\psi}]^T [\theta]^T [A] [\theta] [C_{\psi\psi}] dA \quad (2.182)$$

2.8.13 Virtual Work Done by External Forces

The virtual work done by external forces is given by Eq. (2.68)

$$\delta W_{ext} = + \int_A \{\delta w\}^T (-\rho h \{w_{,tt}\}) dA \quad (2.183)$$

$$+ \int_A \{\delta u\}^T (-\rho h \{u_{,tt}\}) dA \quad (2.184)$$

$$+ \int_A \{\delta v\}^T (-\rho h \{v_{,tt}\}) dA \quad (2.185)$$

$$+ \int_A \{\delta w\}^T (p - p_\infty) dA \quad (2.186)$$

$$+ \int_A \{\delta w\}^T F_d(x, y, t) dA \quad (2.187)$$

2.8.14 Element Mass Matrices

The term ρh can be transformed as

$$\rho h = \frac{\rho h a^4}{D_{110}} \frac{D_{110}}{a^4} = \frac{1}{\omega_o^2} \frac{D_{110}}{a^4} \quad (2.188)$$

where

$$\frac{1}{\omega_o^2} = \frac{\rho h a^4}{D_{110}} \quad (2.189)$$

The constants D_{110} is the first entry of the bending stiffness matrix $[D]$ computed when all of the fibers of the composite layers are aligned in the x direction, and ω_o is a convenient reference frequency. Using Eqs. (2.8) and (2.188), the first inertia term (2.183) appearing in the expression of the virtual work is derived as

$$\begin{aligned} \int_A \{\delta w\}^T (-\rho h \{w_{,tt}\}) dA &= -\frac{1}{\omega_o^2} \frac{D_{110}}{a^4} \int_A (\{\delta w_b\}^T [H_w]^T + \{\delta w_\psi\}^T [H_{w\psi}]^T) ([H_w] \{\ddot{w}_b\} + [H_{w\psi}] \{\ddot{w}_\psi\}) dA \\ &= -\frac{1}{\omega_o^2} \{\delta w_b\}^T \frac{D_{110}}{a^4} \int_A [H_w]^T [H_w] dA \{\ddot{w}_b\} \end{aligned} \quad (2.190)$$

$$-\frac{1}{\omega_o^2} \{\delta w_b\}^T \frac{D_{110}}{a^4} \int_A [H_w]^T [H_{w\psi}] dA \{\ddot{w}_\psi\} \quad (2.191)$$

$$-\frac{1}{\omega_o^2} \{\delta w_\psi\}^T \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_w] dA \{\ddot{w}_b\} \quad (2.192)$$

$$-\frac{1}{\omega_o^2} \{\delta w_\psi\}^T \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_{w\psi}] dA \{\ddot{w}_\psi\} \quad (2.193)$$

The element mass matrices related to the transverse inertia term, can be expressed by

$$[m]_{bb} = \frac{D_{110}}{a^4} \int_A [H_w]^T [H_w] dA \quad (2.194)$$

$$[m]_{b\psi} = \frac{D_{110}}{a^4} \int_A [H_w]^T [H_{w\psi}] dA \quad (2.195)$$

$$[m]_{\psi b} = \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_w] dA \quad (2.196)$$

$$[m]_{\psi\psi} = \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_{w\psi}] dA \quad (2.197)$$

Expanding Eq. (2.184) and (2.185) using the same procedure as for (2.183) by using Eqs.

(2.11) and (2.12) give

$$\int_A \{\delta u\}^T (-\rho h \{u_{,tt}\}) dA = -\frac{1}{\omega_o^2} \{\delta w_m\}^T \frac{D_{110}}{a^4} \int_A [H_u]^T [H_u] dA \{\ddot{w}_m\} \quad (2.199)$$

$$\int_A \{\delta v\}^T (-\rho h \{v_{,tt}\}) dA = -\frac{1}{\omega_o^2} \{\delta w_m\}^T \frac{D_{110}}{a^4} \int_A [H_v]^T [H_v] dA \{\ddot{w}_m\} \quad (2.199)$$

where the mass element matrices related to the in-plane inertia, can be expressed by

$$[m]_m^u = \frac{D_{110}}{a^4} \int_A [H_u]^T [H_u] dA \quad (2.200)$$

$$[m]_m^v = \frac{D_{110}}{a^4} \int_A [H_v]^T [H_v] dA \quad (2.201)$$

For convenience, the following notation is introduced

$$[m]_{mm} = \begin{bmatrix} [m]_m^u & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [m]_m^v \end{bmatrix} \quad (2.202)$$

2.8.15 Quasi-Steady First-Order Aerodynamic Piston Theory

During the supersonic flight operations, the curved panel is subjected to the aerodynamic pressure generated by the surrounding air. The considered aerodynamic pressure in this study is modeled with the quasi-steady first-order piston theory [3]. The theory simulates the aerodynamic pressure acting over the skin of a curved panel subjected to a yawed supersonic airflow, see Fig. 2.7. The theoretical assumptions of the theory are:

- The gas flow is an ideal gas with a constant specific heat.
- The process of energy exchange between the panel and the surrounding gas is considered isentropic (constant entropy).
- The airflow is parallel to the panel surface but with an arbitrary angle.
- The infinitesimal motion of the panel simulates a piston motion.
- The infinitesimal motion of the panel is negligible with respect to the motion of the gas flow.

The aerodynamic pressure appearing in expression (2.186) is given by the quasi-steady first-order piston theory as

$$p - p_\infty = -\frac{2q_a}{\beta} \left(\{w_{o,x} + w_{,x}\} \cos \Lambda + \{w_{o,y} + w_{,y}\} \sin \Lambda + \frac{M_\infty^2 - 2}{M_\infty^2 - 1} \frac{1}{V_\infty} \{w_{o,t} + w_{,t}\} \right) \quad (2.203)$$

Developing Eq. (2.202) gives

$$p - p_\infty = -\frac{2q_a}{\beta} \left(\{w_{,x}\} \cos \Lambda + \{w_{,y}\} \sin \Lambda + \frac{M_\infty^2 - 2}{M_\infty^2 - 1} \frac{1}{V_\infty} \{w_{,t}\} \right) - \frac{2q_a}{\beta} (w_{o,x} \cos \Lambda + w_{o,y} \sin \Lambda) \quad (2.204)$$

The first term of Eq. (2.204) represents the aerodynamic pressure loading due to the curved panel deformation during the flutter motion, while the second term is the loading due to the geometry (curvature) of the curved panel $w_o(x, y)$, see Fig. 2.5 and Fig. 2.6. The latter term is called the static aerodynamic load, (SAL) [13]. The terms of Eq. (2.204) are identified as

$p - p_\infty$ is the aerodynamic pressure loading

V_∞ is the far field air flow velocity without perturbations

M_∞ is the far field Mach number

$q_a = \rho_a \frac{V_\infty^2}{2}$ is the far field dynamic pressure

ρ_a is the air density

Λ is the arbitrary flow angle with respect to the x axis

$w_{o,x}, w_{o,y}$ are the derivatives with respect to x and y of the curved geometry $w_o(x, y)$.

The parameter β is defined as a function of the Mach number and called the Prandtl-Glauert parameter

$$\beta = \sqrt{M_\infty^2 - 1} \quad (2.205)$$

Define the non-dimensional dynamic pressure as

$$\lambda = \frac{2q_a a^3}{\beta D_{110}} \quad (2.206)$$

where a represents the curved panel length.

The non-dimensional aerodynamic damping parameter is defined as

$$g_a = \frac{\rho_a V_\infty (M_\infty^2 - 2)}{\rho h \omega_o \beta^3} = \sqrt{\lambda C_a} \quad (2.207)$$

where the damping parameter C_a is a function of the mass density ratio μ , and the Mach number M_∞ . C_a is defined as

$$C_a = \frac{\mu (M_\infty^2 - 2)^2}{\beta (M_\infty^2 - 1)^2} \quad (2.208)$$

where the mass density ratio μ is defined as

$$\mu = \frac{\rho_a a}{\rho h} \quad (2.209)$$

Substituting Eqs. (2.188) and (2.189) in Eq. (2.204) give

$$\begin{aligned} p - p_\infty = & -\frac{\lambda D_{110}}{a^3} \cos \Lambda \{w_{,x}\} - \frac{\lambda D_{110}}{a^3} \sin \Lambda \{w_{,y}\} - \frac{g_a D_{110}}{\omega_o a^4} \{w_{,t}\} \\ & - \frac{\lambda D_{110}}{a^3} \cos \Lambda w_{o,x} - \frac{\lambda D_{110}}{a^3} \sin \Lambda w_{o,y} \end{aligned} \quad (2.210)$$

Substituting Eq. (2.210) in Eq. (2.186) gives

$$\int_A \{\delta w\}^T (p - p_\infty) dA = -\lambda \cos \Lambda \{\delta w_b\}^T \frac{D_{110}}{a^3} \int_A [H_w]^T [H_w]_{,x} dA \{w_b\} \quad (2.211-1)$$

$$- \lambda \cos \Lambda \{\delta w_\psi\}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_w]_{,x} dA \{w_b\} \quad (2.211-2)$$

$$- \lambda \sin \Lambda \{ \delta w_b \}^T \frac{D_{110}}{a^3} \int_A [H_w]^T [H_w]_{,y} dA \{ w_b \} \quad (2.211-3)$$

$$- \lambda \sin \Lambda \{ \delta w_\psi \}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_w]_{,y} dA \{ w_b \} \quad (2.211-4)$$

$$- \lambda \cos \Lambda \{ \delta w_b \}^T \frac{D_{110}}{a^3} \int_A [H_w]^T [H_{w\psi}]_{,x} dA \{ w_\psi \} \quad (2.211-5)$$

$$- \lambda \cos \Lambda \{ \delta w_\psi \}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_{w\psi}]_{,x} dA \{ w_\psi \} \quad (2.211-6)$$

$$- \lambda \sin \Lambda \{ \delta w_b \}^T \frac{D_{110}}{a^3} \int_A [H_w]^T [H_{w\psi}]_{,y} dA \{ w_\psi \} \quad (2.211-7)$$

$$- \lambda \sin \Lambda \{ \delta w_\psi \}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_{w\psi}]_{,y} dA \{ w_\psi \} \quad (2.211-8)$$

$$- \lambda \cos \Lambda \{ \delta w_b \}^T \frac{D_{110}}{a^3} \int_A [H_w]^T w_{o,x} dA \quad (2.211-9)$$

$$- \lambda \cos \Lambda \{ \delta w_\psi \}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T w_{o,x} dA \quad (2.211-10)$$

$$- \lambda \sin \Lambda \{ \delta w_b \}^T \frac{D_{110}}{a^3} \int_A [H_w]^T w_{o,y} dA \quad (2.211-11)$$

$$- \lambda \sin \Lambda \{ \delta w_\psi \}^T \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T w_{o,y} dA \quad (2.211-12)$$

$$- \frac{g_a}{\omega_o} \{ \delta w_b \}^T \frac{D_{110}}{a^4} \int_A [H_w]^T [H_w] dA \{ w_{b,t} \} \quad (2.211-13)$$

$$- \frac{g_a}{\omega_o} \{ \delta w_\psi \}^T \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_w] dA \{ w_{b,t} \} \quad (2.211-14)$$

$$- \frac{g_a}{\omega_o} \{ \delta w_b \}^T \frac{D_{110}}{a^4} \int_A [H_w]^T [H_{w\psi}] dA \{ w_{\psi,t} \} \quad (2.211-15)$$

$$-\frac{g_a}{\omega_o} \{\delta w_\psi\}^T \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_{w\psi}] dA \{w_{\psi,t}\} \quad (2.211-16)$$

2.8.16 Element Aerodynamic Stiffness Matrices

The element aerodynamic stiffness matrices can be defined from Eq. (2.211) as

$$[a_x]_{bb} = \frac{D_{110}}{a^3} \int_A [H_w]^T [H_w]_{,x} dA \quad (2.212)$$

$$[a_x]_{\psi b} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_w]_{,x} dA \quad (2.213)$$

$$[a_x]_{b\psi} = \frac{D_{110}}{a^3} \int_A [H_w]^T [H_{w\psi}]_{,x} dA \quad (2.214)$$

$$[a_x]_{\psi\psi} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_{w\psi}]_{,x} dA \quad (2.215)$$

$$[a_y]_{bb} = \frac{D_{110}}{a^3} \int_A [H_w]^T [H_w]_{,y} dA \quad (2.216)$$

$$[a_y]_{\psi b} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_w]_{,y} dA \quad (2.217)$$

$$[a_y]_{b\psi} = \frac{D_{110}}{a^3} \int_A [H_w]^T [H_{w\psi}]_{,y} dA \quad (2.218)$$

$$[a_y]_{\psi\psi} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T [H_{w\psi}]_{,y} dA \quad (2.219)$$

2.8.17 Element Aerodynamic Damping Matrices

The aerodynamic damping matrices can be defined from Eq. (2.211) as

$$[g]_{bb} = \frac{D_{110}}{a^4} \int_A [H_w]^T [H_w] dA \quad (2.220)$$

$$[g]_{\psi b} = \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_w] dA \quad (2.221)$$

$$[g]_{b\psi} = \frac{D_{110}}{a^4} \int_A [H_w]^T [H_{w\psi}] dA \quad (2.222)$$

$$[g]_{\psi\psi} = \frac{D_{110}}{a^4} \int_A [H_{w\psi}]^T [H_{w\psi}] dA \quad (2.223)$$

Comparing Eqs. (2-220) to (2.223) with the element mass matrices derived in Eqs. (2.194) to (2.197) reveals that

$$[m]_{bb} = [g]_{bb} \quad (2.224)$$

$$[m]_{b\psi} = [g]_{b\psi} \quad (2.225)$$

$$[m]_{\psi b} = [g]_{\psi b} \quad (2.226)$$

$$[m]_{\psi\psi} = [g]_{\psi\psi} \quad (2.227)$$

The equality between element mass matrices and element damping matrices is useful in allowing the flutter system equations of motion to be formulated as an eigen-value problem. This equivalence is a direct consequence related to the use of the first-order piston theory. Further details developing this assertion will be given in Chapter 3.

2.8.18 Element Static Aerodynamic Load Vector

$$\{p_b^{sal_x}\} = \frac{D_{110}}{a^3} \int_A [H_w]^T w_{o,x} dA \quad (2.228)$$

$$\{p_\psi^{sal_x}\} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T w_{o,x} dA \quad (2.229)$$

$$\{p_b^{sal_y}\} = \frac{D_{110}}{a^3} \int_A [H_w]^T w_{o,y} dA \quad (2.230)$$

$$\{p_{\psi}^{sal}\} = \frac{D_{110}}{a^3} \int_A [H_{w\psi}]^T w_{o,y} dA \quad (2.231)$$

One can notice easily that the static aerodynamic load depends on the geometry of the curved panel $w_o(x, y)$.

2.8.19 Element External Load Vectors

Expanding Eq. (2.187), and noting that $F_d = F_d(x, y, t)$ yields

$$\begin{aligned} \int_A \{\delta w\}^T F_d dA &= \int_A (\{\delta w_b\}^T [H_w]^T + \{\delta w_{\psi}\}^T [H_{w\psi}]^T) F_d dA \\ &= \{\delta w_b\} \int_A [H_w]^T F_d dA + \{\delta w_{\psi}\} \int_A [H_{w\psi}]^T F_d dA \end{aligned} \quad (2.232)$$

where the externally applied mechanical loads can be defined as

$$\{p(t)\}_b = \int_A [H_w]^T F_d dA \quad (2.233)$$

$$\{p(t)\}_{\psi} = \int_A [H_{w\psi}]^T F_d dA \quad (2.234)$$

The nature of the force $\{F_d\}$ could be mechanical, acoustic, etc.

2.9 Element Equations of Motion of the Curved Panel

Proceeding with a preliminary regrouping of the linear stiffness terms depending on the curved geometry $w_o = w_o(x, y)$ in a matrix form results in

$$\begin{bmatrix} [k]_{bb}^{\theta_o} & [k]_{b\psi}^{\theta_o} & [k]_{bm}^{\theta_o} \\ [k]_{\psi b}^{\theta_o} & [k]_{\psi\psi}^{\theta_o} & [k]_{\psi m}^{\theta_o} \\ [k]_{mb}^{\theta_o} & [k]_{m\psi}^{\theta_o} & [0] \end{bmatrix} = \begin{bmatrix} [0] & [k_o]_{b\psi}^1 & [k_o]_{bm}^1 \\ [k_o]_{\psi b}^1 & [k_o]_{\psi\psi}^1 & [k_o]_{\psi m}^1 \\ [k_o]_{mb}^1 & [k_o]_{m\psi}^1 & [0] \end{bmatrix} + \begin{bmatrix} [k_o]_{bb}^2 & [k_o]_{b\psi}^2 & [0] \\ [k_o]_{\psi b}^2 & [k_o]_{\psi\psi}^2 & [0] \\ [0] & [0] & [0] \end{bmatrix} \quad (2.235)$$

Regrouping the element mass, linear and non-linear stiffness and the load terms resulting from the expansion of the virtual work (2.67) in a matrix form gives the element equations of motion of the non-linear fluttering systems as

$$\frac{1}{\omega_o^2} \begin{bmatrix} [m]_{bb} & [m]_{b\psi} & [0] \\ [m]_{\psi b} & [m]_{\psi\psi} & [0] \\ [0] & [0] & [m]_{mm} \end{bmatrix} \begin{Bmatrix} \ddot{w}_b \\ \ddot{w}_\psi \\ \ddot{w}_m \end{Bmatrix}$$

Element Mass Matrix Regrouping

Expressions (2.194) to (2.197) and (2.202)

$$+ \frac{g_a}{\omega_o} \begin{bmatrix} [g]_{bb} & [g]_{b\psi} & [0] \\ [g]_{\psi b} & [g]_{\psi\psi} & [0] \\ [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} \dot{w}_b \\ \dot{w}_\psi \\ \dot{w}_m \end{Bmatrix}$$

Element Aerodynamics Damping Matrix

Regrouping Expressions (2.220) to (2.223)

$$\left(+ \lambda \cos \Lambda \begin{bmatrix} [a_x]_{bb} & [a_x]_{b\psi} & [0] \\ [a_x]_{\psi b} & [a_x]_{\psi\psi} & [0] \\ [0] & [0] & [0] \end{bmatrix} \right.$$

Element Aerodynamic Stiffness Matrix in x

Direction Regrouping Expressions

(2.212) to (2.215)

$$+ \lambda \sin \Lambda \begin{bmatrix} [a_y]_{bb} & [a_y]_{b\psi} & [0] \\ [a_y]_{\psi b} & [a_y]_{\psi\psi} & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Aerodynamic Stiffness Matrix in y

Direction Regrouping Expressions (2.216)

to (2.219)

$$+ \begin{bmatrix} [0] & [0] & [0] \\ [0] & [k]_{\psi\psi} & [k]_{\psi m} \\ [0] & [k]_{m\psi} & [k]_{mm} \end{bmatrix}$$

Element Linear Stiffness Matrix

Regrouping Expressions (2.83) to (2.86)

$$+ \alpha_s \begin{bmatrix} [k]_{bb}^s & [k]_{b\psi}^s & [0] \\ [k]_{\psi b}^s & [k]_{\psi\psi}^s & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Linear Shear Matrix

Regrouping Expressions (2.87) to (2.90)

$$+ \begin{bmatrix} [k]_{bb}^{\theta_o} & [k]_{b\psi}^{\theta_o} & [k]_{bm}^{\theta_o} \\ [k]_{\psi b}^{\theta_o} & [k]_{\psi\psi}^{\theta_o} & [k]_{\psi m}^{\theta_o} \\ [k]_{mb}^{\theta_o} & [k]_{m\psi}^{\theta_o} & [0] \end{bmatrix}$$

Element Linear Stiffness Matrix Due to

Curved Geometry Regrouping Expressions

(2.91) to (2.97) and (2.98) to (2.101)

$$+ \frac{1}{2} \begin{bmatrix} [0] & [n1]_{b\psi} & [n1]_{bm} \\ [n1]_{\psi b} & [n1]_{\psi\psi} & [n1]_{\psi m} \\ [n1]_{mb} & [n1]_{m\psi} & [0] \end{bmatrix}$$

Element Non-linear First-order Stiffness

Matrix Regrouping Expressions (2.138) to

(2.146)

$$+ \frac{1}{2} \begin{bmatrix} [n1]_{bb}^{\theta_o} & [n1]_{b\psi}^{\theta_o} & [0] \\ [n1]_{\psi b}^{\theta_o} & [n1]_{\psi\psi}^{\theta_o} & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Non-linear First-order Stiffness

Matrix Due to Curved Geometry

Regrouping Expressions (2.145) to (2.156)

$$+ \frac{1}{2} \begin{bmatrix} [n1]_{bb}^{N_b} & [n1]_{b\psi}^{N_b} & [0] \\ [n1]_{\psi b}^{N_b} & [n1]_{\psi\psi}^{N_b} & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Non-linear First-order Stiffness

Matrix Including Term $[N_b]$ Regrouping

Expressions (2.159) to (2.162)

$$+ \frac{1}{2} \begin{bmatrix} [n1]_{bb}^{N_m} & [n1]_{b\psi}^{N_m} & [0] \\ [n1]_{\psi b}^{N_m} & [n1]_{\psi\psi}^{N_m} & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Non-linear First-order Stiffness
Matrix Including Term $[N_m]$ Regrouping
Expressions (2.163) to (2.164)

$$+ \frac{1}{2} \begin{bmatrix} [n1]_{bb}^{N_{\theta_o}} & [n1]_{b\psi}^{N_{\theta_o}} & [0] \\ [n1]_{\psi b}^{N_{\theta_o}} & [n1]_{\psi\psi}^{N_{\theta_o}} & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

Element Non-linear First-order Stiffness
Matrix Including Term $[N_{\theta_o}^{\psi b}]$ and $[N_{\theta_o}^{\psi\psi}]$
Regrouping Expressions (2.175) to (2.178)

$$+ \frac{1}{3} \begin{bmatrix} [n2]_{bb} & [n2]_{b\psi} & [0] \\ [n2]_{\psi b} & [n2]_{\psi\psi} & [0] \\ [0] & [0] & [0] \end{bmatrix} \begin{Bmatrix} w_b \\ w_\psi \\ w_m \end{Bmatrix}$$

Element Non-linear Second-order Stiffness
Matrix Regrouping Expressions (2.179) to
(2.182)

$$= \begin{Bmatrix} p_b(t) \\ p_\psi(t) \\ [0] \end{Bmatrix}$$

Element External Load Vector Due to
Mechanical Forces Regrouping
Expressions (2.233) to (2.234)

$$- \lambda \cos \Lambda \begin{Bmatrix} p_b^{sal_x} \\ p_\psi^{sal_x} \\ [0] \end{Bmatrix} - \lambda \sin \Lambda \begin{Bmatrix} p_b^{sal_y} \\ p_\psi^{sal_y} \\ [0] \end{Bmatrix}$$

Element Static Aerodynamic Load Vector
Regrouping Expressions (2.228) to (2.231)

(2.236)

The element equation of motion (2.236), can be written in a compact form as

$$\frac{1}{\omega_o^2} [m] \{\ddot{w}\} + \frac{g_a}{\omega_o} [g] \{\dot{w}\} + (\lambda \cos \Lambda [a_x] + \lambda \sin \Lambda [a_y]) \{w\}$$

$$\begin{aligned}
& + ([k] + \alpha_s [k]^s + [k]^{\theta_o}) \{w\} \\
& + \frac{1}{2} ([n1] + [n1]^{\theta_o} + [n1]^{N_b} + [n1]^{N_m} + [n1]^{N_{\theta_o}}) \{w\} \\
& + \frac{1}{3} [n2] \{w\} \\
& = \{p(t)\} - \lambda \cos \Lambda \{p_{w_{o,x}}\} - \lambda \sin \Lambda \{p_{w_{o,y}}\}
\end{aligned} \tag{2.237}$$

In Eq. (2.237), the matrix $[m]$ denotes the element mass matrix, $[g]$ the element aerodynamic damping, $[a_x]$ the element aerodynamic stiffness matrix with respect to x direction, $[a_y]$ the element aerodynamic stiffness matrix with respect to y direction, $[k]$ the element linear stiffness matrix, $[k]^s$ the element linear shear stiffness matrix, $[k_o]^{\theta_o}$ the element linear stiffness matrix due to the curved geometry, $[n1]$ element non-linear first-order stiffness matrix, $[n1]^{\theta_o}$ element non-linear first-order stiffness matrix due to curved geometry, $[n1]^{N_b}$ element non-linear first-order stiffness matrix computed with the nodal vector $\{w_b\}$, $[n1]^{N_m}$ element non-linear first-order stiffness matrix computed with the nodal vector $\{w_m\}$, $[n1]^{N_{\theta_o}}$ element non-linear stiffness matrix due to the curved geometry and computed with the nodal vectors $\{w_b\}$ and $\{w_\psi\}$, $[n2]$ element non-linear second-order stiffness matrix, $\{p(t)\}$ the mechanical external loads, $\{p_{w_{o,x}}\}$ and $\{p_{w_{o,y}}\}$ the static aerodynamic loads with respect to x and y axis respectively, and $\{w\}$ is the element nodal displacement vector.

2.10 Global Equations of Motion

Performing the assembly process over the complete fluttering system and applying the boundary conditions yields the global system equations of motion as

$$\begin{aligned}
 & \frac{1}{\omega_o^2} [M] \{\ddot{W}\} + \frac{g_a}{\omega_o} [G] \{\dot{W}\} + \left(\lambda \cos \Lambda [A_x] + \lambda \sin \Lambda [A_y] \right) \{W\} \\
 & + \left([K] + [K]^s + [K]^{\theta_o} \right) \{W\} \\
 & \left(+ \frac{1}{2} [N1] + \frac{1}{2} [N1]^{\theta_o} + \frac{1}{2} [N1]^{N_b} + \frac{1}{2} [N1]^{N_m} + \frac{1}{2} [N1]^{N_{\theta_o}} + \frac{1}{3} [N2] \right) \{W\} \\
 & = \{P(t)\} - \lambda \cos \Lambda \{P_{w_{o,x}}\} - \lambda \sin \Lambda \{P_{w_{o,y}}\}
 \end{aligned} \tag{2.238}$$

Eq. (2.238) is used in the following chapters to determine the non-linear flutter response of a variety of curved panels.

Fig. 2.1 MIN 3 element geometry including area coordinates

Fig. 2.2 MIN3 element nodal displacements and element dimensions

Fig. 2.3 **Coordinates details of a point belonging to the curved panel**

Fig. 2.4 Curved panel geometry characterized by $w_o(x, y)$, the height-rise H/h and MIN3 element

Fig. 2.5

Cylindrical panel geometry and dimensions

Fig. 2.6 Spherical panel geometry and dimensions

Fig. 2.7 First-order piston theory pressure modeling

Chapter 3

3 SOLUTIONS PROCEDURES

3.1 Introduction

In the first part of this chapter, the non-linear flutter EOM (2.238) is separated into two equations, which are to be solved sequentially. The outcome of the first equation is the static aerodynamic equilibrium position $\{W\}_s$ at a prescribed non-dimensional dynamic pressure λ , the equation is a time independent static non-linear algebraic equation. The second equation is a self-excited time dependent EOM, which describes the linear flutter dynamic behavior of the curved panel about the static aerodynamic position $\{W\}_s$ at a prescribed non-dimensional dynamic pressure λ . The iterative Newton-Raphson method is applied to the set of non-linear algebraic static equilibrium equations, while an eigen-solution procedure is used to determine the dynamic behaviors from a set of linear ordinary differential equations. In the second part of this chapter, the non-linear flutter EOM is solved directly by a Runge-Kutta fourth-order numerical integration scheme. The non-linear flutter dynamic response is then compared to an existing case in the literature. The latter comparison will shed light on the stunning difference between sinusoidal series based modes and the linear modes of a curved panel. Tools such as time domain response, phase plot, power spectrum density, and bifurcation diagrams associated with the non-linear curved panel flutter dynamic response, will be used to dissect the non-linear flutter response.

3.2 Preliminary Assumptions

The system non-linear flutter curved panel equations of motion described by equations (2.238) are

$$\begin{aligned}
 & \frac{I}{\omega_o^2} [M] \{\ddot{W}\} + \frac{G_a}{\omega_o} [G] \{\dot{W}\} + \left(\lambda \cos \Lambda [A_x] + \lambda \sin \Lambda [A_y] \right) \{W\} \\
 & + \left([K] + [K]^s + [K]^{\theta_o} \right) \{W\} \\
 & \left(+ \frac{1}{2} [N1] + \frac{1}{2} [N1]^{\theta_o} + \frac{1}{2} [N1]^{N_b} + \frac{1}{2} [N1]^{N_m} + \frac{1}{2} [N1]^{N_{\theta_o}} + \frac{1}{3} [N2] \right) \{W\} \\
 & = \{P(t)\} - \lambda \cos \Lambda \{P_{w_{o,x}}\} - \lambda \sin \Lambda \{P_{w_{o,y}}\}
 \end{aligned} \tag{3.1}$$

Equation (3.1) is a set of non-linear ordinary differential equations with respect to time t .

Considering the mechanical load vector null $\{P(t)\} = \{0\}$, the remaining static aerodynamic loads, $\{P_{w_{o,x}}\}$ and $\{P_{w_{o,y}}\}$ are constant time-independent vectors depending only upon the dynamic pressure λ . According to Gray, [68], the solution of such a system of ordinary differential equations with a static load term can be separated as the sum of a time-independent particular solution and a time-dependent homogenous solution. The total deflection can be written as

$$\{W\} = \{W\}_s + \{W\}_t \tag{3.2}$$

The homogenous solution characterizes a self-excited dynamic flutter oscillation $\{W\}_t$ about the static equilibrium position of $\{W\}_s$, while the particular solution characterizes an aerodynamic static equilibrium deflection $\{W\}_s$ under the SAL.

Equation (3.1) can be summarized as

$$\frac{1}{\omega_o^2} [M] \{\ddot{W}\} + \frac{g_a}{\omega_o} [G] \{\dot{W}\} + \left([A_a] + [K_L] + \frac{1}{2} [N1] + \frac{1}{3} [N2] \right) \{W\} = \{P_{sal}\} \quad (3.3)$$

where

$$[A_a] = \lambda \cos \Lambda [A_x] + \lambda \sin \Lambda [A_y] \quad (3.4)$$

$$[K_L] = [K] + [K]^s + [K]^{\theta_o} \quad (3.5)$$

$$[N1] = [N1] + [N1]^{\theta_o} + [N1]^{N_b} + [N1]^{N_m} + [N1]^{N_{\theta_o}} \quad (3.6)$$

$$[N2] = [N2] \quad (3.7)$$

$$\{P_{sal}\} = -\lambda \cos \Lambda \{P_{w_{o,x}}\} - \lambda \sin \Lambda \{P_{w_{o,y}}\} \quad (3.8)$$

Substituting Eq. (3.2) in (3.3) gives

$$\begin{aligned} & \frac{1}{\omega_o^2} [M] (\{\ddot{W}\}_s + \{\ddot{W}\}_t) + \frac{g_a}{\omega_o} [G] (\{\dot{W}\}_s + \{\dot{W}\}_t) \\ & + \left([A_a] + [K_L] + \frac{1}{2} [N1]_{s+t} + \frac{1}{3} [N2]_{s+t} \right) (\{W\}_s + \{W\}_t) = \{P_{sal}\} \end{aligned} \quad (3.9)$$

where the subscript $(s+t)$ denotes that the non-linear stiffness matrices $[N1]_{s+t}$, and $[N2]_{s+t}$ are evaluated with $\{W\}_s$ and $\{W\}_t$ simultaneously. Consequently, the aforementioned non-linear stiffness matrices must be evaluated at the element level then assembled to the system level.

The first-order stiffness matrix $[N1]_{s+t}$ is a matrix linearly dependent on the deflection $\{W\}$, see Eq. (3.2). The resultant matrix is a linear combination of the static and dynamic matrices as

$$[N1]_{s+t} = [N1]_s + [N1]_t \quad (3.10)$$

where the matrix $[N1]_s$ is evaluated with the static deflection $\{W\}_s$ and the matrix $[N1]_t$ is evaluated with the dynamic deflection $\{W\}_t$, respectively.

The second-order stiffness matrix $[N2]_{s+t}$ is a matrix depending on the square of the deflection $\{W\}$. Recalling the terms of the second-order stiffness matrix $[N2]$ at the element level as

$$[n2]_{bb} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \quad (3.11)$$

$$[n2]_{\psi b} = \frac{3}{2} \int_A [C_{\psi \psi}]^T [\theta]^T [A] [\theta] [C_{\psi b}] dA \quad (3.12)$$

$$[n2]_{b\psi} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T [A] [\theta] [C_{\psi \psi}] dA \quad (3.13)$$

$$[n2]_{\psi \psi} = \frac{3}{2} \int_A [C_{\psi \psi}]^T [\theta]^T [A] [\theta] [C_{\psi \psi}] dA \quad (3.14)$$

Based on Eq. (3.2) the slope matrix $[\theta]$ is a linear combination of a static slope matrix $[\theta_s]$ and a dynamic slope matrix $[\theta_t]$ written as

$$[\theta] = [\theta_s + \theta_t] \quad (3.15)$$

The above matrices in Eqs. (3.11) to (3.14) can be rewritten as

$$[n2]_{bb} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta_s + \theta_t]^T [A] [\theta_s + \theta_t] [C_{\psi b}] dA \quad (3.16)$$

$$[n2]_{\psi b} = \frac{3}{2} \int_A [C_{\psi \psi}]^T [\theta_s + \theta_t]^T [A] [\theta_s + \theta_t] [C_{\psi b}] dA \quad (3.17)$$

$$[n2]_{b\psi} = \frac{3}{2} \int_A [C_{\psi b}]^T [\theta_s + \theta_t]^T [A] [\theta_s + \theta_t] [C_{\psi \psi}] dA \quad (3.18)$$

$$[n2]_{\psi \psi} = \frac{3}{2} \int_A [C_{\psi \psi}]^T [\theta_s + \theta_t]^T [A] [\theta_s + \theta_t] [C_{\psi \psi}] dA \quad (3.19)$$

Performing the matrix operations under the integral terms produces the following expressions:

$$[n2_{bb}]_{s+t} = [n2_{bb}]_s + [n2_{bb}]_t + 2[n2_{bb}]_{st} \quad (3.20)$$

$$[n2_{\psi b}]_{s+t} = [n2_{\psi b}]_s + [n2_{\psi b}]_t + 2[n2_{\psi b}]_{st} \quad (3.21)$$

$$[n2_{b\psi}]_{s+t} = [n2_{b\psi}]_s + [n2_{b\psi}]_t + 2[n2_{b\psi}]_{st} \quad (3.22)$$

$$[n2_{\psi\psi}]_{s+t} = [n2_{\psi\psi}]_s + [n2_{\psi\psi}]_t + 2[n2_{\psi\psi}]_{st} \quad (3.23)$$

Assembling the element matrices to the system level, the system second-order matrix $[N2]_{s+t}$ can then be expressed as

$$[N2]_{s+t} = [N2]_s + [N2]_t + 2[N2]_{st} \quad (3.24)$$

3.3 Separation of the EOM into Static and Dynamic Equations

Substituting Eqs. (3.10) and (3.24) into Eq. (3.9) and noticing that the first and second derivative of the static aerodynamic deflection $\{\mathcal{W}\}_s$ are null, the equation of motion (3.9) can be separated into two equations.

First Equation:

$$\begin{aligned} & \frac{1}{\omega_o^2} [M] \{\ddot{\mathcal{W}}\}_t + \frac{g_a}{\omega_o} [G] \{\dot{\mathcal{W}}\}_t + \left([A_a] + [K_L] + \frac{1}{2} [N1]_t + \frac{1}{3} [N2]_t \right) \{\mathcal{W}\}_t \\ & + \left(\frac{1}{2} [N1]_t + \frac{1}{3} [N2]_t + \frac{2}{3} [N2]_{st} \right) \{\mathcal{W}\}_s + \left(\frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s + \frac{2}{3} [N2]_{st} \right) \{\mathcal{W}\}_t = \{0\} \end{aligned} \quad (3.25)$$

Second Equation:

$$\left([A_a] + [K_L] + \frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{\mathcal{W}\}_s = \{\mathcal{P}_{sal}\} \quad (3.26)$$

Equation (3.26) is a set of non-linear algebraic equations, which holds a particular solution $\{W_s\}$, for the system governing Eqs. (3.9). The solution $\{W_s\}$ of Eq. (3.26) is a static aerodynamic deflection determined under a specific static aerodynamic load $\{P_{sal}\}$ at a prescribed dynamic pressure λ and yaw flow angle Λ . The dynamic flutter equation, Eq. (3.25) will be discussed later (see Eq. (3.88)).

3.4 Solution Procedure for the Static Aerodynamic Equation

The Newton-Raphson iteration method is used to solve the non-linear algebraic system Eqs. (3.26). The aforementioned static system can be written as a function of the static aerodynamic deflection $\{W\}_s$ as

$$\Phi(\{W\}_s) = \{0\} \quad (3.27)$$

allowing Eq. (3.26) to be expressed as

$$\Phi(\{W\}_s) = \left([A_a] + [K_L] + \frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \{W\}_s - \{P_{sal}\} = \{0\} \quad (3.28)$$

The function $\Phi(\{W\}_s)$ can be developed as a Taylor series expansion, which gives

$$\Phi(\{W\}_s + \{\Delta W\}_s) = \Phi(\{W\}_s) + \frac{d\Phi(\{W\}_s)}{d\{W\}_s} \{\Delta W\}_s = \{0\} \quad (3.29)$$

Shifting the function $\Phi(\{W\}_s)$ to the right hand side, one has

$$\frac{d\Phi(\{W\}_s)}{d\{W\}_s} \{\Delta W\}_s = -\Phi(\{W\}_s) \quad (3.30)$$

The derivative of $\Phi(\{W\}_s)$ can be developed as

$$\frac{d\Phi(\{W\}_s)}{d\{W\}_s} = \frac{d}{d\{W\}_s} \left[\left([A_a] + [K_L] + \frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \{W\}_s \right] - \frac{d}{d\{W\}_s} \{P_{sal}\} \quad (3.31)$$

The linear stiffness matrix $[K_L]$, and the aerodynamic stiffness matrix $[A_a]$ are constant matrices, thus the derivative with respect to $\{W\}_s$ gives

$$\frac{d}{d\{W\}_s} [([A_a] + [K_L])\{W\}_s] = [A_a] + [K_L] \quad (3.32)$$

The non-linear stiffness matrices $[N1]_s$, and $[N2]_s$ are linearly and quadratically dependent on the static aerodynamic deflection $\{W\}_s$, respectively. Thus the derivative with respect to the static aerodynamic deflection is

$$\begin{aligned} & \frac{d}{d\{W\}_s} \left[\left(\frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \{W\}_s \right] \\ &= \left(\frac{d}{d\{W\}_s} \left(\frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \right) \{W\}_s + \left(\frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \end{aligned} \quad (3.33)$$

To proceed with the derivation of the first term of Eq. (3.33), one should look at the differential of each of the nine terms composing the matrices $[N1]_s$, and $[N2]_s$. Thus one has

$$(1,1) = d \left(\frac{1}{2}[N1]_{bb}^{\theta_o} + \frac{1}{2}[N1]_{bb}^{N_b} + \frac{1}{2}[N1]_{bb}^{N_m} + \frac{1}{2}[N1]_{bb}^{N_{\theta_o}} + \frac{1}{3}[N2]_{bb} \right) \{W_b\}_s \quad (3.34)$$

$$(1,2) = d \left(\frac{1}{2}[N1]_{b\psi}^{\theta^T} + \frac{1}{2}[N1]_{b\psi}^{\theta_o} + \frac{1}{2}[N1]_{b\psi}^{N_b} + \frac{1}{2}[N1]_{b\psi}^{N_m} + \frac{1}{2}[N1]_{b\psi}^{N_{\theta_o}} + \frac{1}{3}[N2]_{b\psi} \right) \{W_\psi\}_s \quad (3.35)$$

$$(1,3) = d \left(\frac{1}{2}[N1]_{bm}^{\theta^T} \right) \{W_m\}_s \quad (3.36)$$

$$(2,1) = d \left(\frac{1}{2}[N1]_{\psi b}^{\theta} + \frac{1}{2}[N1]_{\psi b}^{\theta_o} + \frac{1}{2}[N1]_{\psi b}^{N_b} + \frac{1}{2}[N1]_{\psi b}^{N_m} + \frac{1}{2}[N1]_{\psi b}^{N_{\theta_o}} + \frac{1}{3}[N2]_{\psi b} \right) \{W_b\}_s \quad (3.37)$$

$$(2,2) = d \left(\frac{1}{2}[N1]_{\psi\psi}^{\theta\theta^T} + \frac{1}{2}[N1]_{\psi\psi}^{\theta_o} + \frac{1}{2}[N1]_{\psi\psi}^{N_b} + \frac{1}{2}[N1]_{\psi\psi}^{N_m} + \frac{1}{2}[N1]_{\psi\psi}^{N_{\theta_o}} + \frac{1}{3}[N2]_{\psi\psi} \right) \{W_\psi\}_s \quad (3.38)$$

$$(2,3) = d\left(\frac{1}{2}[N1]_{\psi m}^{\theta^T}\right)_s \{W_m\}_s \quad (3.39)$$

$$(3,1) = d\left(\frac{1}{2}[N1]_{mb}^{\theta}\right)_s \{W_b\}_s \quad (3.40)$$

$$(3,2) = d\left(\frac{1}{2}[N]_{m\psi}^{\theta}\right)_s \{W_{\psi}\}_s \quad (3.41)$$

$$(3,3) = d([0])_s \{W_m\}_s \quad (3.42)$$

The differentiation of Eqs. (3.34) to (3.42) can be performed following a particular scheme of calculations. A sample of these lengthy calculations is given below for the first-order non-linear matrices. The second term of Eq. (3.34) $d([N1]_{bb}^{N_b})_s \{W_b\}_s$ and the third term of Eq. (3.35) $d([N1]_{b\psi}^{N_b})_s \{W_{\psi}\}_s$ can be transformed using Eq. (2.109) at the element level as

$$\frac{1}{2}d([n1]_{bb}^{N_b})_s \{w_b\}_s + \frac{1}{2}d([n1]_{b\psi}^{N_b})_s \{w_{\psi}\}_s = \frac{1}{2} \int_A [C_{\psi b}] d[N_b]_s [C_{\psi b}] dA \{w_b\}_s \quad (3.43)$$

$$+ \frac{1}{2} \int_A [C_{\psi b}] d[N_b]_s [C_{\psi\psi}] dA \{w_{\psi}\}_s \quad (3.44)$$

$$= \frac{1}{2} \int_A [C_{\psi b}] d[N_b]_s ([C_{\psi b}] \{w_b\}_s + [C_{\psi\psi}] \{w_{\psi}\}_s) dA$$

$$= \frac{1}{2} \int_A [C_{\psi b}]^T [\theta]_s^T [B] [C_b] dA \{dw_{\psi}\}_s$$

$$= \frac{1}{2} ([n1]_{b\psi}^{\theta^T})_s \{dw_{\psi}\}_s \quad (3.45)$$

On the other hand, the first term of Eq. (3.35) $d([n1]_{b\psi}^{\theta^T})_s \{W_{\psi}\}_s$ can be also transformed using Eq. (2.109) at the element level as

$$\begin{aligned}
\frac{1}{2} d \left([n1]_{b\psi}^{\theta^T} \right)_s \{w_\psi\}_s &= \frac{1}{2} \int_A [C_{\psi b}]^T d \left([\theta]^T \right)_s [B] [C_b] dA \{w_\psi\}_s \\
&= \frac{1}{2} \int_A [C_{\psi b}]^T [N_b]_s \{dG\}_s dA \\
&= \frac{1}{2} \int_A [C_{\psi b}] [N_b]_s [C_{\psi b}] dA \{dw_b\}_s + \frac{1}{2} \int_A [C_{\psi b}] [N_b]_s [C_{\psi\psi}] dA \{dw_\psi\}_s \\
&= \frac{1}{2} \left([n1]_{bb}^{N_b} \right)_s \{dw_b\}_s + \frac{1}{2} \left([n1]_{b\psi}^{N_b} \right)_s \{dw_\psi\}_s
\end{aligned} \tag{3.46}$$

The above calculations demonstrated that reciprocity or correspondence could be established between the first-order matrices of Eqs. (3.34) to (3.42). The above example demonstrated that doing the overall matrix differentiation leads to the establishment of the following correspondence

$$d \left([n1]_{X1} \right)_s \{w_{X2}\}_s \rightarrow \left([n1]_{X1} \right)_s \{dw_{X2}\}_s \tag{3.47}$$

where indices $X1$ are a combination of b , ψ , or m , and $X2$ is either b , ψ , or m .

For second-order non-linear matrices of Eqs. (3.34), (3.35), (3.37), and (3.38), a similar procedure is used. For example the sixth second-order term of Eq. (3.34),

$d \left(\frac{1}{3} [N2]_{bb} \right)_s \{W_b\}_s$ can be transformed at the element level as

$$\begin{aligned}
d \left([n2]_{bb} \right)_s \{w_b\}_s &= \frac{3}{2} \int_A [C_{\psi b}]^T d \left([\theta]^T [A] [\theta] \right)_s [C_{\psi b}] dA \{w_b\}_s \\
&= 2 \cdot \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T_s [A] d[\theta]_s [C_{\psi b}] dA \{w_b\}_s \\
&= 2 \cdot \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]^T_s [A] [\theta]_s [C_{\psi b}] dA \{dw_b\}_s \\
&= 2 \left([n2]_{bb} \right)_s \{dw_b\}_s
\end{aligned} \tag{3.48}$$

using the following identity

$$d\left([\theta]_s^T [A] [\theta]_s\right) = 2[\theta]_s^T [A] d[\theta]_s \quad (3.49)$$

For second-order non-linear matrices, the above-performed calculations could be summarized by the following general formula

$$d(n2_{XX})_s \{w_X\}_s = 2(n2_{XX}) d\{w_X\}_s \quad (3.50)$$

The same procedure can be used for the rest of the second-order terms of Eq. (3.35), (3.37), (3.38), leading its differential outcome being twice the matrix $[n2]$.

Expressing the above differentiated first-order and second-order matrices at the system level, respectively, leads to the first term of Eq. (3.33) to be expressed as

$$\left(\frac{d}{d\{W\}_s} \left(\frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \right) \{W\}_s = \frac{1}{2} [N1]_s + \frac{2}{3} [N2]_s \quad (3.51)$$

Replacing Eq. (3.51) in Eq. (3.33) leads to

$$\frac{d}{d\{W\}_s} \left[\left(\frac{1}{2} [N1]_s + \frac{1}{3} [N2]_s \right) \{W\}_s \right] = [N1]_s + [N2]_s \quad (3.52)$$

The static aerodynamic load vector $\{P_{sal}\}$ is independent of the static-aerodynamic deflection, thus its derivative is null.

$$\frac{d}{d\{W\}_s} \{P_{sal}\} = 0 \quad (3.53)$$

Putting together Eqs. (3.32), (3.52), and (3.53) into (3.31) leads to

$$\frac{d\Phi(\{W\}_s)}{d\{W\}_s} = [K_{tan}]_s = [A_a] + [K_L] + [N1]_s + [N2]_s \quad (3.54)$$

The static aerodynamic stiffness matrix resulting from the derivative term in the Taylor series expansion of Eq. (3.29) is called the tangent matrix $[K_{tan}]_s$.

3.4.1 Solution Procedure for the Static Aerodynamic Deflection $\{W\}_s$

In order to determine the static-aerodynamic large deflection $\{W\}_s$, the Newton-Raphson iteration method will be employed. Taking $[K_{tan}]_s$ into Eq. (3.30) gives

$$[K_{tan}]_s \{\Delta W\}_s = -\Phi(\{W\}_s) \quad (3.55)$$

Implementing $\Phi(\{W\}_s)$ of Eq. (3.28) into Eq. (3.55) leads to the following scheme

$$[K_{tan}]_s \{\Delta W\}_s = \{P_{sal}\} - \left([A_a] + [K_L] + \frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \{W\}_s \quad (3.56)$$

Equation (3.56) can be rewritten as

$$[K_{tan}]_s \{\Delta W\}_s = \{\Delta P_{sal}\} \quad (3.57)$$

Where the unbalanced force vector is

$$\{\Delta P_{sal}\} = \{P_{sal}\} - \left([A_a] + [K_L] + \frac{1}{2}[N1]_s + \frac{1}{3}[N2]_s \right) \{W\}_s \quad (3.58)$$

The system Eqs. (3.57) is ready to be solved for the incremental static aerodynamic deflection $\{\Delta W_s\}$ using the Newton-Raphson iterative procedure. Considering the i^{th} iteration, the system Eq. (3.57) can be written in an incremental form as

$$[K_{tan}]_s^i \{\Delta W\}_s^{i+1} = \{\Delta P_{sal}\}_s^i \quad (3.59)$$

where

$$\{\Delta P_{sal}\}_s^i = \{P_{sal}\} - \left([A_a] + [K_L] + \frac{1}{2}[N1]_s^i + \frac{1}{3}[N2]_s^i \right) \{W\}_s^i \quad (3.60)$$

The superscript i denotes that the non-linear stiffness $[N1]_s^i$ and $[N2]_s^i$ are evaluated with the static aerodynamic deflection $\{W\}_s^i$ determined at the i^{th} iteration step. The updated static aerodynamic deflection $\{W\}_s^{i+1}$ can be determined from the incremental vector

$\{\Delta W\}_s^{i+1}$, and the updated static aerodynamic deflection $\{W\}_s^i$ resulting from the previous iteration i . This strategy leads to

$$\{W\}_s^{i+1} = \{W\}_s^i + \{\Delta W\}_s^{i+1} \quad (3.61)$$

The aim of the Newton-Raphson iteration method is to decrease gradually the unbalanced vector $\{\Delta P_{sal}\}_s^i$ in order to decrease the incremental static deflection to a minimal value accounting for the tangent matrix $[K_{tan}]_s$. At the convergent deflection $\{W\}_s$, $[K_{tan}(\{W\}_s)]$ is ultimately the tangent (slope) defined by

$$K_{tan}(\{W\}_s) = \frac{d(\{P_{sal}\})}{d\{W\}_s} \quad (3.62)$$

An initial guess vector accounting for the initial starting deflection is needed to start the Newton-Raphson iteration method. The initial deflection was chosen to be the first natural mode shape of the curved panel tempered by a constant $C = 0.1 \times h$, one has

$$\{W\}_s^{initial} = C \times \{1^{st} \text{ mode-shape}\} \quad (3.63)$$

3.4.2 Solution Procedure for the Dynamic Small Deflection $\{W\}_t$

The non-linear flutter behavior of the curved panel can be determined by solving the non-linear ordinary differential Eq. (3.25) for $\{W\}_t$. In order to obtain Eq. (3.25) in terms of $\{W\}_t$, only the fourth term in the equation needs to be rearranged.

$$\left(\frac{1}{2}[N1]_t + \frac{1}{3}[N2]_t + \frac{2}{3}[N2]_{st} \right) \{W\}_s = \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{bmatrix} \begin{Bmatrix} \{W_b\}_s \\ \{W_\psi\}_s \\ \{W_m\}_s \end{Bmatrix} \quad (3.64)$$

The nine constituting terms of the matrices composing Eq. (3.64) are

$$(1,1)\{W_b\}_s = \left(\frac{1}{2}[N1]_{bb}^{\theta_o} + \frac{1}{2}[N1]_{bb}^{N_b} + \frac{1}{2}[N1]_{bb}^{N_m} + \frac{1}{2}[N1]_{bb}^{N_{\theta_o}} + \frac{1}{3}[N2]_{bb} \right)_t \{W_b\}_s + \left(\frac{2}{3}[N2]_{bb} \right)_{st} \{W_b\}_s \quad (3.65)$$

$$(1,2)\{W_\psi\}_s = \left(\frac{1}{2}[N1]_{b\psi}^{\theta^T} + \frac{1}{2}[N1]_{b\psi}^{\theta_o} + \frac{1}{2}[N1]_{b\psi}^{N_b} + \frac{1}{2}[N1]_{b\psi}^{N_m} + \frac{1}{2}[N1]_{b\psi}^{N_{\theta_o}} + \frac{1}{3}[N2]_{b\psi} \right)_t \{W_\psi\}_s + \left(\frac{2}{3}[N2]_{b\psi} \right)_{st} \{W_\psi\}_s \quad (3.66)$$

$$(1,3)\{W_m\}_s = \left(\frac{1}{2}[N1]_{bm}^{\theta^T} \right)_t \{W_m\}_s \quad (3.67)$$

$$(2,1)\{W_b\}_s = \left(\frac{1}{2}[N1]_{\psi b}^{\theta} + \frac{1}{2}[N1]_{\psi b}^{\theta_o} + \frac{1}{2}[N1]_{\psi b}^{N_b} + \frac{1}{2}[N1]_{\psi b}^{N_m} + \frac{1}{2}[N1]_{\psi b}^{N_{\theta_o}} + \frac{1}{3}[N2]_{\psi b} \right)_t \{W_b\}_s + \left(\frac{2}{3}[N2]_{\psi b} \right)_{st} \{W_b\}_s \quad (3.68)$$

$$(2,2)\{W_\psi\}_s = \left(\frac{1}{2}[N1]_{\psi\psi}^{\theta\theta^T} + \frac{1}{2}[N1]_{\psi\psi}^{\theta_o} + \frac{1}{2}[N1]_{\psi\psi}^{N_b} + \frac{1}{2}[N1]_{\psi\psi}^{N_m} + \frac{1}{2}[N1]_{\psi\psi}^{N_{\theta_o}} + \frac{1}{3}[N2]_{\psi\psi} \right)_t \{W_\psi\}_s + \left(\frac{2}{3}[N2]_{\psi\psi} \right)_{st} \{W_\psi\}_s \quad (3.69)$$

$$(2,3)\{W_m\}_s = \left(\frac{1}{2}[N1]_{\psi m}^{\theta^T} \right)_t \{W_m\}_s \quad (3.70)$$

$$(3,1)\{W_b\}_s = \left(\frac{1}{2}[N1]_{mb}^{\theta} \right)_t \{W_b\}_s \quad (3.71)$$

$$(3,2)\{W_\psi\}_s = \left(\frac{1}{2}[N1]_{m\psi}^{\theta} \right)_t \{W_\psi\}_s \quad (3.72)$$

$$(3,3)\{W_m\}_s = ([0])_t \{W_m\}_s \quad (3.73)$$

Indices s and t can be rearranged according to the below described scheme of Eqs. (3.74)

to (3.80). Taking for example the second term of Eq. (3.65) $\left([N1]_{bb}^{N_b} \right)_t \{W_b\}_s$ and the third

term of Eq. (3.66) $\left([N1]_{b\psi}^{N_b}\right)_t \{W_\psi\}_s$, they can be transformed using Eq. (2.108) at the element level as

$$\frac{1}{2} \left([n1]_{bb}^{N_b}\right)_t \{w_b\}_s + \frac{1}{2} \left([n1]_{b\psi}^{N_b}\right)_t \{w_\psi\}_s = \frac{1}{2} \int_A [C_{\psi b}] [N_b]_t [C_{\psi b}] dA \{w_b\}_s \quad (3.74)$$

$$+ \frac{1}{2} \int_A [C_{\psi b}] [N_b]_t [C_{\psi\psi}] dA \{w_\psi\}_s \quad (3.75)$$

$$= \frac{1}{2} \int_A [C_{\psi b}] [N_b]_t ([C_{\psi b}] \{w_b\}_s + [C_{\psi\psi}] \{w_\psi\}_s) dA \quad (3.76)$$

$$= \frac{1}{2} \int_A [C_{\psi b}]^T [N_b]_t \{G\}_s dA \quad (3.77)$$

$$= \frac{1}{2} \int_A [C_{\psi b}]^T [\theta]_s^T \{N_b\}_t dA \quad (3.78)$$

$$= \frac{1}{2} \int_A [C_{\psi b}]^T [\theta]_s^T [B] [C_b] dA \{w_\psi\}_t \quad (3.79)$$

$$= \frac{1}{2} \left([n1]_{b\psi}^{\theta^T}\right)_s \{w_\psi\}_t \quad (3.80)$$

Following the same type of scheme for the first-order matrices with index t from Eq. (3.65) to Eq. (3.73), it can be concluded for the assembled curved panel system that

$$[N1]_t \{W\}_s = [N1]_s \{W\}_t \quad (3.81)$$

For second-order matrices, the matrix indices s and t can be rearranged according to the below scheme. For example rewriting the sixth second-order matrix of Eq. (3.65), $\left([N2]_{bb}\right)_t \{W_b\}_s$, and the seventh second order matrix of Eq. (3.66), $\left([N2]_{b\psi}\right)_t \{W_\psi\}_s$, they can be transformed at the element level as

$$\left([n2]_{bb}\right)_t \{w_b\}_s + \left([n2]_{b\psi}\right)_t \{w_\psi\}_s = \frac{3}{2} \int_A [C_{\psi b}]^T ([\theta]^T [A] [\theta])_t ([C_{\psi b}] \{w_b\}_s + [C_{\psi\psi}] \{w_\psi\}_s) dA$$

$$= \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]_t^T [A] [\theta]_t \{G\}_s dA \quad (3.82)$$

noticing that the matrix product $[\theta]_t \{G\}_s$ can be written as

$$[\theta]_t \{G\}_s = \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix}_t \left\{ \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} \end{bmatrix}_s \right\} = \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix}_s \left\{ \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} \end{bmatrix}_t \right\} = [\theta]_s \{G\}_t \quad (3.83)$$

Expression (3.82) can be rewritten as

$$\begin{aligned} ([n2]_{bb})_t \{w_b\}_s + ([n2]_{b\psi})_t \{w_\psi\}_s &= \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]_t^T [A] [\theta]_s \{G\}_t dA \\ &= \frac{3}{2} \int_A [C_{\psi b}]^T [\theta]_t^T [A] [\theta]_s ([C_{\psi b}] \{w_b\}_t + [C_{\psi\psi}] \{w_\psi\}_t) dA \\ &= ([n2]_{bb})_{st} \{w_b\}_t + ([n2]_{b\psi})_{st} \{w_\psi\}_t \end{aligned} \quad (3.84)$$

Applying the same type of scheme to all second-order matrices with indices t and st , the second-order stiffness matrices can be rearranged at the system level as

$$[N2]_t \{W\}_s = [N2]_{st} \{W\}_t \quad (3.85)$$

$$[N2]_{st} \{W\}_s = [N2]_s \{W\}_t \quad (3.86)$$

Implementing Eqs. (3.81), (3.85), and substituting Eq. (3.86) into Eq. (3.64) yields

$$\left(\frac{1}{2} [N1]_t + \frac{1}{3} [N2]_t + \frac{2}{3} [N2]_{st} \right) \{W\}_s = \left(\frac{1}{2} [N1]_s + \frac{1}{3} [N2]_{st} + \frac{2}{3} [N2]_s \right) \{W\}_t \quad (3.87)$$

Implementing Eq. (3.87) in the curved panel system equation of motion (3.25) leads to

$$\begin{aligned} \frac{1}{\omega_o^2} [M] \{\ddot{W}\}_t + \frac{g_a}{\omega_o} [G] \{\dot{W}\}_t + \left([A_a] + [K_L] + \frac{1}{2} [N1]_t + \frac{1}{3} [N2]_t \right) \{W\}_t \\ + ([N1]_s + [N2]_s + [N2]_{st}) \{W\}_t = \{0\} \end{aligned} \quad (3.88)$$

Examining Eq. (3.88), the first three terms describe the flutter of flat plates ($\{W_s\}=0$) and the last term implies that the flutter motion is about the aerostatic equilibrium position under the SAL. The self-excited panel flutter behavior is indicated by the null force vector. This exact formulation for curved panels at supersonic flow is derived the first time in the literature. Critical dynamic pressure can be predicted using type 1 analysis [10], that is the application of linear structural theory to (3.88) once the converged aerostatic equilibrium position has been determined from Eqs. (3.59) and (3.61). The linear response of the panel can be deduced from Eq. (3.88) with the assumption of $\{W\}_t \ll 1$ by dropping the terms of second order of $\{W\}_t$ and higher. The linearized equation of motion (3.88) can be written as

$$\frac{1}{\omega_o^2} [M] \{\ddot{W}\}_t + \frac{g_a}{\omega_o} [G] \{\dot{W}\}_t + ([A_a] + [K_L] + [N1]_s + [N2]_s) \{W\}_t = \{0\} \quad (3.89)$$

Introducing Eq. (3.54) into Eq. (3.89) leads to the final form of the linearized dynamic equation of motion of the curved panel

$$\frac{I}{\omega_o^2} [M] \{\ddot{W}\}_t + \frac{g_a}{\omega_o} [G] \{\dot{W}\}_t + [K_{tan}]_s \{W\}_t = \{0\} \quad (3.90)$$

The flutter system equations of motion (3.90) is composed of constant matrices; consequently, it represents a linear self-excited dynamic system oscillating about the static aerodynamic deflection $\{W\}_s$ of the curved panel under a prescribed dynamic pressure λ , a flow yawing angle A , and a static aerodynamic load $\{P_{sal}\}$. The matrices $[N1]_s$ and $[N2]_s$ account for the added stiffness related to the static deflection $\{W\}_s$ under the aforementioned loads $\{P_{sal}\}$ and λ . In the next paragraph Eq. (3.90) will be

rearranged to fit into a consistent eigen-value problem. The required consistency translates in meeting the critical condition

$$[M_b] = [G] \quad (3.91)$$

developed in Eqs. (2.224) to (2.227). To this end Eq. (3.90) should be written only as a function of the bending and rotational displacement vectors $\begin{Bmatrix} W_b \\ W_\psi \end{Bmatrix}$. The in-plane-inertia

term

$$[M_m] \{\ddot{W}_m\} \quad (3.92)$$

is neglected. Solution of the eigen-value problem leads to the determination of the flutter onset of the deflected curved panel under the static aerodynamic load $\{P_{sal}\}$ and a defined critical dynamic pressure λ .

3.5 Preliminary Process for Eigen-Solutions

In order to establish a consistent eigen-values problem to solve the flutter onset, it is necessary to transform the equations of motion (3.90) in functions of the first two global nodal displacements, bending and rotational vectors $\{W_b\}$ and $\{W_\psi\}$.

3.5.1 Equations of Motion Expressed in $\{W^b\}$

For the handiness of the upcoming calculation the global-nodal displacement vector

$$\{W\} = \begin{Bmatrix} W_b \\ W_\psi \\ W_m \end{Bmatrix} \quad (3.93)$$

is subdivided into two global-nodal displacement vectors

$$\{W\} = \begin{Bmatrix} W^b \\ W^m \end{Bmatrix} \quad (3.94)$$

where

$$\{W^b\} = \begin{Bmatrix} W_b \\ W_\psi \end{Bmatrix} \quad (3.95)$$

and

$$\{W^m\} = \{W_m\} \quad (3.96)$$

Neglecting the in-plane inertia term (3.92) in Eq. (3.90), the structure node DOF system equations of motion for a curved panel can be rewritten in reduced form as

$$\begin{aligned} & \frac{1}{\omega_o^2} \begin{bmatrix} M_b & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{W}^b \\ \ddot{W}^m \end{Bmatrix} \\ & + \frac{g_a}{\omega_o} \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{W}^b \\ \dot{W}^m \end{Bmatrix} \\ & + \left(\begin{bmatrix} A_a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_b & K_B \\ K_B^T & K_m \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_b^{\theta_o} & K_B^{\theta_o} \\ K_B^{\theta_o^T} & 0 \end{bmatrix} \right. \\ & + \begin{bmatrix} N1_b & N1_{bm} \\ N1_{mb} & 0 \end{bmatrix}_s + \begin{bmatrix} N1_b^{\theta_o} & 0 \\ 0 & 0 \end{bmatrix}_s + \begin{bmatrix} N1^{N_b} & 0 \\ 0 & 0 \end{bmatrix}_s + \begin{bmatrix} N1^{N_m} & 0 \\ 0 & 0 \end{bmatrix}_s + \begin{bmatrix} N1^{N_{\theta_o}} & 0 \\ 0 & 0 \end{bmatrix}_s \\ & \left. + \begin{bmatrix} N2_b & 0 \\ 0 & 0 \end{bmatrix}_s \right) \begin{Bmatrix} W^b \\ W^m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.97) \end{aligned}$$

A detailed definition of the terms of the reduced Eq. (3.97) can be found in Appendix A.

Equation (3.97) can be expanded and split into two equations. The first equation can be written as

$$\frac{1}{\omega_o^2} [M_b] \{\ddot{W}^b\} + \frac{g_a}{\omega_o} [G] \{\dot{W}^b\} + [A_a] \{W^b\}$$

$$\begin{aligned}
& + [K_b] \{W^b\} + [K_B] \{W^m\} + [K_s] \{W^b\} + [K_b^{\theta_o}] \{W^b\} + [K_B^{\theta_o}] \{W^m\} \\
& + [N1_b]_s \{W^b\} + [N1_{bm}]_s \{W^m\} + [N1_b^{\theta_o}]_s \{W^b\} + [N1^{N_b}]_s \{W^b\} + [N1^{N_m}]_s \{W^b\} + [N1^{N_{\theta_o}}]_s \{W^b\} \\
& + [N2_b]_s \{W^b\} = \{0\}
\end{aligned} \tag{3.98}$$

The second equation can be written as

$$[K_B^T] \{W^b\} + [K_m] \{W^m\} + [K_B^{\theta_o^T}] \{W^b\} + [N1_{mb}]_s \{W^b\} = \{0\} \tag{3.99}$$

Expressing the in-plane displacement vector $\{W^m\}$ functions of the bending and rotational vector $\{W^b\}$ gives

$$\{W^m\} = -[K_m]^{-1} [K_B^T] \{W^b\} - [K_m]^{-1} [K_B^{\theta_o^T}] \{W^b\} - [K_m]^{-1} [N1_{mb}]_s \{W^b\} \tag{3.100}$$

Substituting the in-plane displacement vector $\{W^m\}$, of Eq. (3.100) into Eq. (3.99) gives

$$\begin{aligned}
& \frac{1}{\omega_o^2} [M_b] \{\ddot{W}^b\} + \frac{\mathcal{G}_a}{\omega_o} [G] \{\dot{W}^b\} \\
& + [A_a] \{W^b\} \\
& + [K_b] \{W^b\} \\
& - [K_B] [K_m]^{-1} [K_B^T] \{W^b\} \\
& - [K_B] [K_m]^{-1} [K_B^{\theta_o^T}] \{W^b\} \\
& - [K_B] [K_m]^{-1} [N1_{mb}]_s \{W^b\} \\
& + [K_s] \{W^b\} \\
& + [K_b^{\theta_o}] \{W^b\} \\
& - [K_B^{\theta_o}] [K_m]^{-1} [K_B^T] \{W^b\}
\end{aligned}$$

$$\begin{aligned}
& -[K_B^{\theta_o}][K_m]^{-1}[K_B^{\theta_o^T}]\{W^b\} \\
& -[K_B^{\theta_o}][K_m]^{-1}[N1_{mb}]_s\{W^b\} \\
& +[N1_b]_s\{W^b\} \\
& -[N1_{bm}]_s[K_m]^{-1}[K_B^T]\{W^b\} \\
& -[N1_{bm}]_s[K_m]^{-1}[K_B^{\theta_o^T}]\{W^b\} \\
& -[N1_{bm}]_s[K_m]^{-1}[N1_{mb}]_s\{W^b\} \\
& +[N1_b^{\theta_o}]_s\{W^b\} \\
& +[N1^{N_b}]_s\{W^b\} \\
& +[N1^{N_m}]_s\{W^b\} \\
& +[N1^{N_{\theta_o}}]_s\{W^b\} \\
& +[N2_b]_s\{W^b\} = \{0\}
\end{aligned} \tag{3.101}$$

In the transformed Eq. (3.101), one can notice that the three stiffness terms involving the in-plane displacement vector $\{W^m\}$ in Eqs. (3.98) were substituted by nine composed stiffness terms involving the displacement vector $\{W^b\}$ in Eq. (3.101). Four terms out of nine are composed linear stiffness, three of them include terms related to the curved panel geometry, and they are

$$-[K_B][K_m]^{-1}[K_B^T] \tag{3.102}$$

$$-[K_B^{\theta_o}][K_m]^{-1}[K_B^{\theta_o^T}] \tag{3.103}$$

$$-[K_B][K_m]^{-1}[K_B^{\theta_o^T}] \tag{3.104}$$

$$- [K_B^{\theta_o}] [K_m]^{-1} [K_B^T] \quad (3.105)$$

terms (3.104) and (3.105) are mutually transposed.

The transformation also included four known composed non-linear first-order stiffness terms depending on the static deflection $\{W\}_s$, and they are

$$- [N1_{bm}]_s [K_m]^{-1} [K_B^T] \quad (3.106)$$

$$- [K_B] [K_m]^{-1} [N1_{mb}]_s \quad (3.107)$$

$$- [K_B^{\theta_o}] [K_m]^{-1} [N1_{mb}]_s \quad (3.108)$$

$$- [N1_{bm}]_s [K_m]^{-1} [K_B^{\theta_o^T}] \quad (3.109)$$

two known composed non-linear stiffness terms are related to the curved panel geometry.

Terms (3.106), (3.107) and (3.108), (3.109) are mutually transposed. The transformation also introduced a single known non-linear composed second-order stiffness term depending on the static deflection $\{W\}_s$, and it is:

$$- [N1_{bm}]_s [K_m]^{-1} [N1_{mb}]_s \quad (3.110)$$

After splitting Eq. (3.97), and neglecting the in-plane inertia term (3.92), the system equations of motion are finally expressed in terms of the bending and rotational displacement vector $\{W^b\}$ in reduced form as

$$\frac{1}{\omega_o^2} [M]^b \{\ddot{W}^b\} + \frac{g_a}{\omega_o} [G]^b \{\dot{W}^b\} + ([A_a]^b + [K_L]^b + [N1]_s^b + [N2]_s^b) \{W^b\} = \{0\} \quad (3.111)$$

where the aerodynamic stiffness matrices are given by

$$[M]^b = [M_b] \quad (3.112)$$

$$[G]^b = [G] \quad (3.113)$$

$$[A_a]^b = \lambda \cos \lambda [A_x] + \lambda \sin \lambda [A_y] \quad (3.114)$$

the linear stiffness matrices are given by

$$[K_L]^b = [K_b] + [K_s] + [K_b^{\theta_o}] - [K_B][K_m]^{-1}[K_B^T] - [K_B][K_m]^{-1}[K_B^{\theta_o}] - [K_B^{\theta_o}][K_m]^{-1}[K_B^T] - [K_B^{\theta_o}][K_m]^{-1}[K_B^{\theta_o}] \quad (3.115)$$

and the non-linear stiffness known matrices depending on the static deflection $\{W\}_s$ are given by

$$[N1]_s^b = [N1_b]_s + [N1_b^{\theta_o}]_s + [N1^{N_b}]_s + [N1^{N_m}]_s + [N1^{N_{\theta_o}}]_s - [N1_{bm}]_s [K_m]^{-1} [K_B^T] - [K_B][K_m]^{-1} [N1_{mb}]_s - [K_B^{\theta_o}][K_m]^{-1} [N1_{mb}]_s - [N1_{bm}]_s [K_m]^{-1} [K_B^{\theta_o}] \quad (3.116)$$

$$[N2]_s^b = [N2_b]_s - [N1_{bm}]_s [K_m]^{-1} [N1_{mb}]_s \quad (3.117)$$

One can define a new tangent matrix after neglecting the in-plane inertia term as

$$[K_{tan}]_s^b = [A_a]^b + [K_L]^b + [N1]_s^b + [N2]_s^b \quad (3.118)$$

Equation (3.111) can be rewritten as:

$$\frac{1}{\omega_o^2} [M]^b \{\ddot{W}^b\} + \frac{g_a}{\omega_o} [G]^b \{\dot{W}^b\} + [K_{tan}]_s^b \{W^b\} = \{0\} \quad (3.119)$$

3.5.2 Eigen-Analysis

Equation (3.119) can be formulated as an eigen-value problem. Accounting for Eq. (3.91) it can be written as

$$\kappa [M]^b \{\phi_b\} = [K_{tan}]_s^b \{\phi_b\} \quad (3.120)$$

where

$$\kappa = -\left(\frac{\Omega}{\omega_o}\right)^2 - g_a \left(\frac{\Omega}{\omega_o}\right) \quad (3.121)$$

is the non-dimensional eigen-value of the eigen-system (3.120). The parameter Ω is defined as:

$$\Omega = \alpha + i\omega \quad (3.122)$$

where α is the panel damping rate, and ω is the oscillating frequency. The eigen-system of Eq. (3.120) is used to determine the flutter dynamic pressure λ_{cr} of the gradually deformed curved system under the dynamic pressure λ . In principle, flutter occurs when the damping rate starts to become positive.

3.6 Non-Linear Post-Flutter Panel Response

To investigate the non-linear post-flutter time response, a straightforward and efficient approach is to solve directly the system Eq. (3.3) by transferring it into modal coordinates. This modal method is considered in the present paragraph.

3.6.1 Neglecting the In-plane Inertia Term in the EOM

Neglecting the in-plane inertia term in Eq. (2.237) due to in-plane high frequency [69], and implementing the transformed nodal vectors of Eqs. (3.94) to (3.96) into the structure node DOF system equation of motion (3.3), one has the following reduced form equation

$$\frac{1}{\omega_o^2} \begin{bmatrix} M_b & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{W}^b \\ \ddot{W}^m \end{Bmatrix}$$

$$\begin{aligned}
& + \frac{g_a}{\omega_o} \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{W}^b \\ \dot{W}^m \end{Bmatrix} \\
& + \left(\begin{bmatrix} A_a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_b & K_B \\ K_B^T & K_m \end{bmatrix} + \begin{bmatrix} K_s & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_b^{\theta_o} & K_B^{\theta_o} \\ K_B^{\theta_o^T} & 0 \end{bmatrix} \right. \\
& + \frac{1}{2} \begin{bmatrix} N1_b & N1_{bm} \\ N1_{mb} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} N1_b^{\theta_o} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} N1^{N_b} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} N1^{N_m} & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} N1^{N_{\theta_o}} & 0 \\ 0 & 0 \end{bmatrix} \\
& \left. + \frac{1}{3} \begin{bmatrix} N2_b & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} W^b \\ W^m \end{Bmatrix} = \begin{Bmatrix} P_{sal}^b \\ 0 \end{Bmatrix} \tag{3.123}
\end{aligned}$$

where

$$\{P_{sal}\} = \begin{Bmatrix} P_{sal}^b \\ 0 \end{Bmatrix} \tag{3.124}$$

The expanded matrices forms of the reduced matrices of Eq. (3.123) are found in Appendix A. Equation (3.123) can be expanded and split into two equations. The first equation can be written as:

$$\begin{aligned}
& \frac{1}{\omega_o^2} [M_b] \{\ddot{W}^b\} \\
& + \frac{g_a}{\omega_o} [G] \{\dot{W}^b\} \\
& + [A_a] \{W^b\} \\
& + [K_b] \{W^b\} + [K_B] \{W^m\} + [K_s] \{W^b\} + [K_b^{\theta_o}] \{W^b\} + [K_B^{\theta_o}] \{W^m\} \\
& + \frac{1}{2} [N1_b] \{W^b\} + \frac{1}{2} [N1_{bm}] \{W^m\} + \frac{1}{2} [N1_b^{\theta_o}] \{W^b\} + \frac{1}{2} [N1^{N_b}] \{W^b\} + \frac{1}{2} [N1^{N_m}] \{W^b\} \\
& + \frac{1}{2} [N1^{N_{\theta_o}}] \{W^b\} + \frac{1}{3} [K2_b] \{W^b\} = \{P_{sal}^b\} \tag{3.125}
\end{aligned}$$

The second equation can be written as

$$[K_m] \{W^m\} + [K_B^T] \{W^b\} + [K_B^{\theta_o^T}] \{W^b\} + \frac{1}{2} [N1_{mb}] \{W^b\} = \{0\} \quad (3.126)$$

Expressing the in-plane displacement vector $\{W^m\}$ in functions of the bending and rotation vector $\{W^b\}$ gives

$$\{W^m\} = -[K_m]^{-1} [K_B^T] \{W^b\} - [K_m]^{-1} [K_B^{\theta_o^T}] \{W^b\} - \frac{1}{2} [K_m]^{-1} [N1_{mb}] \{W^b\} \quad (3.127)$$

Substituting the in-plane displacement vector $\{W^m\}$ of Eq. (3.127) into the expanded Eq. (3.125) gives

$$\begin{aligned} & \frac{1}{\omega_o^2} [M_b] \{\ddot{W}^b\} + \frac{g_a}{\omega_o} [G] \{\dot{W}^b\} \\ & + [A_a] \{W^b\} \\ & + [K_b] \{W^b\} \\ & + [K_s] \{W^b\} \\ & - [K_B] [K_m]^{-1} [K_B^T] \{W^b\} \\ & + [K_b^{\theta_o}] \{W^b\} \\ & - [K_B^{\theta_o}] [K_m]^{-1} [K_B^{\theta_o^T}] \{W^b\} \\ & - [K_B] [K_m]^{-1} [K_B^{\theta_o^T}] \{W^b\} \\ & - [K_B^{\theta_o}] [K_m]^{-1} [K_B^T] \{W^b\} \\ & + \frac{1}{2} [N1_b] \{W^b\} \\ & - \frac{1}{2} [N1_{bm}] [K_m]^{-1} [K_B^T] \{W^b\} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}[K_B][K_m]^{-1}[N1_{mb}] \{W^b\} \\
& +\frac{1}{2}[N1_b^{\theta_o}] \{W^b\} \\
& -\frac{1}{2}[K_B^{\theta_o}][K_m]^{-1}[N1_{mb}] \{W^b\} \\
& -\frac{1}{2}[N1_{bm}][K_m]^{-1}[K_B^{\theta_o^T}] \{W^b\} \\
& +\frac{1}{2}[N1^{N_b}] \{W^b\} \\
& +\frac{1}{2}[N1^{N_m}] \{W^b\} \\
& +\frac{1}{2}[N1^{N_{\theta_o}}] \{W^b\} \\
& -\frac{1}{4}[N1_{bm}][K_m]^{-1}[N1_{mb}] \{W^b\} \\
& +\frac{1}{3}[K2_b] \{W^b\} = \{P_{sal}^b\}
\end{aligned} \tag{3.128}$$

The system equations of motion (3.128) are then expressed in terms of the bending /rotational displacement vector $\{W^b\}$ as

$$\frac{1}{\omega_o^2}[M]^b \{\ddot{W}^b\} + \frac{g_a}{\omega_o}[G]^b \{\dot{W}^b\} + ([A_a]^b + [K_L]^b + [K1]^b + [K2]^b) \{W^b\} = \{P_{sal}^b\} \tag{3.129}$$

where $[M]^b$, $[G]^b$, $[A_a]^b$, and $[K_L]^b$ are already detailed in Eqs. (3.112), (3.113), (3.114), and (3.115), respectively.

The non-linear stiffness matrices are given by

$$[K1]^b = \frac{1}{2}[N1_b] + \frac{1}{2}[N1_b^{\theta_o}] + \frac{1}{2}[N1^{N_b}] + \frac{1}{2}[N1^{N_m}] + \frac{1}{2}[N1^{N_{\theta_o}}]$$

$$\begin{aligned}
& -\frac{1}{2}[N1_{bm}][K_m]^{-1}[K_B^T] - \frac{1}{2}[K_B][K_m]^{-1}[N1_{mb}] - \frac{1}{2}[K_B^{\theta_o}][K_m]^{-1}[N1_{mb}] \\
& -\frac{1}{2}[N1_{bm}][K_m]^{-1}[K_B^{\theta_o^T}]
\end{aligned} \tag{3.130}$$

$$[K2]^b = +\frac{1}{3}[N2_b] - \frac{1}{4}[N1_{bm}][K_m]^{-1}[N1_{mb}] \tag{3.131}$$

and the load vector is given by

$$\{P_{sal}^b\} = \{P_{sal}^b\} \tag{3.132}$$

3.6.2 Equations in Modal Coordinates

Assuming that the panel nodal displacements $\{W^b\}$ can be expressed as a linear combination of a selected set of natural modes and associated modal coordinates

$$\{W^b\} = \sum_{r=1}^n q_r(t) \{\phi_b\}^{(r)} = [\Phi] \{q\} \tag{3.133}$$

where the number n of a selected set of in-vacuo natural modes is much smaller than the number of structural node DOF. The selected normal modes $\{\phi_b\}^{(r)}$ are obtained from the linear eigen-value problem as

$$\left(\frac{\omega_r^2}{\omega_o^2}\right)[M]^b \{\phi_b\}^{(r)} = [K_L]^b \{\phi_b\}^{(r)} \tag{3.134}$$

In the expression of the non-linear matrix $[K1]^b$ and $[K2]^b$, the matrices

$$[N1_b] \tag{3.135}$$

$$[N1_b^{\theta_o}] \tag{3.136}$$

$$[N1^{N_b}] \tag{3.137}$$

$$[N1^{N_{\theta_o}}] \quad (3.138)$$

$$[N1_{bm}][K_m]^{-1}[K_B^T] \quad (3.139)$$

$$[K_B][K_m]^{-1}[N1_{mb}] \quad (3.140)$$

$$[K_B^{\theta_o}][K_m]^{-1}[N1_{mb}] \quad (3.141)$$

$$[N1_{bm}][K_m]^{-1}[K_B^{\theta_o^T}] \quad (3.142)$$

$$[N1_{bm}][K_m]^{-1}[N1_{mb}] \quad (3.143)$$

$$[N2_b] \quad (3.144)$$

are all functions of the unknown bending/rotational DOF $\{W^b\}$. They can be evaluated with the bending/rotational natural modes $\{\phi_b\}^{(r)}$ according to expression (3.133). Consequently, implementing Eq. (3.133) into Eqs. (3.135) to (3.144), the non-linear first-order and second-order matrices can be expressed as a sum of products between modal coordinates and non-linear modal stiffness as

$$[N1_b] = \sum_{r=1}^n q_r [N1_b]^{(r)} \quad (3.145)$$

$$[N1_{bm}] = \sum_{r=1}^n q_r [N1_{bm}]^{(r)} \quad (3.146)$$

$$[N1_{mb}] = \sum_{r=1}^n q_r [N1_{mb}]^{(r)} \quad (3.147)$$

$$[N1_b^{\theta_o}] = \sum_{r=1}^n q_r [N1_b^{\theta_o}]^{(r)} \quad (3.148)$$

$$[N1^{N_b}] = \sum_{r=1}^n q_r [N1^{N_b}]^{(r)} \quad (3.149)$$

$$[N1^{N_{\theta_0}}] = \sum_{r=1}^n q_r [N1^{N_{\theta_0}}]^{(r)} \quad (3.150)$$

and

$$[N2_b] = \sum_{r=1}^n \sum_{s=1}^n q_r q_s [N2_b]^{(rs)} \quad (3.151)$$

The non-linear first-order modal stiffness matrices with the superscript (r) are evaluated with the correspondent linear natural mode $\{\phi_b\}^{(r)}$. The non-linear second-order modal matrix with the superscript (r,s) are evaluated with a combination of the correspondent linear natural modes $\{\phi_b\}^{(r)}$ and $\{\phi_b\}^{(s)}$ simultaneously. Consequently, after the assembly process, those non-linear modal matrices are constants matrices. The first-order non-linear stiffness matrix $[N1^{N_m}]$ is linearly dependent on the in-plane displacement $\{W^m\}$ vector of Eq. (3.127). Substituting the bending/rotational displacement $\{W^b\}$ by its expression in Eq. (3.133), and the non-linear first-order stiffness matrix $[N1_{mb}]$ by its expression in Eq. (3.147), the in-plane displacement vector $\{W^m\}$, can be stated as

$$\begin{aligned} \{W^m\} = & -[K_m]^{-1} [K_B^T] \left(\sum_{r=1}^n q_r(t) \{\phi_b\}^{(r)} \right) \\ & - [K_m]^{-1} [K_B^{\theta_o^T}] \left(\sum_{r=1}^n q_r(t) \{\phi_b\}^{(r)} \right) \\ & - \frac{1}{2} [K_m]^{-1} \left(\sum_{r=1}^n q_r [N1_{mb}]^{(r)} \right) \left(\sum_{s=1}^n q_s(t) \{\phi_b\}^{(s)} \right) \end{aligned} \quad (3.152)$$

Transferring the outside matrices under the summation sign, and merging the last two parenthesis terms, let

$$\begin{aligned}
\{W^m\} = & -\sum_{r=1}^n q_r(t) [K_m]^{-1} [K_B^T] \{\phi_b\}^{(r)} \\
& -\sum_{r=1}^n q_r(t) [K_m]^{-1} [K_B^{\theta_o^T}] \{\phi_b\}^{(r)} \\
& -\frac{1}{2} \sum_{r=1}^n \sum_{s=1}^n q_r(t) q_s(t) [K_m]^{-1} [N1_{mb}]^{(r)} \{\phi_b\}^{(s)}
\end{aligned} \tag{3.153}$$

From Eq. (3.153), one can define two in-plane mode shapes corresponding the r^{th} bending/rotational mode $\{\phi_b\}^{(r)}$, and the joint r^{th} and s^{th} mode $\{\phi_b\}^{(rs)}$. The two in-plane modes are given as

$$\{\phi_m\}^{(r)} = [K_m]^{-1} \left([K_B^T] + [K_B^{\theta_o^T}] \right) \{\phi_b\}^{(r)} \tag{3.154}$$

$$\{\phi_m\}^{(rs)} = [K_m]^{-1} [N1_{mb}]^{(r)} \{\phi_b\}^{(s)} \tag{3.155}$$

Consequently, the in-plane displacement vector can be written as

$$\{W^m\} = -\sum_{r=1}^n q_r(t) \{\phi_m\}^{(r)} - \frac{1}{2} \sum_{r=1}^n \sum_{s=1}^n q_r(t) q_s(t) \{\phi_m\}^{(rs)} \tag{3.156}$$

Since the in-plane displacement vector $\{W^m\}$ is linearly and quadratically depending on the modal coordinates $\{q(t)\}$, the non-linear first-order stiffness matrix $[N1^{N_m}]$ can be expressed as

$$[N1^{N_m}] = -\sum_{r=1}^n q_r(t) [N1^{N_m}]^{(r)} - \frac{1}{2} \sum_{r=1}^n \sum_{s=1}^n q_r(t) q_s(t) [N2^{N_m}]^{(rs)} \tag{3.157}$$

The non-linear modal matrices $[N1^{N_m}]^{(r)}$ and $[N2^{N_m}]^{(rs)}$ are evaluated with the in-plane natural modes $\{\phi_m\}^{(r)}$, and $\{\phi_m\}^{(r,s)}$. They are consequently constant matrices.

The equations of motion (3.129) are transformed to the following reduced non-linear system in the modal coordinates

$$\frac{1}{\omega_o^2} [\bar{M}]^b \{ \ddot{q} \} + \frac{g_a}{\omega_o} [\bar{G}]^b \{ \dot{q} \} + \left([\bar{K}_L]^b + [\bar{K}_q]^b + [\bar{K}_{qq}]^b \right) \{ q \} = \{ \bar{P}_{sal} \}^b \quad (3.158)$$

where the diagonal modal mass matrix is given as

$$[\bar{M}]^b = [\Phi]^T [M]^b [\Phi] \quad (3.159)$$

The modal aerodynamic damping matrix is given as

$$[\bar{G}]^b = [\Phi]^T [G]^b [\Phi] \quad (3.160)$$

The modal linear stiffness matrix is given as

$$[\bar{K}_L]^b = [\Phi]^T [K_L]^b [\Phi] \quad (3.161)$$

$$= [\Phi]^T \begin{pmatrix} [A_a] \\ + [K_b] \\ + [K_s] \\ + [K_b^{\theta_o}] \\ - [K_B][K_m]^{-1}[K_B^T] \\ - [K_B][K_m]^{-1}[K_B^{\theta_o T}] \\ - [K_B^{\theta_o}][K_m]^{-1}[K_B^T] \\ - [K_B^{\theta_o}][K_m]^{-1}[K_B^{\theta_o T}] \end{pmatrix} [\Phi]$$

The quadratic terms in modal coordinates are given as

$$[\bar{K}_q]^b \{q\} = [\Phi]^T \left(\frac{1}{2} \sum_{r=1}^n q_r(t) \begin{pmatrix} +[N1_b]^{(r)} \\ +[N1_{b_o}^{\theta_o}]^{(r)} \\ +[N1^{N_b}]^{(r)} \\ +[N1^{N_{\theta_o}}]^{(r)} \\ -[N1^{N_m}]^{(r)} \\ -[N1_{bm}]^{(r)} [K_m]^{-1} [K_B^T] \\ -[K_B] [K_m]^{-1} [N1_{mb}]^{(r)} \\ -[K_B^{\theta_o}] [K_m]^{-1} [N1_{mb}]^{(r)} \\ -[N1_{bm}]^{(r)} [K_m]^{-1} [K_B^{\theta_o T}] \end{pmatrix} \right) [\Phi] \{q\} \quad (3.162)$$

The cubic terms in modal coordinates are given as

$$[\bar{K}_{qq}]^b \{q\} = [\Phi]^T \left(\sum_{r=1}^n \sum_{s=1}^n q_r(t) q_s(t) \begin{pmatrix} -\frac{1}{4} [N1_{bm}]^{(r)} [N_m]^{-1} [N1_{mb}]^{(s)} \\ +\frac{1}{3} [N2_b]^{(rs)} \\ -\frac{1}{4} [N2^{N_m}]^{(rs)} \end{pmatrix} \right) [\Phi] \{q\} \quad (3.163)$$

Finally, the modal static aerodynamic load is given as

$$\{\bar{P}_{sal}^b\} = [\Phi]^T \{P_{sal}^b\} \quad (3.164)$$

The modal Eq. (3.158) will be used to determine the post-flutter time history response of the curved panel at different height-rise H/h . Such ordinary non-linear equations provide out-of-plane displacement and velocity responses. Consequently, these time history responses were used to shed light on the non-linear frequency content of the post-flutter response.

3.7 Modal Participation Definition

Since the curved panel geometry is characterized by the panel non-flatness, the modal participation definition of each r^{th} mode $\{\phi_b\}^{(r)}$ has to account for the asymmetry of the panel. Defining for each modal solution variable $q_r(t)$ two quantities characterizing the asymmetric limit-cycle solution as

$$q_{\max}^r = \text{maximum} (q_r(t)) \quad (3.165)$$

and

$$q_{\min}^r = \text{minimum} (q_r(t)) \quad (3.166)$$

the modal participation of each mode can be defined as

$$\text{Participation of the } r^{th} \text{ mode} = \frac{|q_{\max}^r - q_{\min}^r|}{\sum_{r=1}^n |q_{\max}^r - q_{\min}^r|} \quad (3.167)$$

Equation (3.167) defines without ambiguity the modal participation of each mode $\{\phi_b\}^{(r)}$ in the flutter of curved panels.

3.8 Multimode Fourth-Order Runge-Kutta Method

In this paragraph the steps of this numerical integration method used to solve non-linear ordinary differential equations, originally developed by Runge and Kutta will be employed. The method referred to as the classical fourth-order four-stage Runge-Kutta method has a local truncation error that is proportional to h^5 , h being the time step. Thus, the method is two orders of magnitude better than the well known Euler formula, and is relatively simple to use. The method is also sufficiently accurate to handle many non-linear structural problems efficiently.

Consider the Taylor expansion of the sought function $y(t)$

$$y(t_{i+1}) = y(t_i) + h \left[\dot{y}(t_i) + \dots + y^{(p)}(t_i) \frac{h^{(p-1)}}{p!} \right] + y^{(p+1)}(\zeta_i) \frac{h^{(p+1)}}{(p+1)!} \quad (3.168)$$

where $t_i \leq \zeta_i \leq t_{i+1}$, and h is a time step interval. $y^{(p+1)}(t)$ is continuous over the bounded interval $[t_i, t_{i+1}]$, so the quantity

$$y_n^{(p+1)}(\zeta_i) \frac{h^{(p+1)}}{(p+1)!} = O(h^{(p+1)}) \quad (3.169)$$

If one assumes $[t_i, t_{i+1}] = [0, h]$, expression (3.168) can be written as

$$y(t+h) = y(t) + hf' + \frac{h^2}{2!} f'' + \frac{h^3}{3!} f''' + \dots \quad (3.170)$$

The fourth order Runge-Kutta method consists of finding an approximation function $y(t)$ satisfying the equation

$$\dot{y} = f(t, y(t)) \quad (3.171)$$

where the slope function f is first estimated at the beginning of the interval $[t_i, t_{i+1}]$, and then estimated two times at the middle of the interval, and finally estimated at the end of the interval. The resultant effective slope is then the mean weighted slope at the four different points in the time step interval $[t_i, t_{i+1}]$. The weight coefficients are $\left\{ \frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6} \right\}$

so that the two midpoint values contribute dominantly to the value of the effective slope. Taking a weighted mean of these four estimates and thus canceling the higher order terms up to order h^4 , the sought function can be approximated by

$$y(t+h) = y(t) + h \bar{f}' + O(h^5) \quad (3.172)$$

Numerically, the steps of the sought function can be expressed by the following successive approximations:

$$k_1 = h f(t_i, y_i(t_i)) \quad (3.173)$$

$$k_2 = h f(t_i + \frac{h}{2}, y_i(t_i) + \frac{1}{2} k_1) \quad (3.174)$$

$$k_3 = h f(t_i + \frac{h}{2}, y_i(t_i) + \frac{1}{2} k_2) \quad (3.175)$$

$$k_4 = h f(t_i + h, y_i(t_i) + k_3) \quad (3.176)$$

the total approximation is then expressed as

$$y_{i+1} = y_i(t_{i+1}) = y_i(t_i) + \frac{1}{6} k_1 + \frac{2}{6} k_2 + \frac{2}{6} k_3 + \frac{1}{6} k_4 + O(h^5) \quad (3.177)$$

3.8.1 Runge-Kutta State -Space Scheme

In order to solve with the Runge-Kutta numerical scheme a non-linear ordinary differential equation of type (3.158) given as

$$\frac{1}{\omega_o^2} [\bar{M}]^b \{ \ddot{q} \} + \frac{\bar{g}_a}{\omega_o} [\bar{G}]^b \{ \dot{q} \} + \left([\bar{K}_L]^b + [\bar{K}_q]^b + [\bar{K}_{qq}]^b \right) \{ q \} = \{ \bar{P}_{sal} \}^b \quad (3.178)$$

with the assumed initial conditions

$$\begin{aligned} \{ q(0) \} &= \{ q_o \} \\ \{ \dot{q}(0) \} &= \{ \dot{q}_o \} \end{aligned} \quad (3.179)$$

it is necessary to transform the modal Eq. (3.178) to a state-space form such as

$$\begin{cases} \{\mathcal{Q}_1\} = \{q\} \\ \{\mathcal{Q}_2\} = \{\dot{q}\} \end{cases} \quad (3.180)$$

Deriving Eq. (3.180) and accounting for Eq. (3.178), the system of Eqs. (3.180) can be written as a state space system as

$$\begin{cases} \{\dot{\mathcal{Q}}_1\} = \{\dot{q}\} = \{\mathcal{Q}_2\} \\ \{\dot{\mathcal{Q}}_2\} = \{\ddot{q}\} = -\omega_o g_a ([M]^b)^{-1} [G]^b \{\dot{q}\} - \omega_o^2 ([M]^b)^{-1} ([\bar{K}_L]^b + [\bar{K}_q]^b + [\bar{K}_{qq}]^b) \{q\} \\ \quad + \omega_o^2 ([M]^b)^{-1} \{\bar{P}_{sal}\}^b \end{cases} \quad (3.181)$$

Equation (3.181) can be transformed to a state-space form as

$$\begin{Bmatrix} \{\dot{\mathcal{Q}}_1\} \\ \{\dot{\mathcal{Q}}_2\} \end{Bmatrix} = \begin{bmatrix} [0] \\ -\omega_o^2 ([M]^b)^{-1} ([\bar{K}_L]^b + [\bar{K}_q]^b + [\bar{K}_{qq}]^b) - \omega_o g_a ([M]^b)^{-1} [G]^b \end{bmatrix} \begin{Bmatrix} \{\mathcal{Q}_1\} \\ \{\mathcal{Q}_2\} \end{Bmatrix} + \begin{bmatrix} [0] \\ \omega_o^2 ([M]^b)^{-1} \end{bmatrix} \{\bar{P}_{sal}\}^b \quad (3.182)$$

Denoting that

$$\{\dot{\mathcal{Q}}_1\} = \{\mathcal{Q}_2\} = \{f_1(\mathcal{Q}_1, \mathcal{Q}_2)\} \quad (3.183)$$

and

$$\begin{aligned} \{\dot{\mathcal{Q}}_2\} &= -\omega_o g_a ([M]^b)^{-1} [G]^b \{\mathcal{Q}_2\} - \omega_o^2 ([M]^b)^{-1} ([\bar{K}_L]^b + [\bar{K}_q]^b + [\bar{K}_{qq}]^b) \{\mathcal{Q}_1\} + \omega_o^2 ([M]^b)^{-1} \{\bar{P}_{sal}\}^b \\ &= \{f_2(\mathcal{Q}_1, \mathcal{Q}_2)\} \end{aligned} \quad (3.184)$$

one can notice the similarity between the functions f_1 , f_2 and f stated in Eq. (3.171).

3.8.2 Considerations when Applying Runge-Kutta Scheme

When applying the time-numerical integration Runge-Kutta method, it is important that the selection of the sampling time $h = \Delta t$ comply with the Nyquist-Shannon Theorem. The theorem itself states that when sampling a time response at discrete time

intervals, the sampling frequency must be twice greater than the highest frequency of the response itself. This insures the construction of a time response that reflects the dynamic behavior of the system.

$$\Delta t \leq \frac{1}{2f_{cutoff}} \quad (3.185)$$

where f_{cutoff} is the highest frequency buried in the signal. If the sampling frequency is less than the stated limit, then high frequencies in the output response that are above half the sampling rate will be "aliased" and will appear in the resulting spectra output response as lower frequencies. To avoid the aliasing phenomenon, the time step Δt was reduced until two consecutive solutions produced the same response.

3.9 Motion Categories Investigation

Non-linear dynamic systems have been proven to exhibit surprising and complex motions. The most prominent and striking examples of those new categories of motions include bifurcations and chaos. In the present investigation, these motions are mainly determined by a set of single tuning parameters, such as the height-rise parameter H/h , or the aerodynamic pressure λ . The chaotic motion is defined as an unpredictable long time behavior arising in a deterministic dynamical system because of sensitivity to initial conditions. In the present case, which is the flutter of curved panels the sensitivity is due essentially to the aforementioned tuning parameters. Prelude to the chaotic motion are the bifurcations which consist in a qualitative change in the dynamic behavior of the system like a sudden appearance of limit-cycles oscillations (LCO's or Hopf bifurcation), or a disappearance of the system equilibrium (Chaos or Saddle bifurcation) or a simple or

complex change in the stability parameters of the dynamic system under the variation of the tuning parameters (double bifurcations corresponding to two relative maximum peaks). Furthermore, the aforementioned motion categories of bifurcations and chaos need to be identified, characterized and classified. For chaotic motion investigation, Moon [70] and Dowell [71] recommended a set of analysis tools like time history, phase plot, and power spectral density. In addition Moon [72] proposed another set of analysis tools like the Poincare map, the Henon map, the Feigenbaum number, the Horseshoe map, and intermittency. In the present work an emphasis will be put on the first three mentioned tools.

3.9.1 Time History Responses

The time history response offers a description of the non-linear dynamic behavior of the system function of time. For most of the conducted time history analysis, the maximum and minimum deflection of the flutter system is given by a numerical time history computed function. For harmonic oscillations, the time response can show paths of motions (periodic, chaotic, etc), but the analysis by itself does not provide additional and useful information like for instance the frequency contents of the observed vibrations.

3.9.2 Phase Plots

The phase plot is an orbit in the displacement/velocity plan that describes the relationship between the displacement and the velocity of a point system. The usefulness of the phase plot is to detect pattern that can assert the periodicity of the signal. If the

illustrated orbit is closed the periodicity of the signal is affirmative, while if the orbit is not closed, the periodicity of the signal cannot be affirmed. Furthermore a phase plot inner-loops and loops intersections within the same period informs the reader about the complexity of the motion-taking place in the system. Intersection reflects the system goes through the same point during the same period unlike the pure harmonic behavior. Phase plots also provide information about the correlation between displacement and velocity. One can detect easily the maximum displacement and its velocity at that particular point. Times responses, and phase plots do not provide information about the frequency spectrum of the signal.

3.9.3 Power Spectral Density

The power spectral density (PSD) is a tool that detects the frequencies content of the flutter response, usually called the frequency spectrum. In that sense it represents a powerful informative tool that reports to the investigator the category of motion that the system is undergoing under the influence of the tuning parameters. If the PSD is a succession of clear distinct peaks, that is an indication that the flutter time response is a superposition of a known set of non-linear frequencies. This suggests that the time history response of the fluttering system is either a harmonic one or a periodic one. Conversely, if the PSD plot is a narrowband or a broadband frequency range, it is a clear suggestion that the flutter time history response is falling in the category of the chaotic-erratic motion type. The latter case indicates that the flutter time history response is a broadband superposition of infinity of frequencies within the band itself. The usefulness of the PSD rely also on the fact, that the investigator has a tool that can follow the evolution of the

natural frequencies of the system under the dynamic pressure influence λ . The evolution of the dynamic behavior of the system can be fully monitored.

3.9.4 Bifurcations Diagrams

Bifurcations diagrams provide an overview of the different states of behavior that non-linear dynamic systems settle in. One can also state that a bifurcation is a change from N -point attractors to $2N$ -point attractors, which occurs when the tuning parameter dynamic pressure λ is progressing. In the field of structural dynamics many bifurcation types or categories were identified. The field by itself is still wide open for the identification of new types and categories of bifurcations as the investigations of more complex structures is underway. Among the most important well-known bifurcation is the Hopf Bifurcation, which occurs when two complex conjugate eigen-values simultaneously become positive leading to the event of non-linear oscillatory states usually called limit-cycles. In the classical case the typical bifurcation is a single period corresponding to single maximum and minimum deflections. As the tuning parameter increases, a so-called period doubling is observed corresponding to the existence of two relative maximum and two relative minimum. By increasing further more the tuning parameter, relative maximum and minimum deflections increase in number and amplitude paving the way to a chaotic-erratic motion. For the multitude types and categories of bifurcations the reader is referred to Abraham and Shaw [73] and Abraham et al. [74]. The construction of bifurcation's diagrams using flutter time history response conjugated with dynamic pressure λ as a tuning parameter is described in Fig. 3.2. The first step consists of sorting the relative minimum and maximum peaks in a manner that

each point is related to the corresponding deflection amplitude W/h for a given dynamic pressure λ . The second step is a projection of the relative peak amplitudes over a line corresponding to the given dynamic pressure λ ; the results is a scaling along the constant dynamic pressure vertical axe of all the sorted peaks according to their amplitudes. The third and final steps consist of putting side by side the vertical line corresponding to each dynamic pressure used as tuning parameter.

3.10 Strain and Stress Computation

A comparison between the curved panel maximum principal stresses and material yielding stress is necessary herein to prevent a panel breakdown due to large deflection. The comparison ensures the computation of a limiting large deflection $\{W/h\}_{limit}$ beyond which the panel may fail. The limiting large deflection $\{W/h\}_{limit}$ defines a no-crossing limiting deflection in the bifurcation diagrams.

Since the curved panel is isotropic, the stress computation is based on the panel deflection time histories. The panel modal displacement $\{q\}$ computed with the 4th order Runge-Kutta numerical scheme for a specified dynamic pressure λ and panel height-rise H/h is used to determine the panel large deflection components

$$\{W\}^T = \{[W_b] [W_\psi] [W_m]\} \quad (3.186)$$

The computation of the stresses uses the stress-strain relations outlined in Eq. (2.51).

$$\{\sigma\}_k = [\overline{Q}]_k \{\varepsilon\} \quad (3.187)$$

The strain is obtained from Eq. (2.33)

$$\{\varepsilon\} = \{\varepsilon_m^o\} + \{\varepsilon_b^o\} + \{\varepsilon_{w_o}^o\} + \bar{z}\{\kappa\} \quad (3.188)$$

where the membrane strains $\{\varepsilon_m^o\}$ are given by Eq. (2.39)

$$\{\varepsilon_m^o\} = [C_m] \{w_m\} \quad (3.189)$$

the von-Karman large deflection strains $\{\varepsilon_b^o\}$ are given by Eq. (2.42)

$$\{\varepsilon_b^o\} = \frac{1}{2} [\theta] ([C_{\psi b}] \{w_b\} + [C_{\psi \psi}] \{w_\psi\}) \quad (3.190)$$

the strains due to the panel curved geometry $\{\varepsilon_{w_o}^o\}$ are given by Eq. (2.44)

$$\{\varepsilon_{w_o}^o\} = [\theta_o] ([C_{\psi b}] \{w_b\} + [C_{\psi \psi}] \{w_\psi\}) \quad (3.191)$$

and the curvature bending strains $\{\kappa\}$ are computed with Eq. (2.46)

$$\{\kappa\} = [C_b] \{w_\psi\} \quad (3.192)$$

The element $\{w_b\}$, $\{w_\psi\}$, $\{w_m\}$ are obtained from the system deflection vector $\{W\}^T$ shown in Eq. (3.186). The matrices $[C_m]$, $[C_{\psi b}]$, $[C_{\psi \psi}]$, $[C_b]$ are functions of element local coordinates. The transformation of the stresses, Eq. (3.187) into the principal coordinates is performed via the use of Eq. (2.55). For the k^{th} layer of a laminated composite panel, the strains are then expressed as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \tau_{12} \end{Bmatrix}_k = [T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \tau_{xy} \end{Bmatrix}_k \quad (3.193)$$

For isotropic curved panels, stresses are obtained using the basic constitutive equations.

Since the Finite Element formulation is a displacement based formulation, the strains and the stresses are not continuous between an element and its surrounding neighboring elements. The C^0 type of class elements, herein the MIN3 element, has only a continuous displacement in contrast with a C^1 class element that has a continuous displacement and

slope like the BFS element. In addition Barlow [77] and Cook et al. [78] showed that for a displacement based Finite Element formulation, the assumed displacement interpolation functions operate as a set of constraints on the system that results in an excessively panel element stiffening behavior. To correct the excessive stiffening behavior and improve the accuracy of the element panel strains and stresses computation, they applied a low-order Gaussian quadrature rule that tends to soften the panel element and corrects its strain energy. Therefore, the strains and stresses are computed at $N-1$ Gauss points of an element. N is the required Gaussian quadrature that ensures the computation of those strains and stresses. Moreover to improve the further the accuracy of the strains, and the stresses, respectively, a strain averaging operation is performed at the local neighboring shared nodal elements.

Once the stress distributions are obtained, the principal stresses are then computed and the maximum shear stress criterion or the von Mises criterion are used to predict if the panel is going to fails or not.

Fig. 3.1 **Four point slope estimates of the Runge – Kutta method**

Fig. 3.2 **Relative maximums and minimums projection for a given λ**

Chapter 4

4 RESULTS AND DISCUSSIONS: FREQUENCY DOMAIN METHOD

4.1 Introduction

The results discussed in the present chapter tackle many issues regarding the non-linear flutter behavior of curved panels. The chapter is divided in two parts; the first part is dealing with a 2-D flutter system, while the second part is dealing with 2-D flutter systems. Each one of the 2-D and 3-D parts encompasses two approaches relevant to the non-linear flutter study. The first is a frequency domain approach that determines the exact value of the flutter onset. The second part is a time domain approach that determines the flutter time history response. Among the key issues scrutinized in the aforementioned two parts are the comparison and validation of the computed non-linear finite element results with existing results obtained with the PDE/Galerkin method. A complete and exhaustive multi-modal study was performed. It used a gradual assortment of mode shapes from sinusoidal modes, to natural modes, and to aerostatic mode shapes to account for the system flutter deflection approximation. The present chapter also allows a large space to dissect the non-linear flutter dynamic behavior of 2-D and 3-D flutter systems using tools such as bifurcation diagrams, time history responses, and power spectrum densities. Issues like the influence of the curved panel geometry, the mode shape assortment, the number of modes for convergence of the non-linear flutter responses, and the influence of the boundary conditions were fully addressed in the present chapter.

4.2 2-D Cylindrical Curved Geometry

The 2-D curved geometry has a constant radius R_x in the x direction and an infinite radius $R_y = \infty$ in the y direction. Its geometry may be approximated by the expression of the relative height-raise $w_o(x)$, corresponding to each abscissa $\bar{x}$ defined by Dowell [13] as

$$w_o(x) = -\frac{1}{2} \left(\frac{(\bar{x} - a/2)^2}{R_x} \right) \quad (4.1)$$

Eliminating R_x in Eq. (4.1) by applying the Pythagorean theorem to the triangle specified by the angle α , one have

$$R_x^2 = X^2 + (a/2)^2 \quad (4.2)$$

$$(X + H)^2 = X^2 + (a/2)^2 \quad (4.3)$$

where R_x , X , H , and α are defined in Fig. 4.1. Extracting X from Eq. (4.3) gives the following equation

$$X = \frac{a^2}{8H} - \frac{H}{2} \quad (4.4)$$

Using the expression of the curvature

$$R_x = X + H \quad (4.5)$$

and substituting Eq. (4.4) into Eq. (4.5) gives the curvature R_x in function of the height-raise H of the curved as

$$R_x = \frac{a^2}{8H} + \frac{H}{2} \quad (4.6)$$

Given the curved approximation $H \ll a^2$, Eq. (4.6) can be approximated by

$$R_x = \frac{a^2}{8H} \quad (4.7)$$

Substituting Eq. (4.7) into Eq. (4.1) gives the relative height-rise $w_o(x)$ as a function of $\bar{x}$ and H the maximum height-rise only

$$w_o(\bar{x}) = -H \left(\frac{(\bar{x} - a/2)^2}{(a/2)^2} \right) \quad (4.8)$$

by plotting the curved panel geometry described by Eq. (4.8), it gives a surface underneath the $z = 0$ plan represented in Fig. 4.2. To obtain the needed curved surface above the $z = 0$ plan, Eq. (4.8) has to be modified. The modification, which is a translation, gives the cylindrical curved surface represented in Fig. 4.3 and related to Eq. (4.9).

$$w_o(x) = H \left(1 - \frac{(x - a/2)^2}{(a/2)^2} \right) \quad (4.9)$$

4.3 3-D Doubly Curved Geometry

The equation of a 3-D curved panel with constant radius R_x in the x direction and a constant radius R_y in the y direction is defined Dowell [13]

$$w_o(\bar{x}, \bar{y}) = -\frac{l}{2} \left(\frac{(\bar{x} - a/2)^2}{R_x} + \frac{(\bar{y} - b/2)^2}{R_y} \right) \quad (4.10)$$

Following the same procedure outlined in paragraph 4.2, one can write the constant radius R_x in the x direction and the constant radius R_y in the y direction as:

$$R_x = \frac{a^2}{8H} \quad (4.11)$$

$$R_y = \frac{b^2}{8H} \quad (4.12)$$

The resultant equation of a spherical curved is obtained by substituting Eq. (4.11) and Eq. (4.12) in Eq. (4.10) which give the relative height-rise $w_o(x, y)$ function of x and y as

$$w_o(x, y) = H \left(1 - \frac{(x - a/2)^2}{(a/2)^2} - \frac{(y - b/2)^2}{(b/2)^2} \right) \quad (4.13)$$

a representation of the geometry related to Eq. (4.13) is displayed in Fig. 4.4. It goes without saying that Eq. (4.8) and Eq. (4.13) are easy and practical to use in the non-linear finite element code.

4.4 Derivative of the Cylindrical and Spherical Curved Geometry

The derivatives of the curved geometry of Eq. (4.9) and Eq. (4.13) are needed to compute the SAL generated by the curved geometry itself. The derivative with respect to x and the derivative with respect to y are expressed as

$$w_{o,x} = \frac{4H}{a^2} (a - 2x) \quad (4.14)$$

$$w_{o,y} = \frac{4H}{b^2} (b - 2y) \quad (4.15)$$

One can notice that the derivative of the curved geometry are only depending upon the parameter height-rise H rendering easy and practical their coding.

4.5 Static Aerodynamic Load (SAL)

Dowell [13] first introduced the term static aerodynamic load to describe the additional static pressure due to the intrinsic geometry of the curved panel. The

expression of the static aerodynamic load shown as the third term in Eq. (2.203) appears naturally as the result of the Marguerre shell theory. Fig. 4.5 shows the progress of the static aerodynamic load along the x -axis at various dynamic pressures, while Fig. 4.6 shows the evolution along the x -axis of the static aerodynamic load function of the height-rise H/h at a prescribed dynamic pressure $\lambda = 700$. From the aforementioned figures, it can be concluded that the SAL varies with both the dynamic pressure and the height-rise of the panel.

4.6 Frequency Domain Solutions for Flutter Stability Boundaries

The widely used light weighted curved panels in aerospace and aircraft structures are subjected to a variety of aeroelastic loads during supersonic flight. It is critical for safety considerations to thoroughly investigate the flutter stability boundaries of these curved panels in order to avoid the consequences of a dramatic structural failure. In the forthcoming paragraphs the results related to the frequency domain eigen-method will be presented in details. The developed flutter eigen-analysis involves the curved panel's tangent stiffness matrix of Eq. (3.120), which accounts for the stiffness of the deflected curved panel under the static aerodynamic load $\{P_{sal}\}$ at a prescribed dynamic pressure λ . The curved panel's deflection is called the aerostatic deflection. The eigen-solution method happened to be very effective to determine the flutter critical dynamic pressure λ_{cr} of the 2-D and 3-D curved panels. In the following paragraphs, results related to the various steps for the determination of the flutter boundaries by the eigen-solution method will be presented for 2-D and 3-D curved panels.

4.6.1 Aerostatic Deflection of 2-D Cylindrical Panels

A 2-D isotropic cylindrical panel with the following dimensions: $a = 22.86 \text{ cm}$ (9.0 in.), $b = \infty$, $h = 0.020232 \text{ cm}$ (0.008 in.) is investigated for the pre-flutter deflection shape and critical dynamic pressure. The panel dimensions were identical to the ones taken by Anderson [40]. The material properties are: $E = 1.0341 \times 10^{11} \text{ Pa}$ ($15 \times 10^6 \text{ psi}$), $\nu = 0.0$, $\rho = 8518.5 \text{ kg/m}^3$ ($0.0007971 \text{ lb} \times \text{s}^2/\text{in.}^4$). The damping parameter is $C_a = 0.01$. The boundary conditions were set to a simply supported panel along the two edges parallel to y -axis and free along the x -axis to simulate a 2-D panel. Immovable in-plane boundary conditions $u(0, y) = u(a, y) = 0$ were considered along the two edges parallel to y -axis. The cylindrical panel was discretized with 1×79 mesh size encompassing 158 MIN3 elements represented in Fig. 4.7. The aerostatic deflections at various dynamic pressure λ were investigated for the height-rises of $H/h = 1, 1.625, 3$ and 5. A critical milestone toward the computation of the curved panel stability boundaries is the determination of the pre-flutter aerostatic deflection shape $\{W_s\}$. Prior to the determination of the aerostatic deflection shape it is essential to evaluate the tangent matrix $[K_{\text{tan}}]$ using the static deflection $\{W_s\}$ from the previous step in the Newton-Raphson iterative process, till convergence is attained. The tangent matrix, which represents the stiffness of the deflected panel under the static aerodynamic load at a prescribed dynamic pressure, is obtained by feeding the updated $\{W_s\}$ into the first-order $N_1(W_s)$, and second-order $N_2(W_s^2)$ stiffness matrices. The aerostatic deflection is determined using Eq. (3.59) and Eq. (3.61). It is obvious that the flutter response for the curved panels would behave differently from the dynamic behavior of flat plates. Whereas there is no perceptible

static deflection for flat plates in the pre-flutter region, the curved panel exhibits a clear static deflection depending on the dynamic pressure λ and the height-rise H/h . The figures Fig. 4.9, and Fig. 4.11 - Fig. 4.13 show a gradual evolution of the aerostatic deflection shape as the dynamic pressure λ increases. Figure 4.8 and Figure 4.10 depict the shape of the aerostatic equilibrium deflection $\{W_s\}$ only about the curved panel geometry $w_o(x, y)$. The maximum panel's peak amplitude is gradually shifting toward the trailing edge with less static deflection as the dynamic pressure increases, Fig. 4.9, and Fig. 4.11 to Fig. 4.13. As shown in the forthcoming section 4.6.3, for height-rise of $H/h = 1$ the flutter onset occurs at $\lambda_{cr} = 254$, Fig. 4.9 shows that within the range of dynamic pressure $0 \leq \lambda \leq 253$, the cylindrical panel never snaps through over the opposite side before the flutter occurs. At the flutter onset, the panel's maximum aerostatic deflection amplitude is located at $x/a = 0.74$. As the height-rise of the curved panel is progressively raised to $H/h = 1.625$ in Fig. 4.11, a portion of the aerostatic deflection starts to move toward the negative z -axis. Just before the flutter begins at the dynamic pressure $\lambda = 231$, the aerostatic deflection shape is composed of three unsymmetrical undulations. Beyond the height-rise limit of $H/h = 1.625$, the aforementioned static behavior for $H/h = 1.625$ is not happening anymore. The aerostatic deflection shape never crosses the z negative axis before the flutter begins as shown in Fig. 4.12 and Fig. 4.13. The investigation of the pre-flutter static behavior of 2-D cylindrical panels showed that the height-rise domain span of $0 < H/h \leq 5$ can be subdivided into two distinct areas. The height-rises domain span of $0 < H/h \leq 1.625$, where the aerostatic deflection shape is allowed to cross the negative z -axis, and the height-rise domain span $1.625 < H/h \leq 5$, where the aerostatic deflection shape is impeded from crossing the negative z -axis. The existence of the two

aforementioned subdivisions hints toward the occurrence of sudden static snap-through in the second subdivision. Further investigations conducted in paragraph 4.6.3 reinforced the assumption of occurrence of static snap-through.

4.6.2 Aerostatic Deflection of 3-D Cylindrical Panels

A 3-D simply supported isotropic cylindrical panel with the following dimensions: $a = 30.48 \text{ cm}$ (12.0 in.), $b = 30.48 \text{ cm}$ (12.0 in.), $h = 0.1016 \text{ cm}$ (0.04 in.) is studied for the pre-flutter aerostatic deflection shape. The material properties are: $E = 7.1 \times 10^{10} \text{ Pa}$ ($5 \times 10^6 \text{ psi}$), $\nu = 0.3$, $\rho = 2700.0 \text{ kg/m}^3$ ($0.00025234 \text{ lb} \times \text{s}^2/\text{in.}^4$). The damping parameter is $C_a = 0.01$. The panel was discretized with a mesh size of 16×16 encompassing 512 MIN3 elements. The aerostatic deflection was investigated for height-rises $H/h = 1, 3$ and 5 at various yaw flow angles. Since the aerostatic deflection shape is not uniform across the y -axis in the 3-D case, the $y = b/2$ axis was chosen as representative of the deflection shape. The aerostatic deflection shape reported in Fig. 4.14 for $H/h = 1$ and yaw flow angle $\Lambda = 0^\circ$ at different dynamic pressure within the pre-flutter region shows that the gradual increasing dynamic pressure exerted over the cylindrical panel leading edge has forwardly deformed the panel. The $(x, y = b/2)$ line is forced downward and the aerostatic deflection maximum amplitude is located at $x/a = 0.76$ for a dynamic pressure $\lambda = 491$ just before the flutter onset. For height-rises $H/h = 1, 3$, and 5 , the cylindrical panels are not experiencing negative aerostatic deflection as shown in Fig. 4.15. The figure features the aerostatic deflection just before the flutter begins. It shows also that the panel peak amplitudes are less tilted toward the trailing edge as the height-rise H/h increases. The panel is resisting the SAL as its height goes up.

When the yaw flow angle varies the aerostatic deflection becomes unsymmetrical and the cylindrical panel $y = b/2$ line is less prompt for deformation as illustrated in Fig. 4.16. Figures Figure 4.17 to Fig. 4.20 illustrate 3-D views of both aerostatic equilibrium deflection $\{W_s\}$ about the curved panel geometry $w_o(x, y)$ and the aerostatic deflection for height-rises $H/h = 1$ and $H/h = 5$ at various yaw flow angles. The depicted figures demonstrate a gradual stiffening affecting the panels under the combined effect of curvature H/h and yaw flow angle λ . They were plotted just before flutter begins. As mentioned earlier, cylindrical panel with high height-rise $H/h = 5$ are less subjected to deformation than the ones with a relative low height-rise $H/h = 1$.

4.6.3 Flutter Investigation of 2-D Cylindrical Panels

In the previous paragraph the evaluation of the tangent stiffness matrix $[K_{tan}]_s$ rendered possible the determination of the aerostatic deflection shape of the curved panel. Knowing the tangent stiffness matrix for each dynamic pressure λ featured in Eq. (3.118), an eigen-value problem, Eq. (3.120), is established to determine the evolution of the natural frequencies of the aerostatic deflected panel as the dynamic pressure λ increases. The investigation of the natural frequencies of the deflected curved panel under the influence of SAL as a function of the dynamic pressure could lead to the coalescence of frequencies of two aerostatic mode shapes allowing subsequently the determination of the flutter onset. It is shown that the concept of frequency coalescence cannot occur when the curved panel is subjected to static snap through. An isotropic 2-D simply supported cylindrical panel similar to the one chosen in section A is investigated for flutter onset for height-rises $H/h = 1, 1.625, 3, \text{ and } 5$.

Flutter Onset Convergence Study

The flutter onset study of cylindrical panel requires a mesh refinement procedure that aims to produce accurate critical dynamic pressure λ_{cr} characterizing the panel. To that end, the panel lowest natural frequency and critical dynamic pressure were checked for various gradually increasing mesh sizes. Table 4.1 shows clearly a rapid convergence process of the lowest natural frequency and the critical dynamic pressure as the mesh size increases. Herein for the present 2-D case, a mesh size 79×1 was chosen to guaranty an accurate computation of the critical dynamic pressure.

Flutter Investigation of 2-D Cylindrical Panels

The infinite 2-D cylindrical panel with height-rise $H/h = 1$ possess a flutter coalescence curve as illustrated in Fig. 4.21. The coalescence of aerostatic mode 1 and mode 2 at the dynamic pressure $\lambda_{cr} = 254$ occurs when the damping rate becomes positive as illustrated in Fig. 4.22. Figure 4.19 details the mechanisms leading to the coalescence of the two aforementioned aerostatic modes. The non-dimensional frequency ratio ω/ω_{01} , where ω is the frequency of either aerostatic mode 1 or 2 and ω_{01} is the fundamental natural frequency of the cylindrical panel, is plotted versus the non-dimensional dynamic pressure λ . The figure shows a gradual softening of the aerostatic mode 1 then a hardening within the softening region, whereas the aerostatic mode 2 experiences a light hardening then a steep softening toward the softening region. Notice that the coalescence occurs in the softening region at $\omega/\omega_{01} = 0.983$, it is evident that a slight softening process has occurred.

If the height-rise of the curved panel is raised to $H/h = 1.625$, the aerostatic modes coalescence process is represented in Fig. 4.24. It shows that the frequency associated with the first aerostatic mode experiences a sharp decrease approaching nearly the zero frequency, then bounce back at the turning point $\lambda = 221$. Figure 4.21 also shows a sudden decrease of the damping rate at the same dynamic pressure. In this particular case, a large softening process took place till the dynamic pressure $\lambda = 221$, then a surprising hardening process before flutter as illustrated in Fig. 4.26. The non-dimensional frequency ratio at the turning point is approaching zero dramatically at $\omega/\omega_{01} = 0.0917$. At this critical turning point, the curved panel stiffness is nearly on the verge of collapsing, then suddenly it starts to increase. The aforementioned softening/hardening process can be explained by the aerostatic deflections illustrated in Fig. 4.11. At $\lambda = 221$ the panel exhibits unexpectedly a negative deflection, which increases radically its stiffness. Moreover the flutter onset arises at $\omega/\omega_{01} = 0.4589$ indicating that the flutter process begins at a frequency smaller than one-half the fundamental natural frequency ω_{01} of the panel.

Augmenting the panel height-rise beyond the critical limit of $H/h = 1.625$ leads to the type of non-coalescent curve represented in Fig. 4.27 for $H/h = 3$, where the panel loses its stiffness just before higher aerostatic modes are set to coalesce. The first and second aerostatic modes attain the critical $\omega/\omega_{01} = 0.0$ instability allowing a sudden buckling of the panel.

4.6.4 Flutter Investigation of 3-D Cylindrical Panels

Flutter Onset Convergence Study

In the case of 3-D curved panel, the flutter onset convergence study is conducted in similar fashion as the one performed for the 2-D case. A mesh refinement procedure that aims to produce accurate critical dynamic pressure λ_{cr} characterizing the panel is presented in Table 4.2. The Table shows clearly a slow convergence process convergence process of the lowest natural frequency and the critical dynamic pressure as the mesh size increases. A 16×16 mesh size was necessary to reach an acceptable value of the critical dynamic pressure. The requirement for the aforementioned high mesh is obviously accompanied with higher computational cost.

Flutter Investigation of 3-D Cylindrical Panels

The study of the static processes leading to the flutter of 3-D cylindrical panel is conducted herein in a similar fashion as the preceding 2-D case. A 3-D cylindrical panel like the one considered in paragraph 4.6.2 is investigated for pre-flutter behavior and flutter onset at various height-rises. Figure 4.24 shows that the aerostatic mode 1 and mode 2 coalescence take place in the hardening region although mode 1 experiences a small amount of softening. For a height-rise of $H/h = 3.5$ the cylindrical panel start to soften as shown in Fig. 4.29 and the aerostatic mode coalescence starts to take place in the softening region. The zigzags affecting aerostatic mode 2 are essentially due to modes 2 and 3 interactions. For higher height-rise in the present case $H/h = 5$, the damping rate becomes first positive from the frequency coalescence of aerostatic modes 6 and 7 as shown in Fig. 4.30 and Fig. 4.31. The positive value of the damping rate corresponding to

modes 6 and 7 at the coalescence point is insignificant with respect to the damping rate corresponding to modes 1 and 2. The instability introduced by the coalescence of modes 6 and 7 could be benign in the dynamic pressure interval $486 \leq \lambda < 603$ if the panel is exposed to the flow for a short time, whereas, the instability introduced by the coalescence of modes 1 and 2 is almost ten times bigger and could damage the panel even if exposed to the flow for a short time. The deflection shape shown in Fig. 4.15 for $H/h = 5$ and $\Lambda = 0^\circ$ just before flutter reveals that the panels enter in the flutter mode without snapping through. It appears that the snapping through shown in the in the 2-D case is restricted from happening in the 3-D case by the additional boundary constraints along the borders of the curved panel. Under the conjugate effects of the dynamic pressure and the SAL loads, the generated softening intrinsically changes the panel stiffness sufficiently to trigger higher mode coalescence.

4.7 Critical Dynamic Pressure Analysis for 2-D and 3-D Panels

A practical aspect related to the present study is the analysis of the flutter critical dynamic pressure versus the curved panel height-rise to determine the flutter stability boundaries. Fig. 4.32 illustrates the variation of the critical dynamic pressure as function of the cylindrical panel height-rise. The abbreviations FDM and TDM stand for Frequency and Time Domain Methods, respectively. A damping parameter of $C_a = 0.1$ was chosen for comparison purposes. The figure compares the critical dynamic pressure by the FDM and TDM methods to existing values in the literature [13]. It shows clearly that when the finite element (FE) TDM is computed based on the 4 lowest sinusoidal mode shapes, the values of the critical dynamic pressure compares relatively well with

the values shown in the literature. For the present FDM, even though within the small interval $0 \leq H/h \leq 1$, the method compares relatively well with Dowell's values; it diverges for higher values of height-rise H/h . The divergence could be essentially explained by the evident dissimilarity of the curved panel natural modes with sinusoidal flat plate based modes for high height-rise. The FDM shows essentially that critical dynamic pressure occurs at higher values for larger height-rises. However, high height-rises have a detrimental effect on the critical dynamic pressure for 2-D cylindrical panels. For 3-D cylindrical panels, Fig. 4.33 shows that very low curvatures have a small detrimental impact on the critical dynamic pressure, and higher curvatures are beneficial with an optimum height-rise. However, around the height-rise $H/h = 3.75$ aerostatic modes coalescence start to occur between mode 6 and mode 7 which again impact negatively on the critical dynamic pressure.

Fig. 4.1 Cylindrical panel curvature R_x and height-raise H/h

Fig. 4.2

Cylindrical panel underneath the $z = 0$ plan

Fig. 4.3

Cylindrical panel above the $z=0$ plan

Fig. 4.4 Spherical panel with the $z = 0$ plan

Fig. 4.5 **Static aerodynamic load along x axis at various dynamic pressures for**
3-D cylindrical panel of height-rise $H/h = 1$

Fig. 4.6 **Static aerodynamic load along x axis at dynamic pressure $\lambda = 700$ and various height-rises for 3-D cylindrical panel**

**Fig. 4.7 2-D curved panel discretized with a mesh size of 79×1 encompassing
158 MIN3 elements**

Figure 4.8 Aerodynamic static equilibrium deflection $\{W_s\}$ of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1$

Fig. 4.9 Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1$

Figure 4.10 Aerodynamic static equilibrium deflection $\{W_s\}$ of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1.625$

Fig. 4.11 Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 1.625$

Fig. 4.12 Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 3$

Fig. 4.13 Aerostatic deflection of an isotropic simply supported 2-D cylindrical panel with height-rise $H/h = 5$

Fig. 4.14 3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel with a height-rise of $H/h = 1$ and yaw flow angle $\Lambda = 0^\circ$

Fig. 4.15 3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel at yaw flow angle $\Lambda = 0^\circ$ and dynamic pressure $\lambda = \lambda_{cr}$

Fig. 4.16 3-D aerostatic deflection shape at $y = b/2$ of an isotropic simply supported cylindrical panel with a height-rise $H/h = 1$ and dynamic pressure $\lambda = 350$

Figure 4.17 3-D aerodynamic static equilibrium deflection $\{W_s\}/h$ of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure

$$\lambda = \lambda_{cr}$$

Figure 4.18 3-D aerostatic deflection shape of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$

Figure 4.19 3-D aerodynamic static equilibrium deflection $\{W_s\}/h$ of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure

$$\lambda = \lambda_{cr}$$

Fig. 4.20 3-D aerostatic deflection shape of an isotropic simply supported cylindrical panel at various yaw flow angle and dynamic pressure $\lambda = \lambda_{cr}$

Fig. 4.21 Flutter coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 1$ with damping parameter $Ca = 0.01$

Fig. 4.22 Damping rate versus dynamic pressure for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$

Fig. 4.23 Aerostatic mode softening/hardening for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$

Fig. 4.24 Flutter coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 1.625$ with damping parameter $Ca = 0.01$

Fig. 4.25 Damping rate versus dynamic pressure for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1.625$ and damping parameter $Ca = 0.01$

Fig. 4.26 Aerostatic mode softening/hardening for a 2-D isotropic simply supported cylindrical panel of height-rise $H/h = 1.625$ and damping parameter $Ca = 0.01$

Fig. 4.27 Flutter non-coalescence curve for a 2-D simply supported cylindrical panel with a height-rise of $H/h = 3$ with damping parameter $Ca = 0.01$

Fig. 4.28 Aerostatic mode softening/hardening for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and damping parameter $Ca = 0.01$

Fig. 4.29 Aerostatic mode softening/hardening for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 3.5$ and damping parameter $Ca = 0.01$

Fig. 4.30 Damping rate versus non- D dynamic pressure for a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 5$ and damping parameter $Ca = 0.01$

Fig. 4.31 Aerostatic mode softening/hardening for a 3-D isotropic simply supported curved panel of height-rise $H/h = 5$ and damping parameter $Ca = 0.01$

Fig. 4.32 Critical dynamic pressure versus panel height-rise for a 2-D isotropic simply supported cylindrical panel with damping parameter $Ca = 0.1$

Fig. 4.33 Critical dynamic pressure versus panel height-rise for a 3-D isotropic simply supported cylindrical panel with damping parameter $Ca = 0.01$ and yaw flow angle $A = 0^\circ$

Frequency Domain Method		
Critical dynamic Pressure Convergence Study		
2-D Isotropic Panel: $a = 9$ in., $b = 1$ in., $h = 0.008$ in., $H/h = 1$		
Mesh Size	Frequency, Hz (mode 1)	λ_{cr}
analytical	17.0575 (Dowell)	259 (4 modes)
19×1	16.6575	254.2017
39×1	16.6325	253.5516
59×1	16.6283	253.4085
79×1	16.6268	253.3591

Table 4.1 Critical dynamic pressure convergence study for a 2-D simply supported cylindrical panel of height-rise $H/h = 1$

Frequency Domain Method		
Critical Dynamic Pressure Convergence Study		
3-D Isotropic Panel: $a = b = 12 \text{ in.}$, $h = 0.04 \text{ in.}$, $H/h = 1$		
Mesh Size	Frequency, Hz (mode 1)	λ_{cr}
7×7	87.7773	512.5783
9×9	87.0846	503.1149
11×11	86.7121	497.7782
13×13	86.4901	494.5888
15×15	86.3477	492.5371
16×16	86.2950	491.7719
17×17	86.2511	491.1224

Table 4.2 Critical dynamic pressure convergence study for a 3-D simply supported cylindrical panel of height-rise $H/h = 1$

Chapter 5

5 MODE SHAPE CATEGORIES

The mode shapes are an important issue in the present study. In the time domain procedure, the flutter deflection approximation is a linear combination of a chosen set of mode shapes, whereas, in the frequency domain procedure, the mode shapes are an important issue in finding the critical dynamic pressure corresponding to the flutter onset. It turns out when conducting the present study that there are three categories of mode shapes for curved panels. For simply supported curved panels, for instance, it ranges from a simple set of sinusoidal mode shapes, natural mode shapes, and to a more elaborated set of mode shapes related to the pre-flutter static deformation of the curved panels called aerostatic mode shapes. Each one of the aforementioned sets of mode shapes are belonging to a specified category defined as (a) the sinusoidal category, (b) the natural category, and (c) the aerostatic category. It will be shown in the forthcoming paragraphs that the flutter onset is intimately related to the mode shape type of category and their modal participation. An exhaustive study will be conducted herein to show how these mode shapes are constructed, and how they evolve with parameters like the height-rise H/h , dynamic pressure, and the dimensionality of the curved panel 2-D or 3-D. Appended to the present study, an analysis of the linear frequencies associates with the first and second categories of mode shapes was conducted. It points out the similarities and dissimilarities between the different types of mode shapes. It is worthy to mention that the aforementioned mode shape can be related to a set of influential parameters. In the (a) category, the mode shapes are not subjected to any parameter, they are simply assumed for the simply supported panels. In the (b) category the mode shapes are

ultimately defined by the height-rise H/h of the curved panel closely associated with its geometry. Finally in the (c) category, the mode shapes are defined and influenced by three parameters, the height-rise H/h of the curved panel, the dynamic pressure λ , and the yaw flow angle A over it.

5.1 First Category: Sinusoidal Mode Shapes

Historically, 2-D and 3-D sinusoidal mode shapes were used as a linear combination in the deflection series, which approximate the flutter response of 2-D and 3-D flat plates in the non-linear PDE/Galerkin analytical approach. It was obvious to attempt to pursue the same approach when the fluttering system is a curved panel. In his two famous papers, Dowell [13, 14] used the PDE/Galerkin analytical approach and gave a worth of information concerning the non-linear flutter response of 2-D and 3-D curved plates. One of the first tasks of the present work was to reproduce Dowell's results using the versatile finite element approach in order to validate the written code associated with it. For every finite element approach it is important to validate the linear approach by matching the analytical and numerical linear frequencies as a first step. Dowell's formulas for sinusoidal frequencies are given in Appendix C. In the forthcoming chapters sinusoidal mode shapes are used for comparison purposes, and for the computation of the stability boundary margins.

5.2 Second Category: Natural Mode Shapes

The reader might be puzzled by the chosen nomenclature that clearly suggests two types or categories of mode shapes, the sinusoidal ones, and the natural ones. It will be

gradually demonstrated through the forthcoming paragraphs that the aforementioned separation is critical for the study of the non-linear flutter response of curved panels. It should be also stated clearly that the existence of two distinctive categories of mode shapes for curved panels impacts others areas of vibration studies such as free, forced vibrations, and thermal effects on curved panels. The present paragraph is dedicated to provide a clear definition of the natural mode shapes of a curved panel, and to compare the two categories of mode shapes at the mode shape and linear frequency level. The finite element linear eigenvalues and eigen-vectors are directly computed using the eigen-value problem of Eq. (3.134). The eigen-solution is a very specific unique set of eigen-values and eigen-vectors fingerprinting exclusively the particular geometry of the curved panel. It should be pointed out that there is a clear contrast between flat plates eigen-solutions and curved panels eigen-solutions. While the first one depends only on one parameter, which is the dimensional ratio a/b ; the second depends on two parameters, the dimensional ratio and the panel curvatures (i.e. height-rise H/h).

5.2.1 2-D Natural Mode Shapes

Plotting and comparing the two categories of modes shapes, the sinusoidal ones and the natural ones, for a 2-D case, one can notice very clearly the differences in the shape for high height-rises H/h . The 2-D mode shapes are featured in Fig. 5.1 and Fig. 5.2. It can be seen from Fig. 5.1, where the sinusoidal and the natural modes are plotted that there is two significant differences between the two aforementioned mode shapes. The first difference is in the arrangement of the natural mode shapes from the lowest frequency to the highest frequency. It is seen that the natural mode corresponding to the

lowest frequency have two opposite equal half waves, while the second natural mode have only one undetermined half wave shape. This contrasts drastically with the isotropic flat plate arrangement of sinusoidal mode shapes also reported on the same figure. The second difference can be immediately noticed, the odd natural mode shapes are not anymore perfect sinusoidal waves. As the height-rise H/h of the curved panel increases the differences between the two categories of mode shapes are more evidently accentuated, Fig. 5.2. It is astonishing to notice that the even natural mode shapes are unchanged and correspond exactly to their sinusoidal counterparts.

The linear natural frequencies of the 2-D isotropic cylindrical panel with the following dimensions: $a = 22.86 \text{ cm}$ (9.0 in.), $b = \infty$, $h = 0.020232 \text{ cm}$ (0.008 in.) were compared with the linear sinusoidal frequencies computed by the analytical PDE/Galerkin method developed by Dowell. The material properties are: $E = 1.0341 \times 10^{11} \text{ Pa}$ ($15 \times 10^6 \text{ psi}$), $\nu = 0.0$, $\rho = 8518.5 \text{ kg/m}^3$ ($0.0007971 \text{ lbxs}^2/\text{in.}^4$). The comparison is shown in Table 5.1. It can be seen for the first and third computed modes, that there is a growing disagreement between the sinusoidal frequencies computed by Dowell's PDE/Galerkin method and the natural frequencies computed by the finite element code as the height-rise H/h of the cylindrical panel increases. The numerical disagreement in the frequencies is evidently interpreted by the growing difference in the silhouette shape between the sinusoidal and natural mode shapes as shown in Fig. 5.1 and Fig. 5.2. It is remarkable to notice from the same that the even modes are not affected by the height-rise H/h of the panel. The frequencies associated with the even modes stay amazingly almost constant.

5.2.2 3-D Natural Mode Shapes

Computation of the natural frequencies by Dowell's formula in Eq. (C.6) and by the finite element method for a simply supported doubly curved panel with the following dimensions: $a = 22.86 \text{ cm}$ (9.0 in.), $b = 22.86 \text{ cm}$ (9.0 in.), $h = 0.02032 \text{ cm}$ (0.008 in.) is performed for comparison purposes. The material properties are: $E = 1.0341 \times 10^{11} \text{ Pa}$ ($15 \times 10^6 \text{ psi}$), $\nu = 0.3$, $\rho = 8518.5 \text{ kg/m}^3$ ($0.000797 \text{ lb} \times \text{s}^2/\text{in.}^4$). The panel was discretized with a mesh size of 20×20 encompassing 800 MIN3 elements. Many fundamental differences were noted as seen in Table 5.2. Whereas the frequencies numerical trend is the same as they increase, the disparity between the sinusoidal frequencies and the natural frequencies is more evident for the 3-D case. Table 5.2, shows a relative disagreement within a range of 4 % to 5 % between Dowell and the finite element computed natural frequencies for modes (3,1) and (4,1). Conversely, It is well noticed that the small disagreement does not concern the first and the second lowest frequencies where disagreements could reach 28% and 12%, respectively. It is evident that the mode shapes of the doubly curved panel are not matching the correspondent sinusoidal ones even for small height-rises as shown in Fig. 5.3 and Fig. 5.4. The comparison between the mode shapes in the aforementioned figures tells us that the curvature parameter introduces a natural dissymmetry in the mode shapes. As the curvature becomes more important the dissymmetry of the mode shapes becomes more pronounced. It can be concluded at this stage of the analysis that the mode shapes are sensitive to the curvature parameter, consequently also the linear frequencies.

5.3 Third Category: Aerostatic Mode Shapes

The aerostatic mode shapes correspond to the eigen-vector sets coming from the outcome of the eigen-value problem of Eq. (3.120). They are the mode shapes of the deflected curved panel subjected to the SAL $\{P_{sal}\}$, the aerodynamic pressure λ and the yaw flow angle. Their shape evolution as the aerodynamic pressure and the height-rise increase is presented herein for the first time in the literature. The aerostatic mode shapes will be investigated for 2-D and 3-D isotropic simply supported cylindrical panels with height-rises $H/h = 1, 3$ and 5 at different dynamic pressure. The different plots show the aerostatic mode shapes just before the beginning of the flutter instability. In the present paragraph it will be clearly demonstrated that a structural system undergoing external influences still have a set of computable mode shapes, moreover the mode shapes themselves integrate intrinsically those external influences.

5.3.1 2-D Aerostatic Mode Shapes

The 2-D aerostatic mode shapes are investigated for isotropic simply supported cylindrical panels similar to the one investigated in paragraph 4.6.1. The following height-rises are investigated:

- $H/h = 1$

Figure 4.32 features a comparison between the sinusoidal mode shapes and the aerostatic mode shapes of a cylindrical panel. The sinusoidal mode shapes are not influenced by any external loads. They are perfectly symmetric ranging from a half sine wave for mode one to four half sine waves for mode four. Conversely the aerostatic mode shapes associated with the cylindrical panel under the SAL $\{P_{sal}\}$ and dynamic pressure

λ show an intrinsic influence of the aforementioned parameters on the nature of their shape. Basically under the gradual conjugated increase of the SAL and the dynamic pressure the aerostatic mode shapes undergo a steady flattening process of maximums and minimums starting from the leading edge of the curved panel. This is evidently seen in Fig. 5.5. Obviously the influence of the two aforementioned parameters alters the symmetry associated with the curved panel sinusoidal modes generating consequently the aerostatic mode shapes. The presented modes in Fig. 5.5 tentatively lead to conclude that the nature of the aerostatic mode shapes depends exclusively upon the nature of the external loads. But is that assertion true?

- $H/h = 3$

Surprisingly in this paragraph it will be demonstrated that the assertion upon which the form of aerostatic mode shapes is exclusively determined by the nature of the external load is necessary but not sufficient. Figure 5.6 shows a clear disagreement with that assertion. The first and the second aerostatic mode shapes corresponding to a cylindrical panel of a height-rise $H/h = 3$ seems to integrate intrinsically the resistance of the structure to the external loads flattening effect. This new phenomenon shown here for the first time in the literature could explain the sudden snaps-through type of static instability. The present case unlike the first one for $H/h = 1$, shows unequivocally, that the form of the aerostatic mode shapes depends upon a well-defined balance between the internal structural force and the external loads.

- $H/h = 5$

Beside the aforementioned resistance phenomenon described in the previous paragraph, the present case illustrated in Fig. 5.7 shows an extra property of the aerostatic mode shapes. The property could be qualified as half sine-waves multiplication. The particularity here is that those half waves are significantly asymmetric.

5.3.2 3-D Aerostatic Mode Shapes at $\Lambda = 0^\circ$

The aerostatic mode shapes corresponding to a 3-D curved panel show the similar behavior trends found in 2-D curved panels. However the 3-dimensionality adds some interesting complex dynamic behaviors. In-order to show those complex behaviors, the evolution of the shape form of the 3-D aerostatic mode shapes function of the aerodynamic pressure λ and the panel height-rise H/h will be fully investigated in the forthcoming paragraphs.

- $H/h = 1$

For a 3-D near-plate system corresponding to the height-rise of $H/h = 1$ the shape evolution toward the flutter onset of the aerostatic mode shape are presented in Fig. 5.8. It shows the progress of the aerostatic mode shape 1 (left column) and the aerostatic mode shape 2 (right column) toward the flutter coalescence point defined by the critical aerodynamic pressure $\lambda_{cr} = 491$. It is clearly noticeable that the aerostatic mode shapes are greatly influenced by a conjugated balance between the SAL and the aerodynamic pressure. The figure shows clearly this influence by the fact that the aerostatic mode shapes are leaning right toward the trailing edge as the dynamic pressure increases. Those

particular mode shapes migrate from a symmetric mode shape defined in-vacuo to an asymmetric mode shape subjected to the influence of the aforementioned parameters. It is remarkable to notice that the progress of the curved system toward the flutter coalescence point is accompanied by the evolution of the two initially dissimilar aerostatic mode shapes toward the same flutter mode. Subsequently it is important to mention that for future non-linear flutter response research investigation the utilization of the aerostatic mode shapes in the deflection series approximation could lead to better and accurate results. It can be affirmed that the aerostatic mode shapes have intrinsically integrated the ability to characterize or approach more efficiently the physics of the flutter system. It can be also stated that the aerostatic mode shape is a localized mode shape depending locally on the SAL $\{P_{sal}\}$ and the aerodynamic pressure λ .

- $H/h = 3$

In the previous 3-D curved panel system of height-rise $H/h = 1$, the deflection series approximation could be made up of a well defined set of aerostatic mode shapes, essentially those referenced as the stream-wise mode shapes $(n, 1)$, where $n = \{1, 2, \dots, \infty\}$. But is the aforementioned assertion is true for higher height-rises? Figure 4.36 is a denial to that assertion. It shows that for a specified dynamic pressure, in this particular case $\lambda = 250$, the second aerostatic mode shape (right column, second row) could not be integrated in the deflection series approximation. Consequently, it can be stated that for a curved panel with high height-rises an exclusive set of aerostatic mode shapes exist for each dynamic pressure. Those sets can be very different from an aerodynamic pressure to another. The ability of the aerostatic mode shapes to change dramatically their shape

form participating to non-participating in the approximating deflection series is shown here for the first time in the literature. The previous assertion has dramatic consequences on the computation of the non-linear flutter response, it implies that for each dynamic pressure λ , a thorough investigation of the participating aerostatic mode shapes has to be performed. Obviously this represents an added computational difficulty because a close look-up at the form of the aerostatic mode shapes has to be conducted for each dynamic pressure, which is difficult and computationally costly in time.

- $H/h=5$

The present case illustrated in Fig. 5.10 brings up more complexity to the flutter analysis of the system. First of all, the flutter instability occurs at higher modes between the 6th and the 7th aerostatic mode shapes. At starting for a dynamic pressure $\lambda = 0$, the aforementioned aerostatic mode shapes seem non-participating mode shapes. Even during the dynamic pressure increase process, none of the depicted aerostatic mode shapes in Fig. 5.10 look like a participating stream-wise mode shape $(n, 1)$ type. Surprisingly, even at the flutter onset, the aerostatic mode shapes for $\lambda = 486$ look like a non-participating mode. From the previous observation it can be clearly stated that for cylindrical panels with high height-rises unsymmetrical mode shapes do participate in the flutter non-linear response. The participation of unsymmetrical mode shapes in the flutter response is demonstrated here for the first time in the literature. Obviously, the fact to include the unsymmetrical mode shapes in the deflection series approximation leads to a dramatic increase in the computational time.

Fig. 5.1 **Lowest four natural mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rises of $H/h = 0$ and $H/h = 2$**

Fig. 5.2 Lowest four natural mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rises of $H/h = 0$ and $H/h = 5$

mode (1,1)

mode (2,1)

mode (3,1)

mode (4,1)

Fig. 5.3 **Lowest four natural mode shapes of a 3-D isotropic simply supported cylindrical panel of height-rise of $H/h = 0$**

mode (2,1)

mode (3,1)

pseudo-mode (2,1)

mode (4,1)

Fig. 5.4 **Lowest four natural mode shapes of a 3-D isotropic simply supported cylindrical panel of height-rise of $H/h = 4$**

Fig. 5.5 Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 1$

Fig. 5.6 Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 3$

Fig. 5.7 Four lowest aerostatic mode shapes of a 2-D isotropic simply supported cylindrical panel of height-rise of $H/h = 5$

Fig. 5.8 First and second aerostatic mode shape progression toward flutter onset of 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 1$ and yaw flow angle $A = 0^\circ$

Fig. 5.9 First and second aerostatic mode shape evolution toward flutter onset of a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 3$ and yaw flow angle $A = 0^\circ$

Fig. 5.10 Sixth and seventh aerostatic mode shape evolution toward flutter onset of a 3-D isotropic simply supported cylindrical panel of height-rise $H/h = 5$ and yaw flow angle $\Lambda = 0^\circ$

Modes		1	2	3	4
$H/h = 1$	Dowell	17.51	25.76	58.21	103.04
	FEM	16.64	24.61	55.65	98.55
	 % 	4.97	4.46	44.40	4.35
$H/h = 2$	Dowell	33.19	25.76	58.97	103.04
	FEM	30.89	24.68	56.83	98.59
	 % 	6.92	4.19	3.63	4.32
$H/h = 3$	Dowell	49.26	25.76	60.20	103.04
	FEM	42.94	24.79	60.69	98.64
	 % 	12.84	3.76	0.81	4.27
$H/h = 4$	Dowell	65.44	25.76	61.89	103.04
	FEM	48.98	24.94	70.65	98.71
	 % 	25.16	3.18	14.14	4.20
$H/h = 5$	Dowell	81.65	25.76	64.00	103.04
	FEM	51.01	25.13	84.53	98.78
	 % 	37.53	2.54	32.00	4.13

**Table 5.1 2-D curved panel natural frequencies in Hz computation with Dowell
[14] and Finite Elements Method**

Modes		(1,1)	(2,1)	(3,1)	(4,1)
<i>H/h = 1</i>	Dowell	20.59	34.88	63.45	105.26
	FEM	24.83	34.92	61.40	101.02
	 % 	20.61	0.12	3.22	4.03
<i>H/h = 2</i>	Dowell	34.98	44.91	69.47	108.99
	FEM	44.73	47.68	69.17	105.69
	 % 	27.86	6.17	0.42	3.03
<i>H/h = 3</i>	Dowell	50.57	57.88	78.48	114.95
	FEM	64.44	63.35	80.18	113.04
	 % 	27.44	9.44	2.17	1.66
<i>H/h = 4</i>	Dowell	66.51	72.23	89.58	122.80
	FEM	82.34	80.09	93.00	122.58
	 % 	23.81	10.89	3.81	0.18
<i>H/h = 5</i>	Dowell	82.61	87.27	102.10	132.21
	FEM	98.49	97.20	107.16	133.84
	 % 	19.23	11.38	4.96	1.24

Table 5.2 Natural frequencies in Hz comparison using Dowell [14] and finite element method for a 3-D doubly curved panel

Chapter 6

6 TIME DOMAIN ANALYSIS

Historically, non-linear time domain analyses of curved panels were performed by Dowell [13] using the PDE/Galerkin method. He developed a type 3 analysis based on Bolotin [17] equations of motion for 2-D and 3-D curved panels of constant curvature in the x and/or y directions. He showed that a gradual increase of the stream-wise curvature reduces the flutter critical dynamic pressure, and increases the flutter amplitude for 2-D curved panels. He pointed-out the critical role played by the SAL in defining the critical dynamic pressure. He also emphasized the need to employ non-linear structural theory accounting for the deflection of the curved panels under the SAL. A thorough analysis of the pre/post-flutter time response for 2-D and 3-D curved panels under supersonic yawed flow angles is carried out herein. The flutter pre/post-time response inquiry is achieved through tools such as the fourth-order Runge-Kutta numerical method, bifurcation diagrams, phase and power spectrum density plots.

6.1 Convergence Study

6.1.1 Modal participation

A convergence study is conducted using an increasing number of natural mode shapes to verify the mode convergence of the non-linear flutter response. The study is performed in a manner to ensure that all the participating modes in the flutter response are accounted for. Modal participations for 2-D panels and 3-D panels are thoroughly investigated for each panel height-rise H/h according to the following 2-D scheme case.

A 2-D curved panel with dimensions similar to the one used in section 4.6.1 with a mesh size of 79×1 and height-rise $H/h = 5$ was chosen to perform the present analysis. The mesh-size in the x direction were chosen high enough to eliminate the concerns about mesh refinement, ensuring that the highest mode, in this particular case the 18th streamwise mode has enough DOF to characterize all the intrinsic modes to its construction. The dynamic pressure at which the study is performed is $\lambda = 800$. The chosen dynamic pressure is the highest value for which a 2-D bifurcation curve was built for a height-rises $H/h = 5$. The modal participation values displayed in Table 6.1 show that assuming four modes to investigate the flutter response for a dynamic pressure $\lambda = 800$ is not enough to obtain modal convergence. The table shows that the 10 lowest modes are quite sufficient to achieve convergence of the flutter response. When carried out for 8, 10, 14, and 18 modes, the modal participation coefficients demonstrate clearly that the 3rd, 4th, 7th, and 10th lowest modes are the most important participating modes in the flutter response. Mode 3 is the dominant mode, it accounts for more than half of the total LCO deflection (64.87 %). Modes 4, 7, and 10 contribute 23.62 %, 6.76 % and 3.36 % to the LCO deflection, respectively. Surprisingly modes 1 and 2 have almost no participation to the LCO's (0.02 % and 0.35 %). It is clearly seen that higher modes 3, 4, 7 and 10 do participate significantly in the flutter response as predicted by Dowell [14].

In conclusion, it appears according to Table 6.1 that using 10 natural modes to capture the flutter response is sufficient. However, in order to investigate the flutter response for 2-D and 3-D cases, and be on the safe side 10 to 16 or more natural modes will be used as needed to anticipate the participation of eventual higher modes.

6.1.2 Time Integration Step

Another important convergence issue is the time integration step ΔT necessary to ensure the convergence of the flutter time and velocity responses. Those time and velocity responses are computed by the application of the 4th order Runge-Kutta numerical integration scheme to the non-linear flutter modal EOM. An initial guess integration time step ΔT is first chosen for each height-rise H/h , then halved as many times as needed to ensure the convergence of two successive flutter time responses. The selected time integration steps ΔT 's ensuring the convergence of the flutter response are given out in the forthcoming bifurcation sections for 2-D or 3-D panels at various height-rise H/h . For instance, for a 2-D panel with a height-rise $H/h = 1$ a time integration step of $\Delta T = 10^{-4}$ seconds is first tried, then halved and rounded to $\Delta T = 5 \times 10^{-5}$ seconds, $\Delta T = 2.5 \times 10^{-5}$ seconds, $\Delta T = 10^{-5}$ seconds, and $\Delta T = 5 \times 10^{-6}$ seconds. Plotting the flutter time responses for the first three time integration steps, Fig. 6.1, it is seen that $\Delta T = 10^{-4}$ second is amply sufficient to ensure a convergent amplitude and period at the high dynamic pressure $\lambda = 800$.

6.2 2-D Panel Time Domain Analysis

The frequency domain methodology was used for the determination of the pre-flutter dynamic behavior and the panel flutter stability boundary. To target the post-flutter dynamic behavior at the onset and beyond, one has to solve directly Eq. (3.158) by numerical integration. The output of the numerical integration is a time history and velocity response of the fluttering system. The numerical integration is performed using the fourth-order Runge-Kutta numerical scheme, Eqs (3.173) to (3.177). To investigate

the post-flutter characteristics, a 2-D simply supported cylindrical system similar to the one described in section 4.6.1 is used. The first 16 stream-wise flutter modes were used in the modal truncation approximation, their use required a high-order mesh size (79x1). In the following paragraphs, flutter investigation tools such as, bifurcation diagram, time history response, phase lagging and power spectrum density plots are thoroughly used.

6.2.1 Bifurcation Diagrams for 2-D Panels with $H/h = 1$

Bifurcation diagrams are meant to show in one single figure the evolution of the dynamic behavior of the fluttering system as the dynamic pressure λ increases. The numerical integration is performed using the lowest 16 natural modes. The time integration step is $\Delta T = 10^{-4}$ seconds, and 200,000 thousands time integration steps are computed. Fig. 6.2 shows the bifurcation diagram for the near plate curved systems of height-rise $H/h = 1$. Unlike the flat plate system, the transverse displacement, here at $x/a = 81.25\%$ for a curved panel in the pre-flutter region shows a gradual static displacement in the range of $0 \leq \lambda \leq 254$. Within the aforementioned static range, the out of plane static displacement $\{W\}$ amplitude is continuously and steadily changing, it has a static maximum located at $\lambda = 112$, and $W_{peaks}/h = 0.095$. The static minimum is located at $\lambda = 253$, and $W_{peaks}/h = 0.0965$ just before the flutter onset. It is also noticed that within the static region a small fluttering section pops up. This section is located between $188 \leq \lambda \leq 210$. The nature of the flutter within the small section is discussed in the forthcoming paragraphs. The curved panel starts to flutter at the critical dynamic pressure $\lambda_{cr} = 255$. The bifurcation diagram shows that the nature of the flutter oscillations belongs to the flutter motion category of limit-cycles, then chaotic. The same diagram shows also a

heavy asymmetry characterizing the aforementioned oscillations. Minimum and maximum oscillations are not evenly displayed about the $y = 0$ axis as it is the case for flat plates. This complication related to the added curvature prompted the elaboration of a new bifurcation diagram that exhibits simultaneously the out of plane minimum and maximum amplitude shown in Fig. 6.2. Concerning the critical dynamic pressure, it is worthy to mention that there is an almost perfect matching between the two methods.

6.2.2 Flutter Time History Analysis for 2-D Panels with $H/h = 1$

To get deep insights into the pre and post-flutter behavior of the cylindrical panels, flutter time history responses were plotted in Fig. 6.3 and Fig. 6.4. The two figures show panoply of patterns of the flutter time history response as the dynamic pressure increases. In the pre-flutter region, it shows a clear static behavior, though interrupted by complex limit-cycle oscillations around the dynamic pressure $\lambda = 200$. The capture of the aforementioned flutter behavior within the static region underlines the complexity of the dynamics involved in curved panels and point-out the vital necessity to plot the bifurcation diagrams in order to capture such phenomenon. The post-flutter region shows clear evidence of simple and complex limit cycle and random oscillations. The simple oscillations are of periodic type, while the complex ones show different patterns of relative maximums and minimums. In all cases except for $\lambda = 200$, the flutter oscillations are downwardly biased to the negative y -axis for the particular selected location at $x/a = 81.25\%$. For high dynamic pressure, in the present case $\lambda = 750$ the positive amplitude as pointed out by Dowell [13] is of the order of the thickness h , while the negative amplitudes are of the order of two-panel thickness h . The time history responses show

also that the post-flutter oscillating non-linear frequencies keep rising as the dynamic pressure increases. At $\lambda = 342$ and $\lambda = 650$ the time history responses can be connected directly with the bifurcation diagram of Fig. 6.2 where bifurcation patterns change in the neighborhood of the aforementioned dynamic pressures, particularly the bifurcation doubling around $\lambda = 558$.

6.2.3 Phase Plots for 2-D Panels with $H/h = 1$

Digging deeper in the analysis of flutter behavior of cylindrical panels, the phase plots allow the investigation of the type of flutter response and the nature of the lagging between the out-of-plane displacement and its corresponding velocity. Single point phase plots denote a static behavior except in the neighborhood of the dynamic pressure $\lambda = 200$, where a limit-cycle arises characterize the complexity of the flutter behavior in the static region. For limit cycles, in this particular case for dynamic pressures $\lambda = 200, 262, 342, 450$ and $\lambda = 650$ shown in Fig. 6.5, the phase plots are unsymmetrical closed loop multiple ellipse-like, denoting the evidence of the periodicity of the flutter process and its complexity. Those LOC's are complex attractors where the panel motion settles for a particular dynamic pressure. Throughout the bifurcation diagrams of Fig. 6.2, one can spot how numerous these attractors are. The bifurcation diagrams are really the fingerprints of the flutter motion associated with the panels. The inner and outer closed loops in the phase plots reveal the multiplicity of relative maximums and minimums in the time history response. Bifurcation emergence is directly related to the multiplication of inner or outer loops in the phase plots. One can also denote the asymmetry of the ellipse with respect to the $\dot{W} / h = 0$, this brings the fact that the descending process along

the z-axis of the cylindrical panel is totally dissimilar from the ascending process in time. The aforementioned facts are difficult to detect directly from a simple time history response. The phase plots bring also another subtlety of the flutter processes, it shows the patterns of acceleration, deceleration and patterns of constant velocity experienced by the cylindrical panel. Around the dynamic pressure of $\lambda = 750$, the cylindrical panel starts to show pre-random and random patterns. As the dynamic pressure increases the phase plots are not anymore closed loops. They are open random loops, a chaotic motion takes place, Fig. 6.6.

6.2.4 Power Spectral Density for 2-D Panels with $H/h = 1$

The power spectral density plots investigate the frequency content of the panel flutter time history response. The frequency spectrum enumerates explicitly all the participating frequencies composing the flutter response. Figure 6.6 shows the variety of frequency spectrums associated with the time history response of a cylindrical panel at different dynamic pressure λ and the importance of the first three or four embedded lowest frequencies in the flutter time response. Those non-linear frequencies are the dominant ones. For the particular flutter section occurring in the midst of the static region around $\lambda = 200$, the dominant frequency is represented by the peak corresponding to the second lowest one at 23.19 Hz. The ratio of the dominant one to the second lowest one is 1.958 indicating that its participation is moderately small. The LCO popping up could simply be a transient phenomenon. For the dynamic pressure of $\lambda = 262$ just after the hopf bifurcation, the first lowest frequency is the dominant one at 15.87 Hz. It is important to connect the frequency computed by the TDM (15.87 Hz) with the frequency computed with the FDM (16.62 Hz) at the flutter onset. Their evident closeness indicates that the

two methods are firstly validating one another, and secondly complementing one another in the sense that one emphasizes pre-flutter behavior, whereas the second emphasizes post-flutter behavior. Herein the first lowest frequency at 15.87 Hz is largely dominant with a ratio to the second lowest one equal to 168. The ratio of the third frequency to the first one is almost 3 indicating clearly that the flutter phenomenon has a free vibration nature. As the dynamic pressure increases the dominant frequency keep shifting up as seen for $\lambda = 342$ (18.31 Hz), 450 (23.19 Hz), 650 (26.85 Hz), and $\lambda = 750$ (29.29 Hz) in Fig. 6.7. The figure also suggests that high dynamic pressures bring more participating modes and the frequency spectrum becomes more and more a broadband frequency spectrum. Rising further the dynamic pressure over $\lambda = 750$ a chaotic motion takes place.

6.2.5 Bifurcation Diagrams for 2-D Panels with $1 < H/h \leq 1.625$

It was seen that for a height-rise of $H/h = 1$, the post flutter response showed LCO at the flutter onset, and chaotic motion scattered with LCO's for remote dynamic pressure. Since the investigation of cylindrical panels by the FDM demonstrated that the panel experiences a softening effect, it is worthy to wonder how the post-flutter response of a softened panel will be? To plot the bifurcation diagram of such a system, numerical integration is performed using the lowest 12 stream-wise natural modes. The time step is $\Delta t = 10^{-4}$ seconds, and 300,000 time integration steps are computed. The bifurcation diagram for the height-rise $H/h = 1.625$ at $x/a = 81.25\%$, Fig. 6.8 shows a static behavior in the pre-flutter region and a chaotic motion with scattered LCO in the post-flutter region. The pre-flutter interval, $0 \leq \lambda \leq 230$, can be subdivided into two portions according to the FDM results. The softening portion, $0 \leq \lambda \leq 221$, and the hardening

portion, $221 < \lambda \leq 230$. Those results can be clearly correlated with the TDM. The results show clearly a jump phenomenon (snap-through) at $\lambda \approx 221.235$. The panel gets slightly in the negative W/h negative region, then jumps asymptotically along the vertical $\lambda = 221.235$ from $W/h = -0.0312$ at $\lambda = 221.23$ to $W/h = -0.0774$ at $\lambda = 221.24$. The aforementioned jump modified drastically the stiffness properties of the panels, that is why the FDM shows a sudden hardening effect after a steep softening, Fig. 4.24, and Fig. 4.25. Consequently, the snaps through suspected but not proven with the FDM are herein fully demonstrated with the TDM, Fig. 6.9. In the static interval the two methods are complementary. The FDM has brought up clearly the softening effect, whereas, the TDM has brought up clearly the snaps through. As a result of the over whole softening effect a chaotic motion takes place in the post-flutter interval. At the flutter onset, despite the sudden hardening the cylindrical panel is still in the soft region, that it might be impossible for it to show an LCO type of motion. Instead, the panel jumps directly into chaotic motion as shown in the time history response in Fig. 6.10 and the phase plots Fig. 6.11. However, it is also seen that within the broad chaotic motion shown in the bifurcation diagram, Fig. 6.8, small portions do switch to complex LCOs. For remote dynamic pressure, the panel stiffens enough to switch to LCOs. It is essential to illustrate the occurring softening effect during the static motion of the panel by plotting the power spectral density using the time history response at the flutter onset. The FDM gave a flutter onset occurring at $\lambda_{cr} = 230$ and a fluttering onset frequency of 11.25 Hz. The TDM method gives through the data plotted in Fig. 6.12 a broadband frequency domain where the frequencies 9.76 Hz, 28.07 Hz, and 53.71 Hz are the dominant ones in decreasing influence order at the flutter onset $\lambda_{cr} = 228$. The computed two results match

pretty well for the critical dynamic pressure, but since the TDM revealed that most of the flutter behavior in the post-flutter region is random, it is impossible to have only one dominant frequency as suggested by the FDM. However, the frequency suggested by the FDM did show up very closely in the frequency spectrum as the first dominant frequency, 9.76 Hz.

6.2.6 Bifurcation Diagrams for 2-D Panels with $H/h > 1.625$

Static or flutter responses after the critical height-rise $H/h = 1.625$ show another type of dynamic behavior. Bifurcation diagrams were plotted for $H/h = 2.4$, 3, and 5. Time responses are convergent for time integration step $\Delta t = 10^{-4}$ seconds for $H/h = 2.4$, and 3. For $H/h = 5$, the convergence of the time response is obtained for a time integration step of $\Delta t = 10^{-5}$ seconds. 30,000 and 300,000 time integration steps are computed for $H/h = 3$ and $H/h = 5$, respectively. It shows now the existence of a transition interval where static, LCO, or chaotic motion can alternate randomly. It is very remarkable that the transition zone ends around $\lambda \approx 200$, which is almost the value of the critical dynamic pressure predicted by the FDM for the cases pictured in Fig. 6.13. LCO's and random oscillations are transient phenomenon because small changes in the dynamic pressure can settle again the panel in the aerostatic modes. In the transition zone as well as in the established LCO region, it is remarkable that the nature of the limit cycles reduces to the simplest form of oscillation, periodic as shown in Fig. 6.14. Those periodic LCO's act as backbone attractors where the panel settles. They are clearly featured in the bifurcation diagrams as almost continuous upper limit and lower limit lines with very slow increasing amplitude. As the height-rise increases new lines of settlement appears or new

LCO attractors appear as shown for $H/h = 3$ and $H/h = 5$ in Fig. 6.13. Those LCO are still simple periodic type of oscillations, they only differ in amplitudes and frequencies, Fig. 6.15. It is striking to notice that LCO's alternation is accompanied with alternating softening/hardening as well. As the dynamic pressure increases, the dominant frequency shift from 651.85 Hz for $\lambda = 564$, up to 996.03 Hz for $\lambda = 566$, and down to 251.46 Hz for $\lambda = 572$. The multiplicity of these alternating LCO attractors is intimately related to the flutter participating mode sequences. For $\lambda = 564$, the dominant mode is mode 2 with a participation level at 52.28 %, for $\lambda = 566$, the dominant mode is mode 4 with a participation level at 62.91 %, and for $\lambda = 572$, the dominant mode is mode 1 with a participation level at 67.26 %. It can be concluded that for post-near plate height-rises, $H/h > 1.625$, cylindrical panels develop a discrete number of LCO attractors. As the height-rise increases the number of attractors is increased.

6.3 3-D Panel Time Domain Analysis

Solving for the post-flutter response of 3-D cylindrical panels is quite involving. For near-plate cylindrical systems a procedures similar to those of flat plates are still applicable. The assertion that the overall cylindrical panel deflection is a combination of stream-wise modes $(n, 1)$, $n = \{1, \dots, \infty\}$ is a still a valid assumption for near plate-cylindrical systems. For cylindrical systems with high height-rises, the aforementioned assumption is challenged by the fact that there is no guaranty that a mode keeps its stream-wise $(n, 1)$ shape for all dynamic pressures λ . The search for stream-wise participating mode becomes local, it depends on each dynamic pressure as shown in the frequency domain section 4.7.3.2. This is a quite involving problem necessitating the

development of new software codes and the availability of a tremendous computational power. For the present study, we will continue to assume the cylindrical panel deflection as a linear combination of a defined number of a stream-wise set of natural mode shapes even for high height-rises H/h . However, the aforementioned post-flutter response will be challenged by integrating non stream-wise modes in the truncated deflection series to investigate if ever they have a modal participation. Solution of the modal Eq. (3.158) is performed using the fourth-order Runge-Kutta numerical scheme presented in Eqs. (3.173) to (3.177). The post-flutter response characteristics are investigated for a 3-D isotropic stream-wise cylindrical panel of dimensions $12.0 \times 12.0 \times 0.04$ in. ($30.48 \times 30.48 \times 0.1016$ cm) with simply supported edges along the x and y directions. Immovable in-plane boundary conditions $u(0, y) = u(a, y) = v(x, 0) = v(x, b) = 0$ are considered. The cylindrical panel is modeled using 16×16 mesh size representing 512 MIN3 elements. The number of structure DOF is 735 for the curved system after applying the boundary conditions. The material density is $\rho = 0.00025234 \text{ lb} \times \text{s}^2 / \text{in.}^4$ ($2700 \text{ kg} / \text{m}^3$), the Young's modulus of elasticity is $E = 5 \times 10^6 \text{ psi}$ ($7.1 \times 10^{10} \text{ Pa}$) and the Poisson ratio $\nu = 0.3$. The lowest eight stream-wise modes are used in the truncated deflection series approximation to compute the first bifurcation diagram, then the sixteen lowest natural modes are considered to compute the second bifurcation diagram for comparison purposes. Numerical integration is performed using a time step of $\Delta t = 5 \times 10^{-5}$ seconds, and 300,000 time-step discretization points are computed. The panel exposure time is 15 seconds. In the forthcoming paragraphs, flutter characteristics such as, bifurcations, time response, phase lagging, and power spectrum density are thoroughly investigated for height-rises $H/h = 1, 3$ and 5 , and for yaw flow angles $A = 0^\circ$, and $A = 45^\circ$.

6.3.1 Bifurcation Diagrams for 3-D Panel with $H/h = 1$ and $\Lambda = 0^\circ$

Unlike the flat plates, cylindrical panel bifurcation diagrams show a clear dissymmetry as illustrated in Fig. 6.16 and Fig. 6.17. The transverse displacement $\{W\}$ monitored in here along the $y = b/2$ centerline at $x/a = 75\%$ shows a dissymmetry with respect to $W_{peak}/h = 0$ axis. In general the pre-flutter region shows a gradual static displacement before flutter. For the case involving the eight lowest stream-wise natural modes, the pre-flutter region stretches between the dynamic pressures $0 \leq \lambda \leq 498$, the static maximum deflection is located at $\lambda = 230$, and $W_{peaks}/h = 0.099$, and the flutter onset begins at $\lambda_{cr} = 500$. For the case involving the sixteen lowest natural modes, the pre-flutter region is within the dynamic pressure range of $0 \leq \lambda \leq 488$. Within the static range, the out of plane static displacement $\{W\}$ amplitude is steadily increasing then slightly decreasing, it has a static maximum located at $\lambda = 228$, and $W_{peaks}/h = 0.095$. The static minimum is located at $\lambda = 0$. The cylindrical panel experiences a Hopf bifurcation at the critical dynamic pressure $\lambda_{cr} = 490$, the type of flutter corresponds to periodic or complex periodic LCO's. The two bifurcation diagrams show also a heavy asymmetry characterizing the aforementioned oscillations. Minimum and maximum oscillations are not evenly distributed about the $W_{peaks}/h = 0$ axis as for the flat plates. LCO's are biased toward the negative z -axis. It is interesting to notice that in both cases, the bifurcation diagrams are very similar in shape despite some fundamental differences. It is worthy to notice that the critical dynamic pressure computed by the time domain method $\lambda_{cr} = 491$ is very close to the critical dynamic pressure computed using the 16 lowest modes $\lambda_{cr} = 490$. The close agreement hints that it is necessary to account for non-stream-wise modes

in the flutter deflection truncated series approximation for cylindrical panel. This is a major difference with flat plates. Yet this has to be confirmed by investigating the modal participation of non stream-wise modes. Moreover, Fig. 6.17 shows two sudden shifts occurring at $\lambda = 938$ and $\lambda_{cr} = 1512$. They indicate sudden change in the curved panel dynamic behavior. These changes have yet to be explained.

6.3.2 Modal Participation for 3-D Panel with $H/h = 1$ and $\Lambda = 0^\circ$

As pointed out in the previous paragraph it is necessary to investigate the participation of non stream-wise mode in the flutter deflection of curved panels. Table 6.2 shows that for low dynamic pressure herein $\lambda = 492$, the panel dynamic behavior is very close to flat panels. The participating modes are only stream-wise modes with the exception of mode 11 who has a very small contribution of 1.28 %, see Appendix B. The first, the second, the fifth and the ninth modes represent the major contribution to the flutter deflection. As the dynamic pressure increases to $\lambda = 800$ and $\lambda = 950$, the sixth, the seventh, the eighth, the eleventh, and the fourteenth modes become more involved in the flutter deflection. It is remarkable to notice that the participating non stream-wise modes are either odd symmetrical higher modes or diagonal wavy new modes proper to cylindrical panels. The wavy new modes are only specific to cylindrical panels and can be viewed in appendix B. At the dynamic pressure $\lambda = 1130$, the LCOs show a slight peak amplitude random oscillations. This fact triggers the participation of more non stream-wise modes in the truncated deflection series approximation. Finally the panel dynamic behavior at $\lambda = 1550$ shows that for high dynamic pressures, mode one and mode two are still dominant but losing participation to higher stream-wise modes.

6.3.3 Time History Analysis for 3-D Panel with $H/h = 1$ and $\Lambda = 0^\circ$

One of the important steps to characterize the post-flutter dynamic response is to plot the time response of the fluttering cylindrical panel. Herein, the post-flutter response is characterized by a large Hopf bifurcation and LCO's small jumping denoting a change in their intrinsic nature. Those smoothly or sudden alteration of the LCOs are reported in the time responses illustrated in Fig. 6.18. Basically the bifurcation diagram of Fig. 6.17 shows four types of LCOs. The first type comes into sight at the flutter onset $\lambda = 490$, the oscillations are almost evenly split in the positive and negative z -axis sides. The cylindrical panel behaves first like a flat plate system, then, as the dynamic pressure increases the flutter oscillations become unevenly shifted toward the negative amplitudes. The second type becomes visible when a smooth bifurcation appears in a form of a slight positive amplitude snap through. A specimen of this LCO type is represented for $\lambda = 800$. The third type of LCO is characterized by a sudden shift toward the upper amplitudes and represented for $\lambda = 950$ and $\lambda = 1130$. The last LCO type is also characterized by a small downwardly jump in the amplitude. The LCO's become straight oscillation, see Fig. 6.18 for $\lambda = 1550$.

6.3.4 Phase Plots for 3-D Panel with $H/h = 1$ and $\Lambda = 0^\circ$

The phase plots illustrated in Fig. 6.19 are quite distorted but well-defined closed loops. They are clear evidence of the cylindrical panel periodic LCO's behavior when subjected to a supersonic flow. The phase plots variety of forms over a wide range of dynamic pressures illustrates the complex dynamic behavior associated with the flutter

parameters of cylindrical panels. The small inner loop shown for $\lambda = 800$ accounts for a single dynamic snap through, while the thicker LCO associated with the dynamic pressure $\lambda = 1130$ denotes a random peaks perturbation associated with the LCO behavior. The described LCO's are limit cycle attractors since the curved panel settles in an oscillating configuration for a specified dynamic pressure. In this particular case five types of general attractors were found.

6.3.5 Power Spectral Density for 3-D Panel with $H/h = 1$ and $\Lambda = 0^\circ$

The analysis of the frequency spectrum related to the various aforementioned LCOs show a clear decreasing sequence of frequencies composing the flutter oscillations for a particular dynamic pressure. In most of the cases, the fundamental frequency, which has the highest participation, is the dominant one. The participation of the non-linear harmonics lays way behind. For example at $\lambda = 492$, the power ratio between the first and the second participating frequency is 72 indicating that the first frequency is largely dominant. For $\lambda = 800$ it is only 4.5. It is remarkable to notice that the agreement between FDM and the TDM is not only concerning the critical dynamic pressure, but it extends to the flutter non-linear frequency. The FDM gives for the critical dynamic pressure $\lambda_{cr} = 491$ and a frequency of $f_{cr} = 123.17$ Hz, whereas the TDM gives at the critical dynamic pressure $\lambda_{cr} = 492$ a non-linear frequency of $f_{cr} = 122.07$ Hz. The LCO's frequencies computed by the two aforementioned methods, and corresponding to the flutter onset match almost perfectly. The developed TDM serves again as a validation of the newly developed FDM. As illustrated in Fig. 6.20 shows that the post-flutter non-linear frequencies keep increasing as the dynamic pressure increases. For this particular

cylindrical panel the non-linear frequency shifting corresponds to panel the stiffness hardening process.

6.3.6 General Study of 3-D Panel with $H/h = 1$ and $A = 45^\circ$

To investigate the influence of the yaw flow angle on the flutter critical dynamic pressure, a cylindrical panel similar to the one taken in paragraph 4.9.1 is investigated. The bifurcation diagram illustrated in Fig. 6.21 shows 4 limit-cycle discontinuities. The critical dynamic pressure computed by the TDM, $\lambda_{cr} = 556$, match pretty well the one computed by the FDM, $\lambda_{cr} = 557$. It demonstrates that the 16 lowest mode are sufficient to capture the flutter dynamics for a cylindrical panel with height-rise $H/h = 1$ and yaw flow angle $A = 45^\circ$. It is important to notice that the critical dynamic pressure at flow angle $A = 0^\circ$ is less than the critical dynamic pressure at $A = 45^\circ$. This tells us that the critical dynamic pressure at the aforementioned angle is not really a minimum. To get an idea about which yaw angle has the minimum critical dynamic pressure, a computational scheme with sweeping angles has to be done. The time response illustrated in Fig. 6.22 shows four types of LCO's. It shows essentially that the LCO's are either simply periodic or complex periodic with small ripples characterizing the upper position vibrations. The phase plots given in Fig. 6.23 illustrate the aforementioned complexity of the flutter LCO's. The power spectrum density plots associated with the present configuration are plotted in Fig. 6.24. They illustrate the periodicity of the LCO's and the non-linear frequency shifting, 126.95 at $\lambda = 556$, 141.60 Hz at $\lambda = 690$, 180.66 Hz at $\lambda = 1000$, and 214.84 Hz at $\lambda = 1360$ for the first frequency. It is also very remarkable to point out the almost perfect matching between the flutter onset frequency computed with the FDM,

128.67 Hz, and the TDM, 126.95 Hz. In the transient LCO's appearing on the bifurcation diagram between $534 \leq \lambda \leq 556$, the dynamic of the system does show a softening, 117.98 Hz at $\lambda = 540$, but this transient behavior won't last in time.

6.3.7 General Study of 3-D Panels with $H/h > 1$ and $\Lambda = 0^\circ$

The bifurcation diagram for the height-rise $H/h = 3$ is quite different from the one corresponding to the height-rise $H/h = 1$. The flutter starts with a small LCO then very quickly after a short dynamic pressure interval a permanent chaotic motion affects the cylindrical panel. The first LCO starts within a tiny transition zone at a dynamic pressure $\lambda = 606$. Within the transition zone LCO's and static deflections alternate randomly. The last static deflection is recorded for a dynamic pressure $\lambda = 632$. From $\lambda = 634$ the flutter dynamic behavior is dominant. It is very remarkable to notice that the time domain computed dynamic pressure $\lambda_{cr} = 634$ corresponding to the effective starting of the flutter behavior only matches almost perfectly the critical dynamic pressure $\lambda_{cr} = 633$ computed by the FDM. To achieve this perfect matching, the sixteen lowest natural mode shapes of the curved panel were assumed in the deflection series approximation. A first attempt involving the first eleven stream-wise natural modes gave a higher dynamic pressure at $\lambda_{cr} = 748$. Since the critical dynamic pressure $\lambda_{cr} = 633$ computed with the FDM is exact, it was obvious to check if additional modes other than stream-wise modes could participate in the deflection series approximation. The TDM computation involving the sixteen lowest modes confirmed the participation of at least two extra non-stream-wise natural modes in the curved panel deflection series approximation. Identification of these two modes and the amount of their modal participation will be detailed in the

forthcoming paragraphs. The flutter chaotic motion affecting the cylindrical panel produces a stream of numerous relative peaks reported in the dark region of the bifurcation diagram illustrated in Fig. 6.25. The figure obviously records the trends of maximum and minimum relative peaks without given a clear peak-mean trend, the relative peaks are so numerous that is impossible to extract information other than the maximum peaks upper bound and the minimum peaks lower bound. The LCO's last for a short interval, then a chaotic motion starts to take place. Analyzing the time response of the panel at the flutter onset $\lambda_{cr}= 632$, one can notice that the quasi-complex periodic LCO's cross through the negative side over a very short distance. Most of the vibrations occur in the positive side, Fig. 6.26 and Fig. 6.27. Such dynamic behavior can be attributed to a stiffness strengthening with the height-rise $H/h = 3$. The power spectral density represented in Fig. 6.27 shows the multiplicity of the peaks associated with the flutter response of the panel. Three major frequencies emerge, they are 166.01 Hz, 334.47, 500.48 Hz. The dominant frequency is the first one outpacing the others two by a factor of 2. Power ratio between the first and the second frequency is 1.32, whereas the power ratio between the first frequency and the third frequency is 1.60. It is also noticeable that TDM flutter onset frequency 166.01 Hz is within the range of the frequency given by the FDM 178.53 Hz.

Risings further the height-rise of the cylindrical panel to $H/h = 5$, the use of the TDM becomes quite involving, and computationally very costly. The search for the critical dynamic pressure required several trials with an increasing set number of modes. The modes are arbitrary not only stream-wise. For the modes corresponding to the 16 lowest frequencies, the critical dynamic pressure is $\lambda_{cr}= 568$. For the 24 lowest modes, the

critical dynamic pressure is $\lambda_{cr}= 510$, and for the 33 lowest modes, the critical dynamic pressure is $\lambda_{cr}= 486$. The implication of the lowest 33 modes in the deflection series in the latest case leads to a perfect match with the critical dynamic pressure given by the FDM, $\lambda_{cr}= 486$ occurring by the coalescence of mode 6 and mode 7. The aforementioned perfect matching has to be tempered since it is occurring in a transient zone where LCO's, and random oscillations alternate with the panel static state in Fig. 6.28. The real critical dynamic pressure begins at $\lambda_{cr}= 504$. It could mean that more higher modes have to be added in the deflection series. The computational cost becomes unrealistic. The TDM shows herein its limitation and the FDM shows its computational strength and advantage. However after succeeding in computing several points, the TDM shows also its advantages in revealing the types of flutter time history responses, phase plots, and power spectral density plots shown in Fig. 6.29. The most striking features of the presented transient responses is the positive non-zero mean characterizing the time responses. At the particular indicated location 75 %, the panel is vibrating with a pretty high dominant frequencies 510.25 Hz for $\lambda_{cr}= 486$, and 507.81 Hz for $\lambda_{cr}= 496$ over a very short positive amplitude interval. Very likely the frequency 500.48 Hz at $\lambda_{cr}= 496$ could be linked to the frequency coalescence of modes 6 and 7 computed by the FDM, which occurs at 500.87 Hz. This tells us that the fluttering process occurring at $\lambda_{cr}= 496$ is due essentially to the coalescence of mode 6 and 7. The motion of the cylindrical panel at $\lambda_{cr}= 496$ is resembling to a beating phenomena motion as illustrated in Fig. 6.29. Increasing further the dynamic pressure, the panel experiences a deep and rapid softening effect characterized by a dominant frequency of 197.75 Hz at $\lambda_{cr}= 508$, Fig. 6.30. Notice that the value of frequency 193.75 Hz is very close to the coalescence frequency of mode

1 and 2 computed by the FDM, and the nature of the LCO is completely different from the one described for $\lambda_{cr}= 496$. Important conclusions have to be drawn at this point. The flutter instability due to the coalescence of modes 6 and 7 predicted by the FDM is fully demonstrated herein with TDM. A new phenomenon that we call coalescence shifting is fully established herein. At $\lambda_{cr}= 520$, a non-zero-mean random motion start to take place as illustrated by Fig. 6.30.

6.4 Stress Analysis: Case Study of 2-D Panels

Since in the post-flutter phase the curved panel may experience large LCO's deflections, it is important herein to check that the deflections associated with those LCO's are not enough large to trigger a panel breakdown. The stresses computation is based on the panel's time history. They are computed using Eq. (2.34) and compared to the yielding stresses of the chosen material, herein a type of steel with a yielding strength between 300 and 400 Mpa. A general criterion can be then deduced to determine a panel limiting dynamic pressure exposure.

A 2-D simply supported cylindrical panel similar to one studied by Anderson [40] in section 4.6.1 is investigated for principal stresses for two different height-rises, $H/h = 1$ and $H/h = 5$. The illustration in Fig. 6.31 presents an instant deflection with W_{max}/h experienced by the panel at the highest dynamic pressure $\lambda = 800$ represented on the bifurcation diagram of Fig. 6.2 for a height-rise $H/h = 1$. The corresponding stress distributions for $\bar{z} = \pm h/2$ is given in Fig. 6.32. The figure shows an alternance of tensile and compressive stresses for $\bar{z} = +h/2$ and almost a tensile stress for $\bar{z} = -h/2$. The highest principal stress is experienced in the neighborhood of the highest curvature at

-3.22×10^6 MPa. The aforementioned principal stress is far less than the yielding stress of the steel material. Using the von Mises criterion, it can be concluded that at the dynamic pressure $\lambda = 800$ and the height-rise $H/h = 1$, the panel is not going to fail. For the case where the height-rise of the panel is increased to $H/h = 5$ and the dynamic pressure is taken as $\lambda = 800$. It is shown from the illustration in Fig. 6.33 for the highest LCO deflection experienced by the panel and the illustration shown in Fig. 6.34 by the highest stress distribution, respectively, that the panel experience the highest principal stress near by the curvature at 3.74×10^8 MPa. According to the von Mises criterion, the principal stress is within the critical values of the yielding stress. The cylindrical panel may then fails.

Fig. 6.1 Time integration step convergence for 2-D simply supported cylindrical panel with height-rise $H/h = 1$

Fig. 6.2 **Bifurcation diagram for simply supported 2-D cylindrical panels of**
height-rise $H/h = 1$

Fig. 6.3 Flutter time history responses of a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25\%$ location with LCO patterns

Fig. 6.4 Flutter time history responses of a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at 81.25 % location with random oscillation patterns

Fig. 6.5 Phase plots for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25\%$ location with LCO patterns

Fig. 6.6 **Phase plots for simply supported 2-D cylindrical panel of height-rise**
 $H/h = 1$ at $x/a = 81.25\%$ location with random patterns

Fig. 6.7 Power spectral density for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$ at $x/a = 81.25\%$ location

Fig. 6.8 **Bifurcation diagram for simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$**

Fig. 6.9 Static portion of bifurcation diagram illustrating the snap-through phenomenon for simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$

Fig. 6.11 Flutter phase plots of a simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$ at 81.25 % location

Fig. 6.12 Power spectral density for a simply supported 2-D cylindrical panel of height-rise $H/h = 1.625$ at $x/a = 81.25$ % location

Fig. 6.13 Bifurcation diagrams for simply supported 2-D cylindrical panels of height-rise $H/h = 2.4, 3, \text{ and } 5$

Fig. 6.14 Flutter time history responses, phases plots, and power spectrum density of simply supported 2-D cylindrical panels of height-rise $H/h = 3$, and 5 at $x/a = 81.25\%$ location

Fig. 6.15 Flutter time history responses, and power spectrum density illustrating alternating periodic LCO's of simply supported 2-D cylindrical panels of height-rise $H/h = 5$ at $x/a = 81.25$ % location

Fig. 6.16 Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location involving the 8 lowest stream-wise modes

Fig. 6.17 Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location involving the 16 lowest stream-wise modes

Fig. 6.18 Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.19 Phase plots for a simply supported 3-D cylindrical panel of height-rise

$H/h = 1$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.20 Power spectral plot for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.21 Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location involving the 16 lowest stream-wise modes

Fig. 6.22 Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location

Fig. 6.23 Phase plots for a simply supported 3-D cylindrical panel of height-rise

$H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location

Fig. 6.24 Power spectral plot for a simply supported 3-D cylindrical panel of height-rise $H/h = 1$, flow angle $\Lambda = 45^\circ$ at $x/a = 75\%$ location

Fig. 6.25 Bifurcation diagram for a simply supported 3-D cylindrical panel of height-rise $H/h = 3$ and $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.26 Time history response for a simply supported 3-D cylindrical panel of height-rise $H/h = 3$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.27 Time history response and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 3$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.28 Bifurcation diagrams of simply supported 3-D cylindrical panel of height-rise $H/h = 5$ and $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.29 Time history responses, phase plots, and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 5$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.30 Time history responses, phase plots, and power spectrum density of a simply supported 3-D cylindrical panel of height-rise $H/h = 5$, flow angle $\Lambda = 0^\circ$ at $x/a = 75\%$ location

Fig. 6.31 LCO deflection for a simply supported 2-D cylindrical panel of height-rise $H/h = 1$

Fig. 6.32 **Stress distribution of a simply supported 2-D cylindrical panel of**
height-rise $H/h = 1$

**Fig. 6.33 LCO deflection for a simply supported 2-D cylindrical panel of
height-rise $H/h = 5$**

Fig. 6.34 Stress distribution of a simply supported 2-D cylindrical panel of height-rise $H/h = 5$

Modes	Number of Participating Modes				
	4	8	10	14	18
	Modal Participation in %				
1	0.01	0.02	0.18	0.02	0.02
2	0.26	0.33	0.35	0.34	0.35
3	70.91	67.10	65.27	65.04	64.87
4	28.81	24.11	23.77	23.67	23.62
5		0.20	0.21	0.21	0.21
6		0.10	0.10	0.10	0.10
7		8.07	6.77	6.81	6.76
8		0.07	0.06	0.06	0.06
9			0.05	0.04	0.04
10			3.41	3.38	3.36
11				0.11	0.11
12				0.03	0.02
13				0.13	0.12
14				0.11	0.11
15					0.03
16					0.09
17					0.09
18					0.05

Table 6.1 Modal participation in % at a specified non-dimensional dynamic pressure $\lambda = 800$ for a 2-D curved with height-rise $H/h = 5$

Modes	Dynamic Pressure λ				
	492	800	950	1130	1550
1	43.04	40.16	40.86	38.07	39.16
2	38.51	32.12	34.86	32.89	34.22
3	0.55	0.52	0.70	2.54	0.78
4	0.48	0.38	0.40	1.33	0.35
5	11.30	14.46	10.02	10.35	11.07
6	0.47	3.38	1.77	1.41	2.01
7	0.50	1.16	1.10	2.20	2.02
8	0.74	1.29	1.01	1.97	2.40
9	2.39	4.48	2.18	1.63	4.43
10	0.35	0.50	0.30	1.00	0.59
11	1.28	0.72	2.14	1.08	0.88
12	0.04	0.10	0.23	1.91	0.08
13	0.08	0.13	0.30	2.10	0.24
14	0.13	0.24	2.77	0.35	0.90
15	0.10	0.16	0.99	0.44	0.61
16	0.04	0.20	0.37	0.75	0.27

Table 6.2 **Modal participation values at various dynamic pressures for simply supported square isotropic cylindrical panel at $H/h = 1$, flow angle $\Lambda = 0^\circ$ and $Ca = 0.01$**

Chapter 7

7 SUMMARY AND CONCLUSIONS

The present research is central to build a better understanding of the static/dynamic processes behind the flutter of curved skin panels in yawed supersonic flow environment. The in-depth comprehension of the non-linear phenomena underlying the panel structural performances will enable the advent of promising new efficient and accurate analysis and design tools for new a generation of aircraft, missiles, and spacecraft. To this end, a finite element Frequency Domain Method (FDM) and a Time Domain Method (TDM) were developed to predict the pre/post-flutter response and stability boundaries of 2-D and 3-D curved panels under three major supersonic flow configurations, stream-wise, cross-stream, and arbitrary yawing flows. The proposed approach uses the von Karman non-linear strain-displacement relation, the Marguerre curved plate theory, the first order shear deformation theory, and the quasi-steady piston theory.

The most significant contributions of this work are the FE formulation and the solution procedures. The FE formulation is designed to include any analytically or numerically formulated curved panel geometry based on the Marguerre curved plate theory. The additional stiffness associated with the curved geometry is fully developed. In the solution procedure, the present work established a parallel between the EOM's of flat plates and curved panels. Both of the two flutter problems, flat plates and curved panels, are represented by a similar set of self-excited EOM ($\text{RHS} = 0$). A particular effort has been made to include the influence of the dynamic pressure and SAL in the tangent matrix $[K_{\text{tan}}]$. The curved panel stiffness and deflection shape are accurately determined in the pre-flutter phase under the SAL for a specific dynamic pressure λ using

the Newton-Raphson iteration scheme. The main difficulty that lies in predicting the panel stiffness variability [76] in function of the dynamic pressure λ , the panel height-rise H/h , and the yaw flow angle A parameters, is fully addressed herein through the accurate computation of the tangent stiffness matrix $[K_{\text{tan}}]$. The use of the tangent stiffness matrix featuring the stiffness of the deflected panel under a specific set of parameters $(\lambda, H/h, A)$ leads to the exact self-excited flutter EOM of curved panels for the first time in the literature.

The mode shapes of the deflected curved panels (called herein aerostatic modes) are determined through an eigen-value/eigen-vector problem. They are used to predict the flutter stability boundary margins of the panel under the aforementioned set of parameters $(\lambda, H/h, A)$. The characteristics of the aerostatic modes are fundamentally different from the sine-function mode shapes used in the classic analytical PDE/Galerkin method. The advent of the aerostatic mode shapes clearly demonstrates that the sine-function based PDE/Galerkin method is not accurate when attempting to predict the flutter onset of curved panels. Flutter coalescence and damping-rate diagrams obtained by the coalescence of two distinct aerostatic mode shapes define clearly the stability boundary margins of the curved panel. The use of the aerostatic mode shapes sets a higher value of the flutter onset with respect to the computed value if sine-functions are used for high height-rises in the 2-D case.

The method is also very successful in predicting softening/hardening effects, and sudden static snap-through affecting the panels in the pre-flutter phase. One of the most significant results in the 2-D case, is that when the panel height-rises increase from $H/h = 0$ to $H/h = 1.625$, the panel experiences a gradual softening effect leading to a static snap

through for $H/h = 1.625$. The snap-through triggers a sharp hardening effect. The gradual softening effect could be critical and lead to a serious panel failure. A comparison of the stress distributions throughout the panel with the panel yielding stress is worthwhile in the aforementioned case. Moreover, an unexpected outbreak of flutter could happen at a frequency lower than the lowest panel natural frequency. Beyond the panel height-rise of $H/h = 1.625$, the panel still experiences a softening effect, specifically the first aerostatic mode, but it is suddenly interrupted by the occurrence of a chaotic motion. For the 3-D case, the softening effect could lead to higher mode coalescence triggering an undesirable chaotic motion. The importance and the gravity of those higher modes coalescence could be investigated through the study of the panel the damping rate. The particular study of cylindrical panels performed herein showed globally that curvature has a detrimental effect on the flutter boundaries for 2-D panels, and beneficial for 3-D panels at an optimum height-rise. Influence of the yaw flow angle was also investigated for the 3-D case. The results showed that yaw flow angles are beneficial for the flutter boundaries for a height-rise of $H/h = 1$. For $H/h = 3$, the yaw flow angle is extremely beneficial till an optimum value. Beyond that value higher modes coalescence has a detrimental effect on the flutter boundaries. The FDM also suggested through the study of the damping rate diagrams the existence of transition pre-flutter zones due essentially to higher mode coalescence.

Additionally, studying step by step the evolution of the aerostatic mode shapes function of the dynamic pressure, the method showed very clearly the gradual mode switching effect, where an aerostatic mode under the conjugated effect of the SAL and the dynamic pressure can switch from a stream-wise mode to a non-stream-wise mode.

The demonstrated existence of the mode switching phenomenon complicates further the flutter study of curved panel. In fact the flutter response could no longer be based on a minimal set of streamwise natural modes, because anyone of these modes can switch from streamwise to non-streamwise implicating that all modes should be used in investigating the panel deflection. The computational effort becomes significantly costly.

In the TDM, system EOM in structural node DOF are obtained and then transformed into modal coordinates using a defined set of the curved panel natural modes. The non-linear modal equations are heavily coupled and may contain up to 33 natural modes. The modal EOM are then solved for flutter response by a fourth-order Runge-Kutta numerical scheme. Time histories, phase plots, power spectral densities, and bifurcation diagrams for cylindrical panels under arbitrary flow angles were thoroughly investigated. The method revealed its strength in determining very effectively time/velocity history flutter responses. Those time histories were put together for specific set of parameters (λ , H/h , A) and specific geometries to come up with a bifurcation diagram. Those bifurcation diagrams uncovered many types of flutter dynamic behavior. For 2-D cylindrical panels, flutter dynamic behavior ranged from LCO's for slightly curved panel (near plate panels), to chaotic flutter for medium curvature ($H/h \approx 2.5$), to jumping LCO's for higher curvature ($H/h \approx 5$). Those jumping LCO's commonly called LCO attractors have remarkable characteristics, they consist of simple positive and negative peaks, and their differences reside on the amplitude values only. Moreover small amplitude LCO's have higher frequencies than higher amplitude LCO's. For 3-D cylindrical panels the flutter dynamic behavior varied between two types of dynamic behavior. (1) LCO for slightly curved panel and (2) chaotic behavior for panels with height-rise $H/h \approx 3$ and beyond.

The bifurcation diagrams also uncovered transition zones between the static phase and the flutter phase of the curved panel. In those transition zones static, LCO, and chaotic behavior could alternate randomly as the dynamic pressure varies. Through the power spectral density tool, the TDM and FDM give similar results with respect to the flutter boundary margins, and the flutter onset frequencies for LCO type of flutter behavior. The static snap-through predicted by the FDM are proven and fully demonstrated with the TDM.

Speaking about computational efficiency and performance, the FDM has been proven to be computationally more effective than the TDM. For curved panels the TDM requires the integration of a high number of natural modes to attain convergence for high height-rises, up to 33 modes for a height-rise $H/h = 5$, making the method computationally extremely costly.

Future work would include the effect of aerodynamic heating, in-plane temperature distribution and temperature gradient through the panel thickness, on flutter response, and failure/fatigue life under the combined aerodynamic, thermal and random/acoustic loads.

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9 APPENDICES

A. SYSTEM SPLIT MATRICES

The significance of the matrix terms in expression (3.97), and (3.123) are presented in this appendix

$$[M_b] = \begin{bmatrix} [M]_b & [M]_{b\psi} \\ [M]_{\psi b} & [M]_{\psi\psi} \end{bmatrix} \quad (A1)$$

$$[G] = \begin{bmatrix} [G]_{bb} & [G]_{b\psi} \\ [G]_{\psi b} & [G]_{\psi\psi} \end{bmatrix} \quad (A2)$$

$$[A_x] = \begin{bmatrix} [A_x]_{bb} & [A_x]_{b\psi} \\ [A_x]_{\psi b} & [A_x]_{\psi\psi} \end{bmatrix} \quad (A3)$$

$$[A_y] = \begin{bmatrix} [A_y]_{bb} & [A_y]_{b\psi} \\ [A_y]_{\psi b} & [A_y]_{\psi\psi} \end{bmatrix} \quad (A4)$$

$$[K_b] = \begin{bmatrix} 0 & 0 \\ 0 & [K]_{\psi\psi} \end{bmatrix} \quad (A5)$$

$$[K_B^T] = \begin{bmatrix} 0 & [K]_{m\psi} \end{bmatrix} \quad (A6)$$

$$[K_B] = \begin{bmatrix} 0 \\ [K]_{\psi m} \end{bmatrix} \quad (A7)$$

$$[K_m] = [[K]_{mm}] \quad (A8)$$

$$[K_s] = \begin{bmatrix} [K]_{bb}^s & [K]_{b\psi}^s \\ [K]_{\psi b}^s & [K]_{\psi\psi}^s \end{bmatrix} \quad (A9)$$

$$[K_b^{\theta_o}] = \begin{bmatrix} [K]_{bb}^{\theta_o} & [K]_{b\psi}^{\theta_o} \\ [K]_{\psi b}^{\theta_o} & [K]_{\psi\psi}^{\theta_o} \end{bmatrix} \quad (\text{A10})$$

$$[K_B^{\theta_o^T}] = \begin{bmatrix} [K]_{mb}^{\theta_o} & [K]_{m\psi}^{\theta_o} \end{bmatrix} \quad (\text{A11})$$

$$[K_B^{\theta_o}] = \begin{bmatrix} [K]_{bm}^{\theta_o} \\ [K]_{\psi m}^{\theta_o} \end{bmatrix} \quad (\text{A12})$$

$$[N1_b] = \begin{bmatrix} 0 & [N1]_{b\psi} \\ [N1]_{\psi b} & [N1]_{\psi\psi} \end{bmatrix} \quad (\text{A13})$$

$$[N1_{mb}] = \frac{1}{2} \begin{bmatrix} [N1]_{mb} & [N1]_{m\psi} \end{bmatrix} \quad (\text{A14})$$

$$[N1_{bm}] = \begin{bmatrix} [N1]_{bm} \\ [N1]_{\psi m} \end{bmatrix} \quad (\text{A15})$$

$$[N1_b^{\theta_o}] = \begin{bmatrix} [N1]_{bb}^{\theta_o} & [N1]_{b\psi}^{\theta_o} \\ [N1]_{\psi b}^{\theta_o} & [N1]_{\psi\psi}^{\theta_o} \end{bmatrix} \quad (\text{A16})$$

$$[N1^{N_b}] = \begin{bmatrix} [N1]_{bb}^{N_b} & [N1]_{b\psi}^{N_b} \\ [N1]_{\psi b}^{N_b} & [N1]_{\psi\psi}^{N_b} \end{bmatrix} \quad (\text{A17})$$

$$[N1^{N_m}] = \begin{bmatrix} [N1]_{bb}^{N_m} & [N1]_{b\psi}^{N_m} \\ [N1]_{\psi b}^{N_m} & [N1]_{\psi\psi}^{N_m} \end{bmatrix} \quad (\text{A18})$$

$$[N1^{N_{\theta}}] = \begin{bmatrix} [N1]_{bb}^{N_{\theta_o}} & [N1]_{b\psi}^{N_{\theta_o}} \\ [N1]_{\psi b}^{N_{\theta_o}} & [N1]_{\psi\psi}^{N_{\theta_o}} \end{bmatrix} \quad (\text{A19})$$

$$[N2_b] = \begin{bmatrix} [N2]_{bb} & [N2]_{b\psi} \\ [N2]_{\psi b} & [N2]_{\psi\psi} \end{bmatrix} \quad (\text{A20})$$

$$\{P_{sal}^b\} = -\lambda \cos \Lambda \begin{Bmatrix} P_b^{s_x} \\ P_{\psi}^{s_y} \end{Bmatrix} - \lambda \sin \Lambda \begin{Bmatrix} P_b^{s_y} \\ P_{\psi}^{s_x} \end{Bmatrix} \quad (\text{A21})$$

B. MODE SHAPES FOR HEIGHT-RISE $H/h = 1$

C. 2-D AND 3-D SINUSOIDAL FREQUENCIES FOR CURVED PLATES

1 Sinusoidal Frequency Comparison for 2D-Curved Panels

In his PDE/Galerkin approach, Dowell provided the non-linear, non-dimensional equation of motion of a 2-D curved plate. Out of the PDE/Galerkin analytical equation, a general formula for the computation of the curved plate linear frequencies was pulled out. The formula for finding the linear sinusoidal frequencies of a 2-D isotropic simply supported curved plate is given as:

$$f_s = \frac{1}{2\pi} \sqrt{\frac{(s\pi)^4}{2} + 12\Gamma_x^2 \left(\frac{1 - (-1)^s}{s\pi} \right)^2} C \quad (C.1)$$

where

$$C = \frac{2D}{\rho h a^4} \quad (C.2)$$

is the non-dimensionality making multiplier.

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (C.3)$$

is the flexural rigidity.

$$\Gamma_x = \frac{a^2}{h R_x} \cong 8 \frac{H}{h} \quad (C.4)$$

is the non-dimensional curvature in the x direction.

$$s = \{1, 2, 3, \dots, \infty\} \quad (C.5)$$

is an integer index parameter that define the higher order frequencies of the curved plate. It can be seen from Eq. (C.1) that the first term under the square root sign is correlated to the frequency of a flat plate while the second term accounts for the frequency of the curved plate.

2 Sinusoidal Frequency Comparison for 3D-Curved Panels

For a 3-D isotropic simply supported curved plate with constant curvatures R_x in the x direction and R_y in the y direction, Dowell provided the non-dimensional PDE/Galerkin equation of motion for 3-D curved plate. Out of the PDE/Galerkin analytical equation of motion a general formulas for the computation of the curved plate sinusoidal frequencies can be pulled out. The formula for the sinusoidal frequencies of a 3-D isotropic simply supported curved plate is given as:

$$f_{mn} = \frac{1}{2\pi} \sqrt{\left[\pi^4 \left[m^2 + n^2 \left(\frac{a}{b} \right)^2 \right]^2 + 12(1-\nu^2) \frac{\left[m^2 \Gamma_y + n^2 \Gamma_x \left(\frac{a}{b} \right)^2 \right]^2}{\left[m^2 + n^2 \left(\frac{a}{b} \right)^2 \right]^2} \right]} C \quad (C.6)$$

where

$$C = \frac{D}{\rho h a^4} \quad (C.7)$$

is the non-dimensionality making multiplier

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (C.8)$$

is the flexural rigidity

$$\Gamma_x = \frac{a^2}{h R_x} \quad (C.9)$$

is the non-dimensional curvature in the x direction.

$$\Gamma_y = \frac{b^2}{h R_y} \quad (C.10)$$

is the non-dimensional curvature in the y direction.

$$m, n = \{1, 2, 3, \dots, \infty\} \times \{1, 2, 3, \dots, \infty\} \quad (C.11)$$

are integer index parameters that define unequivocally the mode shapes related to the natural frequencies of the curved panel. It can be noticed from the Eq. (C.6) that the first term of the right hand side of the equation is related to the linear sinusoidal frequencies of a flat plate system, whereas, the second term is related to the influence of the curved plate geometry incorporating a non-dimensional curvature Γ_x in the x direction and a non-dimensional curvature Γ_y in the y direction.