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Fermi Problems: Educated Guesses

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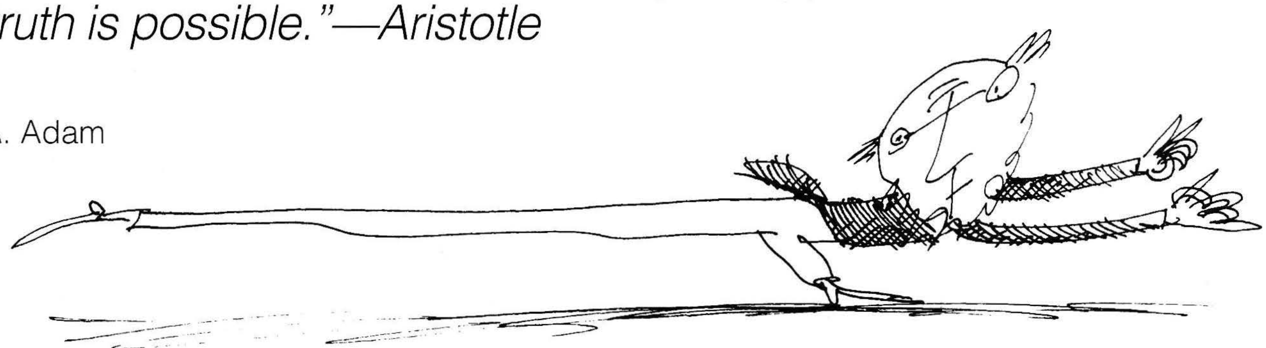
Educated guesses

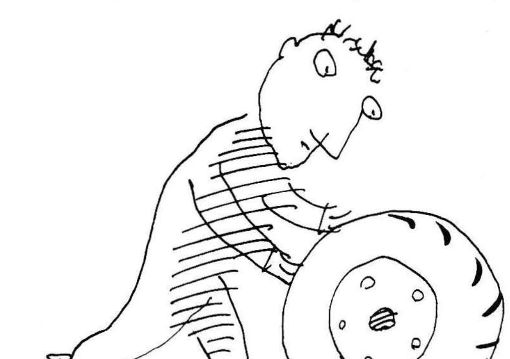
"It is the mark of an instructed mind to rest satisfied with the degree of precision which the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible."—Aristotle

by John A. Adam



WE MAY NOT BE AS ERUdite as Aristotle, or as brilliant as Enrico Fermi, but we can learn to apply elementary reasoning to obtain "ballpark estimates" for problems (subsequently named "Fermi problems") in the manner attributed to that great physicist.

Several years ago a short article by David Halliday appeared in *Quantum* (May 1990). It was called "Ballpark Estimates," and in the context of a specific problem Halliday showed how to obtain order-of-magnitude answers to problems by breaking them down into their components and making appropriate common-sense estimates. The problem was to estimate how many "rubber atoms" are worn from an automobile tire for each revolution of the wheel. We shall consider a slight variant of this problem below, but what I find especially appealing in Halliday's article is the dialogue he provides en route with a typical reader's questions. While not necessarily a prerequisite to this article (having got you to read this far, I don't intend to let you go easily!), I urge you to read it nonetheless.



Of course, the ideas expressed and methods used in such Fermi problems go far beyond physics into the realm of everyday activities (though filling the Earth with sand may not qualify as an *everyday* activity). Two excellent resources I have enjoyed reading and using are *Innumeracy* by John Allen Paulos and *Consider a Spherical Cow* by John Harte. You'll recognize some of the problems cited here if you have already encountered these books. After a while you'll get comfortable with posing and estimating answers to your own Fermi problems. The book by Paulos will be an eye-opener for many: in particular, he shows the power of plausible assumptions coupled with simple calculations. The book by Harte is a good introduction to mathematical modeling (particularly environmental problem solving) with little or no use of calculus. While we're on the subject of interesting books, *The Universe Down to Earth* by Neil de Grasse Tyson has some chapters (1 and 3) relevant to the present article.

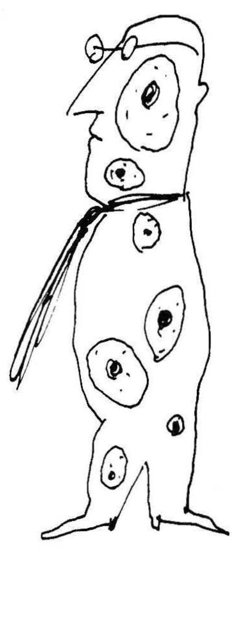
In much of what follows, letters are used to represent typical dimensions or other quantities. This will enable you to obtain your own estimates, though you should resist the temptation to just "plug in" your numbers in the formula without following the prior reasoning. Almost certainly we'll differ on typical sizes of objects (for instance, grains of sand). But almost as certainly we'll choose typical dimensions in the range (for this example) of 10^{-1} mm $\le d \le 2$ mm, so we probably won't differ significantly in our

Art by Sergey Ivanov

subsequent order-of-magnitude answers. Remember that it's to be understood that whenever ratios of dimensional quantities are to be sought, a conversion of units may be necessary in order to compare like quantities. For completeness, actual numerical estimates are given—some of their values may surprise you.

Needless to say, the question will be asked: so what if I know how to estimate the number of grains of sand that would fill Buckingham Palace? (Now there's a thought!) Apart from a spell in jail for attempting to verify such an estimate, it's a great encouragement to realize that a "back of the envelope" type of calculation can be carried out with a modicum of salient information for a "real world problem." Not only might this save a considerable amount of money and computer time on occasion, it might also give you a greater appreciation for the power of arithmetic. I've seen the "lights go on" when intelligent, educated people realize at last the distinction between 10^6 seconds (11½ days) and 10^9 seconds (32 years). Sometimes we need the right pegs to hang numbers (and concepts) on!

Among the simplest estimation problems are those arising from ratios of lengths, areas, and volumes. Thus, if D is a typical linear dimension of a given object (for example, a classroom), and $d < D$ is a typical linear dimension of a smaller object (for example, a piece of popcorn—we'll say popped!), then $N = D^3/d^3$ is the approximate number of smaller objects that would fill the latter. Thus, by using appropriate choices of D and d we can come up with estimates for the following questions.



1. *How many golf balls does it take to fill a suitcase?*

2. *How many pieces of popcorn does it take to fill a room?*

3. *How many soccer balls would fit in an average-size home?*

4. *How many cells are there in a human body?*

5. *How many grains of sand would it take to fill the Earth?*

Related problems involve volumetric measures of fluids.

6. *What is the volume of human blood in the world?*

7. *How many one-gallon buckets are needed to empty Loch Ness (and thus expose the monster)?*

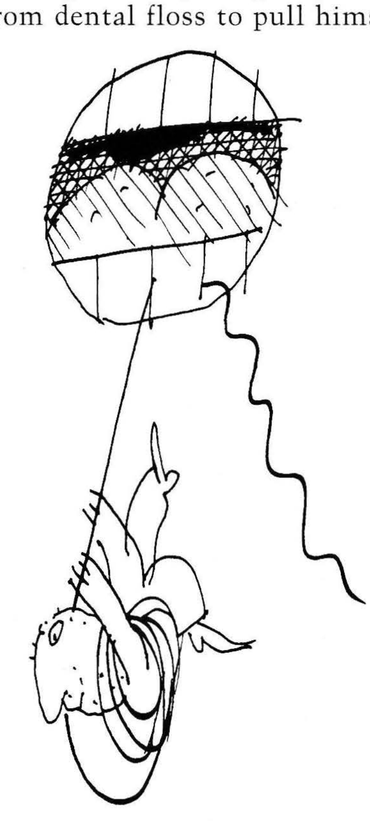
Sometimes everyday objects are obviously represented (or misrepresented) by cubes. Thus, if we are asking how many objects with a typical linear dimension d will fill a space with linear dimensions a, b, c, the formula $N = abc/d^3$ is appropriate. So for problem 1, we might suggest $a = 20$, $b = 24$, $c = 8$, and $d = 1.5$ inches, respectively, so $N \cong 10^3$. For problem 2, suppose $a = 10$ ft, $b = 20$ ft, $c = 15$ ft (classroom size), and $d = 1$ cm. Then, after conversion to metric units, $N \cong 3{,}000 \cdot 30^3 \cong 10^8$. For problem 3, consider $D = 30$ ft and $d = 1$ ft, which gives $N \cong 10^4$. Problem 4 yields 10^{14}, and the answer to problem 6 is less than 1/200 mi^3 (both of these are discussed below). For problem 5, values of $D \cong 10^4$ km and $d = 1$ mm yield $N \cong (10^4 \cdot 10^3 \cdot 10^2 \cdot 10)^3 = 10^{30}$. A cubic Earth, you ask? Don't worry, you'll get over it without falling off (see the comment on problem 14 below). Using the fact that 1 ft^3 of liquid (water, soup, blood, and so on) is about 7.5 gallons, we arrive at $N \cong 10^{12}$ buckets to empty Loch Ness (problem 7). The loch has a volume of approximately 2 mi^3, so $2 \cdot 5{,}280^3 \cdot 7.5 \cong 10^{12}$. And while we're talking about gallons, here's problem 8.

8. *One gallon of paint is used to cover a building of area A. How thick is the coat?*

Clearly, if A is in square feet, then the thickness $d = 1/7.5A$ ft. For the "cubical house" of problem 3 (full of soccer balls by now, you'll recall), $A = 6 \cdot 30^2 \cong 5 \cdot 10^3$ ft^2, so $d \cong 10^{-5}$ ft $\cong 10^{-4}$ in.

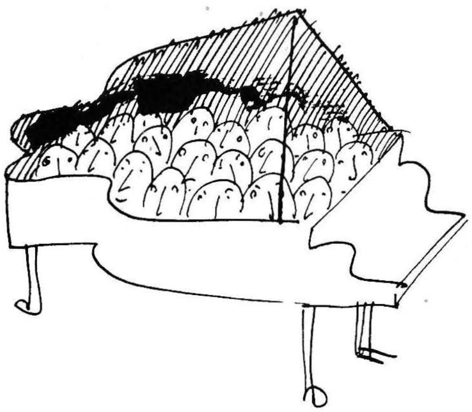
Questions of a more sophisticated nature require, not surprisingly, more terms in the estimation formulas. Thus we have the following problems.

9. *How much dental floss does a convict need?* A recent newspaper article featured the story of an inmate at a correctional center in West Virginia who escaped from the prison grounds by using a rope made from dental floss to pull himself



over the courtyard wall. The rope was estimated to be the thickness of a telephone cord, and the wall was 18 ft high. Taking 4 mm for the diameter of a telephone cord and 1/2 mm for the diameter of the floss, then the number of floss fibers in a cross section is $(4 \div 1/2)^2 \cong 60$, and if each packet of floss contains the standard length of 55 yards, the number of packets required is $N \cong (20 \cdot 60)/(55 \cdot 3) \cong 7$.

10. *Estimate the number P of piano tuners in a certain city or region.* Consider a population in the region totaling N, with an average of p pianos per family (generally $p < 1$).



Suppose that pianos are tuned b times a year on average (generally we expect $0 \le b < 2$), so the number tuned per year is approximately Npb/n_1, where n_1 is the average size of a household. If each tuner tunes n_2 pianos a day ($0 < n_2 < 4$ in general), this corresponds to $250n_2$ pianos per year (for a reasonable working year of $50 \cdot 5$ days). So the number of tuners in the region (city, town, country) is approximately $Npb/250n_1n_2$. Let's pop in some numbers. If, for New York City, say, $N \cong 10^7$, $n_1 = 5$, $b = 0.5$, $p = 0.2$, $n_2 = 2$, then $P \cong (10^7 \cdot 10^{-1})/(250 \cdot 10) \cong 4 \cdot 10^2$—that is, an order of magnitude of 10^2 to 10^3.

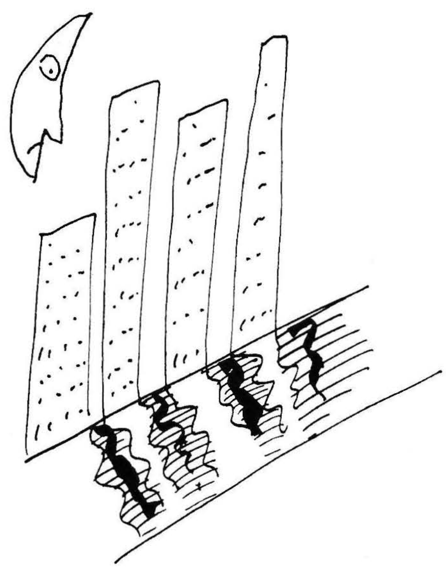
11. *Estimate the number C (for cobbler) of shoe repairers in a city or region.* If such a person spends on average t hours on a repair job in an average working day that's T hours long, T/t is the average number of repairs performed per day. Clearly, some shoes are worth repairing and some are not. Suppose the "average pair of shoes" is repaired on average every n years, leading to a repair rate of $1/n$ per year. For a 250-day working year, our cobbler can perform an average of $250T/t$ repair jobs a year, and

in a population of N, repairs N/n pairs of shoes each year. This leads to an estimate of $Nt/250nT$ cobblers in the region. Thus, if we take as our region this time the whole of the United States (we're being a little ambitious here, of course, but this is a question I'm constantly being asked), then $N \cong 2.5 \cdot 10^8$, $t \cong 1/2$, $T \cong 10$, $n \cong 2$; so $C = (2.5 \cdot 10^8 \cdot 1/2)/250 \cdot 2 \cdot 10) \cong 10^4$.

12. *Estimate how fast human hair grows (on average) in mph.* If the hair is cut every n months (usually $n \le 2$) and the average amount cut off is x inches, then x/n inches per month $\cong x/n \cdot 1/(5{,}280 \cdot 12) \cdot 1/(30 \cdot 24)$ mph $\cong 10^{-8}(x/n)$ mph. If $n = 2$ and $x = 1$, then the rate of hair growth is approximately 10^{-8} mph.

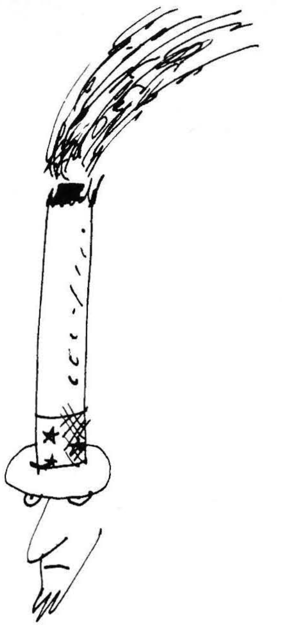
Now back to the blood problem (number 6).

6. (redux) *Estimate the total volume of human blood in the world.* For a population of $5 \cdot 10^9$ with an *average* of 1 gallon of blood per per-



son, $V \cong 5 \cdot 10^9/7.5 \cong 7 \cdot 10^8$ ft^3. This, as Paulos points out, could be contained in a cube of side length $(7 \cdot 10^8)^{1/3} \cong 900$ ft. Putting things a little more prosaically, since Central Park has an area of 1.3 mi^2, all this blood would cover Central Park to a depth of about $(7 \cdot 10^8)/[1.3 \cdot (5{,}280)^2] \cong 20$ ft. Hmm.

13. *Estimate the number of cigarettes smoked annually in the US.* Let f be the fraction of people in the population who smoke and n the average number of cigarettes smoked



per day. Then $N = 2.5 \cdot 10^8 \cdot 365 \cdot fn \cong 10^{11}$, if $f \cong 10^{-1}$ and $n \cong 10$.

14. *The asteroid problem.* In the light of the impact(s) of ex-comet Shoemaker–Levy on Jupiter's outer atmosphere, the question has been raised: could it happen here on Earth? It may have happened already—one theory for dinosaur extinction (not Gary Larson's)[1] is that about 65 million years ago such an encounter occurred—this time with an asteroid. Eventually dust from the impact settled back on the surface of the Earth, having done a superb job of blocking sunlight and thus devastating plant and animal life. According to one hypothesis, about 20% of the asteroid's mass was uniformly deposited over the (now rather inhospitable) surface of the Earth—about 0.02 gm/cm^2. Question: how large was the asteroid? (You may feel that at this point, a more appropriate question would be: "What was the name of the bus driver?" But don't worry, we'll get to that later.) Okay—the mass is clearly about $4\pi R^2 \cdot 0.02 \cdot 5$ if R is the radius of the Earth in centimeters. This must be equated to density times volume for a cube of side length L (this is the simplest geometry to consider: the largest sphere that can be inscribed in a cube of side L differs in

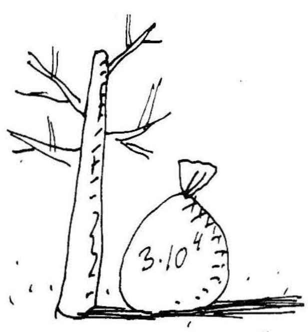
[1]His memorable cartoon shows several tough-looking dinosaurs standing around, smoking cigarettes. The caption reads: "The real reason dinosaurs became extinct."—*Ed.*

volume from that cube by a factor $\pi/6 \cong 1/2$, so this won't affect our order-of-magnitude estimate). Suppose we take a typical rock density of 2 gm/cm^3, so that $2L^3 \cong 0.4\pi R^2$, which gives us $L \cong (0.2\pi R^2)^{1/3}$. Since $R \cong$ 4,000 · 1.6 · 10^5 cm (converting miles to centimeters) = 6.4 · 10^8 cm, then $L \cong 6 \cdot 10^5$ cm, or 6 km (10 km by order of magnitude). This is not unreasonable for an asteroid (even though the dinosaurs may disagree).

15. *Thickness of an oil layer.* Perhaps no one likes to take their medicine. Rumor has it that Benjamin Franklin noted that 0.1 cm^3 of oil (was it cod-liver oil?) dropped on a lake spread to a maximum area of 40 m^2. If d is the thickness of the

layer in meters, then $40d = 10^{-7}$, so $d = 25 \cdot 10^{-10}$ m, or 25 angstroms. Interestingly, this corresponds to a "monomolecular layer" of 10–12 atoms (with atom–space–atom–... for a molecule), which is about right for a molecule of "light" oil.

16. *The number of leaves on a tree.* If r is the typical radius of a tree's canopy, the surface area of the canopy is $4\pi r^2$; and if d is (in the same units as r) a typical leaf size, an estimate for the number of leaves is $4\pi r^2/d^2$. Clearly leaves don't cover the "surface" of the canopy continuously; this does, however, compensate for the fact that there are many leaves on branches inside the canopy.



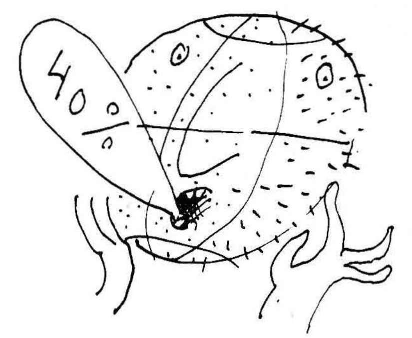
For a small tree (for example, a 15- to 20-year-old yew), the leaf canopy has a radius $r \cong 4$ ft and $d \cong 1$ in, so $N \cong 3 \cdot 10^4$—that is, an order of magnitude of 10^4–10^5 in general, if we include larger trees as well.

17. *Weekly supermarket revenue.* If there are n_1 checkout lines serving an average of n_2 customers per hour, the average customer receipt is x dollars, and the store stays open an average of n_3 hours a day, then in an average week $R \cong 7n_1n_2n_3x$ dollars. If, for example, $n_1 = 10$, $n_2 = 10$, and $n_3 = 14$, then we find that $R \cong 10^5$ dollars.

18. *Daily death rate in a city or region.* If in a city or region of population n_1 the average number of deaths per day (as listed, for example, in the obituary section of the local newspaper) is n_2, we can by a simple proportion get an estimate of the daily death rate d in the country (with a population N). Thus,

$d \cong Nn_2/n_1$. Clearly there are limits to the validity of this crude analysis. Death rates vary considerably from country to country. Nevertheless, one can get "lower bound" estimates for world death rates in a similar fashion. Thus, if $n_1 \cong 10^6$ and $n_2 \cong 30$, then $N \cong 2.5 \cdot 10^8$.

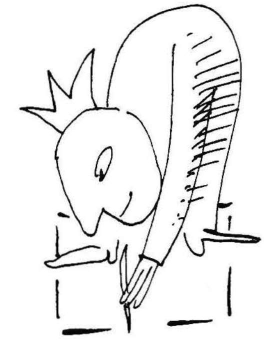
19. *The number of blades of grass on the Earth.* If 40% of the Earth's surface is covered by land, a fraction f_1 of this land is covered by grass. If the average number of blades of grass per square inch is n, then $N \cong (0.4)4\pi R^2 f_1 n$ for R measured in inches. Thus, for $R \cong 4{,}000 \cdot 5{,}280 \cdot 12$, $f_1 \cong 10^{-2}$ or 10^{-1} (this is difficult to estimate without a little research), and $n \cong 20$, then $N \cong 10^{16}$ or 10^{17}.



Now let's return to a variant of the car tire problem.

20. *What is the average depth of tread lost per revolution of a car tire?* This can be answered by a simple proportion: the distance d we require is to a typical tread t (for a new tire) as tire circumference $2\pi R$ is to length of useful mileage L. Thus, $d \cong 2\pi Rt/L$, which for $R = 1$ ft, $L = 5 \cdot 10^4$ mi, $t = 5$ mm corresponds to (after conversions!) to $d \cong 10^{-7}$ mm.

21. *Population square.* If each person on Earth were given enough space to stand comfortably on the ground without touching anyone else, estimate the length of the side of a square that would contain everybody in this way. If we give everyone a square 1/2 m on a side, then the side of the large square is $L \cong (5 \cdot 10^9)^{1/2} \cdot 1/2 \cdot 10^{-3}$ km $\cong 35$ km.

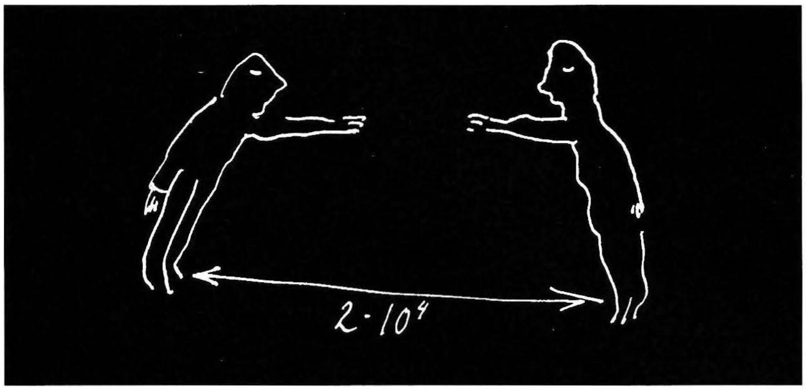


22. *Human surface area and volume.* To estimate these quantities crudely but quickly, consider a cylinder of radius r and height h: if $r \cong 1/2$ ft and $h \cong 6$ ft, then $V = \pi r^2 h \cong 5$ ft^3, and $S = 2\pi rh \cong 20$ ft^2. Since 1 ft $\cong 0.3$ m, $V \cong 0.1$ m^3. Now we're in a position to return to problem 4.

4. (redux) *Estimate the number of cells in the human body.* If we assume an average cell diameter of 10 microns, or 10^{-5} m, then since 1 ft $\cong 0.3$ m, V from problem 22 is

approximately 10^{-1} m^3, so $N \cong 10^{-1}/(10^{-5})^3 \cong 10^{14}$ cells.

23. *The average rate of growth of a child from birth to 18 years.* Over this time span the "speed" equals approximately $(h_{18} - h_0)/18 \cong 1/18$ m/yr $\cong 10^{-3}/(20 \cdot 400 \cdot 20)$ km/h $\cong 10^{-8}$ km/h—that is, about the same order of magnitude as the speed of hair growth! Perhaps we could label children as super- or subfollicular depending on whether or not they grow faster than their hair!



The remaining estimation problems concern SETI (the search for extraterrestrial intelligence) and interstellar launches. The astronomer Frank Drake has done the work for us in providing the famous Drake formula for the number N of extant technical civilizations in the galaxy. Here "technical" can be taken to mean at least as technologically capable as we are on planet Earth. Thus, if n_s equals the mean number of stars in the galaxy, f_p the fraction of these stars with planetary systems, n_p the mean number of planets suitable for life per planetary system, f_b the fraction of planets where life actually evolves, f_i the fraction of those $n_p f_b$ on which intelligent organisms have evolved, f_c the fraction of those intelligent species that have developed communicative civilization, and f_l the mean lifetime of those civilizations in terms of the age of the galaxy, then

$$N \cong n_s f_p n_p f_b f_i f_c f_l.$$

Of the seven quantities on the right of this expression, the first is astronomical in nature and well known to be about $4 \cdot 10^{11}$. The next two numbers are really educated astronomical guesses. The two following (f_b and f_i) are biological in nature, and here we're on pretty shaky ground, because we only have a sample space of one (ourselves!). The final two numbers are sociological in nature, and so in this context they're pure guesswork! Thus it happens that the numbers one puts in are indicative of one's philosophical stance: like it or not, we all have presuppositions about the universe we inhabit. Just for fun, let's see where this leads for the optimist and the pessimist. In both cases we might take $f_p \cong 0.2$ (remember that almost half the stars in our galaxy are thought to be binary systems at least) and $n_p \cong 0.1$. For the remaining four numbers, our optimist takes 1.0, 1.0, 0.5, and $10^6/10^{10} = 10^{-4}$, respectively, yielding $N \cong 10^5$–10^6. Our pessimist, on the other hand, takes the last four numbers to be 0.1, 0.1, 0.1, and $10^4/10^{10} = 10^{-6}$, respectively, yielding $N \cong 10$. Which are you?

At this point, a timely reminder: whether for a debate or a mathematical model or merely an estimate, **the argument is only as good as the weakest assumption built into it**.

24. *Mean distance between two civilizations.* Our galaxy has the shape of a disk 10^5 light-years (LY) in diameter and about 10^4 LY "thick." Obviously stars are concentrated more toward the galactic center, but we can get a crude upper-bound estimate of the mean distance between two civilizations by dividing the volume of the galaxy $[\pi(10^5)^2/4] \cdot 10^4 \cong 10^{14}$ cubic LY by the optimist's figure of $N \cong 10^6$. (Remember that 1 LY $\cong 6 \cdot 10^{12}$ mi is the distance light travels in one year. Work it out for yourself.) Taking the cube root of 10^8 gives us approximately 500 LY. On the other hand, if $N \cong 10$ (the pessimist's estimate), the distance is $2 \cdot 10^4$ LY.

25. *How many launches of interstellar space vehicles might we expect per year?* Suppose that on average each civilization is able to launch s such vehicles per year. If $N \cong 10^6$—our most optimistic estimate—there will be (at steady state) some $10^6 s$ vehicles arriving per year somewhere or other within the galaxy. Suppose there are approximately 10^{11} interesting places to visit (each star!). Then we can expect $10^6 s/10^{11} = 10^{-5} s$ arrivals at a given "interesting place" per year. Suppose it is claimed that here on Earth we receive v such visits per year. The mean launch rate s should then be $10^5 v$ per year, or a total of $10^{11} v$ launches per year within the galaxy. This corresponds to 10^{11}–10^{14} if $v = 1$–10^3. All in all, it seems rather excessive, especially if you try to compute the quantity of material required to make such large numbers of spacecraft!

Oh, yes—one more thing. In problem 14 I asked (among other things) what was the name of the bus driver. There's a good chance it's John. Why? A simple estimate will suffice. Taking a "typical" sample, there are 28 full-time faculty in my department (Mathematics and Statistics). Seven of us have the first name John. From this I draw the *inescapable* conclusion that one person on four (yes, even including women) is named John. Of course, this is only an estimate . . .

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