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John A. Adam

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Solar Magnetoatmospheric Waves—a Simplified Mathematical Treatment

J. A. Adam*

Astronomy Centre, University of Sussex, England

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Summary. The inhomogeneous wave equation for a special class of magnetoatmospheric waves is formally solved, and the principle of stationary phase used to provide information on the group velocity properties of such waves. General results are presented concerning the associated mechanical energy flux. The basic problem considered is relevant to waves initiated by sudden events in the solar atmosphere.

Key words: magnetoatmospheric waves — Group velocity — energy flux

I. Introduction

For simplicity in most of the present analysis we consider an isothermal atmosphere permeated by a horizontal magnetic field which decreases with altitude in such a way as to render the Alfvén velocity constant. This type of atmospheric structure has been useful in previous investigations [e.g. see Nye and Thomas (1976) for further references] in that the governing differential equations of the problem have constant coefficients. The general case of non-constant coefficients has been formulated by Adam (1977b) (hereafter referred to as II), but we consider in more detail this simpler model, believing that the underlying physics of the problem is not significantly changed by so doing.

We deal with a subset of the waveforms resulting from an initial disturbance, namely those waves with wavevectors $\boldsymbol{k}=(0,l,m)$ perpendicular to the horizontal magnetic field $\boldsymbol{B}_0=(B_0,0,0)e^{-z/2H}$. H is the constant density scale-height given originally by Yu (1965)

$$H=\frac{c_0^2+\frac{\gamma}{2}a_0^2}{\gamma g},$$

where c_0 is the velocity of sound, $a_0=B_0/(4\pi\rho_0)^{1/2}$ is the Alfvén velocity, γ is the ratio of specific heats and $\boldsymbol{g}=(0,0,-g)$ is the gravitational acceleration.

In order to render the excitation function as simple as possible, we consider a line source of "force" $\delta(y)\delta(z)\delta(t)$ in two spatial dimensions, representing an instantaneous forcing term in the plane $z=0$, which may be taken as the upper photospheric region for the present purposes. This is not mandatory however, and since we are not examining the nature of trapped waves we do not explicitly define a lower boundary condition. We therefore seek information on the subsequent behaviour of disturbances initiated in the upper solar atmosphere by an instantaneous event. Clearly a more sophisticated temporal behaviour of the source can be easily incorporated in the present analysis, and is perhaps ultimately desirable; since it is not only those waves with periods much larger than the characteristic time-scale of the source that are of interest. However, many of the characteristics of magnetoatmospheric waves depend only on the properties of the medium and not on its excitation, and it is the object of this analysis to investigate such properties. In order to predict the spectrum of waves at given points in the medium however one would require some further knowledge of the source.

II. Basic Solutions

A comprehensive account of the complete magnetohydrodynamic wave equation for the problem has been given by the author (Paper II) and for the values of $\boldsymbol{k}$ considered here the coupled inhomogeneous equations are

$$\left[\frac{\partial}{\partial z}+\frac{g}{c_0^2}\right]p_T-\frac{ga_0^2}{c_0^2}\left(\frac{\partial}{\partial t}\right)^{-2}\frac{\partial^2 p_T}{\partial y^2}+\rho_0\left[n_0^2+\frac{\partial^2}{\partial t^2}+\frac{ga_0^2}{c_0^2}\frac{\partial}{\partial z}\right]\xi_z=A\rho_0\delta(y)\delta(z)\delta(t), \quad (1)$$

$$\left[\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right]p_T-\frac{a_0^2}{c_0^2}\frac{\partial^2 p_T}{\partial y^2}+\rho_0\frac{\partial^2}{\partial t^2}\left[\left(1+\frac{a_0^2}{c_0^2}\right)\frac{\partial}{\partial z}-\frac{g}{c_0^2}\right]\xi_z=0, \quad (2)$$

* Present Address: Department of Applied Mathematics, University of St. Andrews, Fife, Scotland

where A is a constant with dimensions $(LT)^{-2}$. The variable $p_T=(\boldsymbol{B}_0\cdot \boldsymbol{b}/4\pi)+p$ is the *total* pressure perturbation field due to both magnetohydrodynamic and kinetic pressure, and the variable ξ_z is the vertical particle displacement. $\rho_0(z)$ is the equilibrium density distribution. The quantity $n_0^2=-(g/c^2)-(g/\rho_0)(d\rho_0/dz)$, when positive is the square of the Brunt-Väisälä frequency for a static atmosphere [see Newcomb (1961), also Paper II]. This is modified when motions with wavevectors $\boldsymbol{k}$ perpendicular to the magnetic field are considered, since then there is no twisting of the field lines and the magnetohydrodynamic effects are accounted for by replacing c_0, in the absence of magnetic field, with $(a_0^2+c_0^2)^{1/2}$.

Elimination of p_T from the above equations yields the following equation for the field quantity $\phi=\rho_0^{1/2}\xi_z$:

$$\frac{\partial^4\phi}{\partial t^4}-(a_0^2+c_0^2)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi+\tilde{\omega}_a^2\frac{\partial^2\phi}{\partial t^2}-(a_0^2+c_0^2)\tilde{n}_0^2\frac{\partial^2\phi}{\partial y^2}=\rho^{1/2}(0)e^{-\frac{z}{2H}}\delta(y)\delta(z)\delta(t), \tag{3}$$

where $\tilde{\omega}_a^2=(a_0^2+c_0^2)/4H^2$ is the magnetoacoustic cut-off frequency in a stratified atmosphere and

$$\tilde{n}_0^2=\frac{c_0^2n_0^2}{a_0^2+c_0^2}+\frac{a_0^2g}{H(a_0^2+c_0^2)}=\frac{g}{H}-\frac{g^2}{a_0^2+c_0^2} \tag{4}$$

is the modified Brunt-Väisälä frequency mentioned above. Note that a plane-wave solution of Equation (3) of the form $\phi\sim\exp i(ly+mz-\omega t)$ yields the dispersion derived by Yu (1965) for $k_x=0$.

We may express ϕ as a Fourier Integral

$$\phi(y,z,t)=C\int_{-\infty}^{\infty}dl\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}d\omega\,\frac{\exp i(ly+mz-\omega t)\,\phi(l,m,\omega)}{l^2\left(\frac{\tilde{n}_0^2}{\omega^2}-1\right)-m^2+\frac{\omega^2-\tilde{\omega}_a^2}{a_0^2+c_0^2}}, \tag{5}$$

where $C(z)=\{(a_0^2+c_0^2)\omega^2\}^{-1}\exp(z/2H)$.

Let $\omega=\alpha\tilde{n}_0$, $\tilde{\omega}_a=\beta\tilde{n}_0$, then

$$\phi=C\int_{-\infty}^{\infty}dl\int_{-\infty}^{\infty}dm\int_{-\infty}^{\infty}d\alpha\,\frac{\tilde{n}_0\exp i(ly+mz-\alpha\tilde{n}_0t)}{l^2(\alpha^{-2}-1)-m^2+\frac{(\alpha^2-\beta^2)\tilde{n}_0^2}{a_0^2+c_0^2}}. \tag{6}$$

The integration with respect to l may be carried out by considering the integral

$$\int_{\mathscr{C}}e^{ipy}\{p^2(\alpha^2-1)+B^2\}^{-1}dp,\quad B^2=\alpha^2\left[m^2+\frac{\tilde{n}_0^2(\beta^2-\alpha^2)}{a_0^2+c_0^2}\right]$$

round the large semicircle $\mathscr{C}$ in the upper half-plane, α being real. Simples poles lie at the points $p=\pm B(1-\alpha^2)^{-1/2}$; if $\alpha<1$, B real, or $\alpha>1$, B imaginary, these lie on the real axis and the contour must be appropriately indented to incorporate a radiation condition, i.e. to ensure that the waves are outgoing (see Fig. 1).

If on the other hand $\alpha>1$, B real or $\alpha<1$, B imaginary, the poles are purely imaginary with only one inside the contour. For the moment we shall concern ourselves with the real poles, representing propagating magnetoatmospheric waves. For a fuller discussion of the continuous and discrete spectra, refer to Paper II. It will suffice here to note that the continuous spectrum consists of three disjoint intervals on the real ω-axis, cuts between the branch points of ω with domains in ω-space $(-\infty,-\tilde{\omega}_a)$, $(-\tilde{n}_0,\tilde{n}_0)$ and $(\tilde{\omega}_a,\infty)$. This will become evident below. The first and last of these frequency domains represent propagating buoyancy-modified magnetoacoustic waves, whereas the middle domain represents propagating compressibility-modified magnetogravity waves.

Were we to be dealing with the modified pressure field, $\rho_0^{-1/2}p_T$, we would find an extra contribution from the ω-integration resulting from the presence of poles where $l^2-(\omega^2/c_0^2)=0$. The sum of the residues represents the pressure contribution from trapped waves.

Returning to the l-integration we consider first propagating magnetogravity waves ($\alpha<1$, B real). The condition that B should be real is automatically satisfied for $\alpha<1$ provided $k_z^2>0$ (of interest here) and indeed for a small range of negative values of k_z^2. The wavenumber integrals are of the form

$$I_1=\int_{-\infty}^{\infty}dl\int_{-\infty}^{\infty}dm\,\frac{e^{i(ly+mz)}}{l^2(\alpha^2-1)+\alpha^2(D^2+m^2)}, \tag{7}$$

where

$$D^2=\frac{n_0^2(\beta^2-\alpha^2)}{a_0^2+c_0^2}$$

and where a negative parameter $-\alpha^2C$ has been suppressed. Then

$$I_1=-\pi i\int_{-\infty}^{\infty}dm\,\frac{\exp\left\{i\left[\frac{y\alpha(m^2+D^2)^{1/2}}{(1-\alpha^2)^{1/2}}+mz\right]\right\}}{(1-\alpha^2)^{1/2}(m^2+D^2)^{1/2}\alpha}$$

where $y>0$. (8)

Following the treatment of similar integrals by Mowbray and Rarity (1967) consider

$$G_1(r,\theta)=\int_{-\infty}^{\infty} dt\,\frac{\exp\{i[r(t^2+1)^{1/2}\cosh\theta+rt\sinh\theta]\}}{(t^2+1)^{1/2}} \quad (9)$$

and let $t=\sinh\eta$, so that

$$G_1(r,\theta)=\int_{-\infty}^{\infty} d\eta\,\exp\{ir\cosh(\theta+\eta)\} \quad (10)$$

which with the infinite limits is independent of θ, so that $G_1(r,\theta)=G_1(r,0)=i\pi H_0^{(1)}(r)$ represents a Hankel function of the first kind of order zero (Courant and Hilbert, 1953).

Therefore $\alpha<1$ and B real

$$I_1=\pi^2\int_{-\infty}^{\infty} d\alpha\,\frac{\tilde{n}_0 e^{-i\alpha\tilde{n}_0 t}}{(1-\alpha^2)^{1/2}\alpha D}\,H_0^{(1)}\left\{D\left(\frac{y^2\alpha^2}{1-\alpha^2}-z^2\right)^{1/2}\right\}, \quad (11)$$

where we have identified the following quantities:

$$t=\frac{m}{D},\quad r\cosh\theta=\frac{Dy\alpha}{(1-\alpha^2)^{1/2}},\quad r\sinh\theta=Dz.$$

We now turn to propagating magnetoacoustic waves for which $\alpha>1$, B imaginary. For B^2 to be negative when $\alpha>1$ it is required that α^2 should be greater than β^2 by at least the quantity $(a_0^2+c_0^2)m^2n_0^{-2}$, otherwise ω is not sufficiently large to enter the upper branch of the model diagram.

Following the same procedure as above for the l-integration we obtain an integral of the form

$$I_2=\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm\,\frac{e^{i(ly+mz)}}{l^2(\alpha^2-1)+\alpha^2(m^2+D^2)}$$

$$=\pi\int_{-\infty}^{\infty} dm\,\frac{\exp\left\{-\dfrac{y\alpha(m^2+D^2)^{1/2}}{(\alpha^2-1)^{1/2}}+imz\right\}}{(\alpha^2-1)^{1/2}\alpha(m^2+D^2)^{1/2}}. \quad (12)$$

By considering the integral

$$G_2=\int_{-\infty}^{\infty} dt\,\frac{\exp\{-r(t^2+1)^{1/2}\cos\theta+irt\sin\theta\}}{(t^2+1)^{1/2}} \quad (13)$$

we may show as before by putting $t=\sinh\phi$ that

$G_2(r,\theta)=G_2(r,0)=i\pi H_0^{(1)}(ir)$

and hence

$$I_2=\pi^2\int_{-\infty}^{\infty}\tilde{n}_0\,\frac{e^{-i\alpha\tilde{n}_0 t}}{\alpha D(1-\alpha^2)^{1/2}}\,H_0^{(1)}\left\{D\left(\frac{y^2\alpha^2}{1-\alpha^2}-z^2\right)^{1/2}\right\}d\alpha, \quad (14)$$

where in Equations (11) and (14) the positive root of the argument of $H_0^{(1)}$ is to be taken. In this paper we are not in the main concerned with the exact solution of this integral, but rather we are interested in the points of stationary phase in the integral over α which represents ϕ. The integrand is of the form

$$\frac{\tilde{n}_0 e^{-i\alpha\tilde{n}_0 t}}{\alpha D(1-\alpha^2)^{1/2}}\left\{H_0^{(1)}\left[D\left(\frac{y^2\alpha^2}{1-\alpha^2}-z^2\right)^{1/2}\right]\right\}$$

and we note that the argument of $H_0^{(1)}$ is real when

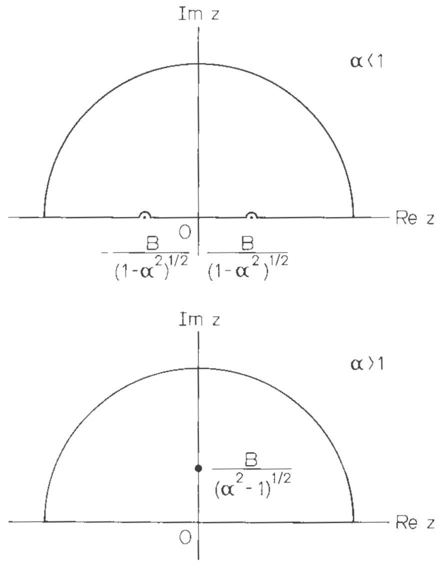

Fig. 1. The complex contours used to evaluate the integral (6)

(i) $D^2>0$ and $\dfrac{y^2\alpha^2}{1-\alpha^2}-z^2>0$

i.e. $\beta>\alpha$ and $\omega>\tilde{n}_0\cos\psi$, where the polar angle $\psi=\tan^{-1}y/z$.

These two conditions give a frequency range of ω for propagating magnetogravity waves modified by compressibility

$\tilde{n}_0>\omega>\tilde{n}_0\cos\psi$.

(ii) $D^2<0$ and $\alpha^2>1$

i.e. propagating magnetoacoustic waves modified by buoyancy for $\omega>\tilde{\omega}_a$.

There exists a third (physically inconsistent) case, namely $D^2<0$ and $(y^2\alpha^2)/(1-\alpha^2)<z^2$ which we reject. Régime (ii) does not appear in the analysis of Mowbray and Rarity (1967) since they considered an incompressible fluid with no magnetic field.

The asymptotic form of the first kind of Hankel function is given in terms of the asymptotic behaviour of the Whittaker function $W_{0,\nu}(-2iz)$ (Spain, 1970) where

$$W_{0,\nu}(z)=\frac{e^{-1/2}}{\Gamma(\nu+1/2)}\int_0^{\infty}\eta^{\nu-1/2}e^{-\eta}\left(1+\frac{\eta}{2}\right)^{\nu-1/2}d\eta \quad (15)$$

for $\mathrm{Re}(z)>0$

to yield

$$H_0^{(1)}(z)\sim\left(\frac{2}{\pi z}\right)^{1/2}\exp\left\{i\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\right\}\{1+O(z^{-1})+\ldots\}. \quad (16)$$

Thus ϕ is proportional to

$$\int_{-\infty}^{\infty} f\left(\alpha,\frac{y}{t},\frac{z}{t}\right)\exp\left\{i\left(D\left[\frac{\alpha^2}{1-\alpha^2}\left(\frac{y}{t}\right)^2-\left(\frac{z}{t}\right)^2\right]^{1/2}-\alpha\tilde{n}_0\right)t\right\}d\alpha, \quad (17)$$

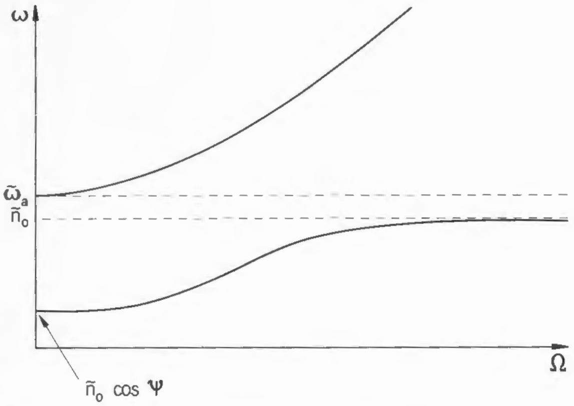

Fig. 2. A schematic representation of the upper and lower frequency behaviour of $\Omega(\omega, \delta)$, for given δ, ψ in the $(\omega-\Omega)$ plane

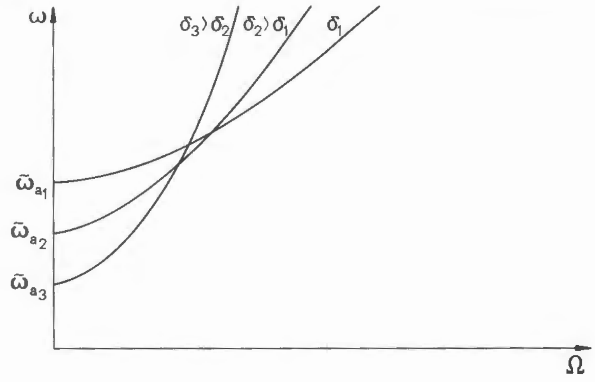

Fig. 3. A schematic representation of the behaviour of $\Omega(\omega, \delta)$ as a function of $\delta = a_0^2/c_0^2$, for given ψ (upper branch)

where t is to be a large parameter and the function f is non-oscillatory. The principle of stationary phase states that the most significant contribution to the integral (17) above comes from the neighbourhood of those points where

$$\frac{d}{d\alpha}\left\{D\left[\frac{\alpha^2}{1-\alpha^2}\left(\frac{y}{t}\right)^2-\left(\frac{z}{t}\right)^2\right]^{1/2}\right\}=\tilde{n}_0 \tag{18}$$

when t is large.

In a slightly different form we may write the exponent as $i(\Omega R-\omega t)$ where $R^2=y^2+z^2$,

$$\begin{aligned}\Omega &= \frac{D(\alpha^2-\cos^2\psi)^{1/2}(a_0^2+c_0^2)^{-1/2}}{(1-\alpha^2)^{1/2}} \\ &= \left\{\frac{(\tilde{\omega}_a^2-\omega^2)(\omega^2-\tilde{n}_0^2\cos^2\psi)}{(\tilde{n}_0^2-\omega^2)(a_0^2+c_0^2)}\right\}^{1/2}\end{aligned} \tag{19}$$

which is a generalisation of a result by Cole and Greifinger (1969) to include magnetic effects, achieved by a different method from theirs. The parameter Ω is real for the propagating ω-bands

$$\tilde{\omega}_a<|\omega|<\infty, \quad \tilde{n}_0>|\omega|>\tilde{n}_0\cos\psi.$$

The points of stationary phase are now given by $R(d\Omega/d\omega)-t=0$. Before proceeding to investigate the group velocity properties of these waves from this information we shall express $\Omega(\omega,\psi)$ as a function of the non-magnetic value of $\Omega(\omega,\psi)$ and the parameter $\delta=(a_0^2/c_0^2)$. This will in principle enable us to see the behaviour of the group velocity of waves in a more realistic atmosphere, in which say a_0 increases with altitude, by considering piecewise-constant regions of this model. Representing as we have done the corresponding "magnetic" frequencies by $\tilde{\omega}_a$, $\tilde{n}_0$, we have that

$$\tilde{\omega}_a^2=\frac{(1+\delta)}{\left(1+\frac{\gamma\delta}{2}\right)^2}\,\omega_a^2=\lambda\omega_a^2$$

and

$$\tilde{n}_0^2=\frac{\left(\gamma-1+\frac{\gamma\delta}{2}\right)}{(\gamma-1)(1+\delta)\left(1+\frac{\gamma\delta}{2}\right)}\,n_0^2=\mu n_0^2,$$

where of course λ and μ are constants.

Thus

$$\Omega=\left\{\frac{(\lambda\omega_a^2-\omega^2)(\omega^2-\mu n_0^2\cos^2\psi)}{(\mu n_0^2-\omega^2)(a_0^2+c_0^2)}\right\}^{1/2}. \tag{20}$$

Note that $0<\lambda\leq 1$, $0<\mu\leq 1$.

We plot the high and low frequency branches of Ω for various values of δ, in Figures 2–4 for a typical value of the polar angle ψ.

The group velocity of waves is defined in real space by $c_g=R/t=d\omega/d\Omega$ from the condition for stationary phase. Hency by studying the upper and lower branches for $\Omega(\omega,\psi,\delta)$, corresponding to buoyancy-modified magnetoacoustic waves and compressibility-modified magnetogravity waves respectively, we can determine the behaviour of $c_g(\omega,\psi,\delta)$. For the upper branch for given ψ, δ we note that the group velocity as defined above rises monotonically from 0 to $(a_0^2+c_0^2)^{1/2}$, the latter value being approached asymptotically from below as $\omega\to\infty$. The group velocity is clearly single-value for all values of ω in this range, as reference to Figure 5 will indicate. In Figure 6 is shown the behaviour of the upper branch as the Alfvén velocity is increased, or equivalently as the wave-group propagates higher into the atmosphere. Since we are only interested here in the qualitative behaviour of $\Omega(\omega,\psi,\delta)$ and hence $c_g(\omega,\psi,\delta)$, Diagrams 2–7 are of a schematic nature, but is is evident that by fixing a_0 and increasing c_0 in a piecewise constant manner similar behaviour (for both branches) will be achieved. Note also the fact that as a_0 (or c_0) increases, $\tilde{\omega}_a$ correspondingly decreases. The means that given an atmospheric structure of the

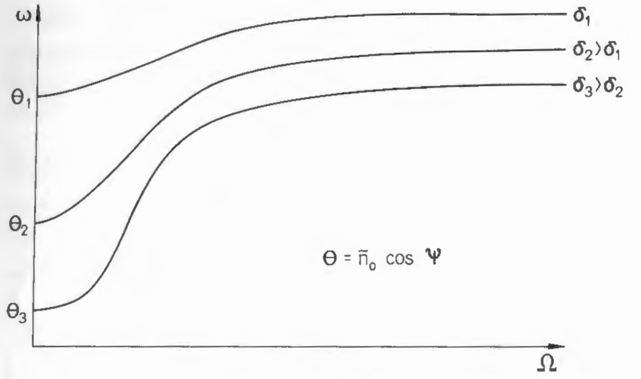

Fig. 4. A schematic representation of the behaviour of $\Omega(\omega, \delta)$ as a function of δ, for given ψ (lower branch). Note the steepening gradient at the inflection point as δ increases

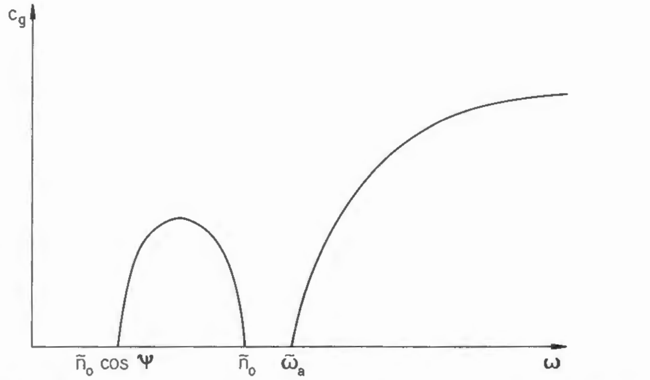

Fig. 5. A schematic representation of the group velocity $c_g(\Gamma, \delta)$ for given ψ, δ in both lower and upper frequency domains

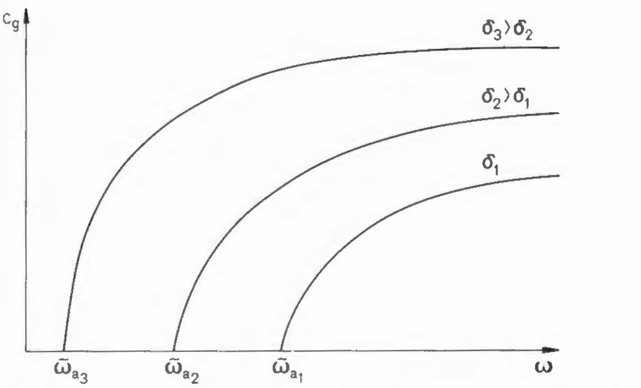

Fig. 6. A schematic representation of c_g as a function of δ for given ψ (upper branch)

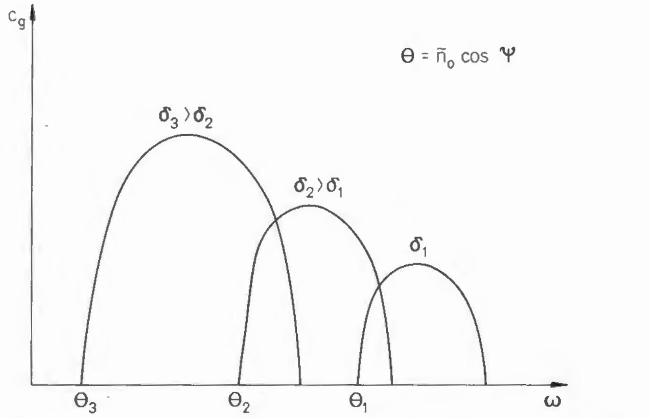

Fig. 7. A schematic representation of c_g as a function of δ for given ψ (lower branch)

kind investigated here (a_0 increasing in a piecewise-constant fashion) waves corresponding to the upper branch, propagating vertically upwards will not be reflected, since from Figure 6 there always exists a real group velocity for any given ω.

Turning now to the lower branch, the behaviour of which is illustrated schematically in Figure 2, 4 and 7, we see that there is a maximum in the group velocity curve, i.e. there is a fastest group of waves, for which $c_{g_{\max}} < (a_0^2 + c_0^2)^{1/2}$. Below $c_{g_{\max}}$ there are *two* groups of waves, one asymptotically approaching $\tilde{n}_0 \cos \psi$, the other, $\tilde{n}_0$. The peak arises because of the inflection point in the $\omega - \Omega$ curve. The locus in physical space of coincident double points of stationary phase defines a front or "caustic" representing the onset of the disturbance (Mowbray and Rarity, 1967; Cole and Greifinger, 1969). In our notation the location of the caustic can therefore be found from the condition

$$\frac{\partial}{\partial \omega}\left(\frac{d\omega}{d\Omega}\right) = 0. \tag{21}$$

As a_0 increases the maximum value of c_g increases and the whole lower branch is shifted to smaller values of ω. Hence it is clear that wave-reflection can occur when for a given ω, the ordinate ceases to intersect a lower branch curve for a given altitude, at which point wave-reflection occurs.

III. The Associated Mechanical Energy Flux

In this section we relax the assumption of constant coefficients as used above and consider the wave energy flux associated with the motion. We retain the condition $k=0$ which makes the work of this section complementary to some of that in Paper II in which the cross-field wavenumber l was considered zero. The analysis is based on techniques used in Paper I (Adam, 1976a).

We make use of the dependent variable transformations

$$\left.\begin{aligned} P_T(l,\omega;z) &= \rho_0^{-1/2}(z) \exp\left(\int^z \alpha_1(z)dz\right) P_T(l,\omega;z) \\ Q(l,\omega;z) &= \phi(l,\omega;z) \exp\left(\int^z \alpha_1(z)dz\right) \end{aligned}\right\} \tag{22}$$

as defined in Paper II. Here $\alpha_1 = g/(c_0^2(1+\delta)) - (1/2H)$. Neglecting the source term this yields the homogeneous operator equation

$$L \cdot R = 0, \tag{23}$$

where L is the operator

$$\begin{bmatrix} \dfrac{d}{dz} & \alpha(z) \\ \beta(z) & \dfrac{d}{dz} \end{bmatrix}$$

and R is the vector

$$\begin{bmatrix} P_T(l,\omega;z) \\ Q(l,\omega;z) \end{bmatrix}$$

and

$$\alpha(z) = n_0^2 - \omega^2 + \frac{g^2 a_0^2}{c_0^2(a_0^2+c_0^2)} = \tilde{n}_0^2 - \omega^2 \tag{24}$$

$$\beta(z) = \frac{1}{a_0^2+c_0^2} - \frac{l^2}{\omega^2}. \tag{25}$$

Equation (23) reduce to the following equation in P_T:

$$\frac{d}{dz}\left[\frac{1}{\alpha(z)}\frac{dP_T}{dz}\right] - \beta(z)P_T = 0. \tag{26}$$

Consider the complex energy-flux function (Souffrin (1966)) $F(y,z,t) = \overline{p_T(y,z,t)\dot{\xi}_z(y,z,t)}$ where, for a particular Fourier component $(p_T, \xi_z) \sim (P_T, Q)e^{i(ly-\omega t)}$ and hence $\mathscr{F}(l,\omega;z) = -i\omega P_T(l,\omega;z)\overline{Q(l,\omega;z)}e^{2\omega_i t}$ where $\mathscr{F}(l,\omega;z)$ is the normal-mode flux function, where the bar denotes complex conjugation and the dot denotes time differentiation $\frac{\partial}{\partial t}$. From Equations (26) and (27) above we derive

$$\frac{d\mathscr{F}(l,\omega;z)}{dz} = i\omega\left\{\frac{|P_T'|^2}{\alpha} + \beta|P_T|^2\right\}. \tag{28}$$

The real part of $\mathscr{F}(l,\omega;z)$ will concern us most since it represents the physical quantity of interest, but for completeness and a further result we retain the reactive part of the flux, $\mathrm{Im}(\mathscr{F})$. Clearly, after a little manipulation, for general ω,

$$\mathrm{Re}\left(\frac{d\mathscr{F}}{dz}\right) = -\omega_i\left\{\frac{\tilde{n}_0^2+|\omega|^2}{|\alpha|^2}|P_T'|^2 + \left(\frac{l^2}{|\omega|^2} + \frac{1}{a_0^2+c_0^2}\right)|P_T|^2\right\} \tag{29}$$

and

$$\mathrm{Im}\left(\frac{d\mathscr{F}}{dz}\right) = \omega_r\left\{\frac{\tilde{n}_0^2-|\omega|^2}{|\alpha|^2}|P_T'|^2 + \left(\frac{1}{a_0^2+c_0^2} - \frac{l^2}{|\omega|^2}\right)|P_T|^2\right\}. \tag{30}$$

We can consider either free or rigid or mixed boundary conditions at $z=0$, $z=d$ (where d may be infinite); the results established below are valid for all cases, since $\mathscr{F}(l,\omega;z)$ vanishes when either $P_T(0)$, $P_T(d)$ or $Q(0)$, $Q(d)$ are zero.

Clearly, Equations (29) and (30) possess singularities when the atmospheric resonance frequency is reached, i.e. when $\omega = \tilde{n}_0$. If we consider first Equation (29), two results can be derived. If ω is purely real (oscillatory motions) then except when $\omega = \tilde{n}_0$, $F'(z) = \mathrm{Re}(\mathscr{F}'(z))$ is zero. It can be shown (Paper I) that if the singularity occurs at $z = z_c$, then the discontinuity in flux, $[F] \propto |P_T'|_c^2/(n_0^2)_c'\omega$. See also Rosencrans (1965). We shall show below that ω must be either real or pure imaginary. Hence our second result is that if ω is imaginary, with $\omega_i > 0$, then F is a differentiable monotonically decreasing function of z in the stable layers. This follows from the fact that if ω is imaginary then α is always real continuous and positive in the stable regions.

We now show that the eigenvalues ω are either real or imaginary i.e. have real squares. Suppose that either ω is complex, in which case no singularity occurs, or that ω is imaginary and ω^2 not in the range of $-\tilde{n}_0^2$ i.e. $\omega^2 > \sup(-\tilde{n}_0^2)$. Then integrating Equations (29) and (30) over $(0,d)$ and applying boundary conditions we find that

$$-\omega_i \int_0^d \left\{\frac{\tilde{n}_0^2+|\omega|^2}{|\alpha|^2}|P_T'|^2 + \left(\frac{l^2}{|\omega|^2} + \frac{1}{a_0^2+c_0^2}\right)|P_T|^2\right\}dz = 0 \tag{31}$$

and

$$\omega_r \int_0^d \left\{\frac{\tilde{n}_0^2-|\omega|^2}{|\alpha|^2}|P_T'|^2 + \left(\frac{1}{a_0^2+c_0^2} - \frac{l^2}{|\omega|^2}\right)|P_T|^2\right\}dz = 0. \tag{32}$$

If $\omega_r\omega_i \neq 0$ we can divide equations by ω_i and ω_r respectively and add the results to obtain

$$\int_0^d \left\{\frac{|\omega|^2}{|\alpha|^2}|P_T'|^2 + \frac{l^2}{|\omega|^2}|P_T|^2\right\}dz = 0$$

which implies $|P_T| = P_T = 0$ so there are no such eigenvalues. We can state these results, which are magnetohydrodynamic generalisations of work by Rosencrans for different boundary conditions and formulation, in the form of a theorem. In Papers I and II these results were generalised to include shear flow, and shear flow along a magnetic field for $\mathbf{k} = (k, 0, m)$ respectively.

Theorem. (a) If $\omega_i = 0$ then $F(z)$ is piecewise constant with possible discontinuities where $\omega = \tilde{n}_0$.

(b) $F(z)$ is a differentiable, monotonically decreasing function of z in the stable layers, if $\omega_i > 0$.

(c) The square of ω is real, i.e. modes are purely oscillatory or exponential in time.

Implicit in the analysis of this section has been the motive of including locally convectively unstable atmospheres ($\tilde{n}_0^2 < 0$) bounded above and below by stable regions.

We note, as did Thomas and Nye (1975), that a magnetic field increasing with altitude may actually stabilitise an otherwise convectively unstable region.

IV. Discussion

Since the model considered here is explicitly concerned with waves propagating radially from a magnetic field, we consider first the relevance of this two-dimensionality to solar atmospheric wave phenomena, and then proceed to discuss the applications of the analysis in sections two and three with this as background.

By considering only those waves which propagate perpendicular to the magnetic field lines we are clearly neglecting some of the magnetohydrodynamic effects associated with more general situations. Much more complex analytic work is required in such a case (Adam, 1977b) and the purpose of this paper is to reduce such complexity while hopefully retaining much of the important physics—viz. the existence of a magnetic field ab initio. There are situations when such a twodimensional model is extremely useful; especially when the problem under consideration contains a certain degree of cylindrical symmetry, viz. magnetic flux tubes in active regions at many levels in the solar atmosphere.

Solar active regions contain a great deal of filamentary and loop structure, and in order to understand some of the phenomena which are important in the local energy balance of such active regions, models of chromospheric and coronal filaments are obviously desirable. In particular, a complete dynamical study of such structures would need to incorporate some aspects of mechanical wave energy deposition in these features. One such model has been suggested by Billings (1966) for a coronal filament, and Pneuman (1972) has investigated more quantitatively the gross energy balance in such a structure. We quote those aspects of Billing's model for which the present paper may be relevant, and suggest that the mechanisms involved may have some consequences at chromospheric levels also. The suggested model is one in which the magnetic field is proportional to the excess density in the tube, each having a gaussian distribution across it. A distribution of this type would result if a flux tube from beneath the solar surface erupted into the corona (Babcock, 1961) and then expanded radially until the outer density of the tube matched that of the surrounding medium.

Billing's suggestion is that a plane hydromagnetic wave entering the tube from below, with wave front normal to the axis, would after a short time have elongated strongly along the tube axis because the wave velocity would in general be largest there (this would of course depend on the specific values of density and field strength along the axis). Thereafter, further progression of the wave may be considered as an advance along the axis and a spreading out from the axis. It is this latter part of the wave behaviour that is qualitatively described by the analysis in Section II above, and for a given initial wave we have seen that (within the range of validity of the linear régime) as time becomes large relative to some given period, there exist *two* groups of magnetogravity waves approaching specified limiting frequencies. This is to be constrasted with the single group of magnetoacoustic waves that are also present. These results are qualitatively true for more realistic behaviour of $a_0(z)$ also, since they are not a consequence only of the particular profiles considered (the latter being chosen for analytic convenience). Some care should be taken in applying this to coronal flux tubes only, since buoyancy forces are then relatively unimportant for wave-motions and presumably the single limiting magnetoacoustic frequency is the relevant one here. However, for the corresponding situation in the chromosphere it would be entirely appropriate to expect the limiting magnetogravity frequencies to be present. The comments above for coronal flux tubes presumably hold also for flare-induced coronal waves, since it appears that gravity is not a significant factor in their subsequent behaviour after being generated (see references in Nye and Thomas, 1975).

The theoretical boundary or "caustic" (representing the onset of these disturbances orthogonal to the flux tubes) can be found from linear theory, but in reality non-linear effects ignored here many swamp such predictions; this makes the observational problem a complex one. Indeed, on the basis of Billing's model, conditions are eventually made favourable for the development of shock fronts progressing radially outward from the axis of the tube. This type of mechanism may well provide a good model of the observed thermal structure of coronal filaments.

In addition to such "first-order" effects (in the sense that the initial wave ultimately becomes the radially-moving disturbance), secondary generation of waves may occur in the form of standing hydromagnetic waves across the diameter of such chromospheric or coronal tubes. This "pulsing" or "sounding-board" type model would give rise to secondary waves propagating outward from the tube, which for the mhd modes considered here would, unfortunately, be very difficult to detect. The corresponding model for the radio régime has been suggested by Rosenberg (1970, 1971) in an effort to explain weak, often recurring quasi-periodic fluctuations superimposed on a Type IV—like continuum (Mclean et al., 1971). Again, the model presupposes the existence of a well-defined magnetic flux-tube with its feet rooted in the photosphere and its top high in the corona, but—additionally—it is situated over a flare region.

In a subsequent paper concerning waves in coronal and chromospheric flux tubes the author hopes to discuss more quantitatively the above phenomena, i.e. radially-moving waves and radially-generated waves in flux tubes, with particular reference to local energy balance. As far as this present analysis is concerned, these are two possible mechanisms for which the above theory may be directly applicable.

Another possible application of the theory in section II is that of regarding the famous "five-minute" oscillations as a "ringing" of the stable photosphere, constantly struck from below by decelerating convective elements. A number of papers on this phenomenon have appeared in the literature [see Stein and Leibacher (1974) for further references] and briefly the mechanism is as follow. The impulse excitation introduces an upward propagating pulse, and because the group velocity of acoustic-type waves tends to zero as $\omega \to \omega_a$ (see Fig. 6) the high frequency components run ahead, and leave behind an oscillating standing-wave wake at the acoustic cut-off frequency (Period ~ 200 s in the photosphere and low chromosphere for zero magnetic field). Since we have generalized the theory to include magnetic effects in a stratified atmosphere the following suggestions may be made

(i) the disagreement between the 200s and 300s periods above may be resolved by noting that for buoyancy—dominated waves, $c_g \to 0$ as

(a) $\omega \to n_0$,

(b) $\omega \to n_0 \cos \psi$.

ψ being the polar angle. We have not yet invoked the magnetic field since this can occur in its absence. The period corresponding to n_0 in the photosphere and low chromosphere is close to 300s.

(ii) In the presence of any horizontal magnetic field (*not* necessarily the special case considered here) since $\tilde{\omega}_a < \omega_a$ and $\tilde{n}_0 < n_0$ this means that the corresponding periods of waves at the cut-off frequencies become larger as a_0 increases. Hence in active regions of this type it may be that the magnetoacoustic cut-off frequency is the appropriate one. The observations are not yet conclusive since periods ranging from 190s–450s in umbrae have been observed (Stein and Leibacher, 1974). It is worth remarking that a decelerating convective element is more likely to generate gravity waves when it penetrates overlying convectively stable regions since the basic phenomenon is unchanged when compressibility is ignored. Thus perhaps a study of this mechanism, with regard to the generation of gravity waves rather than acoustic waves may be appropriate for the five-minute oscillations, although Athay's comments (1976) should be borne in mind.

In connection with the lowering of the cut-off frequencies in the presence of a purely horizontal magnetic field, we note, as did Kuperus (1969) that for modes perpendicular to the field this favours the transmission of a greater part of the acoustic spectrum in magnetic regions. In addition, for wave-motions strictly perpendicular to the magnetic field the corresponding transmission of gravity modes is reduced.

We have also seen how the stationary-phase function Ω contains information on the propagation properties of magnetoatmospheric waves. In particular, examination of the $\Omega - \omega$ diagrams for a sequence of constant Alfvén velocity regions can give reflection heights to any desired degree of accuracy by suitably matching boundaries. This can also be achieved for non-isothermal models, provided that in each case the vertical dimensions of each "slab" are at most rather less than a typical density scale-height.

We now proceed to analyse the consequences of Section III in some detail, and discuss possible applications in the light of observations. The theorem has been derived for *arbitrary* a_0- and c_0-profiles and hence it is a general result for radial modes in a horizontal magnetic field region. Although in some senses the results are known from physical considerations, Rosencrans (1965) proved them rigorously, and the extension of this has been given here. The incorporation of shear in the analysis also has important stability consequences (Adam, 1977a, b).

We have assumed initially that $\omega_r \omega_i$ is non-zero and then proceeded to derive certain results from this. The presence of an $\omega_i \neq 0$ ($\omega_i > 0$ is only of interest here for obvious reasons, but linear inviscid theory applies also to $\omega_i < 0$ with equal validity) clearly implies the existence of a destabilising force which in the absence of shear can only be due to unstable stratification. This will occur in regions of granular convection, bounded above as they are by convectively stable regions (strictly they are bounded below by regions unstable on a supergranular time-scale, but this is not relevant here). Statement (a) states that if in such a system the waves are purely propagating in time ($\omega_i = 0$) then on the basis of adiabatic theory the associated energy flux ($\mathrm{Re}\,\mathscr{F}$) is constant at all altitudes, unless the wave frequency $\omega(=\omega_r)$ becomes equal at some level to the local magnetic Brunt-Väisälä frequency. The discontinuity which occurs in this case can be easily evaluated (see Adam, 1977a, c), and results from the fact that the whole atmospheric region moves in phase at this frequency. By the very nature of the boundary conditions which can be chosen it is apparent that we are dealing with a form of resonant cavity. Specific models based on this type of physical phenomenon do not concern us here since they have been adequately reviewed by Stein and Leibacher. Nevertheless the generality of this analysis shows that the theory here is fundamental to all types of cavity model.

Statement (b) concerns in particular temporally over-stable modes ($\omega_i > 0$) since we are still assuming $\omega_r \omega_i \neq 0$ [but in the light of Statement (c) clearly refers to non-oscillatory growing, convective modes]. Physically, if modes exist such that $\omega_i > 0$ they can only occur when an energy source is available (unstable stratification; and in stable layers the mechanical energy flux associated with such modes decays monotonically with altitude. Thus convective overshoot is ultimately "contained" when penetration occurs into stable regions.

Statement (c) is perhaps the most interesting since it states that within this model *overstable modes do not exist* (nor indeed do exponentially decaying oscillatory modes). Some other means of destabilisation (for example shear or dissipation) must be present for overstability to occur.

In principle a considerable amount of information is contained in Equation (29), particularly with regard to the rate of decay of wave anergy flux with altitude. Given a particular choice of the parameters $a_0(z)$, $c_0(z)$, ω, l etc. we can obtain a solution of the governing wave Equation (26) either analytically (for simple profiles) or numerically, and then use the results in conjunction with (29). For example generality the inhomogeneous analogue of Equation (26) should be solved, incorporating a source term representing, say, convective overshoot. This has in fact been investigated for an optically thin atmosphere by Chen (1974) in the absence of magnetic fields. But in addition, by comparing the magnitudes of $\mathrm{Re}\,\dfrac{d\mathscr{F}}{dz}$ for various choices of ω, a_0, c_0 etc. the relative importance of magnetic field effects, on, say, penetrative convection may be ascertained.

Hence the results of Section III are not all directly of relevant to solar observations, but rather, information derived from them when compared with observations, may provide evidence for the importance or otherwise of convectively generated magnetohydrodynamic wave energy flux in stable magnetoatmospheric regions.

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