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A Method of Modeling Multirate Two-Dimensional Recursive Digital Filters Author

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ABSTRACT

A METHOD OF MODELING MULTIRATE TWO-DIMENSIONAL RECURSIVE DIGITAL FILTERS

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This dissertation presents a method of modeling two dimensional sampled data systems with two sampling rates in each dimension. This methodology is applied to the problem of synthesizing two dimensional velocity filters using one dimensional prototypes and multirate concepts.

The modeling method includes the replacement of scalar z-transforms of signals by vectors of polynomials, and the replacement of scalar z-transforms of impulse responses by matrices of polynomials.

The synthesis of velocity filters is accomplished through the use of coordinate transformations in the z-transform domain which skew the ω_1-ω_2 axes on the unit bidisc. The filters synthesized using these methods operate at sampling rates higher than the input and output data rates. The process of interpolating the input to accommodate the increase in sampling rates is analyzed in the frequency domain.

This overall methodology is applied to the synthesis of fan filters of arbitrary order. Simple synthesis and realization methods are given.

DEDICATION

For Mike, Steve and Vicky

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The author also wishes to thank Linda Levick for typing the final manuscript.

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LIST OF SYMBOLS

a, b, c, d	integers
$a(t_1, t_2)$	a continuous time signal
$\underline{A}, \underline{B}$	matrices of polynomials defined in Chapter Two
$B(f_1, f_2)$, $C(f_1, f_2)$	frequency responses
$\tilde{B}(f_1, f_2)$, $\tilde{C}(f_1, f_2)$	frequency responses
$\underline{d}$	a vector of polynomials defined in Chapter Two
f_1, f_2	directional frequencies
$f_n(z)$	a set of polynomials defined in Chapter Four
$G(f_1, f_2)$, $\tilde{G}(f_1, f_2)$	frequency responses
$g_n(z)$	a set of polynomials defined in Chapter Four
$H(z_1, z_2)$	base rate transfer function
$H(z_1, z_2)$	high rate transfer function
$H(*)$	an operator defined in Chapter Two
$\underline{H}_{a,c}$	matrix-valued point spread function
$\underline{H}(z_1, z_2)$	matrix-valued transfer function
j	an integer
i	$\sqrt{-1}$
i	an integer (in section 2.3 only)
k	an integer

k_1, k_2	oversampling ratios
ℓ, m, n	integers
M	order of the prototype used in Chapter Four
p, q	integers
s	Laplace transform variable
P, Q, R, S	integers
t_1, t_2	continuous time variables
T_1, T_2	sampling intervals
$T(*)$	an operator defined in Chapter Two
$T'(*)$	an operator defined in Chapter Two
t, u, v	integers
$x^*_{p,q}$	interpolated data set
$\tilde{x}_{p,q}$	scalar-valued input
$\tilde{X}(z_1, z_2)$	z-transform of $x_{p,q}$
$\underline{x}_{m,n}$	vector-valued input
$\underline{X}(z_1, z_2)$	z-transform of $\underline{x}_{m,n}$
$\tilde{y}_{p,q}$	scalar-valued output
$\tilde{Y}(z_1, z_2)$	z-transform of $\tilde{y}_{p,q}$
$\underline{y}_{m,n}$	vector-valued output
$\underline{Y}(z_1, z_2)$	z-transform of $\underline{y}_{m,n}$

z_1, z_2 $\hat{z}_1, \hat{z}_2$	z-transform variables
$Z(*)$	z-transform operator
$\alpha_{m,M}$, $\beta_{m,M}$	parameters defined in Chapter Four
β	a vector of parameters defined in Chapter Four
$\lceil * \rceil$	integer greatest lower bound
$[*]_k$	modulo k
ϕ_1, ϕ_2	angles defined in Chapter 3
θ_1, θ_2	angles defined in Chapter 4

Chapter One

Introduction

This dissertation presents a method of modeling two dimensional sampled data systems with two sampling rates in each dimension. This methodology is applied to the synthesis of two dimensional velocity filters using one dimensional prototypes.

Two dimensional digital filters can be characterized in much the same way as one dimensional digital filters. Both are characterized by transfer functions and frequency responses. The impulse response of a two dimensional system is sometimes called a point spread function. However, the synthesis techniques used in one dimensional digital filters cannot be translated directly to two dimensions in part because the fundamental theorem of algebra only applies to polynomials in one variable. Thus, a more complicated stability constraint must be enforced during synthesis, and numerical accuracy problems cannot in general be ameliorated through decomposition of the transfer function.

The literature on the synthesis of recursive two dimensional digital filters can be divided into two major categories. The earlier literature is predominated by papers [1]-[6] which rely on one dimensional synthesis procedures [7] and transform one dimensional systems into two dimensional forms. The more recent literature is

predominated by an interest in iterative optimization procedures such as the Fletcher-Powell algorithm [8] to optimize the frequency response at a number of discrete frequencies [9]-[14].

The methods which rely on one dimensional techniques have produced useful albeit subpotimal synthesis procedures for circularly symmetric [2]-[5] and rectangular [6] responses. With one exception [14], the optimum procedures had to enforce the complicated two dimensional stability constraints [15]-[18]. The optimum procedures were therefore quite expensive in terms of computer time. The existing literature thus provides simple methods for a limited class of responses, and computationally expensive procedures for general responses.

A convenient assumption commonly made in the literature was that the system be described by the convolution

$$y_{m,n} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} h_{p,q} x_{m-p,n-q} \tag{1.1}$$

where $x_{m,n}$ is the input, $y_{m,n}$ is the output, and $h_{p,q}$ is the point spread function. This assumption is called quarter plane causality because the corresponding impulse response is zero except in the quarter plane of the index set $\{(m,n): m \geq 0 \text{ and } n \geq 0\}$.

Murray [19] indicated that quarter plane causality severely restricted the class of stable denominators in the transfer function. Karivaratharajan and Swamy [20] investigated the consequences of a symmetry condition which is exhibited by a large class of useful

responses. Combining the two results, it can be concluded that the symmetry condition, along with the assumptions of quarter plane causality and stability mandate transfer functions with denominators in the form

$$D(z_1,z_2) = D_1(z_1)D_2(z_2). \tag{1.2}$$

This condition is called strict separability. It is a severe restriction which limits the recursive portion of the filtering operation to providing transition bands which are parallel to the two frequency axes. With this knowledge, it can be seen that the procedure given by Hirano and Aggarwal [6] exploits the use of one dimensional prototypes to its practical limit as long as quarter plane causality is assumed. As a consequence of the results of Murray [19] and Karivaratharajan and Swamy [20], the efficacy of two dimensional recursive digital filtering itself became an important question.

The solution has been to extend the class of stable denominators by eliminating the assumption of quarter plane causality. The basis for this extension was given by Justice and Shanks [18] who investigated the less stringent stability conditions for near-half-plane causal filters.

Murray [21] provided a simple suboptimal synthesis procedure for velocity filters which took advantage of an extended stability class. The procedure, however produced transfer functions which were not amenable to parallel decomposition.

Other authors [3 and 13] have used different, although not larger, stability classes to rotate transition bands which were originally parallel to the two frequency axes in the frequency domain. They both used transformations in the form

$$\hat{z}_1 = z_1 \, z_2^{a/b} \tag{1.3}$$

and

$$\hat{z}_2 = z_1^{a/b} \, z_2^{-1} \, . \tag{1.4}$$

The synthesis procedure of Chang and Aggarwal [3] produced transfer functions with denominators in the form

$$D(z_1,z_2) = D_1(\hat{z}_1)D_2(\hat{z}_2). \tag{1.5}$$

This condition insures a great deal of decomposability and it is referred to as directional separability.

In using the transformations of equations (1.3) and (1.4), Garibotto [13] and Chang and Aggarwal [3] all encountered the same difficulty. Both procedures require the filters to operate at sampling frequencies in excess of the frequencies dictated by the sampling theorem [22] and the spectral content of the input.

This dissertation presents a method which eliminates any discrepancies between the clock frequencies of a filter and the sampling rate at the input and output by redefining the filters at the input and output sampling rates. This ability to redefine the sampling rates of a filter warrants an extension of the synthesis procedure given by Chang and Aggarwal [3] to a new class of responses.

In Chapter 2 of this dissertation, a modeling procedure suitable for transforming filters with two sampling rates in each dimension (multirate filters) into filters with only one sampling rate in each dimension (single-rate filters) is developed. Unlike the one dimensional method of modeling multirate digital filters given by Jullien [23], the method presented in Chapter 2 replaces each multirate subsystem with an equivalent matrix representation. In this way, the subsystems can be cascaded via matrix multiplication. Undersampling at the input and output can then be modeled by selecting one element from the system's matrix representation.

Chapter 3 presents an extension of the method of Chang and Aggarwal [3] to produce velocity filters. This extension is warranted by the fact that the method presented in Chapter 2 can be used to isolate the storage required in the realization of the filters from the amount of oversampling required in the prototypes. The problem of interpolating the input was approached by Chang and Aggarwal using time domain methods. Their procedure was based on one dimensional methods, and did not produce a two dimensional extension suitable for velocity filters because the interpolation would not eliminate all of the unwanted passbands. An entirely new approach to interpolation is developed in Chapter 3. This new approach is based on frequency domain considerations and leads to a result which is appropriate for two dimensional filters.

Chapter 3 presents a method using strictly separable prototypes which are synthesized using the method of Hirano and Aggarwal [6].

The strictly separable prototypes are manipulated to produce velocity filters. A numerical example is given.

Chapter 4 presents a synthesis procedure for 90^{o} fan filters [24]. The methods of Hirano and Aggarwal [6] and Chang and Aggarwal [3] are employed using one dimensional Butterworth prototypes of arbitrary order. The multirate modeling method of Chapter 2 and the frequency domain approach to interpolation are then applied to produce trivial results. The triviality of these results are of interest because it contributes to the computational efficiency of the realization. Further, it is conjectured that a number of pole-zero cancellations occur as long as the one dimensional Butterworth prototype is odd-ordered. This conjecture is reasonable, yet unproven. It is shown to contribute to a further improvement in the computational efficiency of the realization.

It is shown that this synthesis process leads to transfer functions which are decomposable into one dimensional all-pass filters. A computationally efficient algorithmic structure for all-pass digital filters will be indicated.

Conclusions, observations and a discussion of further research directions are presented.

Chapter Two

The Multirate Modeling Method

2.1. Introduction

In this chapter a method of modeling multirate digital filters as single rate digital filters is developed. Figure 2.1 is a comparison of these two types of digital filters. The input to the filter $\tilde{H}$ in Figure 2.1(a) is taken to be zero except when $t_1 = mT_1$ and $t_2 = nT_2$.

The method developed in this chapter differs markedly from an analogous one dimensional technique by Jullien [23]. That method uses contour integrals in the z-transform domain to transform entire multirate filters to single-rate filters in one step. The method developed in this chapter will use time domain methods to convert each multirate subsystem to a matrix representation. This mechanism allows the subsystems to be cascaded via matrix multiplication or paralleled via matrix addition, thus simplifying the transformation process.

The methods developed in this chapter warrant an extension of an existing synthesis procedure for two dimensional digital filters [3] to produce a new class of responses.

The methodology developed in this chapter includes the representation of signals as vectors rather than as scalars. In this way, a multiplicity of data points in each sampling interval $\{(t_1,t_2): mT_1 \le t_1 < (m+1)T_1 \text{ and } nT_2 \le t_2 < (n+1)T_2\}$ can be accounted for without referring to the higher sampling rates k_1/T_1 and k_2/T_2.

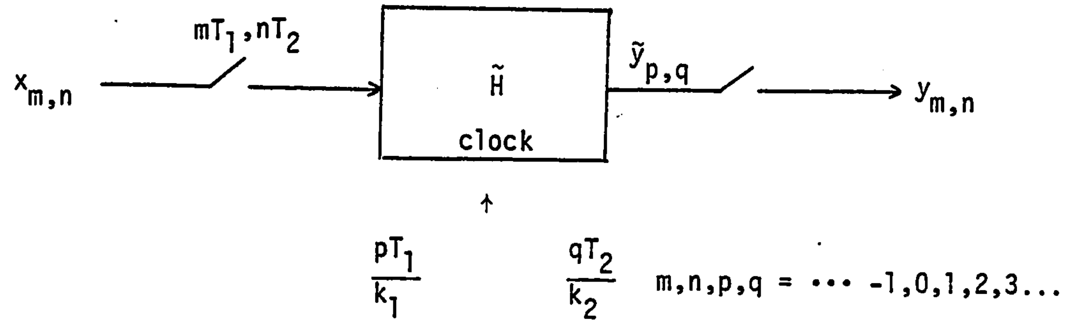

Figure 2.1(a) A Multirate Filter

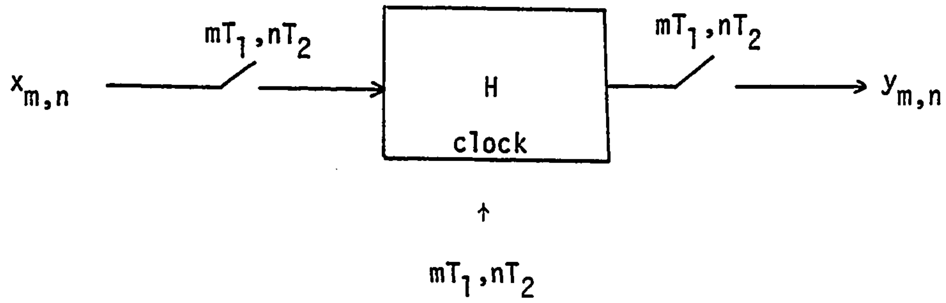

Figure 2.1(b) A Single-Rate Filter

A given transfer function $\tilde{H}(z_1,z_2)$ will be represented as a square matrix of polynomials in z_1 and z_2, $\underline{H}(z_1,z_2)$. The convolution

$$y_{p,q} = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \tilde{h}_{p-j,\, q-\ell}\, \tilde{x}_{j,\ell} \tag{2.1}$$

can then be represented by the equation

$$\underline{Y}(z_1,z_2) = \underline{H}(z_1,z_2)\, \underline{X}(z_1,z_2) \tag{2.2}$$

This form is useful because it completely describes a system which is clocked at frequencies higher than the rates associated with the z-transform variables z_1 and z_2.

The input and output are assumed to be sampled synchronously. Thus,

$$\underline{X}(z_1,z_2) = Z(x_{m,n})\ [1\ 0\ 0\ .\ .\ .\ 0]^T \tag{2.3}$$

and

$$y_{m,n} = [1\ 0\ 0\ .\ .\ .\ 0]\ Z^{-1}(\underline{Y}(z_1,z_2)). \tag{2.4}$$

The overall transfer function at the lower sampling rates is then

$$H(z_1,z_2) = [1\ 0\ 0\ .\ .\ .\ 0]\ \underline{H}(z_1,z_2)\ [1\ 0\ 0\ .\ .\ .\ 0]^T. \tag{2.5}$$

2.2. Representation of Signals

Before redefining the transfer functions on the original sampling grid, it is necessary to represent the signals in a way which is independent of the higher sampling rates.

The method used here is to represent the signals as vector-valued functions of two integer variables. The integer variables

correspond to the larger sampling intervals T_1 and T_2. The ordering of the elements in each vector is somewhat arbitrary.

The ordering used throughout this dissertation is given by the relationship

$$x^{\ell}_{m,n} = \tilde{x}_{p,q} \tag{2.6}$$

where

$$m = \lceil p/k_1 \rceil \tag{2.7}$$

$$n = \lceil q/k_2 \rceil \tag{2.8}$$

and

$$\ell = [p]_{k_1} + k_1 [q]_{k_2} \ . \tag{2.9}$$

The function $[*]_k$ indicates the modulo-k value of the argument. The function * indicates the integer greatest lower bound.

Each triple (ℓ,m,n) is related to one and only one pair (p,q) via

$$p = mk_1 + [\ell]_{k_1} \tag{2.10}$$

and

$$q = nk_2 + \lceil \ell/k_1 \rceil \ . \tag{2.11}$$

The relationship between the indices is therefore isomorphic and thus invertible. The relationship between a scalar-valued signal $\tilde{x}_{p,q}$ and its vector-valued counterpart $\underline{x}_{m,n}$ is therefore invertible. It will be denoted

$$\underline{x} = T(\tilde{x}) \tag{2.12}$$

and

$$\tilde{x} = T^{-1}(x) . \tag{2.13}$$

Figure 2.2 demonstrates an example of the relationship T. Figure 2.2(a) is an arbitrary scalar-valued signal. Its vector-valued counterpart is shown in Figure 2.2(b) where $k_1 = 2$ and $k_2 = 3$.

The formal z-transforms of the two data representations are given by

$$\tilde{X}(z_1,z_2) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \tilde{x}_{p,q}\, z_1^p\, z_2^q = z(\tilde{x}) \tag{2.14}$$

and

$$\underline{X}(z_1,z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \underline{x}_{m,n}\, z_1^m\, z_2^n = Z(\underline{x}) \tag{2.15}$$

2.3. Representations of Transfer Functions

The purpose of this section is to find an operator $H(*)$ such that if

$$\tilde{Y}(z_1,z_2) = \tilde{H}(z_1,z_2)\, \tilde{X}(z_1,z_2) \tag{2.16}$$

$$= Z(\tilde{y}) \tag{2.17}$$

$$\tilde{X}(z_1,z_2) = Z(\tilde{x}) \tag{2.18}$$

$$\underline{y} = T(\tilde{y}) \tag{2.19}$$

and

$$\underline{x} = T(\tilde{x}) \tag{2.20}$$

then

$$\underline{Y}(z_1,z_2) = H(\underline{X}(z_1,z_2)) \tag{2.21}$$

q \ p	0	1	2	3
0	0	1	2	3
1	4	5	6	7
2	8	9	10	11
3	12	13	14	15
4	16	17	18	19
5	20	21	22	23

$\tilde{x}_{p,q}$

Figure 2.2(a) A Scalar-Valued Signal Sampled At $pT_1/2$ and $qT_2/3$

$$\begin{array}{c|cc}
n \backslash m & 0 & 1 \\
\hline
0 & \left|\begin{matrix}0\\1\\4\\5\\8\\9\end{matrix}\right| & \left|\begin{matrix}2\\3\\6\\7\\10\\11\end{matrix}\right| \\
1 & \left|\begin{matrix}12\\13\\16\\17\\20\\21\end{matrix}\right| & \left|\begin{matrix}14\\15\\18\\19\\22\\23\end{matrix}\right|
\end{array}
\qquad \underline{x}_{m,n} = T(\tilde{x}_{p,q})$$

Figure 2.2(b) A Vector-Valued Signal Sampled At mT_1 and nT_2

The operator H replaces scalar multiplication by $\tilde{H}(z_1,z_2)$ and it has vector-valued arguments and vector-valued point images.

The input-output relationship of the system $\tilde{H}$ can be described by the convolution

$$\tilde{y}_{p,q} = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} \tilde{h}_{p-u,q-v}\, \tilde{x}_{u,v} \ . \tag{2.22}$$

The substitutions

$$\tilde{y}_{p,q} = y^{\ell}_{m,n} \tag{2.23}$$

and

$$x_{u,v} = x^{[u]_{k_1} + k_1 [v]_{k_2}}_{\lceil u/k_1 \rceil ,\ \lceil v/k_2 \rceil} \tag{2.24}$$

where

$$p = [\ell]_{k_1} + mk_1 \ , \tag{2.25}$$

and

$$q = \lceil \ell/k_1 \rceil + nk_2 \ , \tag{2.26}$$

lead to

$$y^{\ell}_{m,n} = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} \tilde{h}_{[\ell]_{k_1} + mk_1 - u,\ \lceil \ell/k_1 \rceil + nk_2 - v} \tag{2.27}$$

$$\cdot\ x^{[u]_{k_1} + k_1 [v]_{k_2}}_{\lceil u/k_1 \rceil ,\ \lceil v/k_2 \rceil}$$

The substitutions

$$u = ak_1 + b \tag{2.28}$$

and

$$v = ck_2 + d \tag{2.29}$$

lead to

$$y_{m,n}^{\ell} = \sum_{a=-\infty}^{\infty} \sum_{c=-\infty}^{\infty} \sum_{b=0}^{k_1-1} \sum_{d=0}^{k_2-1} x_{a,c}^{t}$$

$$\cdot \tilde{h}_{k_1(m-a)+[\ell]k_1-b,\,k_2(n-c)+\lceil \ell/k_1 \rceil -d} \tag{2.30}$$

where

$$t = b + k_1 d \ . \tag{2.31}$$

A matrix-valued function of two discrete variables $\underline{H}_{a,c}$ can now be introduced where

$$\underline{h}_{a,c} = \{h_{a,c}^{\ell,t}\} \tag{2.32}$$

and

$$h_{a,c}^{\ell,t} \triangleq \tilde{h}_{k_1 a+[\ell]_{k_1}-[t]_{k_1},\,k_2 c+\lceil \ell/k_1 \rceil - \lceil t/k_1 \rceil} \tag{2.33}$$

to produce

$$\underline{y}_{m,n} = \sum_{a=-\infty}^{\infty} \sum_{c=-\infty}^{\infty} \underline{h}_{m-a,n-c}\, \underline{x}_{a,c} \ . \tag{2.34}$$

In the z-transform domain, equation (2.34) is equivalent to

$$\underline{Y}(z_1,z_2) = \underline{H}(z_1,z_2)X(z_1,z_2) \tag{2.35}$$

where

$$H^{\ell,t}(z_1,z_2) = \sum_{a=-\infty}^{\infty} \sum_{c=-\infty}^{\infty} h_{a,c}^{\ell,t}\, z_1^{a} z_2^{c} \tag{2.36}$$

The matrix $\underline{H}(z_1,z_2)$ will be referred to as a transfer function matrix.

The operator H can now be seen to map vectors of polynomials onto vectors of polynomials with action equivalent to the matrix multiplication by $\underline{H}(z_1,z_2)$.

The relationship between transfer functions and the corresponding transfer function matrices is denoted

$$\underline{H}(z_1,z_2) = T'(\tilde{H}(z_1,z_2)) \tag{2.37}$$

When subsystems $\tilde{H}_1$ and $\tilde{H}_2$ are cascaded, the input-output relationship can be represented by

$$\tilde{Y}(z_1,z_2) = \tilde{H}_1(\tilde{H}_2(X(z_1,z_2))) \tag{2.38}$$

$$= \tilde{H}_2(\tilde{H}_1(\tilde{X}(z_1,z_2))) \tag{2.39}$$

Thus,

$$\underline{Y}(z_1,z_2) = H_1(H_2(\underline{X}(z_1,z_2))) \tag{2.40}$$

$$= H_2(H_1(\underline{X}(z_1,z_2))) \tag{2.41}$$

$$= \underline{H}_1(z_1,z_2)\underline{H}_2(z_1,z_2)\underline{X}(z_1,z_2) \tag{2.42}$$

$$= \underline{H}_2(z_1,z_2)\underline{H}_1(z_1,z_2)\underline{X}(z_1,z_2) \tag{2.43}$$

This leads to the conclusion that the operator T' is distributive. That is

$$T'(\tilde{H}_1\ \tilde{H}_2) = T'(\tilde{H}_1)T'(\tilde{H}_2)\ . \tag{2.44}$$

Further, it leads to the conclusion that multiplication of transfer function matrices is commutative. That is

$$\underline{H}_1\ \underline{H}_2 = \underline{H}_2\ \underline{H}_1\ . \tag{2.45}$$

The commutivity of the transfer function matrices can be confirmed by noting that the transfer function matrix corresponding to a one dimensional digital filter will be a square matrix in the form

$$\underline{A} = \begin{vmatrix} a_0 & z_1 a_{k_1-1} & \cdot & z_1 a_1 \\ a_1 & a_0 & z_1 a_{k_1-1} & z_1 a_2 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{k_1-1} & \cdot & \cdot & a_0 \end{vmatrix} \tag{2.46}$$

where each a_i is a polynomial in z_1.

By carrying out the multiplications $\underline{A}\ \underline{B}$ and $\underline{B}\ \underline{A}$, it can be shown that

$$\underline{A}\ \underline{B} = \underline{B}\ \underline{A} \tag{2.47}$$

as long as $\underline{A}$ and $\underline{B}$ are in the form of equation (2.46).

For a two dimensional digital filter, the transfer function matrix is in the form

$$\underline{H} = \begin{vmatrix} \underline{A}_0 & \underline{A}_{k_2-1} z_2 & \underline{A}_{k_2-2} z_2 & \cdots & \underline{A}_1 \\ \underline{A}_1 & \underline{A}_0 & \underline{A}_{k_2-1} z_2 & & \underline{A}_2 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \underline{A}_{k_2-1} & \cdot & \cdot & \cdots & \underline{A}_0 \end{vmatrix} \tag{2.48}$$

where each $\underline{A}_i$ is a square matrix of polynomials in the form of equation (2.46) and each a_i is a polynomial in z_1 and z_2.

The matrix $\underline{H}$ is in the same form as $\underline{A}$ except that the polynomials a_i are replaced by matrices $\underline{A}_i$ and z_1 is replaced by z_2.

Because

$$\underline{A}_i\ \underline{A}_j = \underline{A}_j\ \underline{A}_i\ , \tag{2.49}$$

matrices in the form of equation (2.48) commute multiplicitavely.

For digital filters in three or more dimensions, the partitioning method used here can be repeated to demonstrate multiplicative commutivity of the corresponding transfer function matrices.

The multiplicative commutivity of the transfer function matrices is significant because it indicates that subfilters can be cascaded without regard to their ordering.

The transfer function matrices corresponding to all-zero multirate filters can be evaluated by first noting that the first column of the transfer function matrix is in the form

$$[d_0, d_1 \ldots d_{(k_1-1)(k_2-1)}]^T$$

where

$$\underline{d}(z_1,z_2) = Z(T(\tilde{h}_{p,q})) \qquad (2.50)$$

and $\tilde{h}_{p,q}$ is the multirate point spread function.

The other columns of the transfer function matrix can be found by imposing the symmetries present in equations (2.46) and (2.48).

The transfer function matrices corresponding to all-pole multirate filters can be evaluated using matrix inversion. This can be shown by noting that if

$$\underline{H} = T'(\tilde{H}) \qquad (2.51)$$

and that if

$$\tilde{H} \neq 0 \qquad (2.52)$$

then

$$T'(\tilde{H}\tilde{H}^{-1}) = T'(1) = \underline{I}$$

$$= T'(\tilde{H})\, T'(\tilde{H}^{-1}) = \underline{H}\, T'(\tilde{H}^{-1}) \tag{2.53}$$

$$= T'(\tilde{H}^{-1})\, T'(\tilde{H}) = T'(\tilde{H}^{-1})\underline{H} \tag{2.54}$$

Thus

$$\underline{H}^{-1} = T'(\tilde{H}^{-1}) \ . \tag{2.55}$$

2.4. Summary

In this chapter a method of modeling multirate digital filters as single-rate digital filters with identical input-output relationships is developed.

The mechanism used in this modeling method includes the representation of discrete-time systems as matrices of polynomials to facilitate computational simplicity in the calculation of the model.

Computationally simple methods are presented for the modeling of all-zero and all-pole filters.

This modeling method is functionally different from the equivalent one dimensional method presented by Jullien [23] in that subsystems can be cascaded with the method presented here. The single-rate filters found by both methods are identical when the method presented in this chapter is collapsed to one dimension.

Chapter Three

A Synthesis Procedure for Velocity Filters

3.1. Introduction

This chapter provides a synthesis procedure for velocity filters [25]. The prototype filters considered in this chapter are multirate filters. The modeling method of the previous chapter is applied to convert the multirate prototypes to single rate velocity filters.

The systems modeled in Chapter 2 are identical to those considered by Chang and Aggarwal [3] except in the way that interpolation is dealt with. Chang and Aggarwal treated these systems in a way which can be depicted in Figure 3.1(a). In Figure 3.1(a) the subsystem H_I is an interpolator which computes data which was available from the continuous time signal $a(t_1,t_2)$, but was lost due to the low sampling rates. The interpolated data set x^* is then the input to a subsystem H_0 which operates at the high clock frequencies. The subsystems H_I and H_0 are designed and realized separately.

In this dissertation, the subsystems H_I and H_0 are lumped together into one system $\tilde{H}$ as shown in Figure 3.1(b). The "augmentor" appends zeros to the input data stream to accommodate the increase in sampling rates. This leads to a greater flexibility in the design of the systems. This greater flexibility permits the role of the interpolator to be expanded. Section 3.4. of this chapter and Chapter 4 of this dissertation exemplify cases where the function of the interpolator H_I takes on a more general form or is not required.

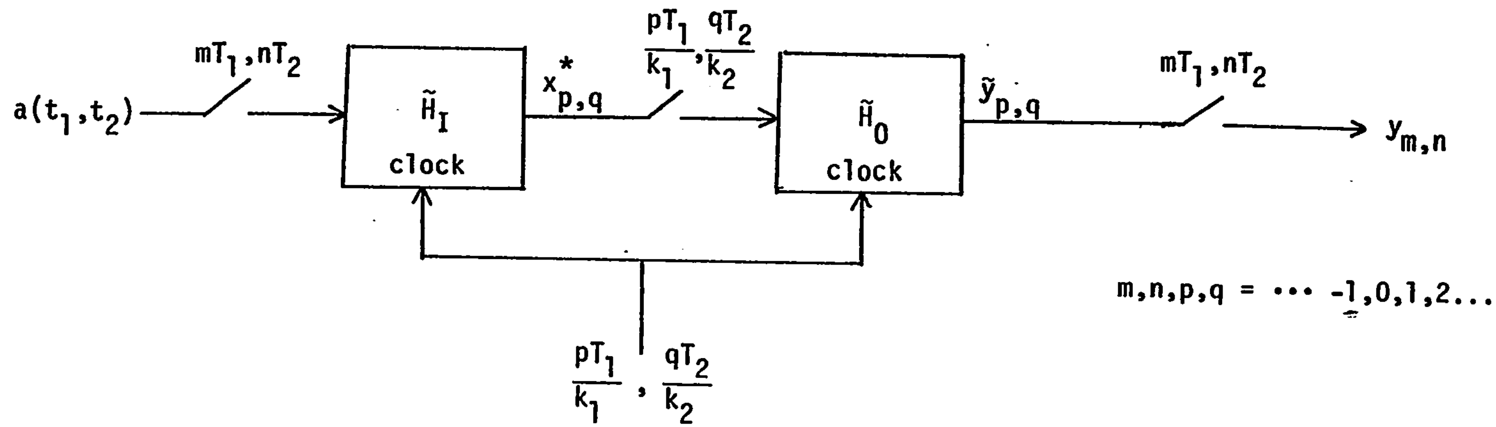

Figure 3.1(a) A Multirate Filter With Explicit Interpolation

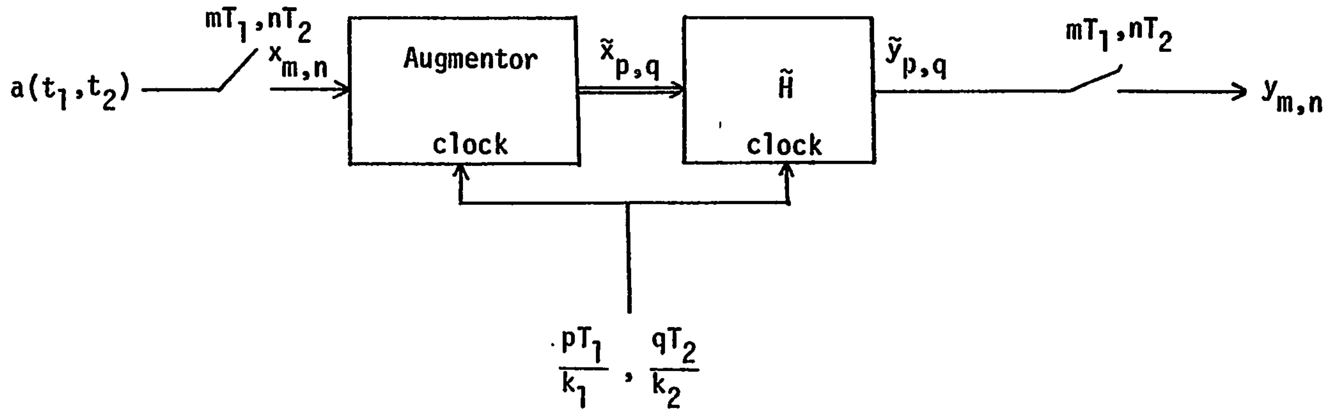

Figure 3.1(b) A Multirate Filter Without Explicit Interpolation

Chang and Aggarwal use index transformations in the form

$$\hat{z}_1 = z_1 z_2^{b/a} \tag{3.1}$$

$$\hat{z}_2 = z_1^{b/a} z_2^{-1} \tag{3.2}$$

where a and b are integers. These transformations rotate the frequency responses of the prototype filters which are designed using reference [6]. This dissertation uses index transformations in the more general form

$$\hat{z}_1 = z_1^P z_2^Q \tag{3.3}$$

$$\hat{z}_2 = z_1^R z_2^S \tag{3.4}$$

where P, Q, R and S are integers. This index transformation is used to warp as well as rotate the frequency responses of the prototype filters. The warping of the frequency responses is of particular interest in the synthesis of velocity filters.

A relationship between the overall frequency response

$$G(f_1,f_2) = \left.\frac{Z(y)}{Z(x)}\right|_{\substack{z_1 = e^{i2\pi f_1 T_1} \\ z_2 = e^{i2\pi f_2 T_2}}} \tag{3.5}$$

and the frequency response of the multirate filter $\tilde{H}$ is derived in this chapter. It leads to a new definition of interpolation in multirate multidimensional systems. This new definition is required for the synthesis of velocity filters.

A synthesis procedure for velocity filters is derived and an example is given.

3.2. A Frequency Domain Approach To Interpolation

In this section, the overall frequency response of the system shown in Figure 3.1(b) is determined from the frequency response of the subsystem $\tilde{H}$. The result is interpreted to produce a new definition of interpolation in multidimensional systems.

The Fourier transform of the sampled signal $x_{m,n}$ exists if $x_{m,n}$ is defined for all m and n. It is given by

$$B(f_1,f_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_{m,n} e^{i2\pi m f_1 T_1} e^{i2\pi n f_2 T_2} . \qquad (3.6)$$

The augmented signal $\tilde{x}_{p,q}$ is defined by

$$\tilde{x}_{p,q} = x_{p/k_1,q/k_2} \qquad (3.7)$$

if k_1 divides p and if k_2 divides q, and

$$\tilde{x}_{p,q} = 0 \qquad (3.8)$$

otherwise. The Fourier transform of $\tilde{x}_{p,q}$ is given by

$$\tilde{B}(f_1,f_2) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \tilde{x}_{p,q} \exp(i2\pi(\frac{f_1T_1}{k_1} + \frac{f_2T_2}{k_2})) . \qquad (3.9)$$

Substituting equations (3.7) and (3.8) produces

$$B(f_1,f_2) = \tilde{B}(f_1,f_2) . \qquad (3.10)$$

It is noted that both frequency responses are periodic with periods $1/T_1$ and $1/T_2$. That is,

$$B(f_1+p/T_1, f_2+q/T_2) = B(f_1,f_2) \tag{3.11}$$

and

$$\tilde{B}(f_1+p/T_1, f_2+q/T_2) = \tilde{B}(f_1,f_2) \; . \tag{3.12}$$

The frequency response of the subsystem $\tilde{H}$ is given by

$$\tilde{G}(f_1,f_2) = \tilde{H}(z_1,z_2) \Bigg|_{z_j = \exp(\frac{i2\pi f_j T_j}{k_j})} \; . \tag{3.13}$$

$$= \frac{\tilde{C}(f_1,f_2}{\tilde{B}(f_1,f_2)}$$

where $\tilde{C}$ is the Fourier transform of $\tilde{y}_{p,q}$, or

$$\tilde{C}(f_1,f_2) \triangleq \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \tilde{y}_{p,q} \exp(i2\pi(\frac{pT_1f_1}{k_1} + \frac{qT_2f_2}{k_2})) \; . \tag{3.14}$$

The effect of undersampling $\tilde{y}_{p,q}$ to produce $y_{m,n}$ can be represented in the frequency domain as an aliasing of $\tilde{C}(f_1,f_2)$ to cause the Fourier transform of $y_{m,n}$

$$C(f_1,f_2) \triangleq \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_{m,n} \exp(i2\pi(mf_1T_1 + nf_2T_2)) \tag{3.15}$$

to be periodic with directional periods $1/T_1$ and $1/T_2$. This aliasing effect can be demonstrated by using the identity

$$k^{-1} \sum_{m=0}^{k-1} \exp(i2\pi mp/k) = \begin{cases} 1 \text{ when } k \text{ divides } p \\ 0 \text{ otherwise} \end{cases} \; . \tag{3.16}$$

Without argument, a two dimensional form of this identity is thus

$$\sum_{m=0}^{k_1-1} \sum_{n=0}^{k_2-1} \frac{\tilde{y}_{p,q}}{k_1 k_2} \exp(i2\pi(\frac{mp}{k_1} + \frac{nq}{k_2})) \tag{3.17}$$

$$= \begin{cases} \tilde{y}_{p,q} & \text{when } k_1 \text{ divides p and } k_2 \text{ divides q} \\ 0 & \text{otherwise .} \end{cases}$$

Equation (3.17) is a representation of the undersampling operation because

$$y_{p/k_1,\, q/k_2} = \begin{cases} \tilde{y}_{p,q} & \text{when } k_1 \text{ divides p and } k_2 \text{ divides q} \\ 0 & \text{otherwise .} \end{cases} \tag{3.18}$$

Substituting equations (3.17), (3.18) and (3.14) into equation (3.15) produces

$$\begin{aligned} C(f_1,f_2) = & \sum_{m=0}^{k_1-1} \sum_{n=0}^{k_2-1} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\tilde{y}_{p,q}}{k_1 k_2} \\ & \cdot \exp\left(\frac{i2\pi pT_1}{k_1}\left(\frac{m}{T_1} + f_1\right)\right) \\ & \cdot \exp\left(\frac{i2\pi qT_2}{k_2}\left(\frac{n}{T_2} + f_2\right)\right) \\ = & \sum_{m=0}^{k_1-1} \sum_{n=0}^{k_2-1} \frac{\tilde{C}(f_1 + \frac{m}{T_1},\, f_2 + \frac{n}{T_2})}{k_1 k_2} \quad . \end{aligned} \tag{3.19}$$

Substituting equations (3.13) and (3.10) into (3.19) produces

$$\begin{aligned} C(f_1,f_2) = & \frac{B(f_1,f_2)}{k_1 k_2} \\ & \cdot \sum_{m=0}^{k_1-1} \sum_{n=0}^{k_2-1} \tilde{G}(f_1 + \frac{m}{T_1},\, f_2 + \frac{n}{T_2}) \ . \end{aligned} \tag{3.20}$$

The overall frequency response of the system is then

$$G(f_1,f_2) = \frac{C(f_1,f_2)}{B(f_1,f_2)}$$

$$= \sum_{m=0}^{k_1-1} \sum_{n=0}^{k_2-1} \frac{\tilde{G}(f_1 + \frac{m}{T_1}, f_2 + \frac{n}{T_2})}{k_1 k_2} . \qquad (3.21)$$

This indicates that the overall frequency response $G(f_1,f_2)$ is the frequency response of the multirate system aliased onto itself to produce a response which is periodic in the frequency domain with periods $1/T_1$ and $1/T_2$.

The function of the interpolator can therefore be described as follows. The interpolator should eliminate any unwanted passbands in $\tilde{G}(f_1,f_2)$ which would otherwise be aliased into the region $\{(f_1,f_2)\colon -1/2T_1 \le f_1 < 1/2T_1, \text{ and } -1/2T_2 \le f_2 < 1/2T_2\}$. The interpolator should also leave the useful passbands undisturbed.

This desciption of the function of the interpolator contrasts with that given by Chang and Aggarwal. Their interpolators provide rectangular low pass responses which would not be adequate for velocity filters. Further, the description of the interpolators used in this dissertation can indicate instances when no interpolation is required, as is demonstrated in Chapter 4.

In this dissertation, the function of the interpolators used by Chang and Aggarwal is generalized. The generalization is expansive enough to make the use of the word "interpolator" inappropriate in some cases. When the filters, H_I, can no longer be properly referred to as interpolators, they are called mask filters.

3.3. Translation of Prototype Filters

This section describes a procedure for the synthesis of velocity filters with the general stopband-passband characteristic shown in Figure 3.2.

The general method is to first synthesize filters with stopband-passband characteristics as shown in Figure 3.3(a). The response shown in Figure 3.3(a) has transition bands tilted at angles of $\emptyset$ and $\emptyset$ with respect to the f_1 axis. The response shown in Figure 3.3(b) is cascaded with the response shown in Figure 3.3(a) to produce the composite response shown in Figure 3.3(c).

The passbands of the response shown in Figure 3.3(c) lie in the region $\{(f_1,f_2)$: $-1/2T_1 \le f_1 < 1/2T_1$, and $-1/2T_2 \le f_2 < 1/2T_2\}$. The input and output can therefore be sampled at the rates $1/T_1$ and $1/T_2$ and the filter can be redefined at these rates using the modeling method given in the previous chapter.

The synthesis of the spectral characteristic indicated in Figure 3.3(a) begins with the synthesis of the prototype characteristic shown in Figure 3.4. This response may be approximated arbitrarily well using one dimensional techniques as indicated by Hirano and Aggarwal [6]. The resulting response can be denoted $\hat{H}_v(\hat{z}_1,\hat{z}_2)$. The substitutions

$$\hat{z}_1 = z_1^P \, z_2^Q \tag{3.22}$$

and

$$\hat{z}_2 = z_1^R \, z_2^S \tag{3.23}$$

warp the frequency response to the general shape shown in Figure

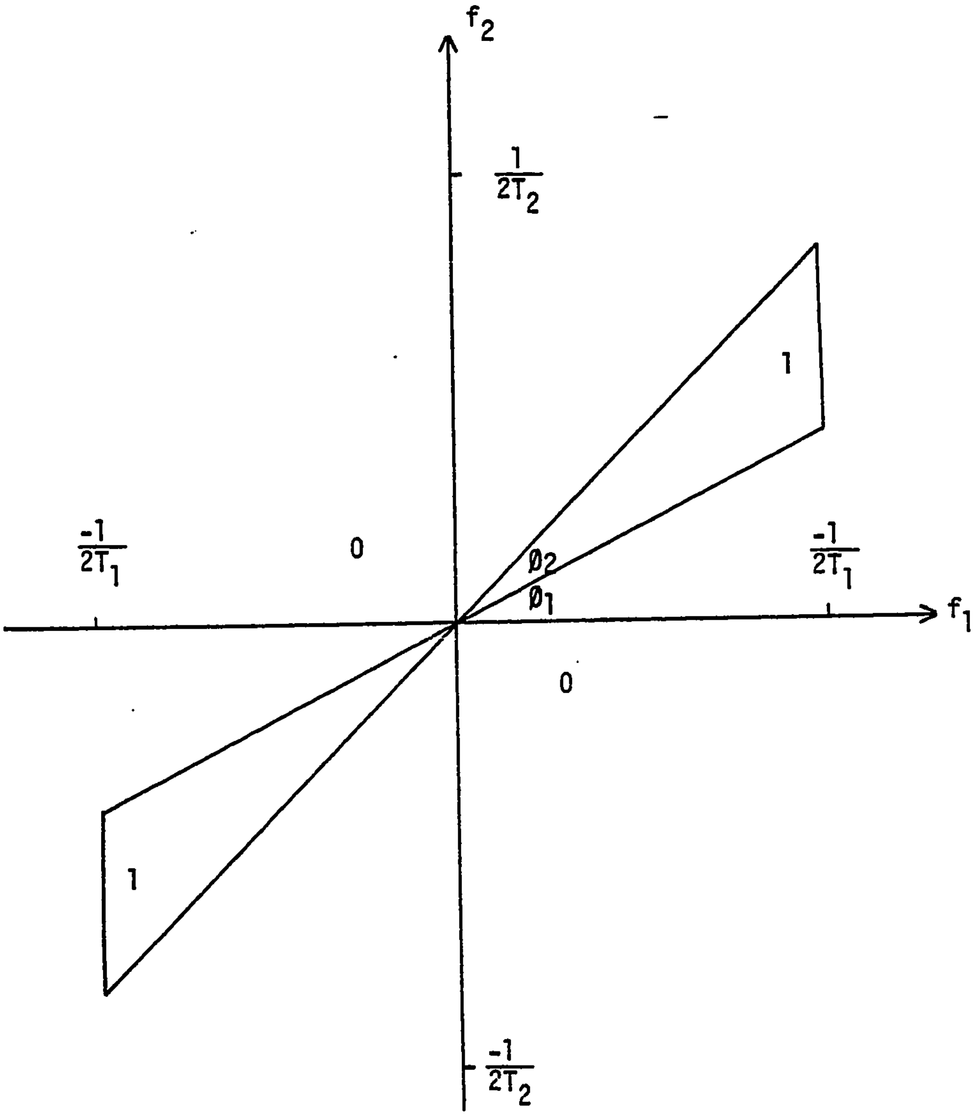

Figure 3.2. Magnitude Response of a Velocity Filter

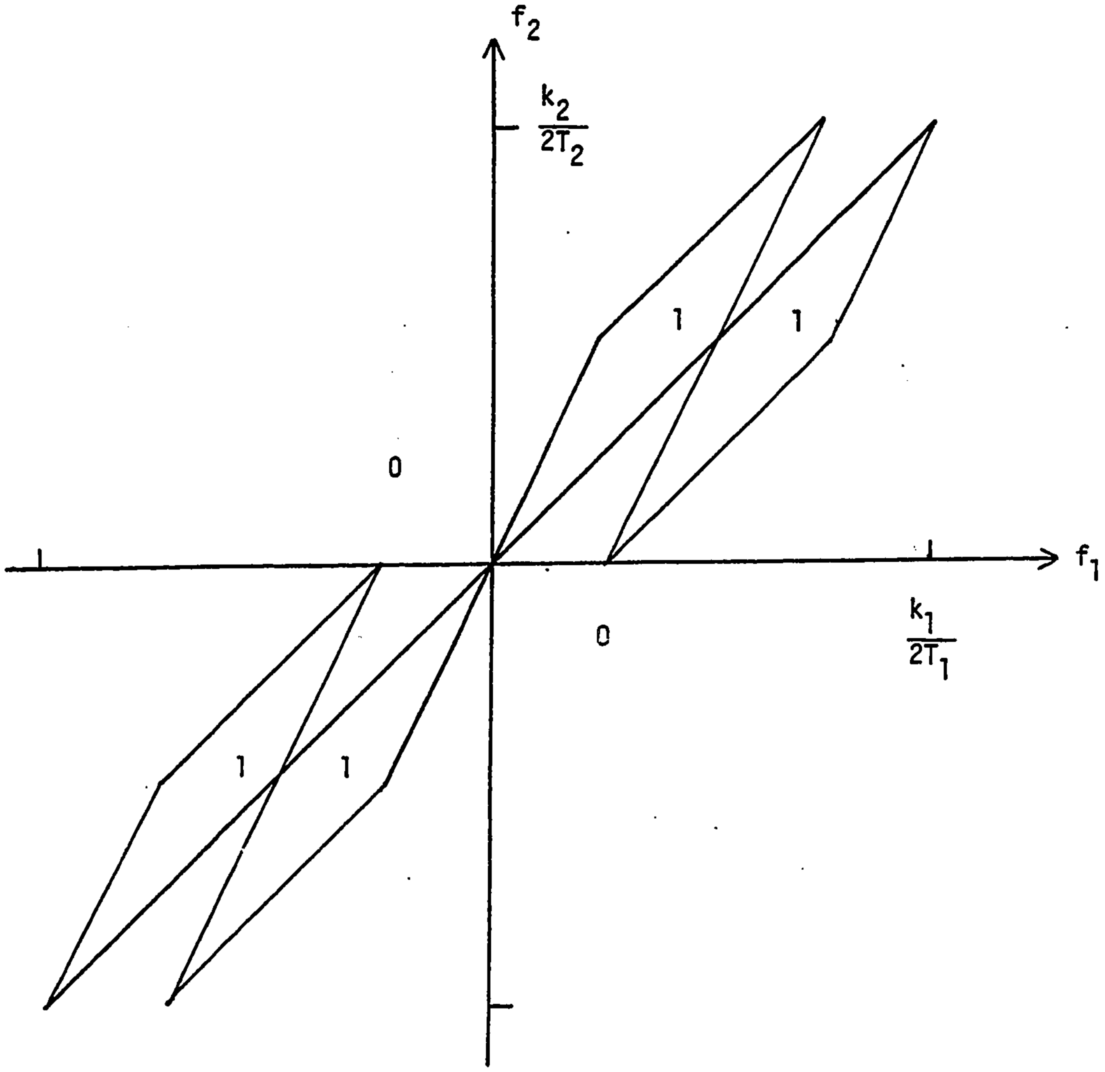

Figure 3.3(a) Magnitude Response of a Velocity Filter Prototype

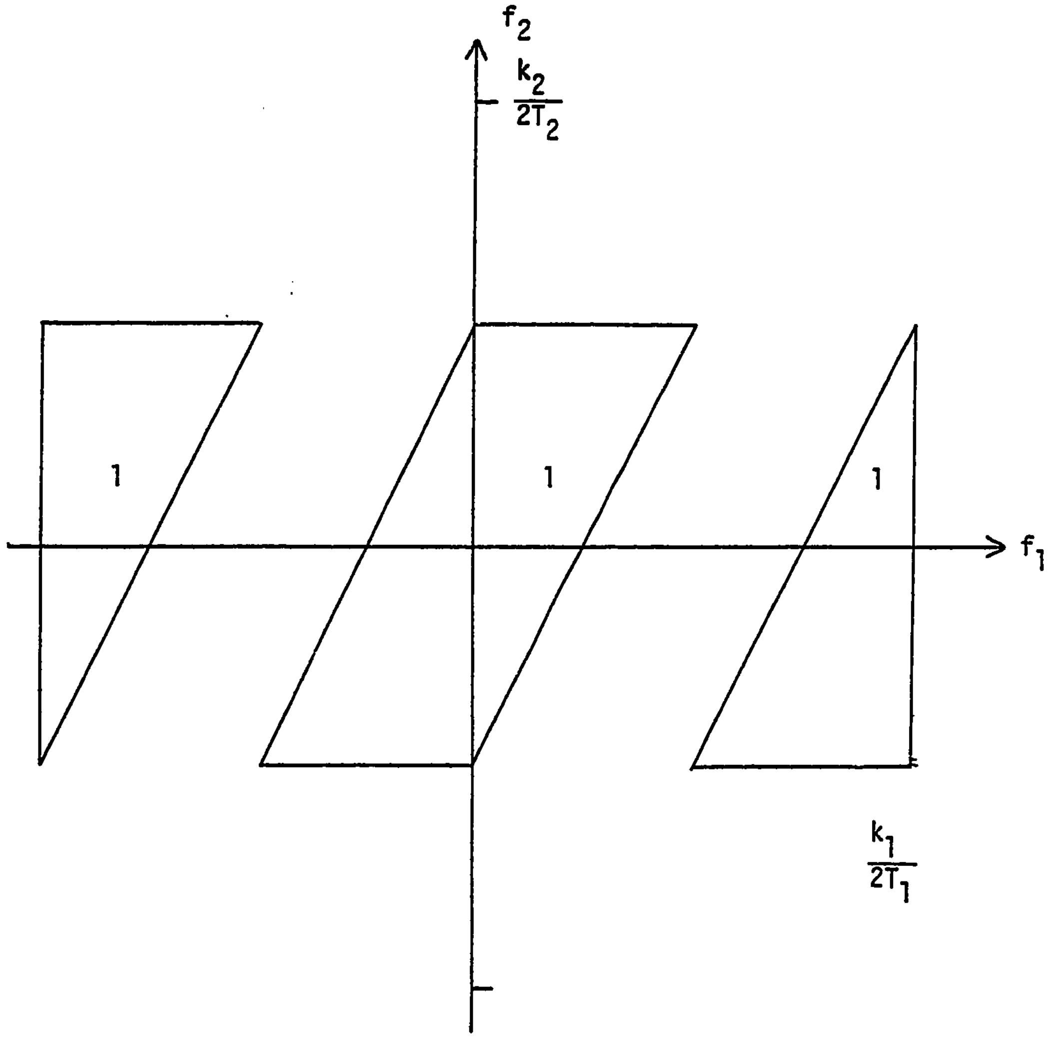

Figure 3.3(b) Magnitude Response of the Interpolator or Mask Filter

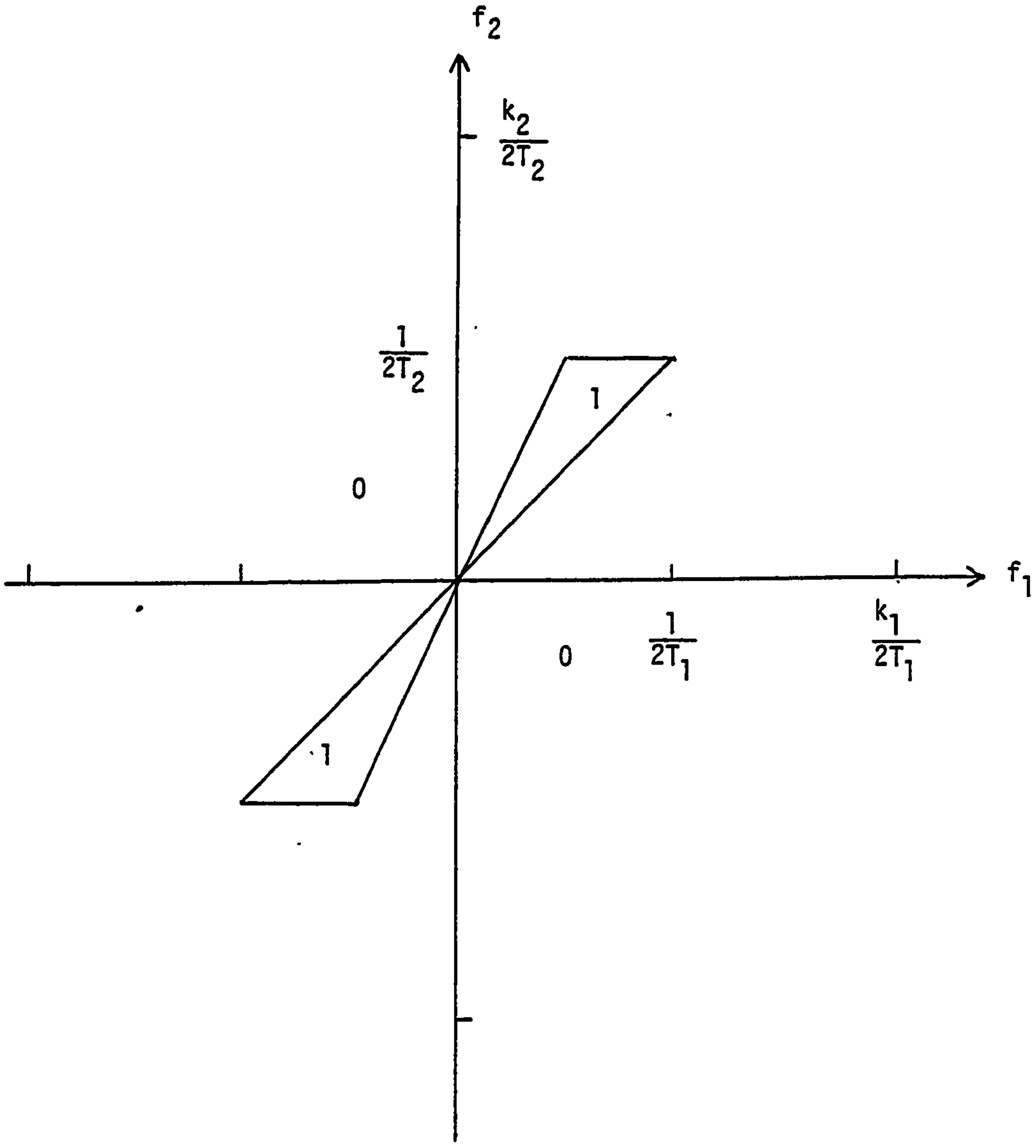

Figure 3.3(c) Magnitude Response of the Oversampled Velocity Filter

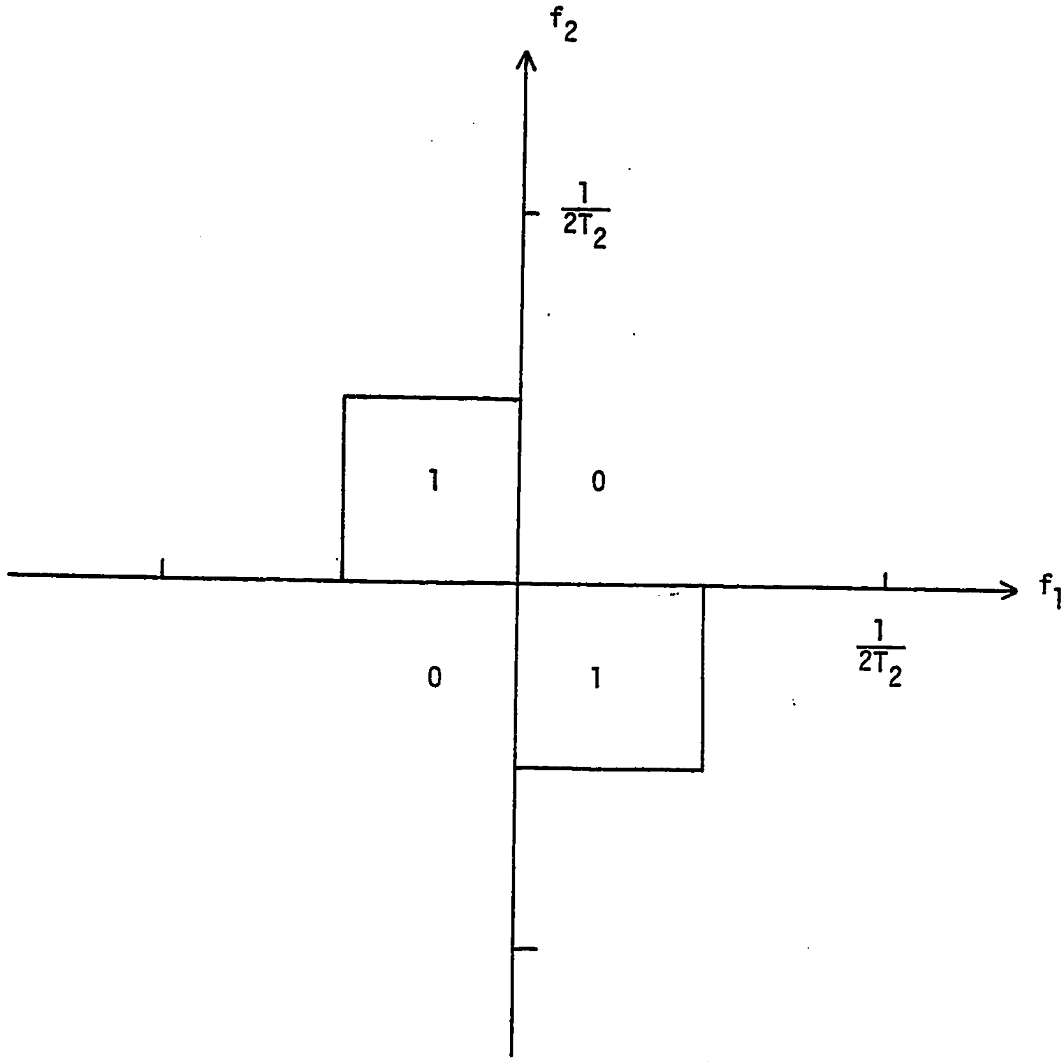

Figure 3.4. Unrotated Prototype Response

3.3(a) where

$$\phi_1 = \tan^{-1}(-P/Q) \tag{3.24}$$

and

$$\phi_2 = \tan^{-1}(-R/S) \; . \tag{3.25}$$

The design of the mask filter is somewhat arbitrary. It must, however, eliminate all passbands except the two triangular passbands as shown in Figure 3.3(c).

The importance of the modeling procedure of Chapter 2 is now evident. The modeling procedure can be used to redefine the filters on the original sampling grid. If the filters designed in this chapter are realized without being redefined on the original sampling grid, the resulting oversampled grid would be $k_1 k_2$ times larger or more dense than the original sampling grid. Thus, the storage of the corresponding interpolated data set x* which might have been a prohibitive factor has been avoided by the modeling approach.

3.4. A Numerical Example

The specifications and prototypes used in this example are illustrated in Figure 3.3 with $\phi_1 = 45^0$ and $\phi_2 = 71.6^0 = \tan^{-1}(3)$.

Using first-order one dimensional prototypes and the procedure of reference [6], the unrotated filter is

$$\hat{H}_v(\hat{z}_1,\hat{z}_2) = \frac{.2114 + .0771\hat{z}_1 - .1219\hat{z}_1^2 + .1667\hat{z}_1\hat{z}_2}{1 - \hat{z}_1 + .333\hat{z}_1^2} \; . \tag{3.26}$$

The substitutions

$$\hat{z}_1 = z_1 z_2^{-1} \tag{3.27}$$

and

$$\hat{z}_2 = z_1^3 z_2^{-1} \tag{3.28}$$

produce

$$\phi_1 = 45^o = \tan^{-1}(1) \tag{3.29}$$

and

$$\phi_2 = 71.6^o = \tan^{-1}(3) \ . \tag{3.30}$$

The warped filter corresponding to Figure 3.3(a) is

$$\begin{aligned}\tilde{H}_v(z_1,z_2) &= \hat{H}_v(z_1 z_2^{-1},\ z_1^3 z_2^{-1}) \\ &= \frac{.2114 + .0771 z_1 z_2^{-1} - .1219 z_1^2 z_2^{-2} + .1667 z_1^4 z_2^{-2}}{1 - z_1 z_2^{-1} + .333 z_1^2 z_2^{-2}}\end{aligned} \tag{3.31}$$

The desired response is Figure 3.3(c) truncated at $f_1 = \pm 1/2T_1$ and $f_2 = \pm 1/2T_2$. This means that the oversampling ratios are

$$k_1 = k_2 = 2 \ . \tag{3.32}$$

The transfer function matrix corresponding to Figure 3.3(a) is

$$\begin{aligned}\underline{H}_v(z_1,z_2) &= T'\{\tilde{H}_v(z_1,z_2)\} \\ &= \frac{\begin{bmatrix} \alpha & 0 & 0 & \beta z_1 z_2^{-1} \\ 0 & \alpha & \beta z_2^{-1} & 0 \\ 0 & \beta z_1 & \alpha & 0 \\ \beta & 0 & 0 & \alpha \end{bmatrix}}{(9 - 3 z_1 z_2^{-1} + z_1^2 z_2^{-2})}\end{aligned} \tag{3.33}$$

where

$$\alpha = 1.9026 + .2310 z_1 z_2^{-1} - .3657 z_1^2 z_2^{-2} + 1.5 z_1^2 z_2^{-1} + .5 z_1^3 z_2^{-2} \tag{3.34}$$

and

$$\beta = 2.5965 - .8658 z_1 z_2^{-1} + 1.5 z_1^2 z_2^{-1} \ . \tag{3.35}$$

The mask filter is generated from a second order rectangular low pass filter with half power points at $f_1 = \pm 1/2T_1$ and $f_2 = \pm 1/2T_2$. The result, using reference [6] is

$$\hat{H}_I(\hat{z}_3, \hat{z}_4) = \frac{(\hat{z}_3 + 1)^2 (\hat{z}_4 + 1)^2}{(3.4142 + .5858\hat{z}_3^2)(3.4142 + .5858\hat{z}_4^2)} \tag{3.36}$$

The substitutions

$$\hat{z}_3 = z_2^{-1} \tag{3.37}$$

and

$$\hat{z}_4 = z_1^2 z_2^{-1} \tag{3.38}$$

produce the warped filter

$$\begin{aligned} \tilde{H}_I(z_1, z_2) &= \hat{H}_I(z_2^{-1}, z_1^2 z_2^{-1}) \\ &= \frac{(z_2^{-1} + 1)^2 (z_1^2 z_2^{-1} + 1)^2}{(3.4142 + .5858 z_2^{-2})(3.4142 + .5858 z_1^4 z_2^{-2})} \ . \end{aligned} \tag{3.39}$$

The transfer function matrix corresponding to the mask filter is then

$$\underline{H}_I(z_1, z_2) = T' \ \{\tilde{H}_I(z_1, z_2)\} \tag{3.40}$$

where $\underline{H}_I$ is shown in Figure 3.5.

The overall transfer function of the single rate filter is given by

$(1 + z_2^{-1})(1 + z_1^2 z_2^{-1})$ $+4\, z_1 z_2^{-1}$	0	$(1 + z_2^{-1}) 2 z_1 z_2^{-1}$ $+2\, z_2^{-1}(1 + z_1^2 z_2^{-1})$	0
0	$(1 + z_2^{-1})(1 + z_1^2 z_2^{-1})$ $+4\, z_1 z_2^{-1}$	0	$(1 + z_2^{-1}) 2 z_1 z_2^{-1}$ $+2\, z_2^{-1}(1 + z_1^2 z_2^{-1})$
$2(1 + z_1^2 z_2^{-1})$ $+2\, z_1(1 + z_2^{-1})$	0	$(1 + z_2^{-1})(1 + z_1^2 z_2^{-1})$ $+4\, z_1 z_2^{-1}$	0
0	$z(1 + z_1^2 z_2^{-1})$ $+2\, z_1(1 + z_2^{-1})$	0	$(1 + z_2^{-1})(1 + z_1^2 z_2^{-1})$ $+4\, z_1 z_2^{-1}$

$$(3.4142 + 5858 z_1^2 z_2^{-1})(3.4142 + 5858 z_2^{-1})$$

Figure 3.5. Mask Filter Transfer Function Matrix

$$[1\ 0\ 0\ 0]\ \underline{H}_V\ \underline{H}_I \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} = H(z_1,z_2) \qquad (3.41)$$

$$= \frac{\left[1.9026 + .2310z_1z_2^{-1} - .3657z_1^2z_2^{-2} + 1.5z_1^2z_2^{-1} + .5z_1^3z_2^{-2}\right] \cdot \left[1 + z_2^{-1} + z_1^2z_2^{-1} + z_1^2z_2^{-2} + 4z_1z_2^{-1}\right]}{(3.4142 + .5858z_1^2z_2^{-1})(3.4142 + .5858z_2^{-1}) \cdot (9 - 3z_1z_2^{-1} + z_1^2z_2^{-2})}$$

The magnitude of $H(e^{i\omega_1}, e^{i\omega_2})$ is illustrated in Figure 3.6.

3.5. Summary

This chapter is an extension of the work of Chang and Aggarwal [3] to velocity filters. The extension to this class of filters is warranted by the ability developed in the previous chapter to replace multirate filters with single-rate filters without regard to the amount of oversampling.

The extension is a result of a generalization of the function of the interpolator, and a generalization of the index transformations used.

The extension of the function of the interpolator makes the use of the word "interpolator" inappropriate. A distinct portion of the filters synthesized in this chapter is reserved for the function previously occupied by interpolator in reference [3], and the new terminology "mask filter" applies to that portion of the filter.

ω_1 [degrees]

ω_2 [degrees]	0	20	40	60	80	100	120	140	160	180
180	.047	.197	.347	.468	.577	.664	.686	.590	.356	.066
160	.100	.291	.463	.613	.728	.761	.661	.432	.103	.269
140	.194	.406	.599	.751	.810	.728	.512	.195	.162	.420
120	.319	.549	.744	.843	.794	.610	.331	.102	.325	.418
100	.483	.719	.866	.866	.727	.494	.234	.243	.345	.306
80	.681	.880	.935	.849	.666	.428	.253	.281	.259	.175
60	.872	.987	.959	.829	.627	.390	.253	.221	.151	.081
40	1.004	1.035	.961	.805	.572	.321	.190	.134	.072	.029
20	1.033	1.047	.945	.747	.462	.208	.112	.070	.029	.006
0	1.080	1.030	.884	.620	.301	.102	.062	.037	.010	.000
-20	1.063	.971	.745	.425	.141	.038	.046	.024	.006	.006
-40	1.064	.831	.538	.228	.022	.052	.046	.023	.014	.029
-60	.872	.623	.323	.074	.053	.074	.055	.038	.043	.081
-80	.681	.408	.145	.040	.102	.099	.081	.079	.107	.175
-100	.483	.227	.016	.113	.145	.140	.138	.160	.220	.306
-120	.319	.090	.088	.176	.203	.214	.239	.294	.372	.418
-140	.194	.029	.170	.249	.292	.333	.392	.463	.501	.420
-160	.100	.112	.252	.344	.419	.495	.568	.594	.509	.269

Figure 3.6. Magnitude Response

The index transformations used in this chapter represent both a modification and a generalization of the index transformations used in reference [3]. The modification is evidenced by the fact that only integer powers of the z-transform variables are used. The extension is evidenced by the fact that the prototype frequency responses are not only rotated, but also warped.

Chapter Four

The Synthesis of 90^{0} Fan Filters

4.1. Introduction

In this chapter the methods of the two previous chapters are applied to the synthesis of a 90^{0} fan filter [20]. The ideal magnitude response is shown in Figure 4.1.

This class of filter is of particular interest because of the utility of the response in geoseismic signal processing and because the resulting designs are extremely efficient. Further, the results are generalized for arbitrarily high orders.

The procedure used is to generate prototypes as shown in Figure 4.2(a) using one dimensional Butterworth low pass transfer functions and reference [6]. The prototypes are then rotated 45^{0} to produce the general response shown in Figure 4.2(b).

The oversampling ratios $k_1 = k_2 = 2$ produce the specified response. The requirement for interpolation, per se, does not exist because of a periodicity in the frequency response of the rotated filter.

4.2. The One Dimensional Prototype

The choice of Butterworth one dimensional filters is critical because it has been demonstrated that it leads to pole-zero cancellations in the two dimensional filter. The half power points

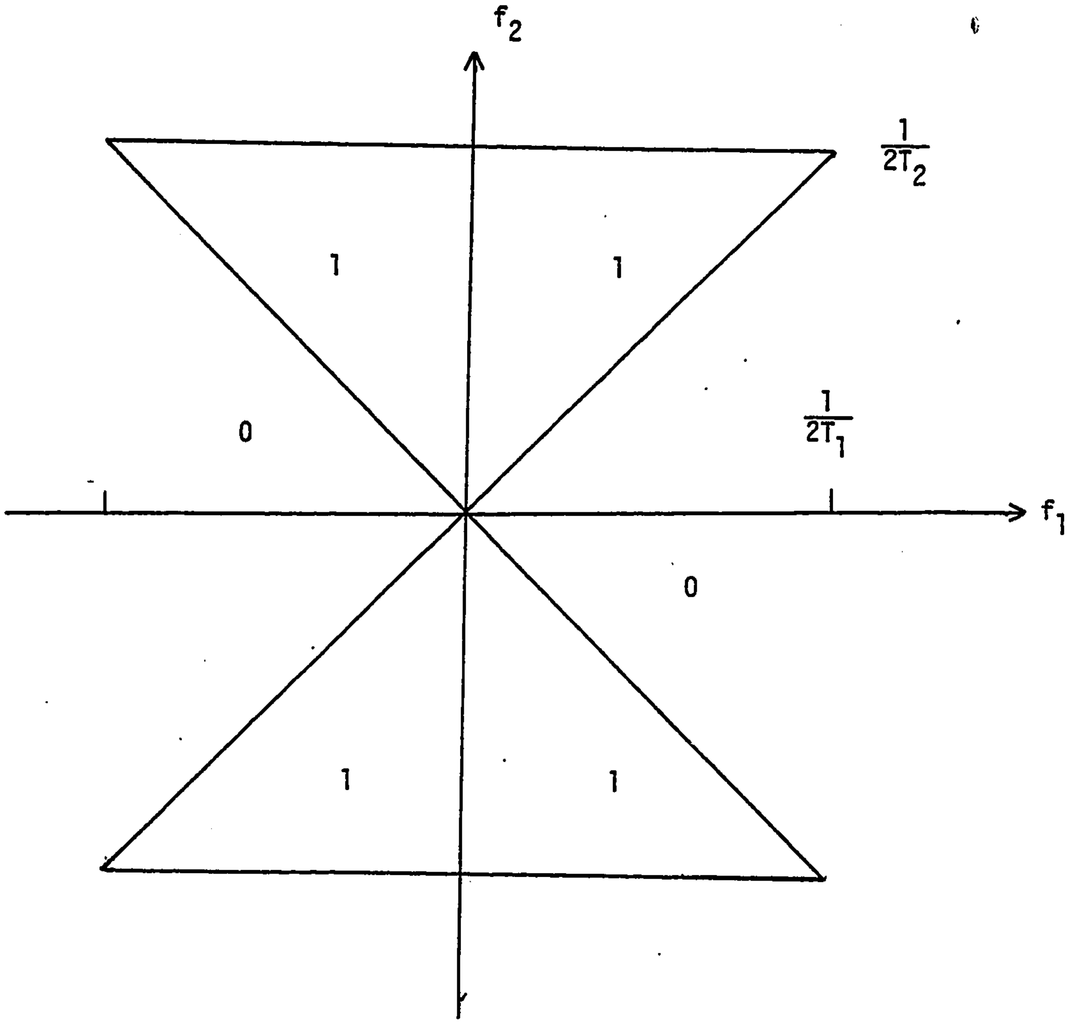

Figure 4.1. Ideal 90° Fan Filter

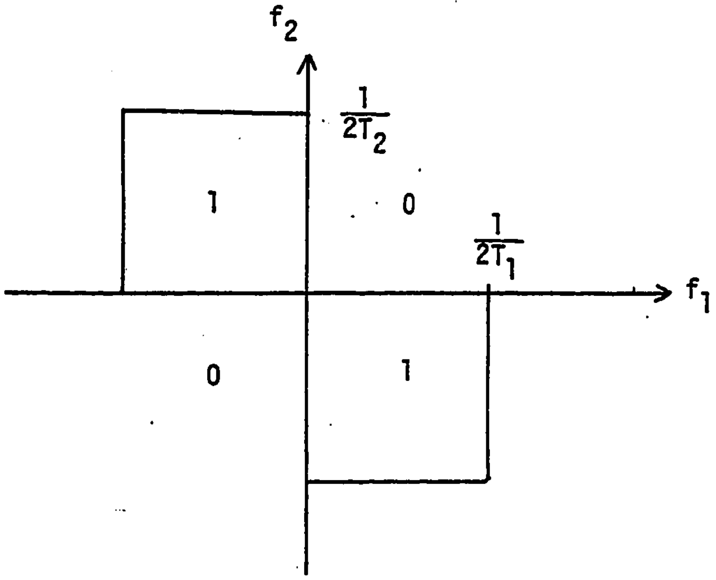

Figure 4.2(a) The Unrotated Prototype

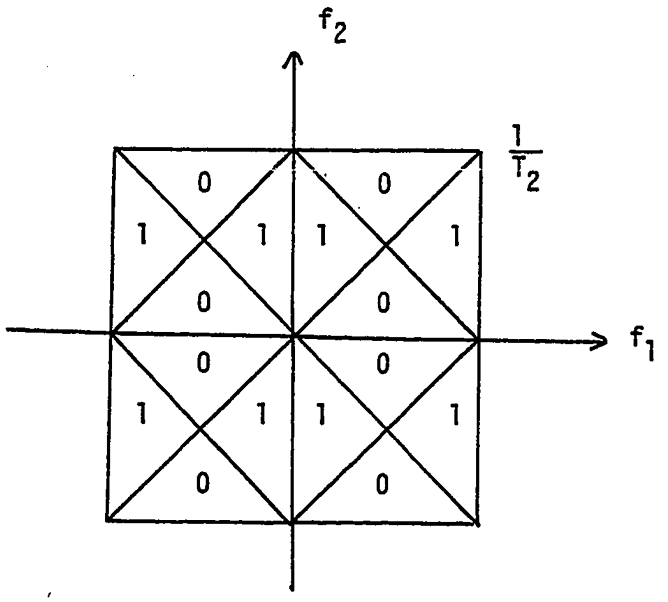

Figure 4.2(b) The Rotated Prototype

of the discrete time prototype are $z = \pm i$. The inverse bilinear transformation of these points produces half power points at $s = \pm i$ in the continuous time prototype.

The corresponding continuous time one dimensional Butterworth filter is

$$H_a^{(M)}(s) = \frac{1}{(s+1)\prod_{m=1}^{\frac{M-1}{2}} (s^2 + s2\cos\theta_{m,M} + 1)} \tag{4.1}$$

where

$$\theta_{m,M} = \frac{m\,\pi}{M} \tag{4.2}$$

for M odd, and

$$H_a^{(M)}(s) = \frac{1}{\prod_{k=1}^{M/2} (s^2 + s2\cos\theta_{m,M} + 1)} \tag{4.3}$$

where

$$\theta_{m,M} = \frac{(2m-1)\pi}{2M} \tag{4.4}$$

for M even.

Upon substituting

$$s = \frac{1-z}{1+z} \tag{4.5}$$

the discrete time filters are described by the transfer functions

$$H_d^{(M)}(z) = \frac{(1 + z)^M}{2\prod_{m=1}^{\frac{M-1}{2}} 2(z^2(1 - \cos\theta_{m,M}) + 1 + \cos\theta_{m,M})} \tag{4.6}$$

for M odd, and

$$H_d^{(M)}(z) = \frac{(1 + z)^M}{\prod_{m=1}^{\frac{M}{2}} 2(z^2(1 - \cos\theta_{m,M}) + 1 + \cos\theta_{n,M})} \tag{4.7}$$

for M even.

When M is odd, it has been observed that a number of pole-zero cancellations occur in the unrotated two dimensional prototype leading to an apparent order reduction and increased computational efficiency. When M is even, no such pole-zero cancellations occur. For this reason, the remainder of this chapter is restricted to the more efficient odd ordered filters.

The substitutions

$$\alpha_{m,M} = 1 + \cos\theta_{m,M} \tag{4.8}$$

and

$$\beta_{m,M} = \frac{1 - \cos\theta_{m,M}}{1 + \cos\theta_{m,M}} \tag{4.9}$$

can be used to produce

$$H_d^{(M)}(z) = \frac{(1 + z)^M}{2^{\frac{M+1}{2}} \prod^{\frac{M-1}{2}} (z^2\beta_{m,M} + 1)\, \alpha_{m,M}} \tag{4.10}$$

4.3. The Unrotated Two Dimensional Prototype

Using the procedure of reference [6], the two dimensional prototype illustrated in Figure 4.2(b) can be realized by

$$H_p^{(M)}(\hat{z}_1,\hat{z}_2) = H_d^{(M)}(i\hat{z}_1)\, H_d^{(M)}(-i\hat{z}_2) + H_d^{(M)}(-i\hat{z}_1)\, H_d^{(M)}(i\hat{z}_2) \tag{4.11}$$

$$= \frac{(1+i\hat{z}_1)^M(1-i\hat{z}_2)^M + (1-i\hat{z}_1)^M(1+i\hat{z}_2)^M}{2^{M+1} \prod_{m=1}^{\frac{M-1}{2}} (\hat{z}_1^2\beta_{m,M}-1)(\hat{z}_2^2\beta_{m,M}-1)\alpha^2_{m,M}}$$

Let $N^{(M)}(\hat{z}_1,\hat{z}_2)$, the numerator of $H_p^{(M)}(\hat{z}_1,\hat{z}_2)$ be defined by

$$N^{(M)}(\hat{z}_1,\hat{z}_2) = (1+i\hat{z}_1)^M(1-i\hat{z}_2)^M + (1-i\hat{z}_1)^M(1+i\hat{z}_2)^M \tag{4.12}$$

Then

$$N^{(M)}(z_1,z_2) = \sum_{m=0}^{M} \sum_{n=0}^{M} \binom{M}{n}(i)^n z_1^n \binom{M}{m}(-i)^m z_2^m + \sum_{m=0}^{M} \sum_{n=0}^{M} \binom{M}{n}(-i)^n z_1^n \binom{M}{m}(i)^m z_2^m \tag{4.13}$$

where $\binom{M}{n}$ is the binomial coefficient.

$$\binom{M}{n} = \frac{M!}{n!(M-n)!} \quad . \tag{4.14}$$

After cancelling the terms with imaginary coefficients and rearranging the summations, $N^{(M)}(z_1,z_2)$ can be written as

$$N^{(M)}(z_1,z_2) = 2 \sum_{n=0}^{\frac{M-1}{2}} \binom{M}{2n} z_1^{2n}(-1)^n \sum_{m=0}^{\frac{M-1}{2}} \binom{M}{2m} z_2^{2m}(-1)^m + 2 \sum_{n=0}^{\frac{M-1}{2}} \binom{M}{2n+1} z_1^{2n+1}(-1)^n \cdot \sum_{n=0}^{\frac{M-1}{2}} \binom{M}{2m+1} z_2^{2m+1}(-1)^m \; . \tag{4.15}$$

With the substitutions

$$f_M(z) = \sum_{n=0}^{\frac{M-1}{2}} \binom{M}{2n} z^{2n}(-1)^n \tag{4.16}$$

and

$$g_M(z) = \sum_{m=0}^{\frac{M-1}{2}} \binom{M}{2m+1} z^{2m}(-1)^m , \tag{4.17}$$

$N^{(M)}(z_1,z_2)$ can be written as

$$N^{(M)}(z_1,z_2) = 2f_M(z_1)f_M(z_2) + 2z_1z_2\, g_M(z_1)g_M(z_2) . \tag{4.18}$$

It has been observed that

$$f_M(z) = \prod_{m=1}^{\mathrm{int}(\frac{M-1}{4})} (\beta_{2m,M}z^2-1) \prod_{n=1}^{\mathrm{int}(\frac{M-1}{4})} (\beta^{-1}{}_{2n-1,M}\, z^2-1) \tag{4.19}$$

and

$$g_M(z) = \prod_{m=1}^{\mathrm{int}(\frac{M-1}{4})} (z^2-\beta_{2m,M}) \prod_{n=1}^{\mathrm{int}(\frac{M-1}{4})} (z^2-\beta^{-1}{}_{2n-1,M}) \tag{4.20}$$

for M = 3,5,7,15 and 19. It is therefore reasonable to use equations (4.19) and (4.20) as unproven conjectures.

Using the conjectured factorizations of f_M and g_M and cancelling zeros produces

$$H_p^{(M)}(\hat{z}_1,\hat{z}_2) = \tfrac{1}{2} \prod_{k=1}^{\mathrm{int}(\frac{M+1}{4})} \frac{(\hat{z}_1^2-\beta_{2k-1,M})(\hat{z}_2^2-\beta_{2k-1,M})}{(\hat{z}_1^2\beta_{2k-1,M}-1)(\hat{z}_2^2\beta_{2k-1,M}-1)} + \frac{\hat{z}_1\hat{z}_2}{2} \prod_{k=1}^{\mathrm{int}(\frac{M-1}{4})} \frac{(\hat{z}_1^2-\beta_{2k,M})(\hat{z}_2^2-\beta_{2k,M})}{(\hat{z}_1^2\beta_{2k,M}-1)(\hat{z}_2^2\beta_{2k,M}-1)} \tag{4.21}$$

4.4. The Rotated Single-Rate Filter

The substitutions

$$\hat{z}_1 = z_1 z_2^{-1} \tag{4.22}$$

and

$$\hat{z}_2 = z_1 z_2 \tag{4.23}$$

rotate the response given by equation (4.21) to the shape indicated by Figure 4.2(b). The resulting transfer function is given by

$$\begin{aligned} H_o(z_1,z_2) &= H_p^{(M)}(z_1 z_2^{-1}, z_1 z_2) \\ &= \tfrac{1}{2} \prod_{k=1}^{\mathrm{int}(\frac{M+1}{4})} \frac{(z_1^2 z_2^{-2} - \beta_{2k-1,M})(z_1^2 z_2^2 - \beta_{2k-1,M})}{(z_1^2 z_2^{-2}\beta_{2k-1,M} - 1)(z_1^2 z_2^2 \beta_{2k-1,M} - 1)} \\ &\quad + \frac{z_1^2}{2} \prod_{k=1}^{\mathrm{int}(\frac{M-1}{4})} \frac{(z_1^2 z_2^{-2} - \beta_{2k,M})(z_1^2 z_2^2 - \beta_{2k,M})}{(z_1^2 z_2^{-2}\beta_{2k,M} - 1)(z_1^2 z_2^2 \beta_{2k,M} - 1)} \end{aligned} \tag{4.24}$$

This transfer function is a function of only z_1^2 and z_2^2. This means that for the oversampling ratios $k_1 = k_2 = 2$, the corresponding transfer function matrix is

$$\begin{aligned} \underline{H}_o &= T'(\tilde{H}_o(z_1,z_2)) \\ &= \underline{I}\, \tilde{H}_o(\sqrt{z_1}, \sqrt{z_2}) \end{aligned} \tag{4.25}$$

where $\underline{I}$ is the 4 x 4 identity matrix. It is also noted that

$$\tilde{G}_0(f_1,f_2) = \tilde{H}_0(z_1,z_2)\Big|_{\substack{z_1 = e^{i\pi f_1 T_1} \\ z_2 = e^{i\pi f_2 T_2}}} \tag{4.26}$$

$$= \tilde{G}_0(f_1+1/T_1,f_2)$$

$$= \tilde{G}_0(f_1,f_2+1/T_2)$$

$$= \tilde{G}_0(f_1+1/T_1,f_2+1/T_2) \ .$$

Referring to equation (3.17), it can be seen that the frequency response of the single-rate filter without interpolation is $\tilde{G}_0(f_1,f_2)$ even with the aliasing due to undersampling. The overall transfer function of the single-rate filter is then

$$H_0(z_1,z_2) = [1\ 0\ 0\ 0]\ \underline{H}_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.27}$$

$$= \tilde{H}_0(\sqrt{z_1},\ \sqrt{z_2})$$

For M = 7,

$$H_o(z_1,z_2) = \frac{(z_1z_2^{-1}-.05210)(z_1z_2-.05210)(z_1z_2^{-1}-.63596)(z_1z_2-.63596)}{2(.05210z_1z_2^{-1}-1)(.05210z_1z_2-1)(.63596z_1z_2^{-1}-1)(.63596z_1z_2-1)} \tag{4.28}$$

$$+ \frac{z_1(z_1z_2^{-1}-.23191)(z_1z_2-.23191)}{2(.23191z_1z_2^{-1}-1)(.23191z_1z_2-1)} \ .$$

The corresponding magnitude response is shown in Figure 4.3.

ω_1 [degrees]

ω_2 [degrees]	0	20	40	60	80	100	120	140	160	180
180	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	1.5	∞
160	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.4	3.8	∞
140	0.0	0.0	0.0	0.0	0.0	0.0	0.4	3.1	11.7	∞
120	0.0	0.0	0.0	0.0	0.0	0.4	3.0	11.0	22.3	∞
100	0.0	0.0	0.0	0.0	0.4	3.0	11.0	21.7	33.6	∞
80	0.0	0.0	0.0	0.4	3.0	11.0	21.7	33.6	45.6	∞
60	0.0	0.0	0.4	3.0	11.0	21.7	33.6	45.6	61.5	∞
40	0.0	0.3	3.0	11.0	21.7	33.6	45.6	61.5	80.1	∞
20	0.0	2.4	10.4	21.2	32.9	45.6	61.4	80.1	105.5	∞
0	0.0	5.4	15.7	27.6	41.2	55.2	74.1	99.5	141.9	∞

$$z_j = e^{i\omega_j}$$

Figure 4.3. Attenuation of a Seventh Order Fan Filter [dB]

4.5. Filter Realization

The transfer function $H_0(z_1,z_2)$ can be decomposed into four one dimensional all-pass sections in the form

$$H_A(z,\underline{\beta}) = \prod_{k=1}^{b} \frac{(z-\beta_k)}{(z\beta_k-1)} \tag{4.29}$$

where the one dimensional sections are realized in the z_1z_2 and $z_1z_2^{-1}$ directions. The vector $\underline{\beta}$ describes the appropriate transfer function.

Transfer functions in the form of equation (4.29) can be realized by the structure shown in Figure 4.4. This structure has the advantage of requiring only b multiplications for an all-pass of order b. It also insures that the filter will be a true all-pass, with phase shift only, irregardless of the quantization of the coefficient values.

If the one dimensional prototypes are of order M, then the two dimensional filter can be realized with M-1 multiplications per point.

4.6. Phase Correction

Multidimensional recursive digital filters can be used to produce a zero phase response by passing the data through the filter in two opposite directions [5]. A simple method of achieving an approximately zero phase response with the 90^o fan filters synthesized in this chapter is possible because of the form of the transfer functions.

The filters synthesized in this chapter are in the form

$$H(z_1,z_2) = \frac{H_1(z_1,z_2) + H_2(z_1,z_2)}{2} \tag{4.30}$$

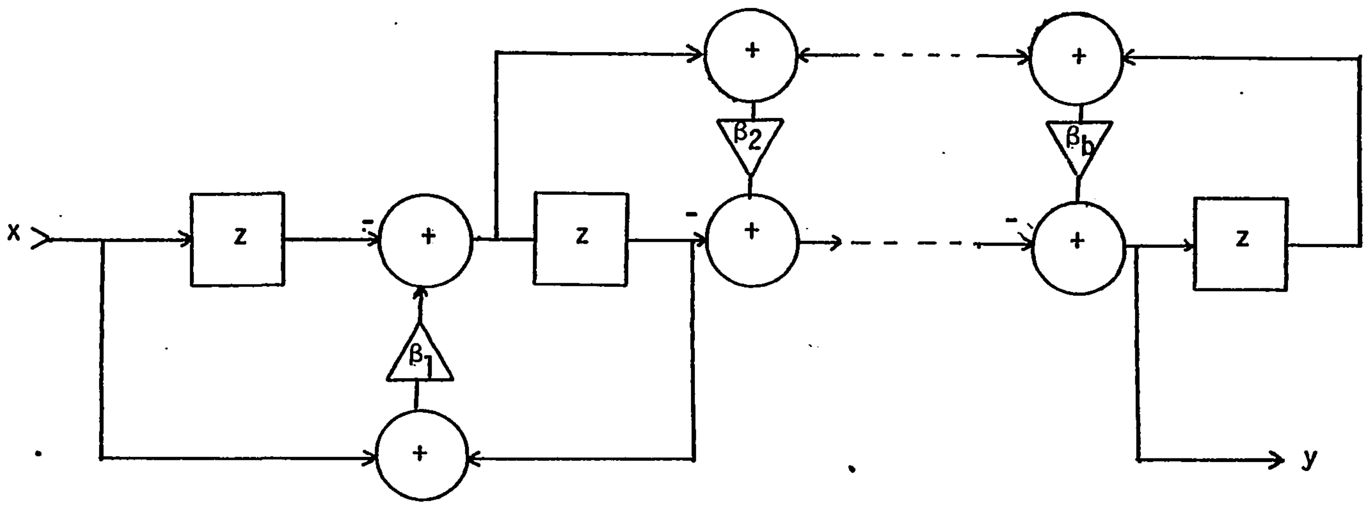

Figure 4.4. All-Pass Filter Realization

where

$$H_1(z_1,z_2) = \prod_{k=1}^{\mathrm{int}(\frac{M+1}{4})} \frac{(z_1z_2^{-1}-\beta_{2k-1,M})(z_1z_2-\beta_{2k-1,M})}{(z_1z_2^{-1}\beta_{2k-1,M}-1)(z_1z_2\beta_{2k-1,M}-1)} \tag{4.31}$$

and

$$H_2(z_1,z_2) = z_1 \prod_{k=1}^{\mathrm{int}(\frac{M-1}{4})} \frac{(z_1z_2^{-1}-\beta_{2k,M})(z_1z_2-\beta_{2k,M})}{(z_1z_2^{-1}\beta_{2k,M}-1)(z_1z_2\beta_{2k,M}-1)} . \tag{4.32}$$

In the passband,

$$|H(z_1,z_2)| \cong 1 . \tag{4.33}$$

This implies that in the passband,

$$H_1(z_1,z_2) \cong H_2(z_1,z_2) \cong H(z_1,z_2) \tag{4.34}$$

In the stopband,

$$|H(z_1,z_2)| \cong 0 . \tag{4.35}$$

This implies that in the stopband,

$$H_1(z_1,z_2) \cong -H_2(z_1,z_2) . \tag{4.36}$$

Also,

$$|H_1(z_1,z_2)| = |H_2(z_1,z_2)| = 1 \tag{4.37}$$

while

$$|z_1| = 1 \text{ and } |z_2| = 1 . \tag{4.38}$$

Because of equation (4.34) the phase in the passband can be set approximately equal to zero by cascading the filter with either $H_1(z_1^{-1},z_2^{-1})$ or $H_2(z_1^{-1},z_2^{-1})$ as shown in Figure 4.5(a). Further, it is

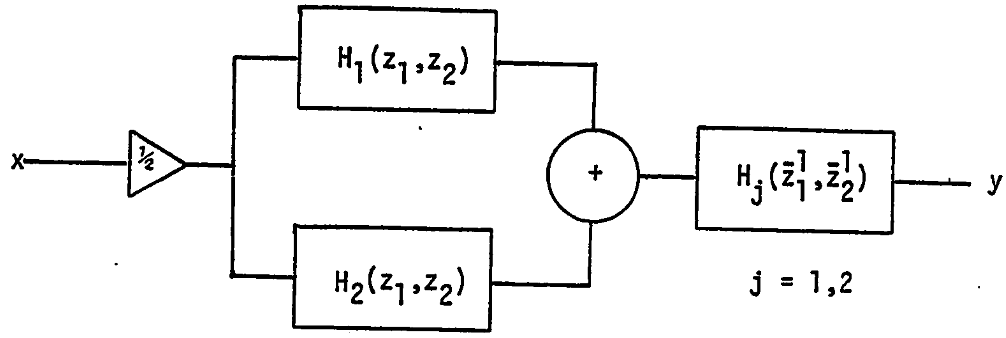

Figure 4.5(a) Phase Corrected Fan Filter

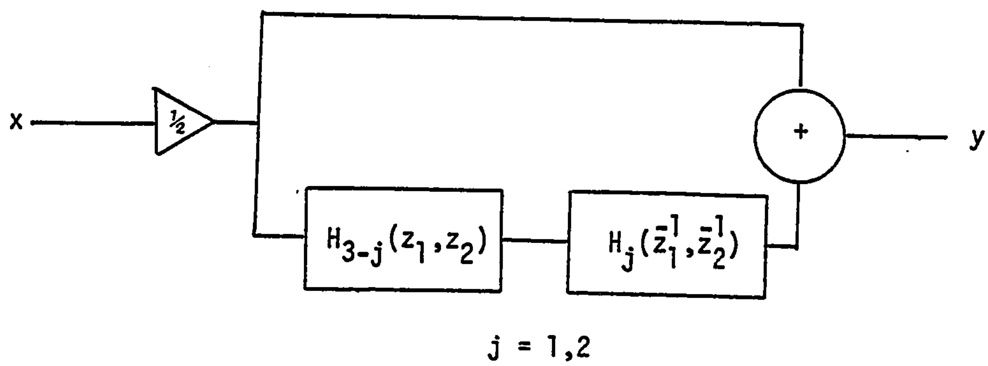

Figure 4.5(b) Simplified Phase Correction

noted that

$$H_1(z_1^{-1},z_2^{-1}) = H_1^{-1}(z_1,z_2) \tag{4.39}$$

and

$$H_2(z_1^{-1},z_2^{-1}) = H_2^{-1}(z_1,z_2) \ . \tag{4.40}$$

The filters shown in Figures 4.5(a) and 4.5(b) therefore produce equivalent responses.

The filter shown in Figure 4.5(b) requires the same amount of arithmetic as the filter without phase correction, but the subsystem $H_1(z_1^{-1},z_2^{-1})$ must be realized in the opposite direction as $H_2(z_1,z_2)$.

The phase response of a seventh order fan filter with phase correction is shown in Figure 4.6.

4.7. Summary

In this chapter the methods of the previous two chapters are applied to the synthesis of 90^0 fan filters. Conventional methods are used throughout, except for trivial cases of the dual-rate to single-rate conversion, and the frequency domain approach to interpolation developed in the two preceding chapters.

The use of odd-ordered Butterworth one dimensional prototypes is found to be fortuitous because of a set of conjectured pole-zero cancellations.

The final form of the transfer function is two parallel directionally separable all-pass filters. In the passbands, the phases of the two all-pass filters are coherent. In the stopbands, the phases differ by 180^0. This suggests a simplification in the method of correcting the phase of the frequency response.

ω_1
[degrees]

ω_2 [degrees]	0	20	40	60	80	100	120	140	160	180
180	0.0	0.0	0.0	0.0	.1	.5	2.4	9.4	32.6	90.0
160	0.0	0.0	0.0	0.0	.3	1.3	5.0	17.5	49.7	90.0
140	0.0	0.0	0.0	.3	1.2	4.7	16.4	45.3	74.9	90.0
120	0.0	0.0	.3	1.2	4.7	16.3	45.0	73.7	85.6	90.0
100	0.0	.3	1.2	4.7	16.3	45.0	73.7	85.3	88.8	90.0
80	0.0	1.2	4.7	16.3	45.0	73.7	85.3	88.8	89.7	90.0
60	0.0	4.4	16.3	45.0	73.7	85.3	88.8	89.7	90.0	90.0
40	0.0	15.1	44.7	73.6	85.3	88.8	89.7	90.0	90.0	90.0
20	0.0	40.3	72.5	85.0	88.7	89.7	90.0	90.0	90.0	90.0
0	0.0	57.4	80.6	87.6	89.5	89.9	90.0	90.0	90.0	90.0

$$z_j = e^{i\omega_j}$$

Figure 4.6. Phase Response of a Seventh Order Fan Filter [degrees]

The fan filters synthesized by Garibotto [13] were serially decomposed into three distinct types of filters. One type of filter achieved the appropriate magnitude response, another type of filter achieved a nearly planar phase response and a third type of filter achieved a hyperbolic phase response.

The filters synthesized in this chapter can directly replace the first and second type of filter synthesized by Garibotto [13]. The synthesis procedure developed in this chapter is much less expensive in terms of computations than the procedure developed by Garibotto [13]. Further, for a given input set, the filters presented in this chapter will operate at sampling frequencies approximately 20% lower than the sampling frequencies of the filters synthesized by Garibotto [13].

There is no universally accepted performance criterion for two dimensional digital filters. However, the filters synthesized in this chapter show roll offs which are comparable to filters which require the same amount of arithmetic, but were synthesized using optimization procedures.

CONCLUSION

This dissertation develops new perspectives for the synthesis of velocity filters. The new material consists of a generalization of the synthesis procedure of Chang and Aggarwal [3], and a vector-matrix method of modeling digital filters with multiple sampling rates. The work presented here permits the synthesis of velocity filters from one dimensional filters using prototypes with multiple sampling rates.

The development provided in Chapter Two describes a method of modeling multirate digital filters as single-rate digital filters with identical input-output relationships. The multirate modeling method can be used to isolate the amount of storage required for intermediate results in the filters from the amount of oversampling present in the prototype filters. The use of a vector-matrix approach permits the subsystems to be cascaded or paralleled via matrix multiplication or addition, respectively. The method presented in Chapter Two is restricted to two dimensions and two pairs of sampling frequencies. It can, however, be generalized to arbitrary numbers of dimensions and sampling frequencies.

In Chapter Three two new approaches are used in the synthesis of velocity filters. The new approaches are warranted because of the ability developed in Chapter Two to replace multirate prototype filters with single-rate filters.

The first new approach is a generalization of the index transformations used by Chang and Aggarwal [3] to warp as well as rotate the unit bidisc. The generalized index transformation leads to transfer functions in integer rather than rational powers of the z-transform variables.

The use of only integer powers of the z-transform variables permits the second new synthesis approach. This approach is the generalization of the function of the interpolators used by Chang and Aggarwal [3] to permit the synthesis of a larger class of filters including velocity filters.

The method of synthesizing velocity filters exemplified in Chapter Three is not completely specified. It is not a closed procedure, and the arbitrary portions of the procedure should eventually be replaced by closed procedures. The transfer functions resulting from the procedure given in Chapter Three should be compared to transfer functions generated by iterative optimization techniques [9-14] on the basis of their ease of realization.

The iterative optimization procedures in the literature [9-14] which can be used for velocity filter synthesis optimize weighted ℓ_p norms and require a great deal of programming effort and computation. Their practical limitations reside in their difficulty of implementation. Their theoretical limitation is that they do not optimize any integral L_p norm, thus leaving the frequency response unspecified except at a number of discrete points. This dissertation is the result of an attempt to supplant iterative optimization methods in the synthesis of two dimensional recursive digital filters. The attempt is

best justified by the simplicity and effectiveness of the synthesis procedure for 90^0 fan filters developed in Chapter Four.

In Chapter Four, the procedures for spectral manipulation developed by Chang and Aggarwal [3] and Hirano and Aggarwal [6] are repeated using one dimensional Butterworth prototypes. The modeling method developed in Chapter Two is used to transform the multirate filter to a single-rate filter. The generalized perspective on interpolation developed in Chapter Three is used to indicate that no interpolation is required. A conjectured set of pole-zero cancellations, which were routinely observed are used to simplify the transfer function. The transfer function is then decomposable into a number of directionally separable all-pass filters.

The decomposition into all-pass filters is significant for three reasons. First, one dimensional all-pass filters can be realized by a special, efficient algorithm. Second, it was shown that the decomposition suggests an efficient method of producing an approximately zero phase response.

Most importantly, the decomposition is significant because of the future research it suggests. It is important to note that decomposition was merely observed. No adequate theory for the decomposition has been identified. If this theory is formulated, it may lead to methods of generalizing the response to a larger class of filters exhibiting the same utility and economy of realization as the filters synthesized in Chapter Four.

This dissertation has resolved a problem peculiar to two dimensional frequency selective filters. The problem is the requirement

to deal with prototype filters with more than one sampling rate in each dimension.

This problem arose in the synthesis of velocity filters because of a requirement to use directionally separable filters, and the solution of the problem in that context occupies this dissertation's third and fourth chapters. The same difficulty also arises in the beamforming problem [26], and the extension of the methodology presented here to that area should be straightforward.

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AUTOBIOGRAPHICAL STATEMENT

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