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Research article

Recent analytic development of the dynamic Q-tensor theory for nematic liquid crystals

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Abstract: Liquid crystals are a typical type of soft matter that are intermediate between conventional crystalline solids and isotropic fluids. The nematic phase is the simplest liquid crystal phase, and has been studied the most in the mathematical community. There are various continuum models to describe liquid crystals of nematic type, and Q-tensor theory is one among them. The aim of this paper is to give a brief review of recent PDE results regarding the Q-tensor theory in dynamic configurations.

Keywords: nematic liquid crystals; Q-tensors; dynamic theory; analytic work; hydrodynamic systems; gradient flows

1. Introduction

Liquid crystals are often considered as the fourth state of the matter besides the gas, liquid and solid, or as an intermediate state between liquid and solid. Liquid crystals are partially ordered materials that can translate freely as conventional fluid, while exhibit certain long-range order below a critical temperature. There are many different types of liquid crystals, the main classes being nematics, smectics and cholesterics. The nematic phase is the simplest among all liquid crystal phases. In this phase molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation [1], which makes the liquid anisotropic. The rod-like nematic liquid crystals are the most widely studied among all liquid crystals. The study of liquid crystals is of great significance for both fundamental scientific researches as well as their widespread applications in industry. Mathematical theories at different levels have been developed to describe nematic liquid crystals, ranging from the microscopic molecular theory to macroscopic continuum ones. Generally speaking, there are several closely related models in the continuum theory to describe nematic liquid crystals [2–4], namely Oseen-Frank theory [5], Ericksen's theory [3], and Landau-de Gennes theory [2, 6]. In these theories, various order parameters are utilized to encode mathematically the local ordering of liquid crystal molecules.

Assume liquid crystals are occupying an open region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. In the Oseen-Frank theory, the order parameter is

$$\vec{n} : \Omega \to \mathbb{S}^2. \tag{1.1}$$

In the Ericksen's theory, the order parameter is

$$(\vec{n}, s) \to \mathbb{S}^2 \times [-1/2, 1]. \tag{1.2}$$

In the Landau-de Gennes theory, the order parameter is

$$Q \to \mathcal{Q} := \{M \in \mathbb{R}^{3\times3},\ M^t = M,\ \mathrm{tr}(M) = 0\}, \tag{1.3}$$

which is an element in the space of Q-tensors. The Oseen-Frank theory can account for point defects but not line/surface defects [7] (roughly speaking, they are discontinuities in the alignment of liquid crystals). The Ericksen's theory incorporates both line/surface defects [8, 9], but not biaxial liquid crystal structures. These two theories are vector theories in that the basic element is a unit vector $\vec{n}$ representing the mean orientation of neighbouring liquid crystal molecules. In the Landau-de Gennes theory, the basic element Q is a 3×3 symmetric, traceless matrix to describe the alignment of liquid crystal molecules. It is the most comprehensive one among these continuum theories [10, 11].

Concerning the mathematical analysis of nematic liquid crystals in the framework of these aforementioned continuum models, there has been a fast development particularly in recent years. The aim of this review is to provide an outline of the relevant PDE analysis on the dynamic Q-tensor theory, many of which are closely related to the author's own research work. Several intrinsically connected topics are presented in separate subsections below, and many unsolved problems are proposed. Due to spatial constraint, this review only aims to present major development of the analytic Q-tensor theory in dynamic settings, but not to provide a comprehensive list of the existing literature.

2. A brief review of recent PDE results on dynamic Q-tensor theory

2.1. The Beris-Edwards hydrodynamic system

The earliest PDE results on the dynamic Q-tensor theory came from [12, 13], where the Cauchy problem of the hydrodynamic flow was studied. The corresponding coupled PDE system, called the Beris-Edwards system [14], consists of incompressible Navier-Stokes equations for the fluid velocity with highly nonlinear anisotropic force terms and nonlinear convection diffusion equations of parabolic type that describe the evolution of the Q-tensor:

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \nu \Delta \vec{u} + \lambda \nabla \cdot (\tau + \sigma), \\ \nabla \cdot \vec{u} = 0, \\ \partial_t Q + \vec{u} \cdot \nabla Q - S(\nabla \vec{u}, Q) = \Gamma H(Q) \end{cases} \tag{2.1}$$

Here $\vec{u}(x,t) : \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}^d$ denotes the velocity field of the fluid, $P : \mathbb{R}^d \times (0, +\infty) \to \mathbb{R}$ represents the hydrostatic pressure, and $Q(x,t) : \mathbb{R}^d \times (0, +\infty) \to \mathcal{Q}$ stands for the order parameter of liquid crystal molecules. The positive constants ν, λ, Γ stand for the fluid viscosity, the competition between kinetic energy and elastic potential energy, and macroscopic elastic relaxation time for the molecular orientation field, respectively. System (2.1) describes the interaction between the incompressible fluid flow and the alignment of liquid crystal molecules. The evolution of the fluid affects the direction of the molecules while changes in molecular alignment also influence the fluid velocity. The free energy of liquid crystal molecules in (2.1) is given by

$$\mathcal{F}(Q) = \int_{\mathbb{R}^d} \Big[\frac{L}{2}|\nabla Q|^2 + f(Q)\Big]\mathrm{d}x, \tag{2.2}$$

where the so called one constant approximation of the Oseen-Frank energy is used, and $L > 0$ is an elastic constant. Meanwhile, the bulk part $f(Q)$ is taken to be a fourth order degree polynomial:

$$f(Q) = \frac{a}{2}\,\mathrm{tr}(Q^2) - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}\,\mathrm{tr}^2(Q^2), \tag{2.3}$$

where a, b, c are assumed to be material and temperature dependent constants. Note that the assumption

$$c > 0 \tag{2.4}$$

is always imposed in all existing literature to ensure the free energy is bounded from below. The tensor $H(Q)$ is defined to be the variational derivative of $\mathcal{F}(Q)$ with respect to Q under both symmetry and traceless constraints:

$$H(Q) = -\frac{\partial \mathcal{F}(Q)}{\partial Q} = L\Delta Q - aQ + b\Big(Q^2 - \frac{1}{3}\mathrm{tr}(Q^2)\mathbb{I}_3\Big) - cQ\,\mathrm{tr}(Q^2). \tag{2.5}$$

And the matrix-valued function $S(\nabla\vec{u}, Q)$ reads

$$S(\nabla\vec{u}, Q) := (\xi D + \omega)(Q + \frac{1}{3}\mathbb{I}_3) + (Q + \frac{1}{3}\mathbb{I}_3)(\xi D - \omega) - 2\xi(Q + \frac{1}{3}\mathbb{I}_3)\,\mathrm{tr}(Q\nabla\vec{u}). \tag{2.6}$$

Here

$$D = \frac{\nabla\vec{u} + \nabla^T\vec{u}}{2}, \qquad \omega = \frac{\nabla\vec{u} - \nabla^T\vec{u}}{2}$$

represent the symmetric and skew-symmetric parts of the rate of strain tensor, respectively. In addition, $S(\nabla u, Q)$ accounts for the rotating and stretching effects on the order parameter Q due to the fluid, as the liquid crystal molecules can be tumbled and aligned by the flow. The constant parameter $\xi \in \mathbb{R}$ depends on the molecular shapes of the liquid crystal, which also measures the ratio between the tumbling and the aligning effects that a shear flow exerts on the liquid crystal director. In particular, when $\xi = 0$ the system (2.1) is called the co-rotational Beris-Edwards system. While when $\xi \neq 0$, it is at times referred to as the non co-rotational Beris-Edwards system, or the full Beris-Edwards system.

The symmetric and skew-symmetric parts of the stress terms in the first equation of (2.1) caused by anisotropy of liquid crystal molecules are given respectively by

$$\tau := -\xi(Q + \frac{1}{3}\mathbb{I}_3)H(Q) - \xi H(Q)(Q + \frac{1}{3}\mathbb{I}_3) + 2\xi(Q + \frac{1}{3}\mathbb{I}_3)\,\mathrm{tr}(QH(Q)) - L\nabla Q \odot \nabla Q, \tag{2.7}$$

$$\sigma := QH(Q) - H(Q)Q. \tag{2.8}$$

It is noted that the Beris-Edwards system (2.1) satisfies a dissipative energy law under various suitable boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u, Q) = -\frac{\nu}{2}\int |\nabla\vec{u}|\,\mathrm{d}x - \lambda\Gamma\int |H(Q)|^2\,\mathrm{d}x. \tag{2.9}$$

Here $E(u, Q)$ denotes the total energy of the hydrodynamic system, which is the sum of the Q-tensor free energy $\mathcal{F}$ and the kinetic energy $1/2 \int |u|^2 \, \mathrm{d}x$.

The Beris-Edwards system (2.1) contains a significant number of highly nonlinear terms and generates considerable analytical difficulties, mainly due to nonlinear coupling with the incompressible Navier-Stokes equations. Since it contains the incompressible Navier-Stokes equations as a subsystem, we may not expect any better well-posedness results than the Navier-Stokes equations. For the co-rotational case $\xi = 0$, the first contribution came from [13], where the authors proved the existence of global weak solutions to the Cauchy problem in $\mathbb{R}^d$ for $d = 2, 3$.

Definition 2.1. *A pair $(\vec{u}, Q)$ is called a weak solution of the problem* (2.1) *in $\mathbb{R}^d$, subject to initial data*

$$\vec{u}(0, x) = \vec{u}_0(x) \in L^2(\mathbb{R}^d), \quad \nabla \cdot \vec{u}_0 = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d), \quad Q(0, x) = Q_0(x) \in H^1(\mathbb{R}^d) \tag{2.10}$$

provided

$$\vec{u} \in L^\infty_{loc}(0, +\infty; L^2) \cap L^2_{loc}(0, +\infty; H^1), \quad Q \in L^\infty_{loc}(0, +\infty; H^1) \cap L^2_{loc}(0, +\infty; H^2),$$

and for compactly supported test functions $\psi \in C^\infty([0, \infty) \times \mathbb{R}^d; \mathbb{R}^d)$, $\nabla \cdot \psi = 0$, $\phi \in C^\infty([0, \infty) \times \mathbb{R}^d; \mathbf{Q})$, it holds

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \big(- \vec{u}\partial_t \psi - \vec{u} \otimes \vec{u} : \nabla \vec{u} + \nu \nabla \vec{u} : \nabla \psi \big) \, \mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^d} u_0(x) \psi(0, x) \, \mathrm{d}x \\
&= L \int_0^\infty \int_{\mathbb{R}^d} \Big[(\nabla Q \odot \nabla Q : \nabla \psi) - (Q \Delta Q - \Delta Q Q) : \nabla \psi \Big] \, \mathrm{d}x\mathrm{d}t, \\
&\int_0^\infty \int_{\mathbb{R}^d} \Big[- Q : \partial_t \phi - \Gamma L \Delta Q : \phi - Q : (\vec{u} \cdot \nabla \phi) - \omega Q : \phi + Q \omega : \phi \Big] \, \mathrm{d}x\mathrm{d}t \\
&= \Gamma \int_0^\infty \int_{\mathbb{R}^d} \Big[- aQ + bQ^2 - \frac{b \, \mathrm{tr}(Q^2)}{d} \mathbb{I}_d - cQ \, \mathrm{tr}(Q^2) \Big] : \phi \, \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^d} Q_0(x) : \phi(0, x) \, \mathrm{d}x.
\end{aligned}$$

Here $A : B$ stands for the inner product between two matrices of the same size.

Theorem 2.1. *For the co-rotational Beris-Edwards system* (2.1)*, there exists a global weak solution subject to the initial condition* (2.10)*.*

The proof of Theorem 2.1 was composed of several steps. First, a regularized sequence $(\vec{u}^{(n)}, Q^{(n)})$ was constructed to an approximate system following the classical Friedrich's scheme. Next, uniform apriori bounds collected from the basic energy dissipative law as

$$\sup_n \|\vec{u}^{(n)}\|_{L^\infty(0,T;L^2) \cap L^2(0,T;H^1)} < \infty, \qquad \sup_n \|Q^{(n)}\|_{L^\infty(0,T;H^1) \cap L^2(0,T;H^2)} < \infty,$$

combined with classical Aubin-Lions compactnes arguments, paved the way to pass limit $n \to +\infty$.

Meanwhile, the authors in [13] also obtained results on higher global regularity of solutions as well as the weak-strong uniqueness for $d = 2$:

Theorem 2.2. *For any $s > 1$ and initial data $(\vec{u}_0, Q_0) \in H^s(\mathbb{R}^2) \cap H^{s+1}(\mathbb{R}^2)$, the co-rotational Beris-Edwards system admits a global strong solution in the sense that*

$$\begin{aligned}
\vec{u} &\in L^2_{loc}(0, +\infty; H^{s+1}(\mathbb{R}^2)) \cap L^\infty_{loc}(0, +\infty; H^s(\mathbb{R}^2)), \\
Q &\in L^2_{loc}(0, +\infty; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{loc}(0, +\infty; H^{s+1}(\mathbb{R}^2)).
\end{aligned}$$

The proof of Theorem 2.2 was mainly based on H^s energy estimates, and the most crucial step to control the highest order nonlinear term was the discovery of the following cancellation relation

$$\int_{\mathbb{R}^d} \mathrm{tr}\,((\omega Q' - Q'\omega)\Delta Q)\,\mathrm{d}x - \int_{\mathbb{R}^d} \nabla \cdot (Q'\Delta Q - \Delta Q Q')\vec{u}\,\mathrm{d}x = 0,$$

for symmetric, smooth matrix-valued functions $Q', Q : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. The Littlewood-Paley approach was used because it allows for initial data $(\vec{u}_0, Q_0) \in H^s(\mathbb{R}^2) \cap H^{s+1}(\mathbb{R}^2)$ for any $s > 1$.

Shortly afterwards, the uniqueness result in [13] was improved in [15] where the author proved uniqueness of global weak solutions for $d = 2$. Besides, an alternative approach that combined the Friedrichs scheme and the Schaefer's fixed point theorem was used in [15] to prove the existence of global weak solutions to (2.1). Next, by using a Fourier-splitting method, the asymptotic behavior of the weak solution to the Cauchy problem in $\mathbb{R}^3$ was given in [16], with some extra assumptions on the initial data and coefficients in (2.3). Further, a partial regularity result of suitable weak solutions in $\mathbb{R}^3$ was achieved in [17]:

Theorem 2.3. *For initial data* $\vec{u}_0 \in \overline{\{v \in C_0^\infty(\mathbb{R}^3),\ \nabla\cdot\vec{v} = 0\}}^{L^2}$ *and* $Q_0 \in H^1(\mathbb{R}^3)$*, there exists a global suitable weak solution* $(\vec{u}, Q)$ *of the co-rotational Beris-Edwards system* (2.1)*. Further,*

$$(\vec{u}, Q) \in C^\infty((0, +\infty) \times \mathbb{R}^3 \setminus \Sigma),$$

where $\Sigma \subset (0, +\infty) \times \mathbb{R}^3$ *is a closed set whose* 1*-dimensional Hausdorff measure is* 0*.*

The existence of weak suitable solutions can be constructed following the classical construction of its counterpart in the incompressible Navier-Stokes equations. And the proof of the partial regularity result in Theorem 2.3 involved several crucial steps. The first key point was to establish a weak maximum principle of Q for the suitable weak solution (u, Q), see also [18]:

Lemma 2.1. *Let* $(u, Q) \in L^2((0, +\infty); H^1) \times L^2((0, +\infty); H^2)$ *be a weak solution of the co-rotational Beris-Edwards system* (2.1)*. If* $Q_0 \in L^\infty(\mathbb{R}^3)$*, then there exists a constant* $C > 0$ *that depends on* $\|Q_0\|_{L^\infty(\mathbb{R}^3)}$ *and* a, b, c *only, such that*

$$|Q(t, x)| \le C, \qquad \forall\,(t, x) \in (0, +\infty) \times \mathbb{R}^3.$$

The second key point was to use a blowing up argument to obtain an ϵ_0-regularity criterion of any suitable weak solution based on a local energy inequality. It states that if the suitable weak solution satisfies

$$r^{-2}\int_{\mathbb{P}_r(t_0,x_0)} (|\vec{u}|^3 + |\nabla Q|^3)\,\mathrm{d}t\mathrm{d}x + \Big(r^{-2}\int_{\mathbb{P}_r(t_0,x_0)} |P|^{\frac{3}{2}}\,\mathrm{d}t\mathrm{d}x\Big)^2 \le \epsilon_0^3,$$

then $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^3$ is a smooth point of (u, Q). The third key point was to achieve the higher order bound of $(\vec{u}, Q)$ near (t_0, x_0), which can be done by intrinsic cancellation property of several coupling nonlinear terms.

On the other hand, the corresponding initial boundary value problems of the co-rotational Beris-Edwards system subject to various boundary conditions for $d = 2, 3$ were studied in [18–20], where the existence of global weak solutions, existence and uniqueness of local strong solutions as well as some regularity criteria were provided.

Concerning the full Beris-Edwards system (2.1) with general $\xi \in \mathbb{R}$, its mathematical analysis is more difficult because when $\xi \neq 0$, there are higher order derivatives in (2.7) as well as (2.6) compared to the case when $\xi = 0$. In [12], the authors proved the existence of global weak solutions for the Cauchy problem in $\mathbb{R}^d$ with $d = 2, 3$ for sufficiently small $|\xi|$. Note that such smallness assumption was necessary for the Cauchy problem otherwise in an infinite domain the L^p a priori estimates cannot be established. The authors also established the existence of global strong solutions for $d = 2$ as that in Theorem 2.2. In their proof a differential inequality relating higher order norms of the solution was rather untraditional in the sense that it was with a double-logarithmic correction. To retrieve this higher order energy differential inequality for $d = 2$, the following logarithmic embedding of $H^{1+\epsilon}$ in L^∞ in conjunction with the precise growth of Sobolev embedding constant of H^1 in any L^p played an essential role:

$$\|Q\|_{L^\infty(\mathbb{R}^2)} \leq \|Q\|_{H^1(\mathbb{R}^2)} \sqrt{\ln\Big(e + \frac{\|\nabla Q\|^2_{H^s(\mathbb{R}^2)}}{\|Q\|_{H^1(\mathbb{R}^2)}}\Big)}, \tag{2.11}$$

$$\|g\|_{L^{2p}(\mathbb{R}^2)} \leq C(p)\sqrt{p}\|g\|^{\frac{1}{p}}_{L^2(\mathbb{R}^2)} \|\nabla g\|^{1-\frac{1}{p}}_{L^2(\mathbb{R}^2)}. \tag{2.12}$$

The uniqueness of weak solutions for $d = 2$ was given in [21]. Later on, the existence and long time behavior of global strong solutions to (2.1) with any arbitrary $\xi \in \mathbb{R}$ under periodic settings for $d = 2$ were proved in [22]. Further, existence of global weak solutions and local well-posedness with higher time-regularity for the initial boundary value problem subject to inhomogeneous mixed Dirichlet/Neumann boundary conditions were given in [23]. Besides, a rigorous derivation from the Beris-Edwards system to the classical Ericksen-Leslie system [24, 25] was provided in [26] by using the Hilbert expansion method. We want to point out that there has been a large number of literature on the analytic study of the simplified/full Ericksen-Leslie system, and interested reader may see [27] and references therein for more details. One unsolved regularity issue for the Beris-Edwards system is

Unsolved research problem 1: Determine the size of the singular set of a suitable weak solution to the non co-rotational Beris-Edwards system (2.1) for any $\xi \neq 0$.

Any suitable weak solution to (2.1) is a weak solution that additionally satisfies a local energy inequality [17], which turns out to be a necessary condition for smoothness of the solution. Compared to the proof in [17] for the co-rotational Beris-Edwards system, one essential difficulty in this research problem is that the weak maximum principle is not valid for Q. Specifically, the L^∞ norm of Q is not ensured to be bounded in terms of that of initial data Q_0 as time evolves for any $\xi \neq 0$. This also has strong physical implications in that when $\xi = 0$, the fluid flow only has tumbling but no stretching effect on liquid crystal molecules, while when $\xi \neq 0$ both effects would be imposed. A closely related but more subtle issue is the preservation of the initial eigenvalue range [28]. It is essentially the behaviour of eigenvalues of the Q-tensors under the fluid dynamics as time evolves.

Such eigenvalue preservation property was proved in [29] for the co-rotational system under periodic settings:

Theorem 2.4. *Let $T > 0$, $\xi = 0$, $a \in \mathbb{R}$, $b > 0$, $c > 0$ and $(\vec{u}_0, Q_0) \in H^1(\mathbb{T}^2) \times H^2(\mathbb{T}^2)$ with $\nabla \cdot \vec{u} = 0$. Suppose the eigenvalues of the initial datum Q_0 satisfy*

$$\lambda_i(Q_0(x)) \in \left[-\frac{b + \sqrt{b^2 - 4ac}}{12c}, \frac{b + \sqrt{b^2 - 4ac}}{6c}\right], \qquad 1 \leq i \leq 3,\ \forall\, x \in \mathbb{T}^2 \tag{2.13}$$

and

$$|a| < \frac{b^2}{3c}.$$

Denote $(\vec{u}, Q)$ *the unique global strong solution to the Beris-Edwards system* (2.1) *on* $[0, T]$ *with initial data* $(\vec{u}_0, Q_0)$. *Then the eigenvalues of* $Q(t, x)$ *stays in the same interval as in* (2.13) *for any* $t \in [0, T]$ *and* $x \in \mathbb{T}^2$.

The result in Theorem 2.4 is also valid in the whole space or for local strong solutions for $d = 3$. The main idea to prove the preservation of the convex hull of eigenvalues of Q_0 for any regular enough solution was based on an operator splitting and a nonlinear Trotter product formula. To be more precise, consider $S(t, \bar{S})$, or $S(t, \cdot)\bar{S} \in \mathcal{Q}$ the flow generated by the ODE system

$$\begin{cases} \frac{d}{dt}S = -aS + b\Big[S^2 - \frac{1}{3}\operatorname{tr}(S^2)\mathbb{I}_3\Big] - c\operatorname{tr}(S^2)S, \\ S(t, \cdot) = \bar{S} \in \mathcal{Q}. \end{cases} \tag{2.14}$$

And for $\bar{R} \in H^2(\mathbb{T}^2 \to \mathcal{Q})$ denote $R(t; s, \bar{R}) = \mathcal{V}(t, s)\bar{R}$ the unique solution to the linear non-autonomous problem:

$$\begin{cases} \partial_t R - \varepsilon\Delta R = -\vec{u}\cdot\nabla R + \omega R - R\omega, \\ R(s, \cdot) = \bar{R}. \end{cases} \tag{2.15}$$

Then it aims to show both the solution $S(t, \bar{S})$ of the ODE system (2.14), and the two parameter evolution system $\mathcal{V}(t, s)$ of the problem (2.15) preserve the above closed convex hull of the range for the initial data in (2.13). And eventually, a nonlinear Trotter product formula provides a way of expressing the solution $Q(t, x)$ of (2.1) as a limit of successive superpositions of solutions to the ODE system (2.14) and the problem (2.15).

If the velocity $\vec{u}$ is neglected, then the Q-equation in (2.1) is reduced to a gradient flow generated by the free energy (2.2). As a consequence, the gradient flow can be "splitted" into a heat flow and an ODE system so that the initial convex hull of eigenvalues are proved to be preserved by both sub-flows, then the combination is performed by a Trotter product formula [28]. But the structure of the Q-equation in (2.1) is much more complicated due to the existence of fluid velocity. To overcome the difficulty, a Maier-Saupe type singular potential was introduced in [29] (see the later subsection for more details on it), whose special analytic properties were exploited. An alternative proof was given in [30] on a smooth, bounded domain with an extra assumption $a \geq 0$.

On the other hand, it was shown in [29] the eigenvalue-range preservation of initial data Q_0 cannot be always valid for strong solutions of the non co-rotational system for all $\xi \neq 0$. The main idea of the proof was to use contradiction argument and consider the so-called high Ericksen number limit, in which the limiting case is a weakly coupled system consisting an incompressible Euler equation for the fluid velocity with a reaction-convection equation for Q. It remains to study

Unsolved research problem 2: Determine whether the eigenvalue-range preservation property is not true for the non co-rotational Beris-Edwards system for any $\xi \neq 0$.

As pointed in [29], we expect in general for any $\xi \neq 0$ one does not have the preservation of initial eigenvalue-range. In fact, for a shear flow in the whole space, the coupled system simply reduces to an ODE system. This situation was analyzed in [31] and the results obtained there allow to conclude that the eigenvalue-range preservation does not hold in general circumstances. But this shear flow contains nonphysical aspects in that it is with infinite energy. Therefore, to address this problem, a crucial step is to construct a family of exact solutions to the system with finite energy.

2.2. The inertial Qian-Shen model

Other than the Beris-Edwards system (2.1), the inertial Qian-Shen model [32] is another hydrodynamic system that has recently aroused the interest of the PDE community. It reads

$$\begin{cases} \partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla P = \dfrac{\beta_4}{2} + \nabla\cdot\big(- L\nabla Q\odot\nabla Q + \beta_1 Q\,\mathrm{tr}(QD) + \beta_5 AD + \beta_6 QD\big) \\ \qquad\qquad +\nabla\Big(\dfrac{\mu_2}{2}(\dot{Q} - [\omega, Q]) + \mu_1[Q, (\dot{Q} - [\omega, Q])]\Big), \\ \nabla\cdot\vec{u} = 0, \\ J\ddot{Q} + \mu_1\dot{Q} = L\Delta Q - aQ + b\Big(Q^2 - \dfrac{1}{d}|Q|^2\mathbb{I}_d\Big) - cQ|Q|^2 - \dfrac{\tilde{\mu}_2}{2}D + \mu_1[\omega, Q]. \end{cases} \tag{2.16}$$

Here in (2.16),

$$\dot{Q} = (\partial_t + \vec{u}\cdot\nabla)Q$$

denotes the material derivative the Q-tensor and for any two $d\times d$ matrices A, B, $[A, B] := AB - BA$ denotes their commutator. Here $J > 0$ in the Q-equation of (2.16) represents the inertial density. It is noted that this inertial term plays a conceivable role when the anisotropic axis is subject to large accelerations, as motivated by the director model [33].

The hydrodynamic system (2.16) can be considered as the Q-tensor version of the full Ericksen-Leslie model (with an inertial term). A rigorous justification from the inertial Qian-Shen model (2.16) to the full Ericksen-Leslis system with an inertial term was performed in [34]. Compared to (2.1), the most specific feature of this model is the presence of the inertial term $J\ddot{Q}$ that appears as a second-order material derivative in the Q-equation. This very term provides a hyperbolic character to the system of equations and is the main source of challenges in its PDE analysis. As an initial attempt to tackle with the system (2.16), a basic energy dissipative law to the inertial Qian-Shen model (2.16) was found in [35]:

Theorem 2.5. *Let $d = 2$ * or 3, and the coefficients in the system* (2.1) *satisfy*

$$\begin{aligned} &\beta_1 \geq 0, \ \ \beta_4 \geq 0, \ \ \mu_1 \geq 0, \\ &\beta_6 - \beta_5 = \mu_2, \\ &\beta_5 + \beta_6 = 0, \\ &\tilde{\mu}_2 = -\mu_2. \end{aligned}$$

Then there exists a constant C_d that depends on $\mu_2, \beta_5, \beta_6, \mu_2$, such that for any classical solutions that decay fast enough at infinity it holds

$$\frac{d}{dt}\int_{\mathbb{R}^d}\Big[\frac{1}{2}|\vec{u}|^2 + \frac{J}{2}|\dot{Q}|^2 + \frac{L}{2}|\nabla Q|^2 + f(Q)\Big]\,\mathrm{d}x \leq 0, \tag{2.17}$$

*In [35] for $d = 2$, Q is assumed to a tensor-valued function into the set of 2-dimensional Q-tensors, that is, symmetric and tracless 2×2 matrices

$$\{M \in \mathbb{R}^{2\times 2}|\ M^t = M,\ \mathrm{tr}(M) = 0\}.$$

provided $\beta_4 > C_d$. Further, for any $T > 0$ the following a priori bounds can be established

$$\vec{u} \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$$
$$Q \in L^\infty(0,T;H^1(\mathbb{R}^d)) \quad \textit{with} \quad \dot{Q} \in L^\infty(0,T;L^2(\mathbb{R}^d)),$$

provided that

1. $d = 2$, $a \geq 0$.
2. *or there exist* $\bar{\mu}_1 = \bar{\mu}_1(a,b,c)$, $J_0 = J_0(\bar{\mu}_1,a,b,c)$ *and* $\tilde{C}_d = \tilde{C}_d(\mu_2,\beta_5,\beta_6) > 0$ *such that* $\mu_1 > \bar{\mu}_1$, $J < J_0$, $\beta_4 > \tilde{C}_d$.

However, one cannot construct weak solutions by virtue of these a priori estimates in Theorem 2.5. The most common approach for the construction of weak solutions is to use the compactness method, that is, to construct approximate sequence of solutions with same energy bounds and then pass to the limit. The main difficulty lies in the stress term $\nabla Q \odot \nabla Q$ in that when $J > 0$ the Q-equation is of hyperbolic nature, and it does not allow to collect enough a priori bounds, especially the $L^2(0,T;H^2(\mathbb{R}^d))$ for Q. Alternatively, the global-wellposedness of the Cauchy problem to (2.16) for small initial data was established in [35]:

Theorem 2.6. *Let $J < J_0$, $\mu_1 > \bar{\mu}_1$, $\beta_4 > \tilde{C}_d$, where $J_0, \bar{\mu}_1, \tilde{C}_d$ are explicitly computable coefficients. Assume $\beta_1 > 0$, $\mu_1 > 0$, $a > 0$, and $(\vec{u}_0, Q_0) \in H^s(\mathbb{R}^d) \times H^{s+1}(\mathbb{R}^d)$ with $s > \frac{d}{2}$ for $d = 2,3$. Then there exists $\varepsilon_0 > 0$, depending on s and d such that if*

$$\eta_0 := \|\vec{u}_0\|_{H^s} + \|Q_0\|_{H^{s+1}} + \|\dot{Q}_0\|_{H^s} < \varepsilon_0,$$

then there exists a unique global strong solution $(\vec{u}, Q)$ of (2.16). Furthermore, there exists $C > 0$ that is independent of the solution such taht

$$\|\vec{u}\|_{L^\infty(0,\infty;H^s(\mathbb{R}^d))} + \|\nabla\vec{u}\|_{L^2(0,\infty;H^s(\mathbb{R}^d))} + \|Q\|_{L^\infty(0,\infty;H^{s+1}(\mathbb{R}^d))} + \|Q\|_{L^2(0,\infty;H^{s+1}(\mathbb{R}^d))}$$
$$\|\dot{Q}\|_{L^\infty(0,\infty;H^s(\mathbb{R}^d))} + \|\dot{Q}\|_{L^2(0,\infty;H^s(\mathbb{R}^d))} \leq C\varepsilon_0.$$

The major difficulty of the proof comes from the second order material derivative $\ddot{Q}$ that reads after expansion

$$\ddot{Q} = \partial_t^2 Q + 2\vec{u}\cdot\nabla\partial_t Q + \partial_t\vec{u}\cdot\nabla Q + (\vec{u}\cdot\nabla\vec{u})\cdot\nabla Q + \vec{u}\nabla^2 Q\vec{u},$$

where $\partial_t\vec{u}$ and $\vec{u}\nabla^2 Q\vec{u}$ are competing terms with regularizing term $L\Delta Q$ in the Q-equation. To overcome the analytic difficulty, the major strategy was to stay as close as possible to standard cancellation with respect to convective derivatives. Furthermore, a higher order commutator estimate at the level of homogeneous Sobolev spaces $\dot{H}^s$ was considered. To prove the existence of classical solutions, the authors managed to achieve a uniform energy estimate for approximate solutions in the form of

$$\Phi'(t) + \Psi(t) \leq C\Phi(t)\Psi(t),$$

where $C > 0$, Φ controls the H^s-norms for the solution and Ψ is integrable function (in time) that involves H^s-norms. Then smallness assumption on $\Phi(0)$ paved the way to uniform bounds for H^s-norms of approximate solutions. The subsequent proof was completed by a compactness argument.

The authors in [35] also constructed one example of the so-called twist wave solution that is a classical solution of the coupled system for which the flow velocity is identically zero [36]. The twist wave solution is a solution of (2.16) with zero flow, such that $\vec{u} = \omega = D = 0$, $\dot{Q} = \partial_t Q$, but it satisfies a nonlinear constraint

$$\nabla P = \nabla \cdot \Big(- \nabla Q \odot \nabla Q + \frac{\mu_2}{2} \partial_t Q + \mu_1 [Q, \partial_t Q] \Big).$$

Afterwards, a finer analysis of coefficient assumptions used in entropy inequality and energy dissipation in Qian-Shen model was performed in [37]. To be more specific, for any integer $s > \frac{d}{2} + 1$ ($d = 2$ or 3), denote

$$E^{in} := \|\vec{u}_0\|_{H^s}^2 + J\|\dot{Q}_0\|_{H^s}^2 + L\|\nabla Q_0\|_{H^s}^2 + a\|Q_0\|_{H^s}^2,$$

then a large initial data/local well-posedness result was established:

Theorem 2.7. *Suppose the coefficients of* (2.16) *satisfy*

$$\begin{cases} a > 0, \ \beta_1 > 0, \ \beta_4 > 0, \ \mu_1 > 0, \\ \beta_5 + \beta_6 = 0. \end{cases} \tag{2.18}$$

If $\beta_6 - \beta_5 = \mu_2$, the initial energy $E^{in} < \infty$, and the so called Condition (H) is valid:

$$(\tilde{\mu}_2 - \mu_2)^2 + 4\mu_2^2 < 8\beta_4\mu_1, \tag{2.19}$$

then the Cauchy problem of (2.16) *admits a unique local strong solution* $(\vec{u}, Q)$ *that satisfies*

$$\begin{cases} \vec{u} \in L^\infty(0, T; H^s(\mathbb{R}^d)) \cap L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^d)), \\ \dot{Q} \in L^\infty(0, T; H^s(\mathbb{R}^d)) \cap L^2(0, T; H^s(\mathbb{R}^d)), \\ Q \in L^\infty(0, T; H^{s+1}(\mathbb{R}^d)). \end{cases} \tag{2.20}$$

And the energy bound depends only on the initial data, the coefficients of the system, and T.

At the same time, the authors in [37] also showed that when the relation $\beta_5 + \beta_6 = 0$ and Condition (H) are not simultaneously satisfied, then smallness assumption must be imposed on initial data to achieve even local well-posedness following energy methods.

Meanwhile, the global existence of dissipative solutions to a dissipative version of Qian-Shen model was established in [38]. Besides, we refer interested readers to [39], where the authors made a comprehensive study of the full Ericksen-Leslie system with an inertial term for $d = 1$. It remains to solve the following two fundamental analytic issues to the inertial Qian-Shen model (2.16)

Unsolved research problem 3: Prove the existence of global weak solutions for the inertial Qian-Shen model.

So far there has been no systematic approach to handle such a hydrodynamic system with hyperbolic structures. New ideas or tools would be necessary.

Unsolved research problem 4: Determine whether singularity exists for solutions of the inertial Qian-Shen model for large enough initial data.

The counterpart of the inertial Qian-Shen model in the director theory is the full Ericksen-Leslie system with an inertial term [33]. In the Poiseuille flow of the full Ericksen-Leslie system with an inertial term, the solution takes the form

$$\vec{u}(x,t) = (0, 0, u(x,t))^t, \qquad \vec{n}(x,t) = (\sin\theta(x,t), 0, \cos\theta(x,t))^t,$$

where u and θ are scalar functions. Consequently, the full Ericksen-Leslie system is reduced to

$$\begin{cases} \rho u_t = \tilde{a} + \big(g(\theta)u_x + h(\theta)\theta_t\big)_x, \\ \nu\theta_t + \gamma_1\theta_t = c(\theta)\big(c(\theta)\theta_x\big)_x - h(\theta)u_x, \end{cases} \tag{2.21}$$

where $\tilde{a} \in \mathbb{R}$ is the gradient of pressure along the flow direction, and

$$\begin{cases} g(\theta) := \alpha_1 \sin^2\theta\cos^2\theta + \frac{\alpha_5-\alpha_2}{2}\sin^2\theta + \frac{\alpha_3+\alpha_6}{2}\cos^2\theta + \frac{\alpha_4}{2}, \\ c(\theta) := \sqrt{K_1\cos^2\theta + K_3\sin^2\theta}, \qquad K_1 \neq K_3, \\ h(\theta) := \frac{\gamma_1+\gamma_2\cos(2\theta)}{2}. \end{cases}$$

However, such singularity formation proof cannot be applied to (2.16) since during the reduction of Q-tensor elastic energy to the Oseen-Frank energy $K_1 \equiv K_3$ even if more quadratic terms of gradient of Q are added (see the discussion after (2.35) for details).

2.3. *Various gradient flow problems associated with the free energy* (2.2) *with one or more elastic constants*

Concerning the free energy (2.2), an interesting topic is the rigorous study of nematic-isotropic phase transition based on a natural Q-tensor gradient flow generated by it. Phase transitions between different phases of liquid crystals give rise to a variety of mathematical questions of great interest. Precisely speaking, for the free energy

$$\mathcal{F}_\varepsilon(Q) = \int_\Omega \Big[\frac{\varepsilon}{2}|\nabla Q|^2 + \frac{1}{\varepsilon}f(Q)\Big]\,\mathrm{d}x, \tag{2.22}$$

where $\Omega \subset \mathbb{R}^d$ is a smooth, bounded domain, $\varepsilon > 0$ denotes the relative intensity of elastic and bulk energy, it is of importance to investigate the small-ε limit of its natural gradient flow dynamics with initial data undergoing a sharp interfacial transition:

$$\begin{cases} \partial_t Q_\varepsilon = \Delta Q_\varepsilon - \dfrac{1}{\varepsilon^2}\Big(aQ_\varepsilon - bQ_\varepsilon^2 + \dfrac{b}{3}|Q_\varepsilon|^2\mathbb{I}_3 + c|Q_\varepsilon|^2 Q_\varepsilon\Big), & \text{in } \Omega\times(0,T), \\ Q_\varepsilon(x,0) = Q_\varepsilon^0(x), & \text{in } \Omega, \\ Q_\varepsilon(x,t) = 0, & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{2.23}$$

It is known that all critical points of $f(Q)$ are uniaxial [65] in the sense that

$$Q = s\Big(\vec{n}\otimes\vec{n} - \frac{1}{3}\mathbb{I}_3\Big), \qquad \text{for some } \vec{n} \in \mathbb{S}^2,\ s \in \mathbb{R}. \tag{2.24}$$

Besides, $f(Q)$ has two families of stable local minimizers corresponding to the following choices of $s = s_\pm$:

$$s_- = 0, \qquad s_+ = \frac{b + \sqrt{b^2 - 24ac}}{4c}. \tag{2.25}$$

And in [40, 41] the following bistable case is considered

$$b^2 = 27ac, \qquad a > 0,\ c > 0, \tag{2.26}$$

which from the physics point of view corresponds to the critical temperature at which the system favors the nematic and isotropic phases equally [2]. In this case, the two families of minimizers associated with (2.25) are the only global minimizers of $f(Q)$ in the sense that

$$f(Q) \geq 0, \text{ and the equality holds if and only if } Q \in \{0\} \cup \mathcal{N},$$

with

$$\mathcal{N} := \left\{ Q \in \mathcal{Q} \middle| Q = s_+\Big(\vec{n} \otimes \vec{n} - \frac{1}{3}\mathbb{I}_3\Big),\ \vec{n} \in \mathbb{S}^2 \right\}. \tag{2.27}$$

In [40, 41], different approaches were utilized to such rigorous proof under the assumption that the initial data of the gradient flow undergoes a sharp transition near a smooth surface. To be more specific, let $T > 0$,

$$I = \bigcup_{t\in[0,T]} (I_t \times \{t\}) \text{ is a smoothly evolving closed surface in } \Omega, \tag{2.28}$$

which starts from a closed smooth surface $I_0 \subset \Omega$. Denote $\Omega^+(t)$ the domain enclosed by I_t, and $d(x, I_t)$ the signed distance from x to I_t that takes positive values in $\Omega^+(t)$ and negative values in $\Omega^-(t) = \Omega \setminus \overline{\Omega^+(t)}$, i.e.,

$$\Omega^+(t) := \{x \in \Omega |\, d(x, I_t) > 0\}, \qquad \Omega^-(t) := \{x \in \Omega |\, d(x, I_t) < 0\}. \tag{2.29}$$

At the same time, we define the distorted parabolic cylinder by

$$\Omega_T^\pm := \bigcup_{t\in(0,T)} \Omega^\pm(t) \times \{t\}. \tag{2.30}$$

Moreover, for any sufficiently small $\delta > 0$, we denote the δ-neighborhood of I_t by

$$I_t(\delta) := \{x \in \Omega :\ |d(x, I_t)| < \delta\}.$$

As a consequence, there exists a suitably small constant $\delta_I \in (0, 1)$ such that the nearest point projection

$$P_I(\cdot, t) : I_t(\delta_I) \longrightarrow I_t$$

is smooth for any $t \in [0, T]$, while the interface I stays away the boundary $\partial\Omega$ for at least δ_I.

To introduce the modulated energy $\mathcal{F}_\varepsilon[Q_\varepsilon|I]$ for the problem (2.23), let us extend the inner normal vector field $\vec{n}_I$ of I_t to a neighborhood by

$$\xi(x, t) := \eta(d(x, I_t))\vec{n}(P_I(x, t), t),$$

where $\eta \in C_c^\infty(\mathbb{R})$ is a cutoff function with the following properties:

$$\begin{cases} \eta(z) = \eta(-z), & \forall\, z \in \mathbb{R}, \\ \eta'(z) \leq 0, & \forall\, z \in [0, +\infty), \\ \eta(z) = 1 - z^2, & \text{for } |z| \leq \frac{\delta_I}{2}, \\ \eta(x) = 0, & \text{for } |z| \geq \delta_I. \end{cases}$$

Further, d^F is defined to be the quasi-distance function

$$d^F(Q) := \inf\left\{\int_0^1 \sqrt{2F(\gamma(t))}|\gamma'(t)|\,\mathrm{d}t\,\middle|\,\gamma \in C^{0,1}([0,1],\mathcal{Q}),\ \gamma(0)\in\mathcal{N},\ \gamma(1)=Q\right\}$$

Consequently, the modulated energy is $\mathcal{F}_\varepsilon$ is defined to be

$$\mathcal{F}_\varepsilon[Q_\varepsilon|I](t) := \int_\Omega \Big[\frac{\varepsilon}{2}|\nabla Q_\varepsilon(\cdot,t)|^2 + \frac{1}{\varepsilon}F(Q_\varepsilon(\cdot,t)) - \xi\cdot\nabla\psi_\varepsilon(\cdot,t)\Big]\,\mathrm{d}t, \tag{2.31}$$

with

$$\psi_\varepsilon(x,t) := (\phi_\varepsilon * d^F)\circ Q_\varepsilon(x,t),$$

and the convolution is considered to be in the Q-tensor space $\mathcal{Q}$ with a standard mollifier ϕ_ε.

The main result in [41] states that

Theorem 2.8. *Suppose the surface I_t evolves by mean curvature flow, and it encloses a simply connected domain $\Omega^+(t)$ for any $t\in[0,T]$. Let the initial data Q_ε^{in} of* (2.23) *be well-prepared such that*

$$\varepsilon\|Q_\varepsilon^0\|_{L^\infty(\Omega)} + \mathcal{F}_\varepsilon[Q_\varepsilon|I](0) \le C_0\varepsilon,$$

where $\mathcal{F}_\varepsilon[Q_\varepsilon|I]$ is the modulated energy of $\mathcal{F}_\varepsilon$ given in (2.31)*, and $C_0>0$ does not depend on ε. Then it holds up to a subsequence that*

$$Q_{\varepsilon_k} \longrightarrow \tilde{Q} := s_\pm\Big(\vec{n}(x,t)\otimes\vec{n}(x,t) - \frac{1}{3}\mathbb{I}_3\Big) \quad \textit{strongly in} \quad C([0,T];L^2_{loc}(\Omega^\pm(t)))$$

where $s_\pm$ are given in (2.25)*. Furthermore, $\vec{n}(x,t)\in H^1(\Omega_T^+;\mathbb{S}^2)$ is a harmonic map heat flow onto $\mathbb{S}^2$ with homogeneous Neumann boundary conditions.*

Theorem 2.8 mainly states that starting from Q_ε^0 with a nematic-isotropic transition from the nematic regime $\Omega^+(0)$ into the isotropic regime $\Omega^-(0)$, the solution Q_ε of the gradient flow (2.23) will converge to $\tilde{Q}\in\mathcal{N}$ that takes values in nematic phase in $\Omega^+(t)$, and to the isotropic phase 0 in $\Omega^-(t)$. The interface between $\Omega^+(t)$ and $\Omega^-(t)$ evolves by mean curvature flow and the limit map $\tilde{Q}$ in $\Omega^+(t)$ is a harmonic map heat flow into $\mathcal{N}$. To prove Theorem 2.8, approaches involving matched asymptotic expansions and spectral gap estimates were used in [40] (with slightly different setting of initial data other than that in Theorem 2.8), while the methods of weak convergence together with modulated energy method were adopted in [41].

However, this topic becomes more challenging if more anisotropic elastic terms are included in the free energy:

$$\int [L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik} + f(Q)]\,\mathrm{d}x. \tag{2.32}$$

Here and after, L_1, L_2, L_3 are material-dependent elastic constants, $\partial_k Q_{ij}$ stands for the k-th spatial partial derivative of Q_{ij}, and Einstein summation convention is used.

Unsolved research problem 5: Under proper assumptions on L_1, L_2, L_3, study the scaling limit of the L^2 gradient flow generated by the free energy (2.32) instead of $\mathcal{F}$.

For the co-rotational Beris-Edwards system with three elastic constants L_1, L_2, L_3 (in which (2.2) is replaced by (2.32)), the Cauchy problem with $d = 3$ was studied in [54], where the authors proved the existence of global weak solutions as well as the existence and uniqueness of global strong solutions provided that the fluid viscosity is sufficiently large. For the full Beris-Edwards system that involves three elastic constants L_1, L_2, L_3 with general $\xi \in \mathbb{R}$, the corresponding initial boundary value problem was studied in [61], where the local well-posedness result was significantly improved. It states that

Theorem 2.9. *For any smooth bounded domain $\Omega \subset \mathbb{R}^3$, consider the full Beris-Edwards system with three elastic constants L_1, L_2, L_3 subject to the initial condition*

$$(\vec{u}_0, Q_0) \in H_0^1(\Omega) \times H^2(\Omega),\ \nabla \cdot \vec{u}_0 = 0$$

and the boundary condition

$$(\vec{u}, Q)|_{\partial\Omega} = (0, Q_0|_{\partial\Omega}).$$

suppose the elastic constants satisfy

$$L_1 > 0, \qquad L_1 + L_2 + L_3 > 0,$$

and the initial data $(\vec{u}_0, Q_0)$ satisfies certain compatibility conditions. Then for some $T > 0$, the system admits a unique strong solution in the sense that

$$\vec{u} \in H^2(0, T; H^{-1}(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)),$$
$$Q \in H^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^3(\Omega)).$$

Moreover, a strong Legendre condition was discovered in the Euler-Lagrange equation associated with the free energy (2.32). Theorem 2.9 is significant in that it greatly improves the spatial regularity of solutions established in [23], and it extends the result in [23] to the anisotropic elastic energy case that is far more involved than the isotropic elastic energy only. The proof of Theorem 2.9 replies on the observation that terms containing third derivatives of Q in stress tensors can be eliminated, and the system can be transformed into a Stokes-type system with positive definite viscosity coefficients.

We proceed to be focused on the free energy with a cubic L_4 term [6, 10, 11]:

$$\tilde{F}(Q) = \int [L_1|\nabla Q|^2 + L_2\partial_j Q_{ik}\partial_k Q_{ij} + L_3\partial_j Q_{ij}\partial_k Q_{ik} + L_4 Q_{lk}\partial_k Q_{ij}\partial_l Q_{ij} + f(Q)]\,\mathrm{d}x \tag{2.33}$$

The appearance of this cubic L_4 term is motivated by the fact that it allows a complete reduction of the elastic energy part in (2.33) to the classical Oseen-Frank energy, by formally taking [28, 46]

$$Q(x) = s_+\Big[\vec{n}(x) \otimes \vec{n}(x) - \frac{1}{3}\mathbb{I}_3\Big], \qquad s_+ \in \mathbb{R}. \tag{2.34}$$

To be more specific, inserting (2.34) into (2.33), we recover the Oseen-Frank energy (with four elastic constants) [5]

$$\int \Big\{K_1(\mathrm{div}\vec{n})^2 + K_2(\vec{n} \cdot \mathrm{curl}\vec{n})^2 + K_3|\vec{n} \times \mathrm{curl}\vec{n}|^2 + (K_2 + K_4)\big[\,\mathrm{tr}(\nabla\vec{n})^2 - (\mathrm{div}\vec{n})^2\big]\Big\}\,\mathrm{d}x, \tag{2.35}$$

where [10]

$$\begin{cases} K_1 &= 2L_1 s^2 + L_2 s^2 + L_3 s^2 - \frac{2}{3} L_4 s^3, \\ K_2 &= 2L_1 s^2 - \frac{2}{3} L_4 s^3, \\ K_3 &= 2L_1 s^2 + L_2 s^2 + L_3 s^2 + \frac{4}{3} L_4 s^3, \\ K_4 &= L_3 s^2. \end{cases}$$

If the L_4 cubic term is not present, say $L_4 = 0$, then this reduction is incomplete because $K_1 \equiv K_3$ while $K_1, \cdots, K_4$ are independent constants. However, the retention of this term makes the free energy $\tilde{\mathcal{F}}$ unbounded from below [44].

The physical relevance of the free energy (2.33) under $2D$ settings was investigated in [28]. That is, Q is assumed to be a tensor-valued function into the space of 2-dimensional Q-tensors

$$\{M \in \mathbb{R}^{2\times 2} \,\big|\, M^t = M, \ \mathrm{tr}(M) = 0\}. \tag{2.36}$$

In particular, an L^2 gradient flow for $d = 2$ into the space (2.36) associated with (2.33) was proposed:

$$\frac{\partial Q}{\partial t} = -\frac{\delta \tilde{F}}{\delta Q} + \lambda \mathbb{I}_2 + \mu - \mu^t, \tag{2.37}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier corresponding to the constraint $\mathrm{tr}(Q) = 0$ and $\mu \in \mathbb{R}^{2\times 2}$ is a Lagrange multiplier associated with the constraint $Q^t = Q$. After expansion it yields

$$\begin{aligned} \frac{\partial Q_{ij}}{\partial t} &= 2L_1 \Delta Q_{ij} - a Q_{ij} - c\ \mathrm{tr}(Q^2) Q_{ij} + (L_2 + L_3)\left(\partial_j \partial_k Q_{ik} + \partial_i \partial_k Q_{jk}\right) \\ &\quad - (L_2 + L_3)\partial_l \partial_k Q_{lk} \delta_{ij} + 2L_4 \partial_l Q_{ij} \partial_k Q_{lk} + 2L_4 \partial_l \partial_k Q_{ij} Q_{lk} \\ &\quad - L_4 \partial_i Q_{kl} \partial_j Q_{kl} + \frac{L_4}{2} |\nabla Q|^2 \delta_{ij}. \end{aligned} \tag{2.38}$$

The problem (2.38) is studied on a bounded domain $\Omega \subset \mathbb{R}^2$ with the following initial data and boundary conditions:

$$Q(x,0) = Q_0(x), \quad \text{and} \quad Q(x,t)|_{\partial\Omega} = \tilde{Q}(x). \tag{2.39}$$

$$Q_0|_{\partial\Omega} = \tilde{Q}.$$

The main aim of [28] was two-folded: show the existence of global weak solutions to the problem (2.38)-(2.39) with $H^1 \cap L^\infty$ initial data that is small in L^∞ and finite time blow up (in L^2) of solutions for specially constructed large initial data. To prove the global existence, the main difficulty comes from the fact that the energy is a priori unbounded from below. But it follows directly from (2.33) that if $\|Q\|_{L^\infty}$ is suitably small, then the cubic L_4 term can be absorbed into the other quadratic terms which are positive definite under mild coercivity assumptions. In this way the H^1-level energy in the gradient flow can be effectively used, provided one can a priori ensure a smallness condition on the L^∞-norm. To this end, a key lemma was established:

Lemma 2.2. *For the* $2D$ *evolution problem* (2.38)-(2.39) *on a bounded smooth domain* $\Omega \subset \mathbb{R}^2$*, suppose*

$$L_1 + L_2 > 0, \qquad L_1 + L_3 > 0. \tag{2.40}$$

For any smooth solution Q there exists an explicitly computable constant η_1 (depending on $L_i, i = 1,\dots,4$) such that if

$$\|\tilde{Q}\|_{L^\infty(\partial\Omega)} \leq \|Q_0\|_{L^\infty(\Omega)} < \sqrt{2\eta_1}$$

and

$$|a| \leq 2c\eta_1,$$

then for any $T > 0$, we have

$$\|Q\|_{L^\infty((0,T)\times\Omega)} \leq \sqrt{2\eta_1}.$$

Lemma 2.2 plays an essential role in the proof of existence of global weak solutions in that it ensures the smallness of $\|Q\|_{L^\infty}$ globally in time, provided it is suitably small at initial time. Together with the energy dissipative law inherited from the gradient flow, one can establish the uniform a priori bounds for $\|Q\|_{L^\infty(0,T;H^1)}$ and $\|Q\|_{L^2(0,T;H^2)}$. But the highly nonlinear terms related to L_4 makes the approximation scheme to establish global weak solutions nonstandard, even if all the necessary a priori bounds are collected. In particular, in order to obtain coercivity of the second order terms, $\|Q\|_{L^\infty}$ must be kept small in the approximation scheme. This is completely nontrivial since Q is a 2×2 matrix instead of a scalar function. To achieve this goal, the following singular potential was introduced:

$$\tilde{f}(Q) := \begin{cases} -\ln(8\eta_2 - |Q|^2), & \text{if } |Q|^2 < 8\eta_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

for some suitably small constant $\eta_2 > 0$. Correspondingly, an extra term

$$\varepsilon\frac{\partial \tilde{f}}{\partial Q},$$

was added to (2.38), which enforced the approximate system to admit $|Q|^2 < 8\eta_2$ almost everywhere. Then using the classical Morrey-Yosida approximation and suitable smoothing argument, a convex, smooth, bounded from below, and monotone increasing sequence $\tilde{f}_n$ ((see [67] for details)) was used to regularize $\tilde{f}$. Together with the Galerkin projection $\mathcal{P}_m : L^2 \to H_m := \operatorname{span}\{\varphi_1, \cdots \varphi_m\}$, where $\{\varphi_1, \cdots \varphi_m, \cdots\}$ is an orthonormal basis of $L^2(\Omega)$ consisting eigenvectors of the Laplacian operator, a three-level approximation scheme was constructed:

$$\frac{\partial Q_m}{\partial t} = \mathcal{P}_m\left\{-\frac{\delta \tilde{F}}{\delta Q} + \lambda\mathbb{I}_2 + \mu - \mu^t\right\} - \varepsilon\mathcal{P}_m\left\{\frac{\partial \tilde{f}_n}{\partial Q}\right\} \tag{2.41}$$

After obtaining the existence of global solutions for the approximation system (2.41), passing to limit with the order $N \to \infty$, $m \to \infty$, $\varepsilon \to 0^+$ led to the desired result.

Theorem 2.10. *There exists an explicitly computable constant η_2 that depends on $L_i, i = 1,\dots,4$ and Ω, such that if $Q_0 \in H^1(\Omega)\cap L^\infty(\Omega)$, $\tilde{Q} \in H^{\frac{3}{2}}(\partial\Omega)$, and the smallness conditions in Lemma 2.2 are replaced by η_2, then the system* (2.38)-(2.39) *admits a unique global weak solution under the coefficient assumption* (2.40). *Moreover, the initial smallness is preserved for all time.*

Under the same $2D$ settings for initial data with small L^∞ norm, the existence and uniqueness of global classical solutions of the Cauchy problem was established in [48], whose result was later improved in [55]. A stable numeric scheme and its convergence analysis were provided in [42], which provided an alternative way to prove the existence of global weak solutions to (2.38)-(2.39). The corresponding biaxial gradient flow in two dimensions was recently studied in [59], where global existence of weak and classical solutions were established under the assumption of small initial data. Concerning the co-rotational Beris-Edwards system whose elastic energy takes the form (2.33), it was proved in [62] that under periodic settings the hydrodynamic system admits a unique global weak solution if the initial L^∞ norm of the Q-tensor is properly small.

Unsolved research problem 6: For the gradient flow (2.37) into the 3-dimensional Q-tensor space $\mathcal{Q}$, prove the global wellposedness of the initial-boundary value problem under the assumption of small $\|Q_0\|_{L^\infty}$.

The essential difference between the gradient flow (2.37) into the 2 and 3-dimensional Q-tensor spaces is that in the former case the system (2.38) can be written as

$$\frac{\partial Q_{ij}}{\partial t} = \zeta\Delta Q_{ij} - aQ_{ij} - c\ \mathrm{tr}(Q^2)Q_{ij} + 2L_4\partial_k(Q_{lk}\partial_l Q_{ij}) - L_4\partial_i Q_{kl}\partial_j Q_{kl} + \frac{L_4}{2}|\nabla Q|^2\delta_{ij}, \tag{2.42}$$

where

$$\zeta := 2L_1 + L_2 + L_3 > 0$$

due to coercivity condition (2.40) in $2D$. As a consequence, a weak maximum principle argument can be used to prove Lemma 2.1 and henceforth the smallness of $\|Q(t)\|_{L^\infty}$ could be a priori kept during time evolution. In the latter case, however, the L_1, L_2, L_3 terms in (2.38) cannot be combined into a single Laplace term ΔQ, so that one cannot establish a counterpart of Lemma 2.1 to ensure the smallness of $\|Q(t)\|_{L^\infty}$ for all time. In [48], a global wellposedness result was given with rather strong assumptions imposed on largeness of L_1 and smallness of initial Q_0 in terms of $C^{2,\alpha}$ norm.

On the other hand, it was shown in [28] that certain solutions exhibit a finite time blowup once the initial smallness assumption in Theorem 2.10 is violated.

Theorem 2.11. *There exists a smooth domain* Ω*, smooth initial data* Q_0*, and a smooth function* $\tilde{Q} : \partial\Omega \to \mathbb{R}$*, under the coefficient assumption* (2.40)*, the problem* (2.38)-(2.39) *does not admit a global smooth solution.*

The key ingredient in the proof of Theorem 2.11 was to show $\|Q(t)\|_{L^2(\Omega)} \to \infty$ in finite time for any smooth solution. To this end, the following hedgehog type ansatz was used

$$Q_{ij}(t,x) = \theta(t,|x|)\Big(\frac{x_i x_j}{|x|^2} - \frac{\delta_{ij}}{2}\Big), \qquad i,j = 1,2.$$

for

$$\Omega = B_{R_1}(0) \setminus B_{R_0}(0) \subset \mathbb{R}^2.$$

The initial data Q_0 was assumed to be smooth and in the form of

$$Q(0) = \theta_0(|x|)\left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \mathbb{I}_2\right),$$

while the boundary data satisfies

$$\theta(t,R_0) = \theta(t,R_1) \geq 0, \quad \forall t > 0.$$

Then to show the finite time blowup of Q, it was equivalent to find the finite time blowup of θ that satisfies

$$\partial_t\theta = L_4\Big(\frac{(\theta')^2}{2} + \frac{\theta\theta'}{r} + \theta\theta'' + \frac{6\theta^2}{r^2}\Big) + \zeta\theta'' + \frac{\zeta\theta'}{r} - \frac{4\zeta\theta}{r^2} - a\theta - \frac{c\theta^3}{2}. \tag{2.43}$$

To proceed, a non-linear differential inequality of the following quantity that blows up in finite time was established:

$$\frac{1}{2}\frac{d}{dt}\int_{R_0}^{R_1}\theta_-^2 r\,dr \geq -\frac{2L_4R_0}{\sqrt{R_1^4-R_0^4}}\Big[\frac{\pi^2}{9(R_1-R_0)^2} - \frac{1}{R_0^2}\Big]\Big(\int_{R_0}^{R_1}\theta_-^2 r\,dr\Big)^{\frac{3}{2}} - |a|\int_{R_0}^{R_1}\theta_-^2 r\,dr + 4\mathcal{F}(0),$$

where

$$\mathcal{F}(t) := \int_{R_0}^{R_1}\Big\{L_4\theta_-\Big[\frac{(\theta_-')^2}{2} - \frac{2\theta_-^2}{r^2}\Big] - \zeta\Big[\frac{(\theta_-')^2}{2} + \frac{2\theta_-^2}{r^2}\Big] - \Big(\frac{a}{2}\theta_-^2 + \frac{c}{8}\theta_-^4\Big)\Big\}\,r\,dr \geq \mathcal{F}(0),$$

played a key role in controlling the high order nonlinearity even though the sign of the energy was not a priori known. Finally, the finite time blowup was ensured by assuming

$$\frac{R_0^2\pi^2}{9(R_1-R_0)^2} > 1.$$

Unsolved research problem 7: For general domains which are not radially symmetric, determine whether the finite time blowup of solutions occurs for large enough initial data.

This problem is much more difficult than the proof of Theorem 2.11 in that the problem (2.38)-(2.39) can no longer be reduced to a scalar equation. Further, boundary data has also to be dealt with in a subtle manner.

2.4. Dynamic problems involving a singular type potential

To further overcome this issue of unboundedness from below caused by the cubic L_4 term, motivated by the work in [56], a Maier-Saupe [64] type singular bulk potential was used in [44]. This potential can be traced back to the mean field theory.

Nematic liquid crystals are a class of condensed matter systems whose rod-like molecules yield rich nonlinear phenomena, including isotropic-nematic phase transitions. Thermotropic nematic liquid crystals possess optical properties that change dramatically with variation of surrounding temperature. Above a certain temperature threshold their molecules are randomly oriented (that corresponds to the isotropic phase), whereas below this threshold they prefer to be aligned locally in a preferred direction (that relates to the nematic phase). The mean field theory, was originally proposed to describe such nematic-isotropic phase transition. In this theory, the liquid crystal molecular alignment was characterized by $\rho(x,\vec{n})$, the density distribution function of the orientation of all molecules at a material point $x \in \Omega \subset \mathbb{R}^3$. The de Gennes Q-tensor, which is defined as the deviation of the second moment of ρ from its isotropic value, is expressed as

$$Q = \int_{\mathbb{S}^2}\Big[\rho(\vec{n})\otimes\rho(\vec{n}) - \frac{1}{3}\mathbb{I}_3\Big]\,\mathrm{d}\sigma(\vec{n}). \tag{2.44}$$

It is indeed the normalized second-order moment of the probability measure ρ on $\mathbb{S}^2$, and it serves as an order parameter in the sense that it vanishes in the isotropic phase. Further, any de Gennes Q-tensor is symmetric, traceless, with all eigenvalues bounded between $-1/3$ and $2/3$. Note that the cases of equality correspond to perfect crystalline nematic alignment, hence they are excluded [67]. Therefore, the eigenvalues of any de Gennes Q-tensor satisfy

$$-\frac{1}{3} < \lambda_i(Q) < \frac{2}{3}, \qquad 1 \le i \le 3. \tag{2.45}$$

By convention, any $Q \in \mathcal{Q}$ that satisfies the eigenvalue inequality (2.45) is called a physical Q-tensor, otherwise it is called unphysical. And we denote the space of physical Q-tensors by

$$\mathcal{Q}_{phy} := \left\{ Q \in \mathcal{Q} \middle| -\frac{1}{3} < \lambda_i(Q) < 1 - \frac{1}{3},\ 1 \le i \le 3 \right\}. \tag{2.46}$$

Following the idea in [56], the singular potential f_{MS} is defined by

$$f_{MS}(Q) \stackrel{\text{def}}{=} \begin{cases} \inf\limits_{\rho \in \mathcal{A}_Q} \displaystyle\int_{\mathbb{S}^2} \rho(\vec{n}) \ln \rho(\vec{n}) \, \mathrm{d}\vec{n}, & -\frac{1}{3} < \lambda_i(Q) < \frac{2}{3}, 1 \le i \le 3 \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.47}$$

where the admissible set $\mathcal{A}_Q$ is

$$\mathcal{A}_Q = \left\{ \rho \in \mathcal{P}(\mathbb{S}^2),\ \rho(\vec{n}) = \rho(-\vec{n}),\ \int_{\mathbb{S}^2} \left[\rho(\vec{n}) \otimes \rho(\vec{n}) - \frac{1}{3}\mathbb{I}_3 \right] \mathrm{d}\vec{n} = Q \right\} \tag{2.48}$$

Roughly speaking, we minimize the Boltzmann entropy over all probability distributions ρ with fixed normalized second moment $Q \in \mathcal{Q}_{phy}$, whose eigenvalues all stay within the interval $(-1/3, 2/3)$. Correspondingly,

$$\psi_B(Q) = f_{MS}(Q) - \frac{\kappa}{2}|Q|^2, \qquad \kappa > 0, \tag{2.49}$$

is used to replace the regular polynomial bulk potential f, where the last term $\frac{\kappa}{2}|Q|^2$ is added to ensure the existence of local energy minimizers:

$$\tilde{\mathcal{E}} := \int \left[L_1 |\nabla Q|^2 + L_2 \partial_j Q_{ik} \partial_k Q_{ij} + L_3 \partial_j Q_{ij} \partial_k Q_{ik} + L_4 Q_{lk} \partial_k Q_{ij} \partial_l Q_{ij} + \psi_B(Q) \right] \mathrm{d}x \tag{2.50}$$

Therefore, a natural enforcement of a physical constraint is imposed on the eigenvalues of the mathematical Q-tensor, and henceforth the free energy (2.33) could be kept under control under mild assumptions on $L_1, \cdots L_4$ [28, 49, 57].

This Maier-Saupe type singular potential $f_{MS}(Q)$ is (see [10, 52, 66] for detailed proofs)

- isotropic, i.e.,
$$f_{MS}(Q) = f_{MS}(B^t Q B), \quad \forall\, B \in SO(3).$$
- It is strictly convex.
- It is smooth in its effective domain $\mathcal{D}(f_{MS})$ where f_{MS} assumes finite value, that is, when
$$-\frac{1}{3} < \lambda_i(Q) < \frac{2}{3}, \qquad 1 \le i \le 3.$$

Moreover, for each given $Q \in \mathcal{Q}_{phy}$, there exists a unique $\rho \in \mathcal{A}_Q$ satisfying

$$f_{MS}(Q) = \int_{\mathbb{S}^2} \rho_Q(\vec{n}) \ln \rho_Q(\vec{n})\, \mathrm{d}\sigma(\vec{n}). \tag{2.51}$$

And ρ_Q is given implicitly by

$$\rho_Q(\vec{n}) = \frac{\exp(\vec{n} \otimes \vec{n} : \mu)}{Z(\mu)}, \qquad \mu \in \mathcal{Q}, \tag{2.52}$$

where

$$Z(\mu) = \int_{\mathbb{S}^2} \exp(\vec{n} \otimes \vec{n} : \mu)\, \mathrm{d}\sigma(\vec{n}) \tag{2.53}$$

satisfies

$$\frac{1}{Z}\frac{\partial Z}{\partial \mu} = Q + \frac{1}{3}\mathbb{I}_3. \tag{2.54}$$

Due to the singular feature of f_{MS} and the implicit formula to express f_{MS}, the relevant analysis is rather challenging in both static and dynamic settings. For the free energy $\tilde{\mathcal{E}}$ with $L_2 = L_3 = L_4 = 0$, the associated Beris-Edwards system was studied analytically in [17,51,52,67]. More precisely, the existence of global weak solutions to a non-isothermal co-rotational Beris-Edwards syste was established in [51,52]. The existence, regularity and strict physicality of global weak solutions of the corresponding isothermal co-rotational Beris-Edwards system under periodic settings was investigated in [67]:

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \nu \Delta \vec{u} + \nabla \cdot (\tau + \sigma), \\ \nabla \cdot \vec{u} = 0, \\ \partial_t Q + \vec{u} \cdot \nabla Q - S(\nabla \vec{u}, Q) = \Gamma \left[L\Delta Q - \frac{\partial f_{MS}}{\partial Q} + \frac{1}{d} \operatorname{tr}(\frac{\partial f_{MS}}{\partial Q})\mathbb{I} + \kappa Q \right]. \end{cases} \tag{2.55}$$

Compared to the co-rotational Beris-Edwards system (2.1) in which $\xi = 0$, the major difference in (2.55) is that the variational derivative of $\tilde{\mathcal{E}}(Q)$ contains a singular term $\frac{\partial f_{MS}}{\partial Q}$ other than regular polynomial terms. Besides the well-posedness result, one interesting topic to study is whether the hydrodynamic system (2.55) respects the physicality property as time evolves. To this end we shall define

Definition 2.2. *A map Q from a spatial domain Ω into $\mathcal{Q}$ is said to be strictly physical, if there exists $\delta > 0$ sufficiently small such that*

$$-\frac{1}{3} + \delta \le \lambda_i(Q(x)) \le \frac{2}{3} - \delta, \quad 1 \le i \le 3, \quad a.e.\ x \in \Omega. \tag{2.56}$$

Further, it is ready to check $Q \in \mathcal{Q}$ is strictly physical if and only if $f_{MS}(Q) \in L^\infty$, while $f_{MS}(Q) \in L^1$ ensures Q is physical. The main result in [67] states that

Theorem 2.12. *Suppose $(\vec{u}_0, Q_0) \in L^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$, $\nabla \cdot \vec{u}_0 = 0$, $d = 2, 3$, and*

$$f_{MS}(Q_0(\cdot)) \in L^1(\mathbb{T}^d) \qquad (\textit{physical initial data}),$$

Then there exists a global weak solution $(\vec{u}, Q)$ of the system (2.55). Furthermore,

$$f_{MS}(Q(t,\cdot)) \in L^\infty(\mathbb{T}^d), \quad \forall\, t > 0 \qquad (\textit{strictly physical at all positive time}).$$

The problem discussed in Theorem 2.12 is an important characteristic of evolution equations of parabolic type, bacause it is always natural to ask whether a solution with L^1 initial data is instantaneously in L^∞ for all $t > 0$. The essential idea in the proof of Theorem 2.12 was that although the evolution equation for the tensor field Q cannot be used to close an equation for the scalar function $f_{MS}(Q)$, a parabolic inequality satisfied by $f_{MS}(Q)$ can be derived so that a maximum principle argument led to strict physicality of weak solutions.

Meanwhile, it was also proved in [67] that under $2D$ settings (for both the spatial dimension and the Q-tensor space) the existence of global strong solutions of the system (2.55) provided the initial data Q_0 is strictly physical. Afterwards, the global existence and partial regularity results of a suitable weak solution to this system were proved in [17].

For the free energy $\tilde{\mathcal{E}}$ with $L_4 = 0$, the relevant analysis is more challenging in both static and dynamic problems in that the appearance of anisotropic L_2, L_3 terms makes the maximum principle argument no longer be valid. Concerning static problems, the regularity results regarding energy minimizers under simpler settings of $\tilde{F}$ were discussed in [47, 50, 53], respectively. Concerning dynamic problems, the authors in [60] studied regularity properties of an L^2 gradient flow generated by $\tilde{\mathcal{E}}$ with $L_4 = 0$ under periodic settings for $d = 2, 3$:

$$\begin{cases} \partial_t Q(t,\cdot) &= -\partial\tilde{\mathcal{E}}(Q(t,\cdot)), \quad t > 0, \\ Q(0,x) &= Q_0(x), \qquad x \in \mathbb{T}^d \end{cases} \tag{2.57}$$

More specifically, it was shown that for rather general initial data the associated L^2 gradient flow admits a unique strong solution after any positive time, and this solution detaches from the physical boundary after a sufficiently large time T_0.

Theorem 2.13. *For $d = 2, 3$ and any initial data*

$$Q_0 \in \overline{\{Q \in L^2(\mathbb{T}^d; \mathcal{Q}_{phy}) \big| \tilde{\mathcal{E}}(Q) < \infty\}}^{L^2(\mathbb{T}^d)}, \tag{2.58}$$

there exists a unique global solution $Q(t,x) : \mathbb{R}^+ \times \mathbb{T}^d \to \mathcal{Q}_{phy}$ of (2.57) such that

$$\begin{aligned} \partial_t Q_{ij} = 2L_1 \Delta Q_{ij} + 2(L_2 + L_3)\partial_j \partial_k Q_{ik} - \frac{2}{3}(L_2 + L_3)\partial_k \partial_l Q_{lk}\delta_{ij} \\ - \frac{\partial f_{MS}}{\partial Q_{ij}} + \frac{1}{3}\operatorname{tr}\Big(\frac{\partial f_{MS}}{\partial Q}\Big)\delta_{ij} + 2\kappa Q_{ij}, \qquad a.e.\ (t,x) \in (0,\infty) \times \mathbb{T}^d. \end{aligned} \tag{2.59}$$

Besides, $Q(t,x)$ is physical for all $t > 0$, a. e. $x \in \mathbb{T}^d$. Further, for any fixed $t_0 > 0$, we have $Q \in L^\infty(t_0, +\infty; H^2(\mathbb{T}^d))$. Moreover, under the stronger assumption

$$L_1 - 3|L_2 + L_3| - \kappa C_{\mathbb{T}^d}^2 > 0,$$

where $C_{\mathbb{T}^d} = (2\pi)^d$ is the Poincaré constant in $\mathbb{T}^d$, there exists $T_0 > 0$ such that the unique solution is strictly physical for all $t \geq T_0$.

The proof of pointwise existence of solutions in Theorem 2.13 relies on a powerful framework provided by Ambrosio-Gigli-Savare to obtain the solution of a gradient flow generated by a free energy (that is λ-convex) under very general assumptions of the initial data. However, there are essential difficulties caused by anisotropic $L_2 + L_3$ terms to establish a uniform-in-time H^2 bound of the solution Q. Without the $L_2 + L_3$ terms in (2.59), the convexity of f_{MS} together with classical $L^1 - L^\infty$ estimate of heat equation would ensure the strict physicality at any positive time and henceforth conventional energy method can be used. But the appearance of L_2+L_3 terms makes (2.59) a non-diagonal parabolic system with a singular potential which trends to infinity logarithmically when Q approaches its physical boundary, so that Q might not stay inside any compact subset of $\mathcal{Q}_{phy}$. To achieve the uniform-in-time H^2 bound of Q, several results from the gradient flow theory by Ambrosio-Gigli-Savare as well as Gamma-convergence of gradient flows by Sandier and Serfaty were combined, and the gradient flow structure of equation (2.59) was carefully exploited.

Furthermore, the estimate of the Hausdorff measure of the singular set where the solution touches the physical boundary in the intermediate stage $(0, T_0)$ was also provided in [60].

Theorem 2.14. *Let $Q(t, x)$ be the unique strong solution of* (2.57) *established in Theorem 2.13. Then for a.e. $t \in (0, T_0)$, the contact set*

$$\Sigma_t := \{x \in \mathbb{T}^d \big| \, Q(t, x) \in \partial\mathcal{Q}_{phy}\} \tag{2.60}$$

where $\partial\mathcal{Q}_{phy}$ denotes the boundary of $\mathcal{Q}_{phy}$, has the following estimate:

- *$dim_{\mathcal{H}}(\Sigma_t) \leq 2$ for $d = 3$.*
- *$dim_{\mathcal{H}}(\Sigma_t) = 0$ for $d = 2$.*

A crucial step in the proof of Theorem 2.14 was to establish the blowup rate of the gradient of f_{MS} as Q approaches its physical boundary. The blowup rate estimates of f_{MS} and ∇f_{MS} were initially studied in [43, 45], while the latest were given in [63]:

Theorem 2.15. *For any $Q \in \mathcal{Q}_{phy}$, assume $\lambda_1(Q) \leq \lambda_2(Q) \leq \lambda_3(Q)$. Then the singular potential f_{MS} is bounded above by*

$$f_{MS}(Q) \leq -\ln 8\sqrt{3} - \frac{1}{2}\ln\Big(\lambda_1(Q) + \frac{1}{3}\Big) - \frac{1}{2}\ln\Big(\lambda_2(Q) + \frac{1}{3}\Big). \tag{2.61}$$

In the meantime, there exists a small computable constant $\delta_0 > 0$, whenever $\lambda_2(Q) + 1/3 < \delta_0$, we have

$$f_{MS}(Q) \geq \ln 16 - 8\ln\pi - \frac{\pi^5}{16} - \frac{1}{2}\ln\Big(\lambda_1(Q) + \frac{1}{3}\Big) - \frac{1}{2}\ln\Big(\lambda_2(Q) + \frac{1}{3}\Big). \tag{2.62}$$

In addition, there exists a small computable constant $\varepsilon_0 > 0$, whenever $\lambda_1(Q) + 1/3 < \varepsilon_0$, it holds

$$\frac{C_1}{\lambda_1(Q) + \frac{1}{3}} \leq \big|\nabla_{\mathbb{Q}} f_{MS}(Q)\big| \leq \frac{C_2}{\lambda_1(Q) + \frac{1}{3}}, \tag{2.63}$$

where $C_1, C_2 > 0$ are explicitly computable constants and

$$\nabla_{\mathbb{Q}} f_{MS} = \frac{\partial f_{MS}}{\partial Q} - \frac{1}{3}\operatorname{tr}\Big(\frac{\partial f_{MS}}{\partial Q}\Big)\mathbb{I}_3,$$

It is noted that the blowup rate of f_{MS} in Theorem 2.15 ranges from $-\alpha\ln(\lambda_1(Q)+1/3)$, $1/2 \leq \alpha \leq 1$, as Q approaches its physical boundary via various directions. The bounds (2.61) and (2.62) are of significance also because they imply

$$f_{MS}(Q) + \frac{1}{2}\ln\Big(\lambda_1(Q) + \frac{1}{3}\Big) + \frac{1}{2}\ln\Big(\lambda_2(Q) + \frac{1}{3}\Big)$$

is a well defined, bounded function in the domain of λ_1, λ_2. Therefore, an accurate numerical approximation of $f_{MS}(Q)$ becomes possible by interpolating this function. On the other hand, the blowup rate estimate of $\nabla_{\mathbb{Q}} f_{MS}$ plays a key role in both dynamic and static problems related to $\tilde{\mathcal{E}}$ whenever anisotropic L_2, L_3 terms are present.

Unsolved research problem 8: Prove the global-wellposeness of the L^2 gradient flow generated by $\tilde{\mathcal{E}}$ provided $\tilde{\mathcal{E}}(Q_0) < \infty$.

Once $L_1, \cdots L_4$ terms are all present in the gradient flow, its relevant analysis becomes more challenging not only because of invalidity of maximum principle argument, but due to the loss of λ-convexity of the free energy $\tilde{\mathcal{E}}$ as well.

Finally, we want to point out that a relatively comprehensive list of references on the analytic study of static Q-tensor theory could be found in [68].

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Conflict of interest

The authors declare there is no conflicts of interest.

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