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Next-to-leading order evolution of color dipoles

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The small-x deep inelastic scattering in the saturation region is governed by the nonlinear evolution of Wilson-line operators. In the leading logarithmic approximation it is given by the Balitsky-Kovchegov equation for the evolution of color dipoles. In the next-to-leading order the Balitsky-Kovchegov equation gets contributions from quark and gluon loops as well as from the tree gluon diagrams with quadratic and cubic nonlinearities. We calculate the gluon contribution to the small-x evolution of Wilson lines (the quark part was obtained earlier).

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I. INTRODUCTION

A general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory, and the only quantum effect is the eikonal phase factor acquired along this propagation path. In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line—the infinite gauge link which is ordered along the straight line collinear to the particle's velocity n^μ:

$$U^\eta(x_\perp) = \mathrm{P}\exp\left\{ig\int_{-\infty}^{\infty} du\, n_\mu A^\mu(un + x_\perp)\right\}. \quad (1)$$

Here A_μ is the gluon field of the target, $x_\perp$ is the transverse position of the particle which remains unchanged throughout the collision, and the index η labels the rapidity of the particle. Repeating the above argument for the target (moving fast in the spectator's frame) we see that particles with very different rapidities perceive each other as Wilson lines, and therefore these Wilson-line operators form the convenient effective degrees of freedom in high-energy QCD (for a review, see Ref. [1]).

Let us consider the deep inelastic scattering from a hadron at small $x_B = Q^2/(2p\cdot q)$. The virtual photon decomposes into a pair of fast quarks moving along straight lines separated by some transverse distance. The propagation of this quark-antiquark pair reduces to the "propagator of the color dipole" $U(x_\perp)U^\dagger(y_\perp)$—two Wilson lines ordered along the direction collinear to the quarks' velocity. The structure function of a hadron is proportional to a matrix element of this color dipole operator,

$$\hat{\mathcal{U}}^\eta(x_\perp, y_\perp) = 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\eta(x_\perp)\hat{U}^{\dagger\eta}(y_\perp)\}, \quad (2)$$

switched between the target states ($N_c = 3$ for QCD). The gluon parton density is approximately

$$x_B G(x_B, \mu^2 = Q^2) \simeq \langle p|\hat{\mathcal{U}}^\eta(x_\perp, 0)|p\rangle|_{x_\perp^2 = Q^{-2}} \quad (3)$$

where $\eta = \ln\frac{1}{x_B}$. (As usual, we denote operators by a "hat.") The energy dependence of the structure function is translated then into the dependence of the color dipole on the slope of the Wilson lines determined by the rapidity η.

Thus, the small-x behavior of the structure functions is governed by the rapidity evolution of color dipoles [2,3]. At relatively high energies and for sufficiently small dipoles, we can use the leading logarithmic approximation (LLA) where $\alpha_s \ll 1$, $\alpha_s \ln x_B \sim 1$ and get the nonlinear Balitsky-Kovchegov (BK) evolution equation for the color dipoles [4,5]:

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \times [\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(y,z) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)]. \quad (4)$$

The first three terms correspond to the linear BFKL evolution [6] and describe the parton emission, while the last term is responsible for the parton annihilation. For sufficiently high x_B the parton emission balances the parton annihilation so the partons reach the state of saturation [7] with the characteristic transverse momentum Q_s growing with energy $1/x_B$ (for a review, see [8]).

As usual, to get the region of application of the leading order evolution equation, one needs to find the next-to-leading order (NLO) corrections. In the case of the small-x evolution equation (4) there is another reason why NLO corrections are important. Unlike the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution, the argument of the coupling constant in Eq. (4) is left undetermined in the LLA, and it is usually set by hand to be Q_s. Careful analysis of this argument is very important from both theoretical and experimental points of view. From the theoretical viewpoint, we need to know whether the coupling constant is determined by the size of the original dipole $|x-y|$ or by the size of the produced dipoles $|x-z|$ and/or $|z-y|$, since we may get a very different behavior

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of the solutions of Eq. (4). On the experimental side, the cross section is proportional to some power of the coupling constant, so the argument determines how big (or how small) the cross section is. The typical argument of α_s is the characteristic transverse momenta of the process. For high enough energies, they are of order of the saturation scale Q_s, which is $\sim 2 \div 3$ GeV for the CERN LHC, so even the difference between $\alpha(Q_s)$ and $\alpha(2Q_s)$ can make a substantial impact on the cross section. The precise form of the argument of α_s should come from the solution of the BK equation with the running-coupling constant, and the starting point of the analysis of the argument of α_s in Eq. (4) is the calculation of the NLO evolution.

Let us present our result for the NLO evolution of the color dipole (hereafter, we use notations $X \equiv x - z$, $X' \equiv x - z'$, $Y \equiv y - z$, and $Y' \equiv y - z'$),

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c - \frac{10}{9}n_f - 2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}[\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_y\} - N_c\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}] + \frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4}+\Big\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\times\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big)[\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_{z'}\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}^\dagger_y\} - \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\hat{U}_{z'}U^\dagger_y\hat{U}_z\hat{U}^\dagger_{z'}\} - (z'\to z)] + \Big\{\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\times\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_z\}\mathrm{Tr}\{\hat{U}_z\hat{U}^\dagger_{z'}\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}^\dagger_y\} + 4n_f\Big\{\frac{4}{(z-z')^4}-2\frac{X'^2Y^2+Y'^2X^2-(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}\times\mathrm{Tr}\{t^a\hat{U}_xt^b\hat{U}^\dagger_y\}[\mathrm{Tr}\{t^a\hat{U}_zt^b\hat{U}^\dagger_{z'}\} - (z'\to z)]\Big]. \tag{5}$$

Here μ is the normalization point in the $\overline{MS}$ scheme and $b = \frac{11}{3}N_c - \frac{2}{3}n_f$ is the first coefficient of the β function. The result of this paper is the gluon part of the evolution; the quark part of Eq. (5) proportional to n_f was found earlier [9,10]. Also, the terms with cubic nonlinearities were previously found in the large-N_c approximation in Ref. [11]. The NLO kernel is a sum of the running-coupling part (proportional to b), the nonconformal double-log term $\sim \ln\frac{(x-y)^2}{(x-z)^2}\ln\frac{(x-y)^2}{(x-z)^2}$, and the three conformal terms which depend on the two four-point conformal ratios $\frac{X^2Y'^2}{X'^2Y^2}$ and $\frac{(x-y)^2(z-z')^2}{X^2Y'^2}$. Note that the logarithm of the second conformal ratio $\ln\frac{(x-y)^2(z-z')^2}{X^2Y'^2}$ is absent.

It should be emphasized that the NLO result itself does not lead automatically to the argument of the coupling constant α_s in Eq. (4). In order to get this argument one can use the renormalon-based approach [12]: first get the quark part of the running-coupling constant coming from the bubble chain of quark loops and then make a conjecture that the gluon part of the β function will follow that pattern. Equation (5) proves this conjecture in the first nontrivial order: the quark part of the β function $\frac{2}{3}n_f$ calculated earlier gets promoted to the full b. The analysis of the argument of the coupling constant was performed in Refs. [9,10], and we briefly review it in Sec. VII for completeness. Roughly speaking, the argument of α_s is determined by the size of the smallest dipole $\min(|x-y|, |x-z|, |y-z|)$.

The paper is organized as follows. In Sec. II we remind the reader of the derivation of the BK equation in the leading order in α_s. In Secs. III and IV, which are central to the paper, we calculate the gluon contribution to the NLO kernel of the small-x evolution of color dipoles: in Sec. III we calculate the part of the NLO kernel corresponding to one-to-three dipoles transition, and in Sec. IV we calculate the one-to-two dipoles part. In Sec. V we assemble the NLO BK kernel, and in Sec. VI we compare the forward NLO BK kernel to the NLO BFKL results [13]. The results of the analysis of the argument of the coupling constant are briefly reviewed in Sec. VII. Appendix A is devoted to the calculation of the UV-divergent part of the one-to-three dipole kernel, and in Appendix B we discuss the dependence of the NLO kernel on the cutoff in the longitudinal momenta.

II. DERIVATION OF THE BK EQUATION

Before discussing the small-x evolution of the color dipole in the next-to-leading approximation, it is instructive to recall the derivation of the leading order (BK) evolution equation. As discussed in the Introduction, the dependence of the structure functions on x_B comes from

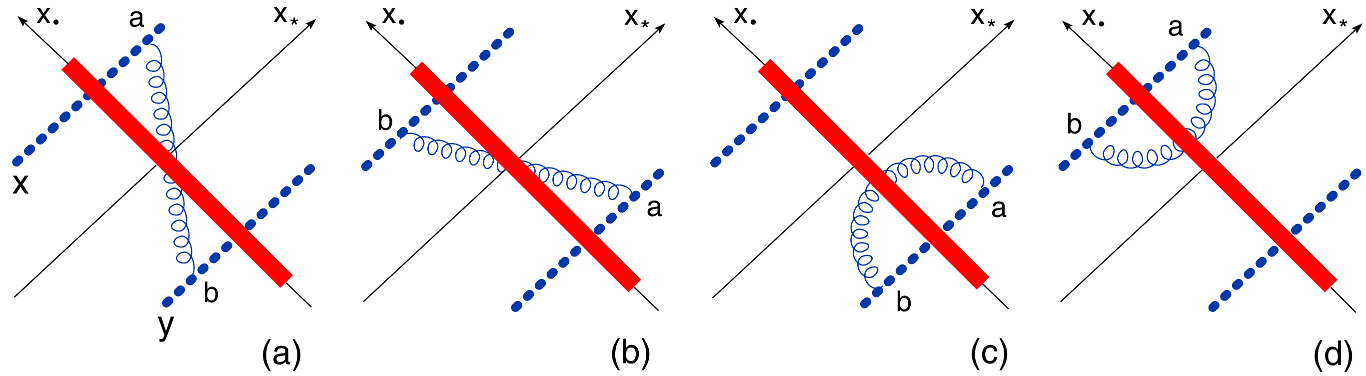

FIG. 1 (color online). Leading order diagrams for the small-x evolution of color dipole. Gauge links are denoted by dotted lines.

the dependence of Wilson-line operators

$$\hat{U}^{\eta}(x_\perp) = \mathrm{P}\exp\left\{ig\int_{-\infty}^{\infty} du n_\mu \hat{A}^\mu(un + x_\perp)\right\},\quad n \equiv p_1 + e^{-2\eta}p_2 \tag{6}$$

on the slope of the supporting line. The momenta p_1 and p_2 are the lightlike vectors such that $q = p_1 - x_B p_2$ and $p = p_2 + \frac{m^2}{s}p_1$, where p is the momentum of the target and m is the mass. Throughout the paper, we use the Sudakov variables $p = \alpha p_1 + \beta p_2 + p_\perp$ and the notations $x_\bullet \equiv x_\mu p_1^\mu$ and $x_* \equiv x_\mu p_2^\mu$ related to the light-cone coordinates: $x_* = x^+\sqrt{s/2}$, $x_\bullet = x^-\sqrt{s/2}$.

To find the evolution of the color dipole (2) with respect to the slope of the Wilson lines in the leading log approximation, we consider the matrix element of the color dipole between (arbitrary) target states and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2 = \eta_1 - \Delta\eta$, leaving the gluons with $\eta < \eta_2$ as a background field (to be integrated over later). In the frame of gluons with $\eta \sim \eta_1$, the fields with $\eta < \eta_2$ shrink to a pancake and we obtain the four diagrams shown in Fig. 1. Technically, to find the kernel in the leading order approximation, we write down the general form of the operator equation for the evolution of the color dipole,

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\,\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \tag{7}$$

(where dots stand for the higher orders of the expansion), and calculate the left-hand side (l.h.s.) of Eq. (7) in the shockwave background

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}. \tag{8}$$

In what follows we replace $\langle\ldots\rangle_{\mathrm{shockwave}}$ by $\langle\ldots\rangle$ for brevity. With future NLO computation in view, we will perform the leading order calculation in the light-cone gauge $p_2^\mu A_\mu = 0$. The gluon propagator in a shockwave external field has the form [11,14]

$$\begin{aligned}\langle\hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\rangle &= \theta(x_*y_*)\delta^{ab}\frac{s}{2}\int d\!\!{}^-\alpha d\!\!{}^-\beta\left(x_\perp\left|\frac{d_{\mu\nu}}{i(\alpha\beta s - p_\perp^2 + i\epsilon)}\right|y_\perp\right)\\ &\quad - \theta(x_*)\theta(-y_*)\int_0^\infty d\!\!{}^-\alpha\,\frac{e^{-i\alpha(x-y)_\bullet}}{2\alpha}\left(x_\perp\left|e^{-i(p_\perp^2/\alpha s)x_*}\left[g^\perp_{\mu\xi} - \frac{2}{\alpha s}(p^\perp_\mu p_{2\xi} + p_{2\mu}p^\perp_\xi)\right]\right.\right.\\ &\quad\times U^{ab}\left[g_\nu^{\perp\xi} - \frac{2}{\alpha s}(p_2^\xi p_\nu^\perp + p_{2\nu}p_\perp^\xi)\right]e^{i(p_\perp^2/\alpha s)y_*}\left|y_\perp\right)\\ &\quad - \theta(-x_*)\theta(y_*)\int_0^\infty d\!\!{}^-\alpha\,\frac{e^{i\alpha(x-y)_\bullet}}{2\alpha}\left(x_\perp\left|e^{i(p_\perp^2/\alpha s)x_*}\left[g^\perp_{\mu\xi} - \frac{2}{\alpha s}(p^\perp_\mu p_{2\xi} + p_{2\mu}p^\perp_\xi)\right]\right.\right.\\ &\quad\times U^{\dagger ab}\left[g_\nu^{\perp\xi} - \frac{2}{\alpha s}(p_2^\xi p_\nu^\perp + p_{2\nu}p_\perp^\xi)\right]e^{-i(p_\perp^2/\alpha s)y_*}\left|y_\perp\right)\end{aligned} \tag{9}$$

where

$$d_{\mu\nu}(k) \equiv g^\perp_{\mu\nu} - \frac{2}{s\alpha}(k^\perp_\mu p_{2\nu} + k^\perp_\nu p_{2\mu}) - \frac{4\beta}{s\alpha}p_{2\mu}p_{2\nu}. \tag{10}$$

Hereafter we use Schwinger's notations $(x_\perp|F(p_\perp)|y_\perp) \equiv \int d\!\!{}^-p\, e^{i(p,x-y)_\perp}F(p_\perp)$ [the scalar product of the four-dimensional vectors in our notations is $x\cdot y = \frac{2}{s}(x_*y_\bullet + x_*y_\bullet) - (x,y)_\perp$]. Note that the interaction with the shockwave does not change the α component of the gluon momentum.

We obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle_{\text{Fig. 1(a)}} = -4\alpha_s\int_0^\infty\frac{d\alpha}{\alpha}\Big(x_\perp\Big|\frac{p_i}{p_\perp^2+\alpha^2e^{-2\eta_1}s} \times U^{ab}\frac{p_i}{p_\perp^2+\alpha^2e^{-2\eta_1}s}\Big|y_\perp\Big) \tag{11}$$

(with power accuracy $\sim\frac{m^2}{s}$ one can replace $n_\mu A^\mu$ by $A_\bullet$). Formally, the integral over α diverges at the lower limit, but since we integrate over the rapidities $\eta>\eta_2$, we get (in the LLA)

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle_{\text{Fig. 1(a)}} = -4\alpha_s\Delta\eta\Big(x_\perp\Big|\frac{p_i}{p_\perp^2}U^{ab}\frac{p_i}{p_\perp^2}\Big|y_\perp\Big) \tag{12}$$

and therefore

$$\langle\hat{U}_x\otimes\hat{U}^\dagger_y\rangle^{\eta_1}_{\text{Fig. 1(a)}} = -\frac{\alpha_s}{\pi^2}\Delta\eta(t^aU_x\otimes t^bU^\dagger_y)\times\int d^2z_\perp\frac{(x-z,y-z)_\perp}{(x-z)^2_\perp(y-z)^2_\perp}U^{ab}_z. \tag{13}$$

The contribution of the diagram in Fig. 1(b) is obtained from Eq. (13) by the replacement $t^aU_x\otimes t^bU^\dagger_y\to U_xt^b\otimes U^\dagger_yt^a$, $x\leftrightarrow y$, and the two remaining diagrams are obtained from Eq. (12) by taking $y=x$ [Fig. 1(c)] and $x=y$ [Fig. 1(d)]. Finally, one obtains

$$\begin{aligned}\langle\hat{U}_x\otimes\hat{U}^\dagger_y\rangle^{\eta_1}_{\text{Fig. 1}} = &-\frac{\alpha_s\Delta\eta}{\pi^2}(t^aU_x\otimes t^bU^\dagger_y+U_xt^b\otimes U^\dagger_yt^a)\times\int d^2z_\perp\frac{(x-z,y-z)_\perp}{(x-z)^2_\perp(y-z)^2_\perp}U^{ab}_z\\ &+\frac{\alpha_s\Delta\eta}{\pi^2}(t^aU_xt^b\otimes U^\dagger_y)\int\frac{d^2z_\perp}{(x-z)^2_\perp}U^{ab}_z\\ &+\frac{\alpha_s\Delta\eta}{\pi^2}(U_x\otimes t^bU^\dagger_yt^a)\int\frac{d^2z_\perp}{(y-z)^2_\perp}U^{ab}_z\end{aligned} \tag{14}$$

so

$$\langle\text{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle^{\eta_1}_{\text{Fig. 1}} = \frac{\alpha_s\Delta\eta}{2\pi^2}\int d^2z_\perp\frac{(x-y)^2_\perp}{(x-z)^2_\perp(y-z)^2_\perp}\times\Big[\text{Tr}\{U_xU^\dagger_z\}\text{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\text{Tr}\{U_xU^\dagger_y\}\Big]. \tag{15}$$

There are also contributions coming from the diagrams shown in Fig. 2 (plus graphs obtained by reflection with respect to the shockwave). These diagrams are proportional to the original dipole $\text{Tr}\{U_xU^\dagger_y\}$, and therefore the corresponding term can be derived from the contribution of Fig. 1 graphs using the requirement that the right-hand side (r.h.s.) of the evolution equation should vanish for $x=y$ since $\lim_{x\to y}\frac{d}{d\eta}\text{Tr}\{U_xU^\dagger_y\}=0$. It is easy to see that this requirement leads to

$$\langle\text{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle^{\eta_1} = \frac{\alpha_s\Delta\eta}{2\pi^2}\int d^2z_\perp\frac{(x-y)^2_\perp}{(x-z)^2_\perp(y-z)^2_\perp}\times[\text{Tr}\{U_xU^\dagger_z\}\text{Tr}\{U_zU^\dagger_y\}-N_c\,\text{Tr}\{U_xU^\dagger_y\}] \tag{16}$$

which is equivalent to the BK equation for the evolution of the color dipole (4).

III. DIAGRAMS WITH TWO GLUON-SHOCKWAVE INTERSECTIONS

A. "Cut self-energy" diagrams

In the next-to-leading order there are three types of diagrams. Diagrams of the first type have two intersections of the emitted gluons with the shockwave, diagrams of the second type have one intersection, and finally diagrams of the third type have no intersections. In principle, there could have been contributions coming from the gluon loop which lies entirely in the shockwave, but we will demonstrate below that such terms are absent (see the discussion at the end of Sec. VI).

For the NLO calculation we use the light-cone gauge $p_2^\mu A_\mu=0$. Also, we find it convenient to change the prescription for the cutoff in the longitudinal direction.

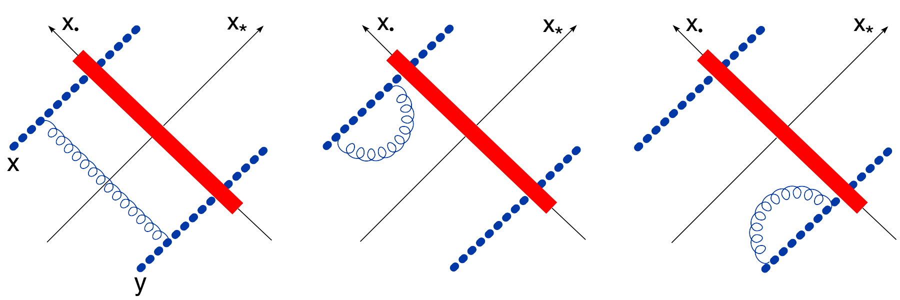

FIG. 2 (color online). Leading order diagrams proportional to the original dipole.

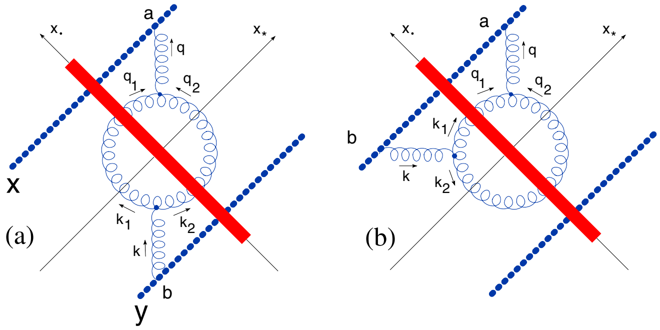

FIG. 3 (color online). Cut self-energy diagram.

We consider the lightlike dipoles (in the p_1 direction) and impose the cutoff on the maximal α emitted by any gluon from the Wilson lines, so

$$U_x^\eta = \mathrm{P}\exp\left[ig\int_{-\infty}^{\infty} du\, p_1^\mu A_\mu^\eta(up_1 + x_\perp)\right],$$
$$A_\mu^\eta(x) = \int d^4k\,\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k). \tag{17}$$

As we will see below, the (almost) conformal result (5) comes from the regularization (17). In Appendix B we will present the NLO kernel for the cutoff with the slope (6). We start with the calculation of Fig. 3(a). Multiplying two propagators (9), two three-gluon vertices, and two bare propagators, we obtain

$$\begin{aligned}
&g^2\int_0^\infty du\int_{-\infty}^0 dv\langle \hat{A}^a_\bullet(up_1+x_\perp)\hat{A}^b_\bullet(vp_1+y_\perp)\rangle\\
&=\frac{1}{2}g^4\frac{s^2}{4}f^{anl}f^{bn'l'}\int d\alpha d\alpha_1 d\beta d\beta' d\beta_1 d\beta_1' d\beta_2 d\beta_2'\int d^2z d^2z'\int d^2q_1 d^2q_2 d^2k_1 d^2k_2 e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp}\\
&\quad\times\frac{4\alpha_1(\alpha-\alpha_1)U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp}}{(\beta-\beta_1-\beta_2+i\epsilon)(\beta'-\beta_1'-\beta_2'+i\epsilon)(\beta-i\epsilon)(\beta'-i\epsilon)}\\
&\quad\times\frac{d_{\bullet\lambda}(\alpha p_1+\beta p_2+q_{1\perp}+k_{1\perp})}{\alpha\beta s-(q_1+q_2)_\perp^2+i\epsilon}\frac{d_{\lambda'\bullet}(\alpha p_1+\beta' p_2+q_{2\perp}+k_{2\perp})}{\alpha\beta' s-(k_1+k_2)_\perp^2+i\epsilon}\\
&\quad\times\frac{d_{\mu\xi}(\alpha_1p_1+\beta_1p_2+q_{1\perp})}{\alpha_1\beta_1 s-q_{1\perp}^2+i\epsilon}\frac{d^\xi{}_{\mu'}(\alpha_1p_1+\beta_1'p_2+k_{1\perp})}{\alpha_1\beta_1's-k_{1\perp}^2+i\epsilon}\\
&\quad\times\frac{d_{\nu\eta}((\alpha-\alpha_1)p_1+\beta_2p_2+q_{2\perp})}{(\alpha-\alpha_1)\beta_2s-q_{2\perp}^2+i\epsilon}\frac{d^\eta{}_{\nu'}((\alpha-\alpha_1)p_1+\beta_2'p_2+k_{2\perp})}{(\alpha-\alpha_1)\beta_2's-k_{2\perp}^2+i\epsilon}\\
&\quad\times\Gamma^{\mu\nu\lambda}(\alpha p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})\Gamma^{\mu'\nu'\lambda'}(\alpha p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})
\end{aligned} \tag{18}$$

where

$$\Gamma_{\mu\nu\lambda}(p,k,-p-k) = (p-k)_\lambda g_{\mu\nu} + (2k+p)_\mu g_{\nu\lambda} + (-2p-k)_\nu g_{\lambda\mu}. \tag{19}$$

In this formula $\frac{1}{\beta-i\epsilon}$ comes from the integration over the u parameter in the l.h.s. and $\frac{1}{\beta-\beta_1-\beta_2+i\epsilon}$ comes from the integration of the right three-gluon vertex over the half-space $x_*>0$. Similarly, we get $\frac{1}{\beta'-i\epsilon}$ from the integration over the v parameter and $\frac{1}{\beta'-\beta_1'-\beta_2'+i\epsilon}$ from the integration of the left three-gluon vertex over the half-space $x_*<0$. The factor $\frac{1}{2}$ in the r.h.s. is combinatorial. Note that in the light-cone gauge one can always neglect the $\beta p_{2\xi}$ components of the momenta in the three-gluon vertex since they are always multiplied by some $d_{\xi\eta}$.

Taking residues at $\beta=\beta'=0$ and $\beta_2=-\beta_1$, $\beta_2'=-\beta_1'$, we obtain

$$g^2\int_0^\infty du\int_{-\infty}^0 dv\langle\hat A^a_\bullet(up_1+x_\perp)\hat A^b_\bullet(vp_1+y_\perp)\rangle$$
$$=\frac{1}{2}g^4\frac{s^2}{4}f^{anl}f^{bn'l'}\int đ\alpha đ\alpha_1 đ\beta_1 đ\beta'_1\int d^2zd^2z'\int đ^2q_1đ^2q_2đ^2k_1đ^2k_2 e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp}4\frac{\alpha_1(\alpha-\alpha_1)}{\alpha^2}$$
$$\times U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1)z-i(q_2-k_2)z'}\frac{(q_{1\perp}+q_{2\perp})_\lambda}{(q_1+q_2)^2_\perp}\frac{(k_{1\perp}+k_{2\perp})_{\lambda'}}{(k_1+k_2)^2_\perp}\frac{d_\mu{}^\xi(\alpha_1p_1+q_{1\perp})}{\alpha_1\beta_1 s-q^2_{1\perp}+i\epsilon}\frac{d_{\xi\mu'}(\alpha_1p_1+k_{1\perp})}{\alpha_1\beta'_1 s-k^2_{1\perp}+i\epsilon}$$
$$\times\frac{d_\nu{}^\eta((\alpha-\alpha_1)p_1+q_{2\perp})}{-(\alpha-\alpha_1)\beta_1 s-q^2_{2\perp}+i\epsilon}\frac{d_{\eta\nu'}((\alpha-\alpha_1)p_1+k_{2\perp})}{-(\alpha-\alpha_1)\beta'_1 s-k^2_{2\perp}+i\epsilon}$$
$$\times\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\times\Gamma^{\mu'\nu'\lambda'}(\alpha_1p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp}).\tag{20}$$

We have omitted terms $\sim\beta p_2$ in the arguments of $d_{\xi\eta}$ since they do not contribute to $d_{\mu\xi}d^{\xi\mu'}$; see Eq. (10). Introducing the variable $u=\alpha_1/\alpha$ and taking residues at $\beta_1=\frac{q_1^2}{\alpha_1 s}$ and $\beta'_1=\frac{k_1^2}{\alpha_1 s}$, we obtain

$$-\frac{g^4}{8\pi^2}f^{anl}f^{bn'l'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\bar uu\int d^2zd^2z'\int đ^2q_1đ^2q_2đ^2k_1đ^2k_2e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp-i(q_1-k_1,z)-i(q_2-k_2,z')}U_z^{nn'}U_{z'}^{ll'}$$
$$\times\frac{(q_{1\perp}+q_{2\perp})_\lambda(k_{1\perp}+k_{2\perp})_{\lambda'}}{(q_1+q_2)^2_\perp(k_1+k_2)^2_\perp}\frac{d_{\mu\xi}(u\alpha p_1+q_{1\perp})d^\xi{}_{\mu'}(u\alpha p_1+k_{1\perp})}{q^2_{1\perp}\bar u+q^2_{2\perp}u}\frac{d_{\nu\eta}(\bar u\alpha p_1+q_{2\perp})d^\eta{}_{\nu'}(\bar u\alpha p_1+k_{2\perp})}{k^2_{1\perp}\bar u+k^2_{2\perp}u}$$
$$\times\Gamma^{\mu\nu\lambda}(u\alpha p_1+q_{1\perp},\bar u\alpha p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})\Gamma^{\mu'\nu'\lambda'}(u\alpha p_1+k_{1\perp},\bar u\alpha p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})\tag{21}$$

where we have imposed a cutoff $\alpha<\sigma$ in accordance with Eq. (17).

Using the formulas,

$$d_{\mu\xi}(u\alpha p_1+q_\perp)d^{\xi\mu'}(u\alpha p_1+k_\perp)=\Big(g^\perp_{\mu\xi}-\frac{2}{s\alpha u}p_{2\mu}q^\perp_\xi\Big)\Big(g_\perp^{\xi\mu'}-\frac{2}{s\alpha u}k_\perp^\xi\Big)$$
$$-(q_1+q_2)^\lambda_\perp\Gamma_{\mu\nu\lambda}(\alpha up_1+q_{1\perp},\alpha\bar up_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp)$$
$$\times\Big(g_\perp^{\mu i}-\frac{2}{s\alpha u}p_2^\mu q_1^i\Big)\Big(g_\perp^{\nu j}-\frac{2}{s\alpha\bar u}p_2^\nu q_2^j\Big)$$
$$=(q^2_{1\perp}-q^2_{2\perp})g^{ij}_\perp+\frac{2}{u}q_1^i(q_1+q_2)^j-\frac{2}{\bar u}(q_1+q_2)^iq_2^j\tag{22}$$

we can represent the contribution of Fig. 3(a) in the form

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\text{Fig. 3(a)}}=-\frac{g^4}{8\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\bar uu$$
$$\times\int đ^2q_1đ^2q_2đ^2k_1đ^2k_2\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,y-z)_\perp-i(k_2,y-z')_\perp}}{(q_1+q_2)^2(k_1+k_2)^2(q_1^2\bar u+q_2^2u)(k_1^2\bar u+k_2^2u)}$$
$$\times\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar u}(q_1+q_2)_iq_{2j}\Big]$$
$$\times\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar u}(k_1+k_2)_ik_{2j}\Big].\tag{23}$$

Throughout the paper we use Greek letters for indices $\mu=0,1,2,3$ [with $g^{\mu\nu}=(1,-1,-1,-1)$] and Latin letters for transverse indices $i=1,2$.

The diagram shown in Fig. 3(b) is obtained by the substitution $e^{-i(k_1+k_2,y_\perp)}\to-e^{-i(k_1+k_2,x)_\perp}$ (the different sign comes from replacing $[-\infty p_1,0]_y$ by $[0,-\infty p_1]_x$). We get

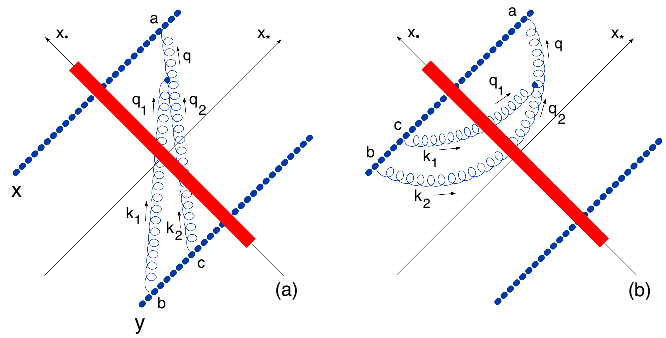

FIG. 4 (color online). Cut vertex diagrams.

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 3(a)+(b)}} = \frac{g4}{8\pi 2}\mathrm{Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\bar{u}u\int \bar{d}^2q_1\bar{d}^2q_2\bar{d}^2k_1\bar{d}^2k_2$$
$$\times\frac{e^{i(q_1+q_2,x)_\perp - i(q_1-k_1,z)-i(q_2-k_2,z')}[e^{-i(k_1+k_2,x)_\perp} - e^{-i(k_1+k_2,y)_\perp}]}{(q_1+q_2)^2(k_1+k_2)^2(q_1^2\bar{u}+q_2^2u)(k_1^2\bar{u}+k_2^2u)}$$
$$\times\left[(q_1^2-q_2^2)\delta_{ij} - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\right]$$
$$\times\left[(k_1^2-k_2^2)\delta_{ij} - \frac{2}{u}k_{1i}(k_1+k_2)_j + \frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\right]. \quad (24)$$

B. "Cut vertex" diagrams

Next, consider the cut vertex diagram in Fig. 4(a). The analog of Eq. (20) has the form

$$g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv\langle \hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+y_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)\rangle$$
$$= 2g^4sf^{mna}\int \bar{d}\alpha_1\bar{d}\beta_1\bar{d}\alpha_2\bar{d}\beta_2\int \bar{d}^2q_1\bar{d}^2q_2\bar{d}^2k_1\bar{d}^2k_2\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_{1\perp},\alpha_2p_1+q_{2\perp},-(\alpha_1+\alpha_2)p_1-(q_1+q_2)_\perp)$$
$$\times\frac{\alpha_1\theta(\alpha_1)\theta(\alpha_2)e^{i(q_1+q_2,x)_\perp}((q_1+q_2)_\perp + 2(\beta_1+\beta_2)p_2)_\lambda}{(\alpha_1+\alpha_2)(\beta_1+\beta_2-i\epsilon)}\frac{(k_1+\frac{2(k_1,q_1)_\perp}{\alpha_1 s}p_2)_\mu(k_2+\frac{2(k_2,q_2)_\perp}{\alpha_2 s}p_2)_\nu}{[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s-(q_1+q_2)^2_\perp+i\epsilon]}$$
$$\times\int d^2zd^2z'\frac{e^{-i(k_1+k_2,y)_\perp}U_z^{mb}U_{z'}^{nc}e^{-i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp}}{k_1^2(k_1^2\alpha_2+k_2^2\alpha_1)(\alpha_1\beta_1s-q^2_{1\perp}+i\epsilon)(\alpha_2\beta_2s-q^2_{2\perp}+i\epsilon)}. \quad (25)$$

Going to variables $\alpha = \alpha_1+\alpha_2$, $u = \alpha_1/\alpha$ and taking residues at $\beta_1+\beta_2 = 0$ and $\beta_1 = \frac{q_1^2}{\alpha_1}$, we get

$$g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv\langle \hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+y_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)\rangle$$
$$= \frac{g^4}{2\pi^2}f^{mna}\int_0^\sigma\frac{d\alpha}{\alpha^2}\int_0^1 duu\int \bar{d}^2q_1\bar{d}^2q_2\bar{d}^2k_1\bar{d}^2k_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}\frac{e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp - i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp}}{(q_1+q_2)^2_\perp(q_1^2\bar{u}+q_2^2u)k_1^2(k_1^2\bar{u}+k_2^2u)}$$
$$\times\left(k_1+\frac{2(k_1,q_1)_\perp}{\alpha us}p_2\right)_\mu\left(k_2+\frac{2(k_2,q_2)_\perp}{\alpha\bar{u}s}p_2\right)_\nu(q_1+q_2)_{\perp\lambda}\Gamma^{\mu\nu\lambda}(\alpha up_1+q_{1\perp},\alpha\bar{u}p_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp) \quad (26)$$

and therefore

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 4(a)}} = -i\frac{g^4}{2\pi^2}f^{mna}\mathrm{Tr}\{t^aU_xt^bt^cU^\dagger_y\}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\bar{u}u\int \bar{d}^2k_1\bar{d}^2k_2\bar{d}^2q_1\bar{d}^2q_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}$$
$$\times\frac{(q_1^2-q_2^2)\delta_{ij} - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\frac{k_{1i}k_{2j}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}$$
$$\times e^{i(q_1+q_2,x)_\perp - i(k_1+k_2,y)_\perp - i(q_1-k_1,z)_\perp - i(q_2-k_2,z')_\perp} \quad (27)$$

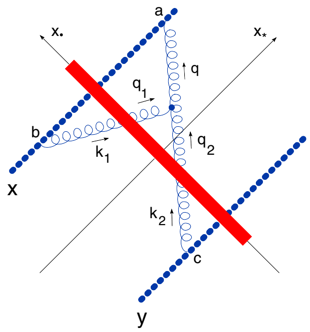

FIG. 5 (color online). Another type of diagram with two gluon-shockwave intersections.

where we have used the formula

$$\left(k_1+\frac{2(k_1,q_1)_\perp}{\alpha us}p_2\right)_\mu\left(k_2+\frac{2(k_2,q_2)_\perp}{\alpha\bar{u}s}p_2\right)_\nu(q_1+q_2)_{\perp\lambda}\Gamma^{\mu\nu\lambda}(\alpha up_1+q_{1\perp},\alpha\bar{u}p_1+q_{2\perp},-\alpha p_1-(q_1+q_2)_\perp)$$
$$=k_{1i}k_{2j}\left[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\right]\tag{28}$$

following from Eq. (22).

The contribution of the diagram shown in Fig. 4(b) differs from Eq. (27) by the substitution $e^{-i(k_1+k_2,y)_\perp}\to e^{-i(k_1+k_2,x)_\perp}$ and by changing the order of t^b, t^c matrices. [Similarly to the case of Fig. 3(b), this prescription follows from replacing $[-\infty p_1,0]_y$ by $[0,\infty p_1]_x$, but now we consider the second term of the expansion in the gauge field.] We get

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig. 4(a)+(b)}}=-i\frac{g^4}{2\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\bar{u}u\int d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}$$
$$\times\frac{(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\frac{k_{1i}k_{2j}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}$$
$$\times e^{i(q_1+q_2,x)_\perp-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}f^{mna}\,\mathrm{Tr}\{t^aU_x[t^ct^be^{-i(k_1+k_2,x)}+t^bt^ce^{-i(k_1+k_2,y)}]U_y^\dagger\}.\tag{29}$$

There is another type of diagram with two gluon-shockwave intersections shown in Fig. 5,

$$g^3\int_0^\infty dt\int_{-\infty}^0du\int_{-\infty}^0dv\langle\hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+x_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)\rangle$$
$$=2g^4s\int d\!\!\!{}^-\alpha_1d\!\!\!{}^-\alpha_2d\!\!\!{}^-\beta_1d\!\!\!{}^-\beta_2\int d^2zd^2z'U_z^{mb}U_{z'}^{nc}\int\frac{d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2}{k_1^2k_2^2}\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2\theta(\alpha_1)\theta(\alpha_2)f^{mna}$$
$$\times\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_1,\alpha_2p_1+q_2,-(\alpha_1+\alpha_2)p_1-(q_1+q_2)_\perp)e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}$$
$$\times\frac{[(q_1+q_2)_\perp+2(\beta_1+\beta_2)p_2]_\lambda}{(\alpha_1+\alpha_2)(\beta_1+\beta_2-i\epsilon)[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s-(q_1+q_2)_\perp^2+i\epsilon]}\frac{k_{1\mu}+\frac{2(k_1,q_1)_\perp}{\alpha_1s}p_{2\mu}}{\alpha_1\beta_1s-q_{1\perp}^2+i\epsilon}\frac{k_{2\nu}+\frac{2(k_2,q_2)_\perp}{\alpha_2s}p_{2\nu}}{\alpha_2\beta_2s-q_{2\perp}^2+i\epsilon}.\tag{30}$$

Taking residues at $\beta_1+\beta_2=0$ and at $\beta_1=\frac{q_1^2}{\alpha_1s}$ and going to variables $\alpha=\alpha_1+\alpha_2$ and $u=\alpha_1/\alpha$, we get

$$g^3\int_0^\infty dt\int_{-\infty}^0du\int_{-\infty}^0dv\langle\hat{A}^a_\bullet(tp_1+x_\perp)\hat{A}^b_\bullet(up_1+x_\perp)\hat{A}^c_\bullet(vp_1+y_\perp)\rangle$$
$$=\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha^2}\int_0^1du\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2U_z^{mb}U_{z'}^{nc}\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}}{k_1^2k_2^2(q_1+q_2)^2(\bar{u}q_1^2+uq_2^2)}\left(k_1+\frac{2(q_1,k_1)_\perp p_2}{s\alpha u}\right)^\mu$$
$$\times\left(k_2+\frac{2(q_2,k_2)_\perp p_2}{s\alpha\bar{u}}\right)^\nu(q_1+q_2)^\lambda\Gamma_{\mu\nu\lambda}(\alpha up_1+q_1,\alpha\bar{u}p_1+q_2,-\alpha p_1-(q_1+q_2)_\perp)$$
$$=\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2U_z^{mb}U_{z'}^{nc}\frac{e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,x-z)_\perp-i(k_2,y-z')_\perp}}{k_1^2k_2^2(q_1+q_2)^2(\bar{u}q_1^2+uq_2^2)}$$
$$\times\left[(q_{1\perp}^2-q_{2\perp}^2)\delta^{ij}-\frac{2}{u}q_1^i(q_1+q_2)^j+\frac{2}{\bar{u}}(q_1+q_2)^iq_2^j\right]k_{1i}k_{2j}\tag{31}$$

and therefore

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\,5}} = i\frac{g^4}{2\pi^2}f^{amn}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1 du\int d^2z d^2z'\int \bar{d}^2q_1\bar{d}^2q_2\int \bar{d}^2k_1\bar{d}^2k_2 e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}U_z^{mm'}U_{z'}^{nn'} \times\frac{e^{i(q_1+q_2,x)_\perp-i(k_1,x)_\perp-i(k_2,y)_\perp}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big]\frac{k_{1i}k_{2j}}{k_1^2k_2^2} \times\mathrm{Tr}\{t^aU_xt^{m'}t^{n'}U_y^\dagger\} \tag{32}$$

where again we have used formula (28).

The sum of the contributions (24), (29), and (32) can be represented as follows:

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\,3+Fig.\,4+Fig.\,5}} \equiv \langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\,6(\,I)+(III)+(V)+(VII)+(IX)}} = \frac{g2}{8\pi 2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1 du\bar{u}u\int d^2zd^2z'\int \bar{d}^2q_1\bar{d}^2q_2\int\bar{d}^2k_1\bar{d}^2k_2e^{-i(q_1-k_{1,z})_\perp-i(q_2-k_{2,z'})_\perp}U_z^{mm'} \times U_{z'}^{nn'}f^{amn}\mathrm{Tr}\Big\{t^a\frac{e^{i(q_1+q_{2,x})_\perp}}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big] \times U_x\Big[t^bf^{bm'n'}\frac{(e^{-i(k_1+k_{2,x})_\perp}-e^{-i(k_1+k_{2,y})_\perp})}{(k_1+k_2)^2(k_1^2\bar{u}+k_2^2u)}\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\Big] -4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_{2,x})_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}-\frac{e^{-i(k_{1,x})_\perp-i(k_{2,y})_\perp}}{\bar{u}uk_1^2k_2^2}t^{m'}t^{n'}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_{2,y})_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}\Big)\Big]U_y^\dagger\Big\}. \tag{33}$$

If we add contributions of the diagrams with the gluon on the right side of the shockwave attached to the Wilson line at the point y instead of x [which differs from Eq. (33) by the substitution $e^{i(q_1+q_2,x)_\perp}\longrightarrow -e^{i(q_1+q_2,y)_\perp}$], we obtain

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{Fig.\,6(I)+(II)+\ldots+(X)}} = \frac{g^2}{8\pi^2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1 du\bar{u}u\int d^2zd^2z'\int\bar{d}^2q_1\bar{d}^2q_2\int\bar{d}^2k_1\bar{d}^2k_2e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}U_z^{mm'}U_{z'}^{nn'} \times f^{amn}\mathrm{Tr}\Big\{t^a\frac{(e^{i(q_1+q_2,x)_\perp}-e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big] \times U_x\Big[t^bf^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar{u}+k_2^2u)}\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\Big] -4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}-\frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{m'}t^{n'}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}\Big)\Big]U_y^\dagger\Big\}. \tag{34}$$

The result (34) can be obtained from the self-energy contribution (24) by the replacement of the term corresponding to the emission of the two gluons via the three-gluon vertex

$$t^bf^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar{u}+k_2^2u)}\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\Big],$$

with a similar contribution containing the "effective vertex"

$$\mathrm{S}^{m'n'}(k_1,k_2;x,y)\equiv t^bf^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar{u}+k_2^2u)}\Big[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\Big] -4ik_{1i}k_{2j}\Big(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}-\frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar{u}uk_1^2k_2^2}t^{m'}t^{n'}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar{u}k_1^2(k_1^2\bar{u}+k_2^2u)}\Big). \tag{35}$$

It can be demonstrated that the sum of the contributions of Figs. 6 (I),…, (IV), (XI),…, (XVI) can be obtained from the self-energy contribution (24) by replacing the gluon vertex

$$t^af^{amn}\frac{(e^{i(q_1+q_2,x)_\perp}-e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar{u}+q_2^2u)}\Big[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\Big] \tag{36}$$

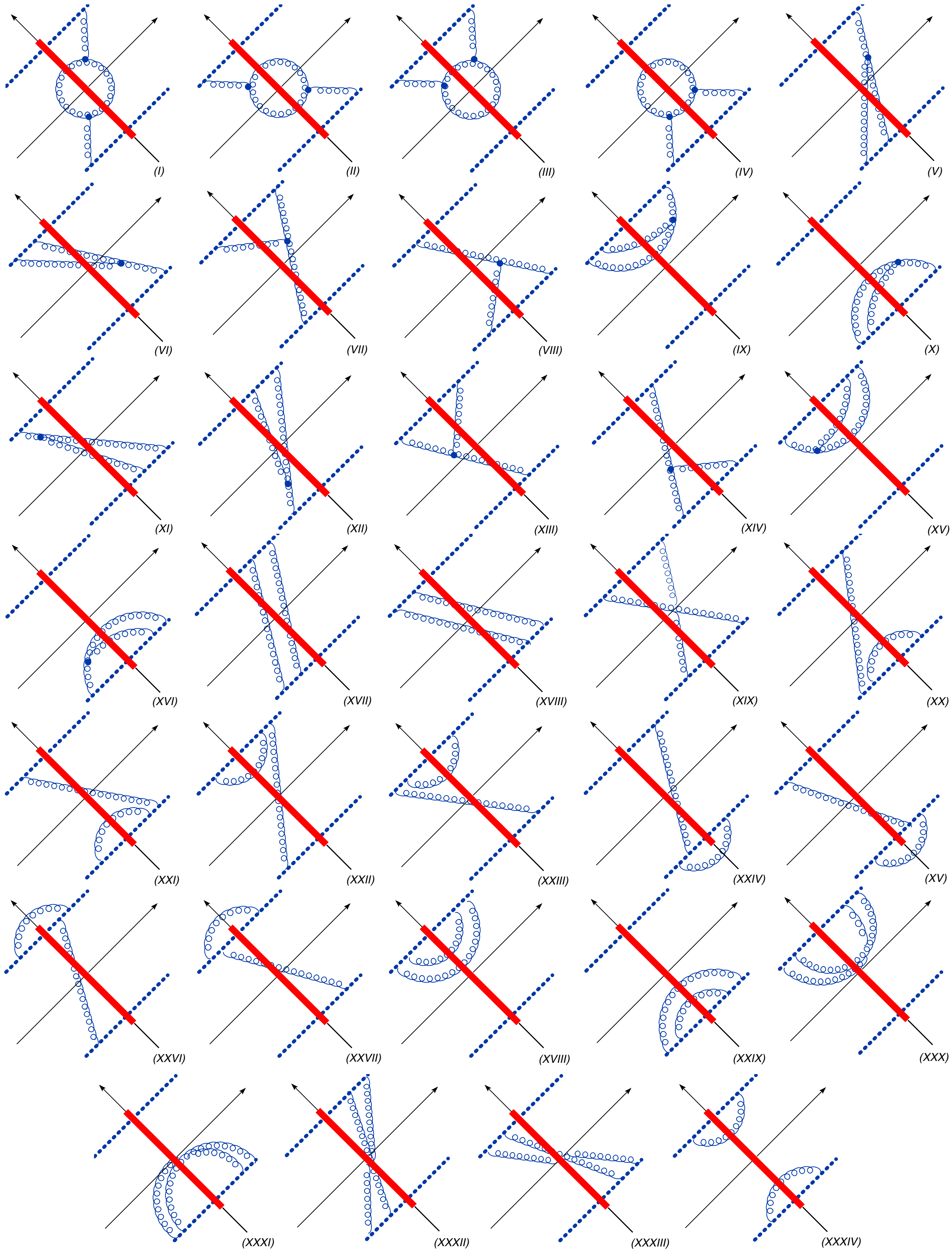

FIG. 6 (color online). Diagrams with two cuts.

with a similar effective vertex

$$t^a f^{amn}\frac{(e^{i(q_1+q_2,x)_\perp}-e^{i(q_1+q_2,y)_\perp})}{(q_1+q_2)^2(q_1^2\bar u+q_2^2u)}\left[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar u}(q_1+q_2)_iq_{2j}\right]$$
$$+4iq_{1i}q_{2j}\left(\frac{t^mt^ne^{i(q_1+q_2,x)_\perp}}{\bar uq_1^2(q_1^2\bar u+q_2^2u)}-\frac{e^{i(q_1,x)_\perp+i(q_2,y)_\perp}}{\bar uuq_1^2q_2^2}t^nt^m+\frac{t^nt^me^{i(q_1+q_2,y)_\perp}}{\bar uq_1^2(q_1^2\bar u+q_2^2u)}\right). \tag{37}$$

Note that (35) is equal to $S^{\dagger mn}(q_1,q_2;x,y)$. Let us consider now the box diagram topology shown in Figs. 6 (XVII)–(XXXIV). The calculation of these diagrams is similar to the above calculation of cut self-energy and cut vertex diagrams, so we present here only the final result:

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\text{Fig. 6(XVII)+...+(XXXIV)}}=\frac{g^2}{2\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\bar uu\int d^2zd^2z'\int đ^2q_1đ^2q_2\int đ^2k_1đ^2k_2e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}$$
$$\times U_z^{mm'}U_{z'}^{nn'}\,\mathrm{Tr}\left\{\left[q_{1i}q_{2j}\left(\frac{t^mt^ne^{i(q_1+q_2,x)_\perp}}{\bar uq_1^2(q_1^2\bar u+q_2^2u)}+\frac{t^nt^me^{i(q_1+q_2,x)_\perp}}{uq_2^2(q_1^2\bar u+q_2^2u)}-\frac{e^{i(q_1,x)_\perp+i(q_2,y)_\perp}}{\bar uuq_1^2q_2^2}t^nt^m\right.\right.\right.$$
$$\left.\left.-\frac{e^{i(q_1,y)_\perp+i(q_2,x)_\perp}}{\bar uuq_1^2q_2^2}t^mt^n+\frac{t^nt^me^{i(q_1+q_2,y)_\perp}}{\bar uq_1^2(q_1^2\bar u+q_2^2u)}+\frac{t^mt^ne^{i(q_1+q_2,y)_\perp}}{uq_2^2(q_1^2\bar u+q_2^2u)}\right)\right]$$
$$\times U_x\left[k_{1i}k_{2j}\left(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar uk_1^2(k_1^2\bar u+k_2^2u)}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,x)_\perp}}{uk_2^2(k_1^2\bar u+k_2^2u)}-\frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar uuk_1^2k_2^2}t^{m'}t^{n'}\right.\right.$$
$$\left.\left.\left.-\frac{e^{-i(k_2,x)_\perp-i(k_1,y)_\perp}}{\bar uuk_1^2k_2^2}t^{n'}t^{m'}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar uk_1^2(k_1^2\bar u+k_2^2u)}+\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,y)_\perp}}{uk_2^2(k_1^2\bar u+k_2^2u)}\right)\right]U_y^\dagger\right\}. \tag{38}$$

This expression agrees with the sum of "box topology" diagrams in Ref. [11].

Now we observe that each three-gluon vertex diagram is equal to its own cross diagram (the same cannot be said for box diagrams). Thus we may redefine the effective vertex (35) in the following way:

$$\mathcal{S}^{m'n'}(k_1,k_2;x,y)=t^bf^{bm'n'}\frac{(e^{-i(k_1+k_2,x)_\perp}-e^{-i(k_1+k_2,y)_\perp})}{(k_1+k_2)^2(k_1^2\bar u+k_2^2u)}\left[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar u}(k_1+k_2)_ik_{2j}\right]$$
$$-i2k_{1i}k_{2j}\left(\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,x)_\perp}}{\bar uk_1^2(k_1^2\bar u+k_2^2u)}+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,x)_\perp}}{uk_2^2(k_1^2\bar u+k_2^2u)}-\frac{e^{-i(k_1,x)_\perp-i(k_2,y)_\perp}}{\bar uuk_1^2k_2^2}t^{m'}t^{n'}-\frac{e^{-i(k_2,x)_\perp-i(k_1,y)_\perp}}{\bar uuk_1^2k_2^2}t^{n'}t^{m'}\right.$$
$$\left.+\frac{t^{m'}t^{n'}e^{-i(k_1+k_2,y)_\perp}}{\bar uk_1^2(k_1^2\bar u+k_2^2u)}+\frac{t^{n'}t^{m'}e^{-i(k_1+k_2,y)_\perp}}{uk_2^2(k_1^2\bar u+k_2^2u)}\right) \tag{39}$$

which corresponds to writing each contribution of the three-gluon vertex diagrams as a sum of two equal terms.

A similar expression can be written for the effective vertex (37), and therefore the sum of all diagrams with two gluon-shockwave intersections can be written as

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\text{Fig. 6}}=\frac{g^2}{8\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\bar uu\int d^2zd^2z'\int đ^2q_1đ^2q_2\int đ^2k_1đ^2k_2e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}U_z^{mm'}U_{z'}^{nn'}$$
$$\times\mathrm{Tr}\{[\mathcal{S}^{\dagger mn}(q_1,q_2;x,y)U_x][\mathcal{S}^{m'n'}(k_1,k_2;x,y)U_y^\dagger]\}. \tag{40}$$

Separating the contributions of different color structures, one obtains

$$\langle \mathrm{Tr}\{\hat U_x \hat U^\dagger_y\}_{\mathrm{Fig.\,6}} = \frac{g^2}{8\pi^2}\int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1 \frac{du}{\bar u u}\int d^2z d^2z' \int đ^2q_1 đ^2q_2 \int đ^2k_1 đ^2k_2 U_z^{mm'} U_{z'}^{nn'}$$
$$\times \mathrm{Tr}\Big\{t^a f^{amn}\frac{(e^{i(q_1,X)+i(q_2,X')} - x\leftrightarrow y)}{(q_1^2\bar u + q_2^2 u)}\Big[\frac{(q_1^2 - q_2^2)\bar u u\delta_{ij} - 2\bar u q_{1i}(q_1+q_2)_j + 2u(q_1+q_2)_i q_{2j}}{(q_1+q_2)^2}$$
$$- u\frac{q_{1i}q_{2j}}{q_1^2} + \bar u\frac{q_{1i}q_{2j}}{q_2^2}\Big] - t^a f^{amn}\frac{q_{1i}q_{2j}}{q_1^2 q_2^2}(e^{i(q_1,X)+i(q_2,Y')} - x\leftrightarrow y)$$
$$+ i\{t^m, t^n\}\frac{q_{1i}q_{2j}}{q_1^2 q_2^2}(e^{i(q_1,X)} - e^{i(q_1,Y)})(e^{i(q_2,X')} - e^{i(q_2,Y')})\Big\}$$
$$\times U_x\Big\{t^b f^{bm'n'}\frac{(e^{-i(k_1,X)-i(k_2,X')} - x\leftrightarrow y)}{(k_1^2\bar u + k_2^2 u)}\Big[\frac{(k_1^2 - k_2^2)\bar u u\delta_{ij} - 2\bar u k_{1i}(k_1+k_2)_j + 2u(k_1+k_2)_i k_{2j}}{(k_1+k_2)^2}$$
$$- u\frac{k_{1i}k_{2j}}{k_1^2} + \bar u\frac{k_{1i}k_{2j}}{k_2^2}\Big] - t^b f^{bm'n'}\frac{k_{1i}k_{2j}}{k_1^2 k_2^2}(e^{-i(k_1,X)-i(k_2,Y')} - x\leftrightarrow y)$$
$$- i\{t^{m'}, t^{n'}\}\frac{k_{1i}k_{2j}}{k_1^2 k_2^2}(e^{-i(k_1,X)} - e^{-i(k_1,Y)})(e^{-i(k_2,X')} - e^{-i(k_2,Y')})\Big\}U_y. \quad (41)$$

This result agrees with Ref. [11].

Performing the Fourier transformation

$$\int đ^2q_1 đ^2q_2 e^{i(q_1,x_1)+i(q_2,x_2)}\frac{q_{1i}q_{2j}}{q_1^2(q_1^2\bar u + q_2^2 u)}$$
$$= -\frac{x_{1i}x_{2j}}{4\pi^2 x_2^2(ux_1^2 + \bar u x_2^2)}\int đ^2q_1 đ^2q_2 e^{i(q_1,x_1)+i(q_2,x_2)}\frac{\delta_{ij}(q_1^2 - q_2^2) - \frac{2}{u}q_{1i}(q_1+q_2)_j + \frac{2}{\bar u}(q_1+q_2)_i q_{2j}}{(q_1+q_2)^2(q_1^2\bar u + q_2^2 u)}$$
$$= \frac{-(x_1^2 - x_2^2)\delta_{ij} + \frac{2}{u}(x_1 - x_2)_i x_{2_j} + \frac{2}{\bar u}x_{1i}(x_1 - x_2)_j}{4\pi^2(x_1 - x_2)^2(ux_1^2 + \bar u x_2^2)} \quad (42)$$

we get

$$\frac{d}{d\ln\sigma}\langle \mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.\,6}} = \frac{\alpha_s^2}{8\pi^4}\int_0^1 du u\bar u\int d^2z d^2z' U_z^{bb'}U_{z'}^{cc'}\,\mathrm{Tr}\Big\{t^a f^{abc}\Big[\frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_iX'_j}{uX^2} - \frac{X_iX'_j}{\bar u X'^2}}{uX^2 + \bar u X'^2} - \frac{X_iY'_j}{\bar u u X^2 Y'^2} - (x\leftrightarrow y)\Big]$$
$$+ i\frac{\{t^b, t^c\}}{\bar u u}\Big(\frac{X_i}{X^2} - \frac{Y_i}{Y^2}\Big)\Big(\frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2}\Big)\Big\}U_x\Big\{t^{a'}f^{a'b'c'}\Big[\frac{\frac{X_{ij}}{(z-z')^2} + \frac{X_iX'_j}{uX^2} - \frac{X_iX'_j}{\bar u X'^2}}{uX^2 + \bar u X'^2} - \frac{X_iY'_j}{\bar u u X^2 Y'^2} - (x\leftrightarrow y)\Big]$$
$$- i\frac{\{t^{b'}, t^{c'}\}}{\bar u u}\Big(\frac{X_i}{X^2} - \frac{Y_i}{Y^2}\Big)\Big(\frac{X'_j}{X'^2} - \frac{Y'_j}{Y'^2}\Big)\Big\}U^\dagger_y \quad (43)$$

where we introduced the notations

$$X_{ij} \equiv (X^2 - X'^2)\delta_{ij} + \frac{2}{u}(z - z')_i X'_j + \frac{2}{\bar u}X_i(z - z')_j, \qquad Y_{ij} \equiv (Y^2 - Y'^2)\delta_{ij} + \frac{2}{u}(z - z')_i Y'_j + \frac{2}{\bar u}Y_i(z - z')_j. \quad (44)$$

C. Subtraction of the $(\mathrm{LO})^2$ contribution

It is easy to see that result (43) for the sum of the diagrams in Fig. 6 diverges as $u \to 0$ and $u \to 1$. If we put a lower cutoff $\alpha > \sigma'$ on the α integrals, we would get a contribution $\sim\ln^2\frac{\sigma}{\sigma'}$ coming from the region $\alpha_2 \gg \alpha_1 > \sigma'$ (or $\alpha_1 \gg \alpha_2 > \sigma'$) which corresponds to the square of the leading order BK kernel rather than to the NLO kernel. To get the NLO kernel we need to subtract this $(\mathrm{LO})^2$ contribution. Indeed, the operator form of the evolution equation for the color dipole up to the next-to-leading order looks like

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\} = K_{\mathrm{LO}}\,\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\} + K_{\mathrm{NLO}}\,\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\} \quad (45)$$

where $\eta = \ln\sigma$. Our goal is to find K_{NLO} by considering the l.h.s. of this equation in the external shockwave background so

$$\langle K_{\rm NLO}\,{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\rm shockwave} = \frac{d}{d\eta}\langle {\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\rm shockwave} - \langle K_{\rm LO}\,{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\rm shockwave}. \tag{46}$$

The subtraction (46) leads to the $[\frac{1}{u}]_+$ prescription for the terms divergent as $\frac{1}{u}$ (and similarly $\frac{1}{\bar u}\to[\frac{1}{\bar u}]_+$ for the contribution divergent as $u\to 1$). Here we define $[\frac{1}{u}]_+$ in the usual way,

$$\int_0^1 du f(u)\Big[\frac{1}{u}\Big]_+ \equiv \int_0^1 du\frac{f(u)-f(0)}{u}, \qquad \int_0^1 du f(u)\Big[\frac{1}{\bar u}\Big]_+ \equiv \int_0^1 du\frac{f(u)-f(1)}{\bar u}. \tag{47}$$

To illustrate this prescription, consider the divergent terms in Eq. (43) proportional to $(X,Y)(Y',z-z')$ or $(X',Y')(Y,z-z')$,

$$\begin{aligned}
\frac{d}{d\eta}\langle {\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle &= \frac{\alpha_s^2}{4\pi^4}\int_0^1\frac{du}{\bar u}\int d^2zd^2z'U_z^{bb'}U_{z'}^{cc'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\
&\times\bigg[\frac{1}{uX^2+\bar uX'^2}\bigg[f^{abc}\,{\rm Tr}\Big\{t^aU_x\Big(\frac{uf^{a'b'c'}t^{a'}}{uY^2+\bar uY'^2}-\frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\Big\}\\
&+f^{a'b'c'}\,{\rm Tr}\Big\{\Big(t^a\frac{uf^{abc}}{uY^2+\bar uY'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\Big\}\bigg]\\
&+\frac{1}{X^2(uY^2+\bar uY'^2)}\,{\rm Tr}\{(f^{abc}t^a+i\{t^b,t^c\})U_xf^{a'b'c'}t^{a'}U^\dagger_y+t^af^{abc}U_x(f^{a'b'c'}t^{a'}-i\{t^{b'},t^{c'}\})U^\dagger_y\}\bigg]\\
&+\frac{\alpha_s^2}{4\pi^4}\int_0^1\frac{du}{u}\int d^2zd^2z'U_z^{bb'}U_{z'}^{cc'}\frac{(X',Y')(Y,z-z')}{(z-z')^2Y^2}\\
&\times\bigg[\frac{1}{uX^2+\bar uX'^2}\bigg[f^{abc}\,{\rm Tr}\Big\{-t^aU_x\Big(\frac{\bar uf^{a'b'c'}t^{a'}}{(uY^2+\bar uY'^2)}+\frac{i}{Y'^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\Big\}\\
&+f^{a'b'c'}\,{\rm Tr}\Big\{\Big(-t^a\frac{\bar uY_iY'_jf^{abc}}{uY^2+\bar uY'^2}+\frac{i}{Y'^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\Big\}\bigg]\\
&+\frac{1}{X'^2(uY^2+\bar uY'^2)}\,{\rm Tr}\{(-f^{abc}t^a+i\{t^b,t^c\})U_xf^{a'b'c'}t^{a'}U^\dagger_y-t^af^{abc}U_x(f^{a'b'c'}t^{a'}+i\{t^{b'},t^{c'}\})U^\dagger_y\}\bigg].
\end{aligned}\tag{48}$$

Note that the second term is equal to the first one after the replacement $u\leftrightarrow\bar u$, $z\leftrightarrow z'$ and $b\leftrightarrow c$, $b'\leftrightarrow c'$.

It is convenient to return back to the notation α_1 and $\alpha_2=\sigma-\alpha_1$ (after $\frac{d}{d\ln\sigma}$ the value of α is set equal to σ).

$$\begin{aligned}
\frac{d}{d\eta}\langle {\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle &= \frac{\alpha_s^2}{2\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2zd^2z'U_z^{bb'}U_{z'}^{cc'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\
&\times\bigg[\frac{\sigma}{\alpha_1X^2+\alpha_2X'^2}\bigg[f^{abc}\,{\rm Tr}\Big\{t^aU_x\Big(\frac{\alpha_1f^{a'b'c'}t^{a'}}{\alpha_1Y^2+\alpha_2Y'^2}-\frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\Big\}\\
&+f^{a'b'c'}\,{\rm Tr}\Big\{\Big(\frac{\alpha_1f^{abc}t^a}{\alpha_1Y^2+\alpha_2Y'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\Big\}\bigg]\\
&+\frac{2i\sigma}{X^2(\alpha_1Y^2+\alpha_2Y'^2)}\,{\rm Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U^\dagger_y-t^af^{abc}U_xt^{b'}t^{c'}U^\dagger_y\}\bigg].
\end{aligned}\tag{49}$$

The corresponding term in $K_{\rm LO}\,{\rm Tr}\{\hat U_x\hat U^\dagger_y\}$ is [see Eq. (4)]

$$K_{\rm LO}\,{\rm Tr}\{\hat U_x\hat U^\dagger_y\} = -\frac{4\alpha_s}{\pi^2}\int d^2z\frac{(x-z,y-z)}{(x-z)^2(y-z)^2}\times{\rm Tr}\{t^a\hat U_xt^b\hat U^\dagger_y\}\,{\rm Tr}\{t^a\hat U_zt^b\hat U^\dagger_z\}. \tag{50}$$

The relevant term in the "matrix element" $\langle K_{\rm LO}\,{\rm Tr}\{\hat U_x\hat U^\dagger_y\}\rangle$ in the external shockwave background comes from $\hat U_x$, $\hat U^\dagger_z$ taken in the leading order in α_s (so that $\hat U_x\to U_x$, $\hat U^\dagger_z\to U^\dagger_z$) and $\hat U_z\otimes\hat U^\dagger_y$ taken in the first order in α_s,

$$\langle\hat U_z\otimes\hat U^\dagger_y\rangle\sim-\frac{\alpha_s}{\pi^2}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2z'\frac{(z-z',y-z')}{(z-z')^2(y-z')^2}\times(t^cU_z\otimes t^{c'}U^\dagger_y+U_zt^{c'}\otimes U^\dagger_yt^c)U_{z'}^{cc'}, \tag{51}$$

or vice versa: $\hat U_x\to U_x$, $\hat U_z\to U_z$ and

$$\langle\hat{U}^\dagger_z \otimes \hat{U}^\dagger_y\rangle \sim \frac{\alpha_s}{\pi^2}\int_0^\sigma \frac{d\alpha_2}{\alpha_2}\int d^2z' \frac{(z-z', y-z')}{(z-z')^2(y-z')^2}(t^{c'}U^\dagger_z \otimes U^\dagger_y t^c + U^\dagger_z t^c \otimes t^{c'}U^\dagger_y)U^{cc'}_{z'}. \tag{52}$$

Here we have used the leading order equations for Wilson lines with arbitrary color indices [4,15]. Substituting Eqs. (51) and (52) in Eq. (50) we obtain

$$\begin{aligned}\langle K_{\rm LO}\,{\rm Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle = \frac{2\alpha_s^2}{\pi^4}\int_0^\sigma \frac{d\alpha_2}{\alpha_2}\int d^2z d^2z' \frac{(X,Y)(Y',z-z')}{X^2Y^2Y'^2(z-z')^2}\\ \times {\rm Tr}\{(if^{a'b'c'}t^ct^bU_xt^{b'}U^\dagger_y - if^{abc}t^aU_xt^{b'}t^{c'}U^\dagger_y)U^{bb'}_zU^{cc'}_{z'}\}.\end{aligned} \tag{53}$$

From Eq. (46) we get

$$\begin{aligned}\langle K_{\rm NLO}\,{\rm Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle &= \frac{\alpha_s^2}{2\pi^4}\int_0^\sigma \frac{d\alpha_2}{\alpha_2}\int d^2zd^2z'U^{bb'}_zU^{cc'}_{z'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\ &\times\Big[\frac{\sigma}{\alpha_1X^2+\alpha_2X'^2}\Big[f^{abc}{\rm Tr}\Big\{t^aU_x\Big(\frac{\alpha_1f^{a'b'c'}t^{a'}}{\alpha_1Y^2+\alpha_2Y'^2} - \frac{i}{Y^2}\{t^{b'},t^{c'}\}\Big)U^\dagger_y\Big\}\\ &+f^{a'b'c'}{\rm Tr}\Big\{\Big(\frac{\alpha_1f^{abc}t^a}{\alpha_1Y^2+\alpha_2Y'^2}+\frac{i}{Y^2}\{t^b,t^c\}\Big)U_xt^{a'}U^\dagger_y\Big\}\Big]\\ &+\frac{2i\sigma}{X^2(\alpha_1Y^2+\alpha_2Y'^2)}{\rm Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U^\dagger_y - t^af^{abc}U_xt^{b'}t^{c'}U^\dagger_y\}\Big]\\ &-\frac{2\alpha_s^2}{\pi^4}\int_0^\sigma\frac{d\alpha_2}{\alpha_2}\int d^2z\frac{(X,Y)(Y',z-z')}{X^2Y^2Y'^2(z-z')^2}{\rm Tr}\{if^{a'b'c'}t^ct^bU_xt^{b'}U^\dagger_y - if^{abc}t^aU_xt^{b'}t^{c'}U^\dagger_y\}U^{bb'}_zU^{cc'}_{z'}\\ &=\frac{\alpha_s^2}{2\pi^4}\int_0^1\frac{du}{\bar{u}}\int d^2zd^2z'U^{bb'}_zU^{cc'}_{z'}\frac{(X,Y)(Y',z-z')}{(z-z')^2Y'^2}\\ &\times\Big[f^{abc}f^{a'b'c'}{\rm Tr}\{t^aU_xt^{a'}U^\dagger_y\}\Big[\frac{2u}{(uX^2+\bar{u}X'^2)(uY^2+\bar{u}Y'^2)} - \frac{2}{X^2Y^2}\Big]\\ &-\frac{i}{Y^2}\Big[\frac{1}{uX^2+\bar{u}X'^2}-\frac{1}{X^2}\Big]f^{abc}{\rm Tr}\{t^aU_x\{t^{b'},t^{c'}\}U^\dagger_y\}+\frac{i}{Y^2}\Big[\frac{1}{uX^2+\bar{u}X'^2}-\frac{1}{X^2}\Big]f^{a'b'c'}\\ &\times{\rm Tr}\{\{t^b,t^c\}U_xt^{a'}U^\dagger_y\}+\frac{2i}{X^2}\Big[\frac{1}{uY^2+\bar{u}Y'^2}-\frac{1}{Y^2}\Big]{\rm Tr}\{t^ct^bU_xf^{a'b'c'}t^{a'}U^\dagger_y - t^af^{abc}U_xt^{b'}t^{c'}U^\dagger_y\}\Big]\end{aligned} \tag{54}$$

which corresponds to the $[\frac{1}{\bar{u}}]_+$ prescription (47) (the same prescription was used in Ref. [11]). Note that the "plus" prescription (47) is a consequence of the "rigid" cutoff $|\alpha| < \sigma$ (17); with the "smooth" cutoff (6) we would get different results—see Appendix B.

D. Assembling the result for 1 → 3 dipoles transition

There are four color structures in the r.h.s. of Eq. (43). Three of them reduce to

$$\begin{aligned}f^{abc}f^{a'b'c'}U^{bb'}_zU^{cc'}_{z'}{\rm Tr}\{t^aU_xt^{a'}U^\dagger_y\} &= \tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_z\}{\rm Tr}\{U_zU^\dagger_{z'}\}{\rm Tr}\{U_{z'}U^\dagger_y\} - \tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\} + (z\leftrightarrow z'),\\ if^{abc}U^{bb'}_zU^{cc'}_{z'}{\rm Tr}(t^aU_x\{t^{b'},t^{c'}\}U^\dagger_y) &= -\tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_z\}{\rm Tr}\{U_zU^\dagger_{z'}\}{\rm Tr}\{U_{z'}U^\dagger_y\} + \tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\} - (z\leftrightarrow z'),\\ if^{a'b'c'}U^{bb'}_zU^{cc'}_{z'}{\rm Tr}(\{t^b,t^c\}U_xt^{a'}U^\dagger_y) &= \tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_z\}{\rm Tr}\{U_zU^\dagger_{z'}\}{\rm Tr}\{U_{z'}U^\dagger_y\} + \tfrac{1}{4}{\rm Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\} - (z\leftrightarrow z').\end{aligned} \tag{55}$$

We will not need the explicit form of the fourth color structure $U^{aa'}_zU^{bb'}_{z'}{\rm Tr}(\{t^a,t^b\}U_x\{t^{a'},t^{b'}\}U^\dagger_y)$ since it is multiplied by the pure LO^2 integral $\int\frac{du}{\bar{u}u} = \int\frac{du}{\bar{u}} + \int\frac{du}{u}$ and does not contribute to the NLO kernel.

Performing integration over u and using the prescription (47), after some algebra we get

$$\int_0^1 du u\bar{u}\bigg[\frac{1}{uX^2+\bar{u}X'^2}\bigg(\frac{X_{ij}}{(z-z')^2}+\frac{X_iX'_j}{uX^2}-\frac{X_iX'_j}{\bar{u}X'^2}\bigg)-\frac{X_iY'_j}{\bar{u}uX^2Y'^2}-(x\leftrightarrow y)\bigg]^2$$
$$=\frac{1}{(z-z')^4}\bigg[-4+2\frac{X^2Y'^2+X'^2Y^2-4\Delta^2(z-z')^2}{X^2Y'^2-X'^2Y^2}\ln\frac{X^2Y'^2}{X'^2Y^2}\bigg]+\bigg(\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg]$$
$$+\frac{(x-y)^2}{(z-z')^2}\bigg[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\bigg]\bigg)\ln\frac{X^2Y'^2}{X'^2Y^2} \tag{56}$$

and

$$\int_0^1 du\bigg[\frac{1}{uX^2+\bar{u}X'^2}\bigg(\frac{X_{ij}}{(z-z')^2}+\frac{X_iX'_j}{uX^2}-\frac{X_iX'_j}{\bar{u}X'^2}\bigg)-\frac{X_iY'_j}{\bar{u}uX^2Y'^2}-(x\leftrightarrow y)\bigg]\bigg(\frac{X_i}{X^2}-\frac{Y_i}{Y^2}\bigg)\bigg(\frac{X'_j}{X'^2}-\frac{Y'_j}{Y'^2}\bigg)$$
$$=-\frac{(x-y)^4}{2X^2Y^2X'^2Y'^2}\ln\frac{X^2Y'^2}{X'^2Y^2}+\frac{(x-y)^2}{2(z-z')^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{X'^2Y^2}\bigg]\ln\frac{X^2Y'^2}{X'^2Y^2} \tag{57}$$

so the two-cut contribution (43) reduces to

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 6}}=\frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\bigg[\bigg\{-\frac{4}{(z-z')^4}+\bigg(2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$
$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg]+\frac{(x-y)^2}{(z-z')^2}\bigg[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\bigg]\bigg)\ln\frac{X^2Y'^2}{X'^2Y^2}\bigg\}$$
$$\times[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}]$$
$$+\bigg\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\bigg(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg)\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}\bigg]. \tag{58}$$

This result agrees with the $1\to 3$ dipoles kernel calculated in Ref. [11].

E. Subtraction of the UV part

The integral in the r.h.s. of Eq. (58) diverges as $z\to z'$. It is convenient to separate the divergent term by subtracting and adding the contribution at $z=z'$:

$$\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}]$$
$$=[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}-(z'\to z)]+[N_c\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_y\}]. \tag{59}$$

For the last line in the r.h.s. of Eq. (58) the subtraction is redundant since

$$\int d^2z'\bigg\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\bigg(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg)\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}=0. \tag{60}$$

The easiest way to prove this is to set $y=0$ and make an inversion $x\to 1/\tilde{x}$, so the integral (60) reduces to

$$\int d^2\tilde{z}'\frac{(\tilde{x}-\tilde{z},\tilde{x}-\tilde{z}')}{(\tilde{x}-\tilde{z}')^2(\tilde{z}-\tilde{z}')^2}\ln\frac{(\tilde{x}-\tilde{z})^2}{(\tilde{x}-\tilde{z}')^2}=0. \tag{61}$$

Thus, we obtain

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\text{Fig. 6}}=\frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\bigg[\bigg(-\frac{4}{(z-z')^4}+\bigg\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$
$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg]+\frac{(x-y)^2}{(z-z')^2}\bigg[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\bigg]\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\bigg)$$
$$\times[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}-(z'\to z)]$$
$$+\bigg\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\bigg(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg)\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}]$$
$$+\frac{\alpha_s^2}{16\pi^4}\int d^2z[N_c\,\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_y\}]$$
$$\times\int d^2z'\bigg[-\frac{4}{(z-z')^4}+\bigg\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg]$$
$$+\frac{(x-y)^2}{(z-z')^2}\bigg[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\bigg]\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\bigg].\tag{62}$$

The first term is now finite while the second term contains the UV-divergent contribution which reflects the usual UV divergency of the one-loop diagrams. To find the second term we use the dimensional regularization in the transverse space and set $d_\perp=2-\epsilon$. Because the Fourier transforms (42) are more complicated at $d_\perp\neq 2$, it is convenient to return back to Eq. (40) and calculate the subtracted term in the momentum representation. The calculation is performed in Appendix A, and here we only quote the final result (B3),

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\text{Fig. 6}}=\frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\bigg[\bigg(-\frac{4}{(z-z')^4}+\bigg\{2\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2-X'^2Y^2]}$$
$$+\frac{(x-y)^4}{X^2Y'^2-X'^2Y^2}\bigg[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg]+\frac{(x-y)^2}{(z-z')^2}\bigg[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\bigg]\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\bigg)$$
$$\times[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}-(z'\to z)]$$
$$+\bigg\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2}+\frac{(x-y)^2}{(z-z')^2}\bigg(\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\bigg)\bigg\}\ln\frac{X^2Y'^2}{X'^2Y^2}\,\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_{z'}\}\mathrm{Tr}\{U_{z'}U^\dagger_y\}]$$
$$-\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z\frac{(x-y)^2}{X^2Y^2}\bigg[\frac{11}{3}\ln\frac{X^2Y^2}{(x-y)^2}\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\bigg]\bigg[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\bigg]\tag{63}$$

where μ is the normalization scale in the $\overline{MS}$ scheme. It is worth noting that the $d_\perp=2-\epsilon$ regularization in the transverse space is independent of the (rigid or smooth) cutoff in the longitudinal direction.

IV. DIAGRAMS WITH ONE GLUON-SHOCKWAVE INTERSECTION

A. "Running-coupling" diagrams

The relevant diagrams are shown in Fig. 7 (plus permutations). Let us start from the sum of the diagrams in Figs. 7(a) and 7(b). It has the form

$$\int_0^\infty du\int_{-\infty}^0 dv\langle\hat A^a_\bullet(up_1+x_\perp)\hat A^b_\bullet(vp_1+y_\perp)\rangle_{\text{Fig. 7(a)+(b)}}$$
$$=g^2N_c\frac{s}{2}\int d\!\!{}^-{}^2k_\perp d\!\!{}^-{}^2k'_\perp\frac{d\!\!{}^-{}^2q_\perp}{q_\perp^2}\int d^2zU_z^{ab}e^{i(q,x-z)_\perp-i(k,y-z)_\perp}\int_0^\infty\frac{d\!\!{}^-\alpha}{\alpha}\int d\!\!{}^-\alpha'\int\frac{d\!\!{}^-\beta d\!\!{}^-\beta'}{\beta-i\epsilon}\bigg[\frac{(q+\frac{2(k,q)_\perp}{\alpha s}p_2)_\lambda}{(k^2+i\epsilon)^2}\frac{d_{\mu\mu'}(k-k')}{(k-k')^2+i\epsilon}$$
$$\times\Gamma^{\mu\nu\lambda}((\alpha-\alpha')p_1+(k-k')_\perp,\alpha'p_1+k'_\perp,-\alpha p_1-k_\perp)\frac{d_{\nu\nu'}(k')}{k'^2+i\epsilon}(k+2\beta p_2)_{\lambda'}\Gamma^{\mu'\nu'\lambda'}$$
$$\times((\alpha-\alpha')p_1+(k-k')_\perp,\alpha'p_1+k'_\perp,-\alpha p_1-k_\perp)-2\frac{(k+2\beta p_2)_\nu(q_\mu+\frac{2(k,q)_\perp}{\alpha s}p_{2\mu})}{(k^2+i\epsilon)^2}\frac{g^{\mu\nu}d^\xi_\xi(k')-d^{\mu\nu}(k')}{k'^2+i\epsilon}\bigg]\tag{64}$$

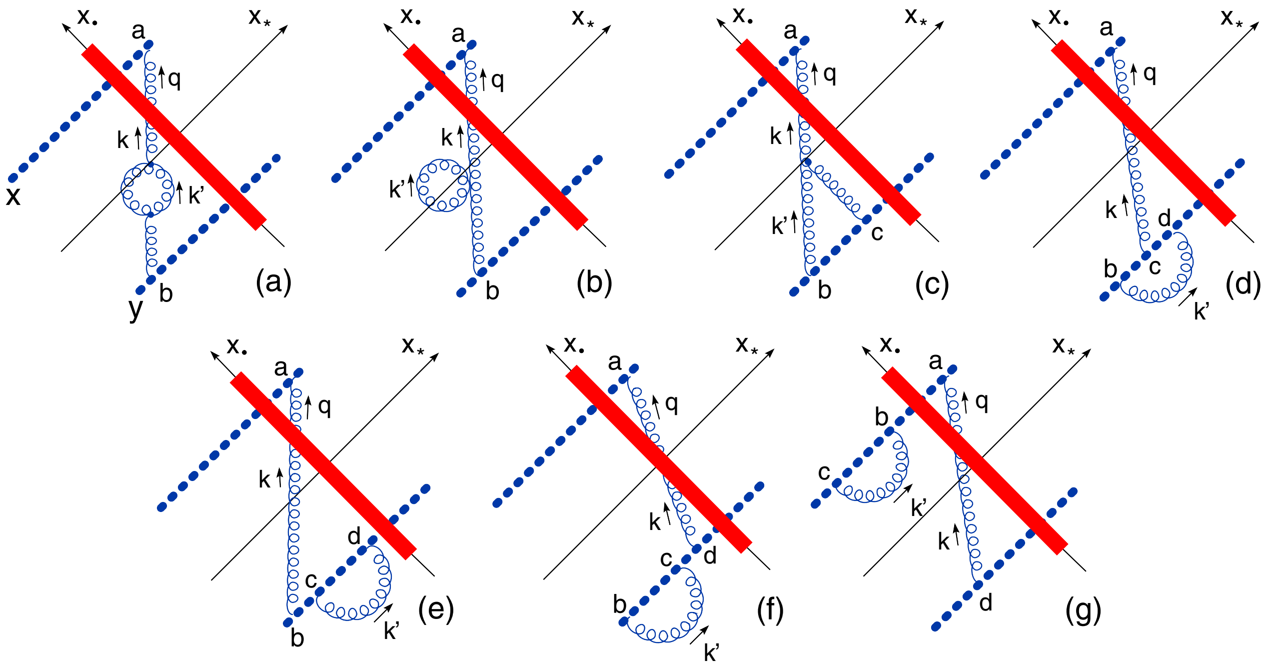

FIG. 7 (color online). Running-coupling diagrams.

where the first term in the square brackets comes from Fig. 7(a) and the second from Fig. 7(b). We use the principal-value prescription for the $1/\alpha'$ terms in $d_{\mu\nu}(k')$ in loop integrals.

To regularize the UV divergence we change the dimension of the transverse space to $2-\varepsilon$. After some algebra one obtains

$$\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(up_1+x_\perp)\hat{A}^b_\bullet(vp_1+y_\perp)\rangle_{\text{Fig. 7(a)+(b)}}$$
$$= g^2N_c\mu^{2\varepsilon}\int {d\!\!{}^-}^{2-\varepsilon}k {d\!\!{}^-}^{2-\varepsilon}k' {d\!\!{}^-}^{2-\varepsilon}q\int {d\!\!{}^-}^{2-\varepsilon}z U^{ab}_z\int_0^\infty {d\!\!{}^-}\alpha\int {d\!\!{}^-}\beta {d\!\!{}^-}\beta'\frac{e^{i(q,X)_\perp-i(k,Y)_\perp}}{\alpha(\beta-i\epsilon)(\alpha\beta s-k_\perp^2+i\epsilon)^2q^2}\frac{s}{2}$$
$$\times\int {d\!\!{}^-}\alpha' {d\!\!{}^-}\beta'\frac{1}{(\alpha'\beta' s-k'^2_\perp+i\epsilon)[(\alpha-\alpha')(\beta-\beta')s-(k-k')^2_\perp+i\epsilon]}\Big\{-\varepsilon[(\alpha-2\alpha')\beta s+k'^2_\perp-(k-k')^2_\perp]$$
$$\times\left[\frac{\alpha-2\alpha'}{\alpha}(k,q)_\perp+(2k'-k,q)_\perp\right]+2\frac{(\alpha-2\alpha')^2}{\alpha}(k,q)_{\perp\beta s}$$
$$+\frac{(\alpha-2\alpha')}{\alpha}(k,q)_\perp(2k'-k,k)_\perp+(\alpha-2\alpha')\beta s(2k'-k,q)_\perp+(q,k)_\perp(k-2k',k-k')_\perp+(q,k')_\perp(k,2k'-k)_\perp$$
$$+\frac{\alpha+\alpha'}{\alpha-\alpha'}\Big[\frac{\alpha-2\alpha'}{\alpha}(q,k)_\perp(k,k-k')_\perp+(q,2k'-k)_\perp(k,k-k')_\perp+(q,k)_\perp(k'^2_\perp-k^2_\perp)+(q,k-k')_\perp(k,2k-k')_\perp$$
$$+(\alpha-2\alpha')(q,k-k')_\perp\beta s+(q,k-k')_\perp(k'^2_\perp-(k-k')^2_\perp)-(q,k')_\perp(k',k-k')_\perp+(q,k-k')_\perp(k-k')^2_\perp\Big]$$
$$+\frac{\alpha'-2\alpha}{\alpha'}\Big[\frac{\alpha-2\alpha'}{\alpha}(q,k)_\perp(k,k')_\perp+(q,2k'-k)_\perp(k,k')_\perp-(q,k')_\perp(k,k+k')_\perp+(q,k)_\perp(k',2k-k')_\perp$$
$$+(\alpha-2\alpha')(q,k')_\perp\beta s+(q,k')_\perp(k,2k'-k)_\perp-(q,k')_\perp k'^2_\perp+(q,k-k')_\perp(k',k-k')_\perp\Big]$$
$$+\frac{\alpha+\alpha'}{\alpha-\alpha'}\frac{\alpha'-2\alpha}{\alpha'}[(k,k')_\perp(q,k-k')_\perp+(q,k')_\perp(k,k-k')_\perp]+[(k-k')^2_\perp+k'^2](q,k)_\perp$$
$$+4\alpha(q,k)_\perp\left[\frac{\alpha-\alpha'}{\alpha'}+\frac{\alpha}{\alpha-\alpha'}\right]\beta s-4\alpha(q,k)_\perp\left[\frac{k'^2_\perp}{\alpha-\alpha'}+\frac{(k-k')^2_\perp}{\alpha'}\right]\Big\} \tag{65}$$

where we have omitted the contribution

$$\int d\!\!\!^-\alpha' d\!\!\!^-\beta' d\!\!\!^-{}^2k' \frac{1}{\alpha'(\alpha'\beta' s - k_\perp'^2 + i\epsilon)} = \int d^4k' \frac{1}{k'^2 + i\epsilon} V.p. \frac{1}{(k', p_2)} = 0. \tag{66}$$

Taking residues at $\beta = 0$ and $\beta' = \frac{k_\perp'^2}{\alpha' s}$ and changing to the variable $u = \frac{\alpha'}{\alpha}$, we obtain

$$\begin{aligned}
&\int_0^\infty du \int_{-\infty}^0 dv \langle \hat{A}^a_\bullet(up_1 + x_\perp)\hat{A}^b_\bullet(vp_1 + y_\perp)\rangle_{\text{Fig. 7(a)+(b)}} \\
&= -\frac{g^2 N_c}{8\pi^2}\mu^{2\varepsilon}\int d\!\!\!^-{}^{2-\varepsilon}k d\!\!\!^-{}^{2-\varepsilon}k' d\!\!\!^-{}^{2-\varepsilon}q \int d^{2-\varepsilon}z U_z^{ab}\int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1 du \frac{e^{i(q,x-z)_\perp - i(k,y-z)_\perp}}{k^4 q^2[k'^2\bar{u} + (k-k')^2 u]} \\
&\quad\times\Big\{-2\varepsilon(2k'-k,k)(q,k'-ku) + (1-2u)(k,q)(2k'-k,k) + (q,k)(k-2k',k-k') + (q,k')(k,2k'-k) \\
&\quad + \frac{1+u}{\bar{u}}[(1-2u)(q,k)(k,k-k') + 2(q,k)k'^2 - 2(k,k')_\perp(q,k')] \\
&\quad - \frac{2-u}{u}[(1-2u)(q,k)(k,k') + 2(q,k)(k-k',k') - 2(q,k')(k-k',k)] \\
&\quad - \frac{(1+u)(2-u)}{u\bar{u}}[(k,k')(q,k-k') + (q,k')(k,k-k')] + (q,k)\Big[(k-k')^2 + k'^2 - 4\frac{k'^2}{\bar{u}} - 4\frac{(k-k')^2}{u}\Big]\Big\}.
\end{aligned} \tag{67}$$

Using the plus prescription (47) to subtract the $(\text{LO})^2$ contribution, we get

$$\begin{aligned}
\frac{d}{d\eta}\langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\}\rangle_{\text{Fig. 7(a)+(b)}} &= -2\alpha_s^2 N_c \mu^{2\varepsilon}\int d^{2-\varepsilon}z\Big[\text{Tr}\{U_x U_z^\dagger\}\text{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c}\text{Tr}\{U_x U_y^\dagger\}\Big] \\
&\quad\times\int d\!\!\!^-{}^{2-\varepsilon}k d\!\!\!^-{}^{2-\varepsilon}k' d\!\!\!^-{}^{2-\varepsilon}q \frac{e^{i(q,X)-i(k,Y)}}{k^2 q^2}\Big[\Big(\frac{(q,2k-k')}{k'^2} - \frac{(q,k+k')}{(k-k')^2}\Big)\ln\frac{(k-k')^2}{k'^2} \\
&\quad + \int_0^1 du \frac{(2-\varepsilon)(q,ku-k')(k,k-2k') + 2(q,k)k^2}{k^2[k'^2\bar{u} + (k-k')^2 u]}\Big].
\end{aligned} \tag{68}$$

Next we calculate the diagram shown in Fig. 7(c).

$$\begin{aligned}
&g^3\int_0^\infty dt\int_{-\infty}^0 du\int_u^0 dv\langle \hat{A}^a_\bullet(tp_1 + x)\hat{A}^b_\bullet(up_1 + y)\hat{A}^c_\bullet(vp_1 + y)\rangle_{\text{Fig. 7(c)}} \\
&= -2ig^4 f^{lbc}\int d\!\!\!^-\alpha d\!\!\!^-\beta d\!\!\!^-\alpha' d\!\!\!^-\beta' d\!\!\!^-\beta'' d\!\!\!^-{}^{2-\varepsilon}q d\!\!\!^-{}^{2-\varepsilon}k d\!\!\!^-{}^{2-\varepsilon}k' e^{(iq,x-z)_\perp - i(k,y-z)_\perp}\frac{\theta(\alpha)U_z^{al}}{q_\perp^2(\alpha\beta s - k^2 + i\epsilon)}\frac{(k'_\perp + 2\beta' p_2)_\mu}{\alpha'\beta' s - k'^2 + i\epsilon} \\
&\quad\times\frac{(q_\perp + \frac{2}{\alpha s}(q,k)_\perp p_2)_\lambda}{\alpha'(\alpha-\alpha')}\frac{((k-k')_\perp + 2\beta'' p_2)_\nu}{(\alpha-\alpha')\beta'' s - (k-k')_\perp^2 + i\epsilon}\frac{\Gamma^{\mu\nu\lambda}(\alpha' p_1 + k'_\perp, (\alpha-\alpha')p_1 + (k-k')_\perp, -\alpha p_1 - k_\perp)}{(\beta' - i\epsilon)(\beta'' + \beta' - i\epsilon)(\beta - \beta' - \beta'' - i\epsilon)}.
\end{aligned} \tag{69}$$

There are two regions of integration over the α's: $\alpha > |\alpha'|$ and $\alpha < |\alpha'|$. Taking relevant residues, we obtain

$$\begin{aligned}
&-\frac{g^4}{2\pi^2}f^{abl}\mu^{2\varepsilon}\int_0^\infty\frac{d\alpha}{\alpha}\int d\!\!\!^-{}^{2-\varepsilon}k d\!\!\!^-{}^{2-\varepsilon}k' d\!\!\!^-{}^{2-\varepsilon}q\int d^2z_\perp U_z^{cl}\frac{e^{i(q,X)-i(k,Y)}}{k'^2 q^2 k^2}\int_0^1 du\Big\{\frac{1}{k'^2\bar{u} + (k-k')^2 u}\Big[\frac{(q,k')}{\bar{u}}[k'^2 + k^2] \\
&\quad + \frac{k'^2}{u}(q,2k-k') - 2(q,k)(k',k-k')\Big] + \frac{1}{(k-k')^2\bar{u} + k^2 u}\Big[\frac{1}{u}[(q,k)k'^2 + (q,k')k^2] - (k',2k-k')(q,k)\Big]\Big\}
\end{aligned} \tag{70}$$

where we have introduced the variable $u = |\alpha'|/\alpha$ as usual. After integration over u with the help of Eq. (47), this reduces to

$$\begin{aligned}
&-\frac{g^4}{2\pi^2}f^{abl}\mu^{2\varepsilon}\int_0^\infty\frac{d\alpha}{\alpha}\int d\!\!\!^-{}^{2-\varepsilon}k d\!\!\!^-{}^{2-\varepsilon}k' d\!\!\!^-{}^{2-\varepsilon}q\int d^{2-\varepsilon}z_\perp U_z^{cl}\Big\{\Big[\frac{(q,k')}{(k-k')^2}[k'^2 + k^2] + (q,k'-2k)\Big]\ln\frac{(k-k')^2}{k'^2} \\
&\quad + \Big[\frac{1}{(k-k')^2}[(q,k)k'^2 + (q,k')k^2] + (q,k)\Big]\ln\frac{(k-k')^2}{k^2} - \int_0^1 du\frac{2(q,k)(k',k-k')}{k'^2\bar{u} + (k-k')^2 u}\Big\}\frac{e^{i(q,X)-i(k,Y)}}{k'^2 q^2 k^2}
\end{aligned} \tag{71}$$

and therefore

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.\,7(c)}} = -\frac{g^4N_c}{8\pi^2}\mu^{2\varepsilon}\int d^{2-\varepsilon}z\Big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big]\int d^{2-\varepsilon}kd^{2-\varepsilon}k'd^{2-\varepsilon}q\frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}$$
$$\times\Big\{\Big[\frac{(q,k')}{(k-k')^2}+\frac{(q,k')k^2}{k'^2(k-k')^2}+\frac{(q,k'-2k)}{k'^2}\Big]\ln\frac{(k-k')^2}{k'^2}-\Big[\frac{(q,k)}{k'^2}+\frac{(q,k)}{(k-k')^2}+\frac{(q,k-k')k^2}{k'^2(k-k')^2}\Big]$$
$$\times\ln\frac{k^2}{k'^2}-2\int_0^1du\frac{(q,k)(k',k-k')}{k'^2[k'^2\bar u+(k-k')^2u]}\Big\}. \tag{72}$$

Next we calculate the sum of the diagrams in Figs. 7(d)–7(f). The contribution of the diagram shown in Fig. 7(d) is

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.\,7(d)}} = \int d\alpha d\alpha' d\beta d\beta'\int d^{2-\varepsilon}kd^{2-\varepsilon}k'd^{2-\varepsilon}q$$
$$\times\int d^{2-\varepsilon}zU_z^{ab}\frac{4g^4\mu^{2\varepsilon}\theta(\alpha)(q,k)_\perp e^{i(q,x-z)_\perp-i(k,y-z)_\perp}\mathrm{Tr}\{t^aU_xt^ct^bt^cU_y^\dagger\}}{\alpha\alpha'(\beta+\beta'-i\epsilon)(\beta-i\epsilon)(\alpha'\beta's-k_\perp'^2+i\epsilon)(\alpha\beta s-k_\perp^2+i\epsilon)q_\perp^2}$$
$$=-\frac{g^4}{\pi^2}\mu^{2\varepsilon}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\int d^{2-\varepsilon}qd^{2-\varepsilon}kd^{2-\varepsilon}k'\int d^{2-\varepsilon}zU_z^{ab}\frac{(q,k)_\perp e^{i(q,x-z)_\perp-i(k,y-z)_\perp}}{uk^2(uk^2+\bar uk'^2)q^2}$$
$$\times\mathrm{Tr}\{t^aU_xt^ct^bt^cU_y^\dagger\} \tag{73}$$

where we took residues at $\beta=\frac{k_\perp^2}{\alpha s}$, $\beta'=\frac{k_\perp'^2}{\alpha' s}$ and introduced the variable $u=\frac{\alpha}{\alpha+\alpha'}$. It should be noted that the cutoff $\alpha<\sigma$ in the r.h.s. of this equation translates into $\int_0^\infty d\alpha d\alpha'\theta(\sigma-\alpha-\alpha')$, while our cutoff (17) corresponds to $\int_0^\infty d\alpha d\alpha'\theta(\sigma-\alpha)\theta(\sigma-\alpha')$. Fortunately, the difference

$$\int_0^\infty\frac{d\alpha d\alpha'}{\alpha'(\alpha k'^2+\alpha' k^2)}[\theta(\sigma-\alpha)\theta(\sigma-\alpha')-\theta(\sigma-\alpha-\alpha')]=\frac{1}{k'^2}\int_0^1\frac{dv}{v}\ln\frac{k'^2+k^2v}{k'^2\bar v+k^2v} \tag{74}$$

does not contain $\ln\sigma$ and hence does not contribute to the NLO kernel. Similarly, one can impose the cutoff $\alpha_1+\alpha_2<\sigma$ instead of the cutoff $\alpha_1,\alpha_2<\sigma$ in other diagrams whenever convenient.

Before calculating the diagrams in Figs. 7(e) and 7(f), it is convenient to make the replacement

$$\int_{-\infty}^0du\int_u^0dv\int_v^0dt\hat A^a(u)\hat A^b(v)\hat A^c(t)\to\frac12\int_{-\infty}^0du\int_u^0dvdt\hat A^a(u)\hat A^b(v)\hat A^c(t) \tag{75}$$

which can be performed since the color indices b and c in $\ldots t^bt^c\ldots$ are contracted. For the diagram in Fig. 7(e) we get

$$\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{Fig.7e}}=2g^4\mu^{2\varepsilon}c_F\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d\alpha d\alpha'd\beta d\beta'\int d^{2-\varepsilon}kd^{2-\varepsilon}k'd^{2-\varepsilon}q\int d^{2-\varepsilon}zU_z^{ab}\frac{e^{i(q,X)_\perp-i(k,Y)_\perp}}{\alpha'(\beta-i\epsilon)^2(k'^2+i\epsilon)}$$
$$\times\Big(\frac{\beta'}{\beta+\beta'-i\epsilon}+\frac{\beta'}{\beta-\beta'-i\epsilon}\Big)\frac{\theta(\alpha)(k,q)_\perp}{\alpha(k^2+i\epsilon)q_\perp^2}$$
$$=4g^4\mu^{2\varepsilon}c_F\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d\alpha d\alpha'd\beta d\beta'\int d^{2-\varepsilon}kd^{2-\varepsilon}k'd^{2-\varepsilon}q\int d^{2-\varepsilon}zU_z^{ab}e^{i(q,X)_\perp-i(k,Y)_\perp}$$
$$\times\frac{\beta'}{\alpha\alpha'(\beta-i\epsilon)(\alpha'\beta's-k_\perp'^2+i\epsilon)(\beta+\beta'-i\epsilon)(\beta-\beta'-i\epsilon)}\frac{\theta(\alpha)(k,q)_\perp}{(\alpha\beta s-k_\perp^2+i\epsilon)q_\perp^2} \tag{76}$$

where $c_F=\frac{N_c^2-1}{2N_c}$. Taking residues at $\beta'=-\beta$, $\beta=\frac{k_\perp^2}{\alpha s}$ at $\alpha'>0$ and $\beta'=\beta$, $\beta=\frac{k_\perp^2}{\alpha s}$ at $\alpha'<0$, we obtain

$$\langle\mathrm{Tr}\{U_xU_y^\dagger\}\rangle_{\mathrm{Fig.\,7(e)}}=\frac{g^4}{\pi^2}c_F\mu^{2\varepsilon}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\int d^{2-\varepsilon}kd^{2-\varepsilon}k'\frac{e^{i(q,X)_\perp-i(k,Y)_\perp}}{uk^2(uk^2+\bar uk'^2)}$$
$$\times\int d^{2-\varepsilon}q\int d^{2-\varepsilon}z\frac{(k,q)_\perp}{q_\perp^2}U_z^{db}\mathrm{Tr}\{t^dU_xt^bU_y^\dagger\}. \tag{77}$$

The diagram in Fig. 7(f) yields

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\,7(f)}} = g^4\mu^{2\varepsilon}\int d\!\!{}^-\alpha d\!\!{}^-\beta d\!\!{}^{-2-\varepsilon}k\int d\!\!{}^-\alpha' d\!\!{}^-\beta' d\!\!{}^{-2-\varepsilon}k'\frac{1}{\alpha'(\beta-i\epsilon)(\beta'-i\epsilon)(\alpha'\beta's-k'^2_\perp+i\epsilon)}2\theta(\alpha)$$
$$\times\int d\!\!{}^{-2-\varepsilon}q\int d^{2-\varepsilon}z e^{i(q,X)_\perp-i(k,Y)_\perp}\frac{(k,q)_\perp}{\alpha(\alpha\beta s-k^2_\perp+i\epsilon)q^2_\perp}U^{ab}_z\mathrm{Tr}\{t^aU_xt^ct^ct^bU^\dagger_y\}$$
$$=-\frac{g^4}{2\pi^2}\mu^{2\varepsilon}\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1\frac{du}{\bar{u}u}$$
$$\times\int d\!\!{}^{-2-\varepsilon}d\!\!{}^{-2-\varepsilon}k'\int d\!\!{}^{-2-\varepsilon}q\int d^{2-\varepsilon}z e^{i(q,x-z)-i(k,y-z)}\frac{(k,q)_\perp}{k^2k'^2q^2_\perp}U^{ab}_z\mathrm{Tr}\{t^aU_xt^ct^ct^bU^\dagger_y\}. \tag{78}$$

Adding Eqs. (73), (77), and (78) and integrating over u using Eq. (47), we get

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\,7(d)+(e)+(f)}} = -\frac{g^4N_c}{4\pi^2}\mu^{2\varepsilon}\int d\!\!{}^{-2-\varepsilon}q d\!\!{}^{-2-\varepsilon}k d\!\!{}^{-2-\varepsilon}k'\int d^{2-\varepsilon}z\frac{(q,k)}{k^2k'^2q^2}$$
$$\times\ln\frac{k^2}{k'^2}e^{i(q,X)-i(k,Y)}\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]. \tag{79}$$

Note that the diagram in Fig. 7(f) does not contribute to the NLO kernel.

The contribution of the last running-coupling diagram shown in Fig. 7(g) has the form

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\,7(g)}} = -\frac{g^4}{2}\mathrm{Tr}\{t^aU_xt^bt^ct^dU^\dagger_y\}\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(up_1+x_\perp)\hat{A}^d_\bullet(vp_1+y_\perp)\rangle$$
$$\times\int_{-\infty}^0 dtdw\langle\hat{A}^b(tp_1+x_\perp)\hat{A}^c(wp_1+x_\perp)\rangle \tag{80}$$

where we have again replaced $\int_{-\infty}^0 dt\int_{-\infty}^t dw\hat{A}^b(t)\hat{A}^c(w)$ by $\frac{1}{2}\int_{-\infty}^0 dtdw\hat{A}^b(t)\hat{A}^c(w)$. Using Eq. (11) we get

$$\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\,7(g)}} = \frac{ig^4}{\pi}\mathrm{Tr}\{t^aU_xt^bt^ct^dU^\dagger_y\}\Big(x\Big|\frac{p_i}{p^2_\perp}U^{ad}\frac{p_i}{p^2_\perp}\Big|y\Big)\int d\!\!{}^-\alpha' d\!\!{}^-\beta'\frac{\beta'}{\alpha'(\beta'^2+\epsilon^2)}\Big(x\Big|\frac{1}{\alpha'\beta's-p^2_\perp+i\epsilon}\Big|x\Big)$$
$$=\frac{g^4}{2\pi^2}\mathrm{Tr}\{t^aU_xt^bt^ct^dU^\dagger_y\}\Big(x\Big|\frac{p_i}{p^2_\perp}U^{ad}\frac{p_i}{p^2_\perp}\Big|y\Big)\Big(x\Big|\frac{1}{p^2_\perp}\Big|x\Big)\int_0^\sigma\frac{d\alpha}{\alpha}\frac{d\alpha'}{\alpha'} \tag{81}$$

which is obviously a $(\mathrm{LO})^2$ term which does not contribute to the NLO kernel.

Combining expressions (68), (72), and (79) we get

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{Fig.\,7}} = -2\alpha_s^2N_c\mu^{2\epsilon}\int d^{2-\varepsilon}z\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]\int d\!\!{}^{-2-\varepsilon}k d\!\!{}^{-2-\varepsilon}k' d\!\!{}^{-2-\varepsilon}q\frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}$$
$$\times\Big\{\Big[\frac{k^2}{k'^2}(q,2k-k')-\frac{k^2}{(k-k')^2}(q,k+k')\Big]\ln\frac{(k-k')^2}{k'^2}$$
$$+\int_0^1du\frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k'^2\bar{u}+(k-k')^2u}+\Big[\frac{(q,k')}{(k-k')^2}+\frac{(q,k')k^2}{k'^2(k-k')^2}+\frac{(q,k'-2k)}{k'^2}\Big]$$
$$\times\ln\frac{(k-k')^2}{k'^2}-\Big[\frac{(q,k)}{k'^2}+\frac{(q,k)}{(k-k')^2}+\frac{(q,k-k')k^2}{k'^2(k-k')^2}\Big]\ln\frac{k^2}{k'^2}-2\int_0^1du\frac{(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}$$
$$+\frac{2(q,k)}{k'^2}\ln\frac{k^2}{k'^2}\Big\}$$
$$=-2\alpha_s^2N_c\mu^{2\varepsilon}\int d^{2-\varepsilon}z\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]\int d\!\!{}^{-2-\varepsilon}k d\!\!{}^{-2-\varepsilon}k' d\!\!{}^{-2-\varepsilon}q\frac{e^{i(q,X)-i(k,Y)}}{k^2q^2}$$
$$\times\Big\{\ln\frac{k^2}{k'^2}\frac{3(q,k')k^2-4(q,k)(k,k')}{k'^2(k-k')^2}+\int_0^1du\frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k^2[k'^2\bar{u}+(k-k')^2u]}$$
$$-\int_0^1du\frac{2(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}\Big\}. \tag{82}$$

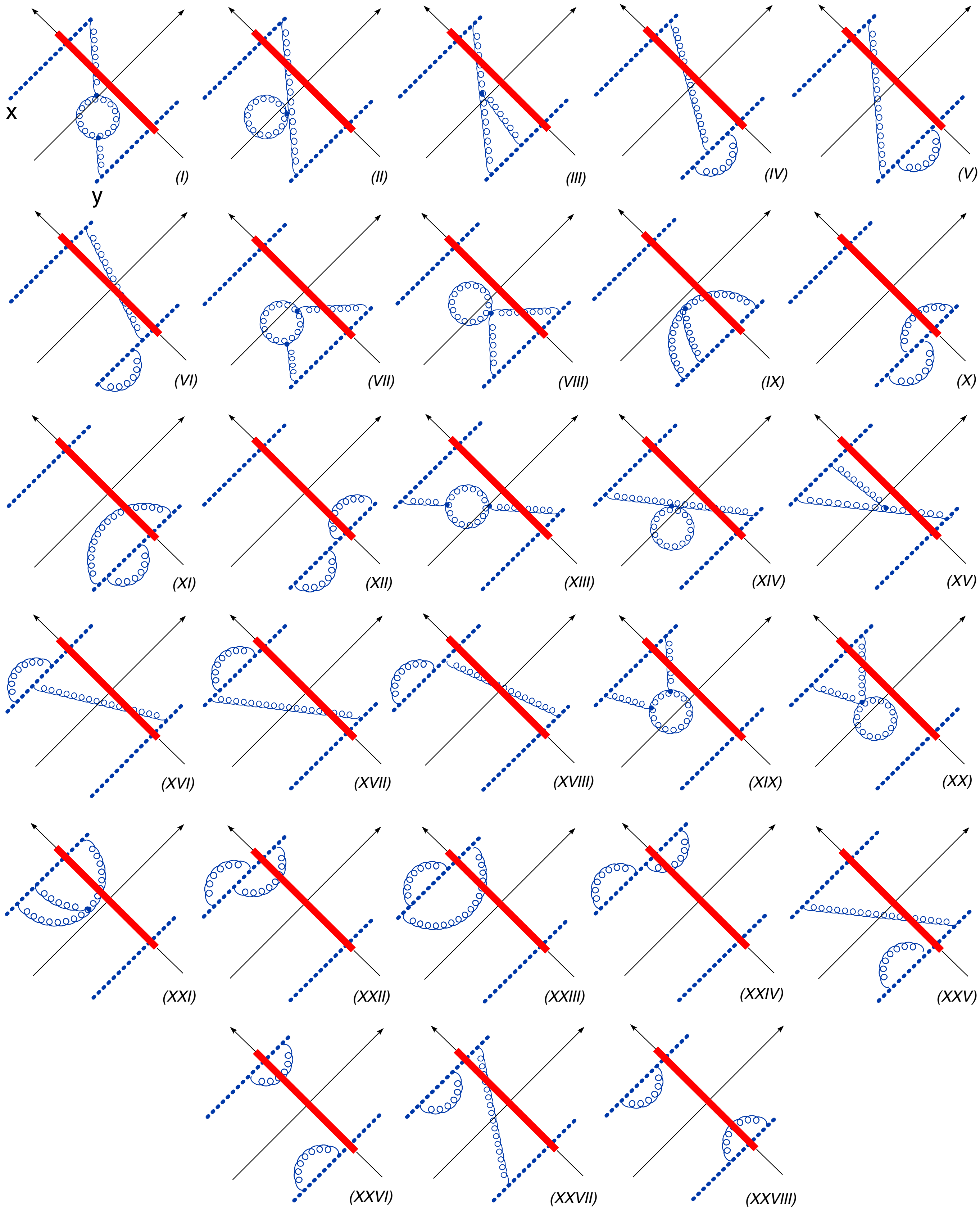

FIG. 8 (color online). The full set of running-coupling diagrams.

Using the integral over k',

$$\int d\!\!{}^-{}^{2-\varepsilon}k'_\perp\left\{\frac{3(q,k')k^2-4(q,k)(k,k')}{k'^2(k-k')^2}\ln\frac{k^2}{k'^2}+\int_0^1 du\frac{(2-\varepsilon)(q,ku-k')(k,k-2k')+2(q,k)k^2}{k^2[k'^2\bar{u}+(k-k')^2u]}\right.$$
$$\left.-\int_0^1 du\frac{2(q,k)(k',k-k')}{k'^2[k'^2\bar{u}+(k-k')^2u]}\right\}=\frac{(q,k)}{4\pi}\left\{\frac{\Gamma(\varepsilon/2)}{(k^2)^{\varepsilon/2}}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\left[\frac{11}{3}-\varepsilon\frac{\pi^2}{6}\right]+\frac{1}{9}\right\},$$

one reduces the r.h.s. of Eq. (82) to

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 7}}=-\frac{\alpha_s^2N_c}{2\pi}\mu^{2\varepsilon}\int d^{2-\varepsilon}z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]\int d\!\!{}^-{}^{2-\varepsilon}k d\!\!{}^-{}^{2-\varepsilon}q e^{i(q,X)-i(k,Y)}\frac{(q,k)}{k^2q^2}$$
$$\times\left\{\frac{\Gamma(\varepsilon/2)}{(k^2)^{\varepsilon/2}}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\left[\frac{11}{3}-\varepsilon\frac{\pi^2}{6}\right]+\frac{1}{9}\right\}. \tag{83}$$

Next we subtract the counterterm

$$-\frac{22}{3}\frac{\alpha_s^2N_c}{\pi\varepsilon}\mu^\varepsilon\int d^{2-\varepsilon}z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]\int d\!\!{}^-{}^{2-\varepsilon}q d\!\!{}^-{}^{2-\varepsilon}k e^{i(q,X)-i(k,Y)}\frac{(q,k)}{q^2k^2} \tag{84}$$

corresponding to the poles $1/\varepsilon$ in the loop diagrams in Fig. 7 (we use the $\overline{MS}$ scheme). We obtain

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 7}}=-\frac{\alpha_s^2N_c}{2\pi}\int d^{2-\varepsilon}z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]$$
$$\times\int d\!\!{}^-{}^{2-\varepsilon}k d\!\!{}^-{}^{2-\varepsilon}k' d\!\!{}^-{}^{2-\varepsilon}q e^{i(q,X)-i(k,Y)}\frac{(q,k)}{k^2q^2}\left\{\frac{11}{3}\ln\frac{\mu^2}{k^2}+\frac{67}{9}-\frac{\pi^2}{3}\right\}. \tag{85}$$

The complete set of running-coupling diagrams is presented in Fig. 8.

The contribution of the diagrams in Figs. 8 (VII)–(XII) differs from Eq. (85) by the replacement $e^{i(q,X)}\to e^{i(q,Y)}$ and sign change. There is also a symmetric set of diagrams (XIII)–(XXIV) obtained by the reflection of diagrams (I)–(XII) with respect to the x_* axis. The result is obtained by the substitution $e^{-i(k,X)}\leftrightarrow e^{-i(k,Y)}$, and therefore the contribution of all diagrams in Fig. 8 takes the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 8(I)+...+(XXIV)}}=\frac{\alpha_s^2N_c}{2\pi}\int d^2z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]$$
$$\times\int d\!\!{}^-{}^2k d\!\!{}^-{}^2k' d\!\!{}^-{}^2q[e^{i(q,X)}-e^{i(q,Y)}][e^{-i(k,X)}-e^{-i(k,Y)}]\frac{(q,k)}{k^2q^2}\left\{\frac{11}{3}\ln\frac{\mu^2}{k^2}+\frac{67}{9}-\frac{\pi^2}{3}\right\}. \tag{86}$$

The remaining diagrams (XXV)–(XXVIII) contribute only to the $(\mathrm{LO})^2$. We have shown this for diagram (XXVII) [Fig. 7(g)]; see Eq. (81). The diagram in Fig. 8 (XXV) is obtained from Eq. (81) by the replacement $x\leftrightarrow y$, and the diagrams (XXVI) and (XXVIII) by replacing $(x|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$ by $(x|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|x)(y|\frac{1}{p_\perp^2}|y)$ and $(y|\frac{p_i}{p_\perp^2}U\frac{p_i}{p_\perp^2}|y)(x|\frac{1}{p_\perp^2}|x)$, respectively. Thus, diagrams (XXV)–(XXVII) do not contribute to the NLO kernel.

There is another set of diagrams obtained by the reflection of diagrams shown in Fig. 8 with respect to the shockwave line. Their contribution is obtained from Eq. (86) by the replacement $q\leftrightarrow k$ in the logarithm, so the final result for the sum of all running-coupling diagrams of Fig. 8 type has the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\text{Fig. 8 total}}=\frac{\alpha_s^2N_c}{2\pi}\int d^2z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]$$
$$\times\int d\!\!{}^-{}^2k d\!\!{}^-{}^2k' d\!\!{}^-{}^2q[e^{i(q,X)}-e^{i(q,Y)}][e^{-i(k,X)}-e^{i(k,Y)}]\frac{(q,k)}{k^2q^2}\left\{\frac{11}{3}\ln\frac{\mu^4}{q^2k^2}+\frac{134}{9}+\frac{2\pi^2}{3}\right\}$$
$$=\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z\left[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\right]$$
$$\times\left\{\frac{(x-y)^2}{X^2Y^2}\left[\frac{11}{3}\ln\frac{X^2Y^2}{\mu^{-4}}+\frac{134}{9}-\frac{2\pi^2}{3}\right]+\frac{11}{3}\left[\frac{1}{X^2}-\frac{1}{Y^2}\right]\ln\frac{X^2}{Y^2}\right\}. \tag{87}$$

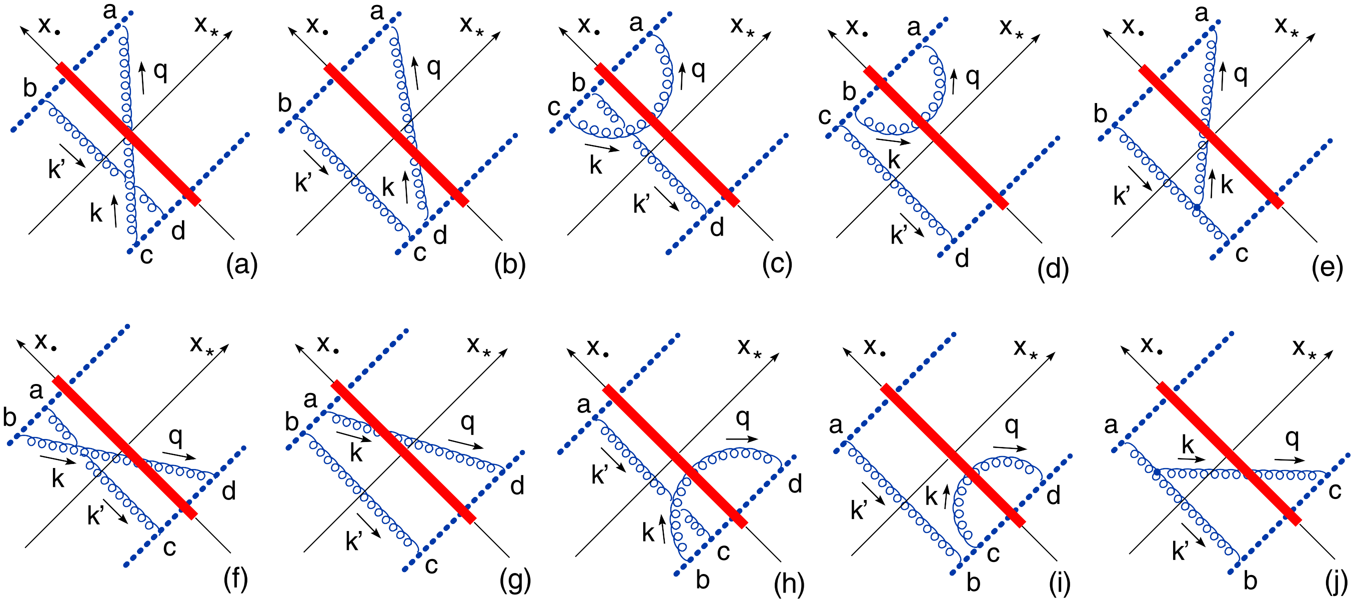

FIG. 9 (color online). $1 \to 2$ dipoles transition diagrams.

B. Diagrams for $1 \to 2$ dipoles transition

There is one more class of diagrams with a one gluon-shockwave intersection shown in Fig. 9. These diagrams are UV convergent so we do not need to change the dimension of the transverse space to $2 - \varepsilon$. First we calculate the diagrams shown in Figs. 9(a) and 9(b).

$$\begin{aligned}
\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 9(a)+(b)}} &= 4\,\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\}\int d^{2-\varepsilon}z \int d\!\!{}^-\alpha_1 d\!\!{}^-\beta_1 d\!\!{}^-\alpha_2 d\!\!{}^-\beta_2 \int d\!\!{}^-{}^{2-\varepsilon}k_1 d\!\!{}^-{}^{2-\varepsilon}k_2 e^{-i(k_1,y)_\perp - i(k_2,y)_\perp} \\
&\quad\times\left[\frac{\delta^{bd}U_z^{ac}}{(\beta_1 - i\epsilon)(\beta_1+\beta_2 - i\epsilon)} + \frac{\delta^{bc}U_z^{ad}}{(\beta_2 - i\epsilon)(\beta_1+\beta_2-i\epsilon)}\right]\theta(\alpha_1)\int d\!\!{}^-{}^2q\, e^{i(q,x-z)_\perp + i(k_1,z)_\perp} \\
&\quad\times\frac{(q_1,k_1)_\perp}{\alpha_1(\alpha_1\beta_1 s - k_{1\perp}^2 + i\epsilon)q_\perp^2}\,\frac{e^{i(k_2,x)_\perp}}{\alpha_2(\alpha_2\beta_2 s - k_{2\perp}^2 + i\epsilon)} \\
&= -\frac{g^4}{\pi^2}\,\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\}\int d^{2-\varepsilon}z\int_0^\sigma \frac{d\alpha}{\alpha}\int_0^1 du \int d\!\!{}^-{}^{2-\varepsilon}k d\!\!{}^-{}^{2-\varepsilon}k' e^{-i(k,y)_\perp - i(k',y)_\perp} \\
&\quad\times\left[\frac{\delta^{bd}U_z^{ac}}{k^2\bar u(k^2\bar u + k'^2 u)} + \frac{\delta^{ab}U_z^{dc}}{k'^2 u(k^2\bar u + k'^2 u)}\right]\int d\!\!{}^-{}^2q\, e^{i(q,x-z)_\perp + i(k,z)_\perp + i(k',x)_\perp}\,\frac{(q,k)_\perp}{q_\perp^2} \\
&= -\frac{g^4}{\pi^2}\,\mathrm{Tr}\{t^a U_x t^b t^c t^d U_y^\dagger\}\int d^2z\int_0^\infty\frac{d\alpha}{\alpha}\int d\!\!{}^-{}^2k d\!\!{}^-{}^2k' d\!\!{}^-{}^2q\,\frac{(q,k)}{q^2}\,e^{i(q,X) - i(k,Y)_\perp + i(k',x-y)_\perp} \\
&\quad\times\frac{\delta^{bc}U_z^{ad} - \delta^{bd}U_z^{ac}}{k^2k'^2}\ln\frac{k^2}{k'^2}
\end{aligned}\tag{88}$$

and therefore

$$\begin{aligned}
\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 9(a)+(b)}} &= -\frac{g^4 N_c}{4\pi^2}\int d^2z\left[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_x U_y^\dagger\}\right] \\
&\quad\times\int d\!\!{}^-{}^2k d\!\!{}^-{}^2k' d\!\!{}^-{}^2q\,\frac{(q,k)}{q^2k^2k'^2}\,e^{i(q,x-z) - i(k,y-z)_\perp + i(k',x-y)_\perp}\ln\frac{k^2}{k'^2}.
\end{aligned}\tag{89}$$

The contribution of the diagrams shown in Figs. 9(c) and 9(d) is obtained from Eq. (89) by the replacement $x \leftrightarrow y$ in the left part of the graph and the sign change so that $e^{-ik(y-z)+i(k',x-y)} \to -e^{-ik(x-z)-i(k',x-y)}$. The sum of the diagrams in Figs. 9(a)–9(d) takes the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig. 9(a)–(d)}} = -\frac{g^4N_c}{4\pi^2}\int d^2z\Big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big] \times\int d\!\!\!{}^-{}^2k\,d\!\!\!{}^-{}^2k'\,d\!\!\!{}^-{}^2q\,\frac{(q,k)}{q^2k^2k'^2}e^{i(q,x-z)}(e^{-i(k,y-z)_\perp - i(k',x-y)_\perp} - x\leftrightarrow y)\ln\frac{k^2}{k'^2}. \tag{90}$$

The next relevant diagram is shown in Fig. 9(e),

$$g^3\int_0^\infty du\int_{-\infty}^0 dv\int_{-\infty}^0 dt\langle\hat{A}^a_\bullet(up_1+x)\hat{A}^b_\bullet(vp_1+x)\hat{A}^c_\bullet(tp_1+y)\rangle_{\text{Fig. 9(e)}}$$
$$= 2g^4\int d\!\!\!{}^-\alpha_1 d\!\!\!{}^-\beta_1 d\!\!\!{}^-\alpha_2 d\!\!\!{}^-\beta_2 d\!\!\!{}^-{}^2k_1^\perp d\!\!\!{}^-{}^2k_2^\perp\frac{\theta(\alpha_1+\alpha_2)f^{bcd}(k^\perp_{1\mu}+2\beta_1p_{2\mu})(k^\perp_{2\nu}+2\beta_2p_{2\nu})}{\alpha_1\alpha_2(\beta_1-i\epsilon)(\alpha_1\beta_1s-k^2_{1\perp}+i\epsilon)(\beta_2-i\epsilon)(\alpha_2\beta_2s-k^2_{2\perp}+i\epsilon)}$$
$$\times\int d^2zU_z^{ad}\int d\!\!\!{}^-{}^2q\frac{e^{i(q-k_1,x-z)_\perp-ik_2(y-z)_\perp}[q+\frac{2(k_1+k_2,q)_\perp}{(\alpha_1+\alpha_2)s}p_2]_\lambda}{q^2[(\alpha_1+\alpha_2)(\beta_1+\beta_2)s-(k_1+k_2)^2_\perp+i\epsilon]}\Gamma^{\mu\nu\lambda}(k_1,k_2,-k_1-k_2). \tag{91}$$

There are three regions of integration over the α's: $\alpha_1,\alpha_2>0$, $\alpha_1>-\alpha_2>0$, and $\alpha_2>-\alpha_1>0$. Going to the variables $\alpha=\alpha_1+\alpha_2$, $u=\alpha_2/\alpha$ in the first region, $\alpha=\alpha_1$, $u=-\alpha_2/\alpha$ in the second, and $\alpha=\alpha_2$, $u=-\alpha_1/\alpha$ in the third, we obtain

$$g^3\int_0^\infty du\int_{-\infty}^0 dv\int_{-\infty}^0 dt\langle\hat{A}^a_\bullet(up_1+x)\hat{A}^b_\bullet(vp_1+x)\hat{A}^c_\bullet(tp_1+y)\rangle_{\text{Fig. 9(e)}}$$
$$=\frac{1}{2\pi^2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1du\int d\!\!\!{}^-{}^2k_1^\perp d\!\!\!{}^-{}^2k_2^\perp f^{bcd}U_z^{ad}\int\frac{d\!\!\!{}^-{}^2q}{q^2}e^{i(q-k_1,x-z)_\perp-i(k_2,y-z)_\perp}$$
$$\times\Big[k^\perp_{1\mu}k^\perp_{2\nu}\Big(q_\lambda+\frac{2(k_1+k_2,q)_\perp}{\alpha s}p_{2\lambda}\Big)\frac{\Gamma^{\mu\nu\lambda}(\alpha\bar{u}p_1+k_1^\perp,\alpha up_1+k_2^\perp,-\alpha p_1-k_1^\perp-k_2^\perp)}{\bar{u}uk^2_{1\perp}k^2_{2\perp}(k_1+k_2)^2_\perp}$$
$$-\bar{u}k^\perp_{1\mu}\Big(k^\perp_{2\nu}+2\frac{(k_1+k_2)^2_\perp}{\alpha\bar{u}s}p_{2\nu}\Big)\Big(q_\lambda+\frac{2(k_1+k_2,q)_\perp}{\alpha\bar{u}s}p_{2\lambda}\Big)\frac{\Gamma^{\mu\nu\lambda}(\alpha p_1+k_1^\perp,-u\alpha p_1+k_2^\perp,-\alpha\bar{u}-k_1^\perp-k_2^\perp)}{uk^2_{1\perp}(k_1+k_2)^2_\perp[u(k_1+k_2)^2_\perp+\bar{u}k^2_{2\perp}]}$$
$$-\bar{u}\Big(k_1^\perp+2\frac{(k_1+k_2)^2_\perp}{\alpha\bar{u}s}p_2\Big)_\mu k^\perp_{2\nu}\Big(q+\frac{2(k_1+k_2,q)_\perp}{\alpha\bar{u}s}p_2\Big)_\lambda\frac{\Gamma^{\mu\nu\lambda}(-u\alpha p_1+k_1^\perp,\alpha p_1+k_2^\perp,-\alpha\bar{u}-k_1^\perp-k_2^\perp)}{uk^2_{2\perp}(k_1+k_2)^2_\perp[u(k_1+k_2)^2_\perp+\bar{u}k^2_{1\perp}]}\Big]. \tag{92}$$

Using the formula

$$\Big(k_1^\perp+\frac{2A}{s}p_2\Big)_\mu\Big(k_2^\perp+\frac{2B}{s}p_2\Big)_\nu\Big(q+\frac{2C}{s}p_2\Big)_\lambda\Gamma^{\mu\nu\lambda}(\alpha_1p_1+k_{1\perp},\alpha_2p_1+k_{2\perp},-(\alpha_1+\alpha_2)p_1-(k_1+k_2)_\perp)$$
$$=-C(\alpha_1-\alpha_2)(k_1,k_2)_\perp-A(\alpha_1+2\alpha_2)(q,k_2)_\perp+B(2\alpha_1+\alpha_2)(q,k_1)_\perp-[(q,k_1)_\perp(k_2,k_1+k_2)_\perp-(k_1\leftrightarrow k_2)]$$

we get

$$\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig. 9(c)}}=\frac{g^4N_c}{8\pi^2}\int d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2\frac{d\!\!\!{}^-{}^2q}{q^2}\int d^2z\int_0^\sigma\frac{d\alpha}{\alpha}\int_0^1du\,e^{i(q-k_1,x-z)-i(k_2,y-z)}$$
$$\times\Big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big]\Big\{\frac{(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)}{\bar{u}uk_1^2k_2^2(k_1+k_2)^2}$$
$$+\frac{(1+\bar{u})(k_1+k_2)^2(q,k_1)-(1+u)(q,k_1+k_2)(k_1,k_2)-\bar{u}[(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)]}{uk_1^2(k_1+k_2)^2(k_2^2\bar{u}+(k_1+k_2)^2u)}$$
$$+\frac{-(1+\bar{u})(k_1+k_2)^2(q,k_2)+(1+u)(q,k_1+k_2)(k_1,k_2)-\bar{u}[(q,k_1)(k_2,k_1+k_2)-(q,k_2)(k_1,k_1+k_2)]}{uk_2^2(k_1+k_2)^2(k_1^2\bar{u}+(k_1+k_2)^2u)}\Big\}. \tag{93}$$

Performing the integration over u [with prescription (47)] we obtain

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.\,9(c)}}=\frac{g^4N_c}{8\pi^2}\int d^2z\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]\int\frac{d\!\!{}^-{}^2k_1d\!\!{}^-{}^2k_2}{k_1^2k_2^2(k_1+k_2)^2}\frac{d\!\!{}^-{}^2q}{q^2}e^{i(q-k_1,X)-i(k_2,Y)}U^{cl}_z$$
$$\times\Big\{(k_1+k_2)^2\Big[(q,k_2)\ln\frac{(k_1+k_2)^2}{k_1^2}-(q,k_1)\ln\frac{(k_1+k_2)^2}{k_2^2}\Big]-(q,k_1+k_2)(k_1^2+k_2^2)\ln\frac{k_1^2}{k_2^2}\Big\}$$
$$=\frac{g^4N_c}{8\pi^2}\int d^2z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\mathrm{Tr}\{U_xU^\dagger_y\}]\int d\!\!{}^-{}^2kd\!\!{}^-{}^2k'\frac{d\!\!{}^-{}^2q}{q^2}e^{i(q-k',x-z)-i(k-k',y-z)}U^{cl}_z$$
$$\times\Big\{\frac{(q,k-k')}{k'^2(k-k')^2}\ln\frac{k^2}{k'^2}-\frac{(q,k')}{k'^2(k-k')^2}\ln\frac{k^2}{(k-k')^2}+\frac{(q,k)}{k^2k'^2}\ln\frac{(k-k')^2}{k'^2}+\frac{(q,k)}{k^2(k-k')^2}\ln\frac{(k-k')^2}{k'^2}\Big\}\qquad(94)$$

where we made the change of variables $k_1\to k'$ and $k_2\to k-k'$.

The sum of the diagrams shown in Figs. 9(a)–9(e) can be represented as

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.\,9(a)-(e)}}=2\alpha_s^2N_c\int d^2z\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]\int d\!\!{}^-{}^2kd\!\!{}^-{}^2k'd\!\!{}^-{}^2qe^{i(q,x-z)}(e^{-i(k,y-z)_\perp-i(k',x-y)_\perp}$$
$$-x\leftrightarrow y)\Big[\frac{(q,k-k')}{k'^2(k-k')^2}+\frac{(q,k)}{k^2(k-k')^2}-\frac{(q,k)}{k^2k'^2}\Big]\ln\frac{k^2}{k'^2}.\qquad(95)$$

Note that the expressions (90) and (94) are IR divergent as $k'\to 0$ but their sum (95) is IR stable. Once again, the contribution of the diagrams in Figs. 9(f)–9(j) is obtained by the replacement $e^{iq(x-z)}\to -e^{iq(y-z)}$, so the contribution of the diagrams of Figs. 9(a)–9(j) has the form

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.\,9(a)-(k)}}=2\alpha_s^2N_c\int d^2z\Big[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}\Big]$$
$$\times\int d\!\!{}^-{}^2kd\!\!{}^-{}^2k'd\!\!{}^-{}^2q(e^{i(q,x-z)}-e^{i(q,y-z)})(e^{-i(k,y-z)_\perp-i(k',x-y)_\perp}-x\leftrightarrow y)$$
$$\times\Big[\frac{(q,k-k')}{k'^2(k-k')^2}+\frac{(q,k)}{k^2(k-k')^2}-\frac{(q,k)}{k^2k'^2}\Big]\ln\frac{k^2}{k'^2}.\qquad(96)$$

Performing the Fourier transformation with the help of the formula

$$\int d\!\!{}^-kd\!\!{}^-k'\frac{e^{-i(k,y)-i(k',x-y)}}{k'^2}\Big(\frac{(k-k')_i}{(k-k')^2}-\frac{k_i}{k^2}+\frac{k_ik'^2}{k^2(k-k')^2}\Big)\ln\frac{k^2}{k'^2}$$
$$=\frac{i}{16\pi^2}\Big(\frac{x_i}{x^2}-\frac{y_i}{y^2}\Big)\ln\frac{(x-y)^2}{x^2}\ln\frac{(x-y)^2}{y^2}+\frac{i}{8\pi^2}\Big(\frac{(x,y)}{y^2}y_i-x_i\Big)\frac{1}{i\kappa}\Big\{\int_0^1du\Big[\frac{\ln u}{u-\frac{(x,y)-i\kappa}{x^2}}-\frac{\ln u}{u-\frac{(x,y)+i\kappa}{x^2}}\Big]$$
$$+\frac{1}{2}\ln\frac{x^2}{y^2}\ln\frac{(x-y,y)+i\kappa}{(x-y,y)-i\kappa}\Big\}+\frac{i}{8\pi^2}\Big(\frac{(x,y)}{x^2}x_i-y_i\Big)\frac{1}{i\kappa}\Big\{\int_0^1du\Big[\frac{\ln u}{u-\frac{(x,x-y)-i\kappa}{(x-y)^2}}-\frac{\ln u}{u-\frac{(x,x-y)+i\kappa}{(x-y)^2}}\Big]$$
$$-\frac{1}{2}\ln\frac{(x-y)^2}{x^2}\ln\frac{(x,y)+i\kappa}{(x,y)-i\kappa}\Big\}\qquad(97)$$

[here $\kappa=\sqrt{x^2y^2-(x,y)^2}$], one obtains

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U^\dagger_y\}\rangle_{\mathrm{Fig.\,9(a)-(j)}}=-\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z[\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{N_c}\mathrm{Tr}\{U_xU^\dagger_y\}]\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}.\qquad(98)$$

Note that the last two terms in the r.h.s. of Eq. (97) do not contribute.

The contribution of the diagram obtained by reflection of Fig. 9 with respect to the shockwave differs from Eq. (95) by the replacement $q\leftrightarrow k$, which doubles the result (98). The final expression for the contribution of all "dipole recombination diagrams" of Fig. 9 type has the form

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig. 9 total}} = -\frac{\alpha_s^2 N_c}{4\pi^3}\int d^2z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}]\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}. \quad (99)$$

V. ASSEMBLING THE NLO KERNEL

Adding results (58), (87), and (99) one obtains the contribution of the diagrams with one and two gluon intersections with the shockwave in the form

$$\begin{aligned}
\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\text{Fig. 6+Fig. 8+Fig. 9}} = {}& \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln(x-y)^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] + \frac{11}{3}\Big[\frac{1}{X^2} - \frac{1}{Y^2}\Big]\ln\frac{X^2}{Y^2} \\
& - 2\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}] \\
& + \frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]} \\
& + \frac{(x-y)^4}{X^2Y'^2 - X'^2Y^2}\Big[\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2} - \frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
& \times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} - (z'\to z)] \\
& + \Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2} + \frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big)\Big\} \\
& \times\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big].
\end{aligned} \quad (100)$$

There are also diagrams without a gluon-shockwave intersection, like the graph shown in Fig. 10. They are proportional to the parent dipole $\mathrm{Tr}\{U_xU_y^\dagger\}$, and their contribution can be found from Eq. (100) using the requirement that the r.h.s. of the evolution equation must vanish at $x = y$ (since $U_xU_x^\dagger = 1$). It is easy to see that replacing $\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}$ by $\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}$ fulfills the above requirement, so one obtains the final gluon contribution to the NLO kernel in the form

$$\begin{aligned}
\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = {}& \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z\Big\{\frac{(x-y)^2}{X^2Y^2}\Big[\frac{11}{3}\ln(x-y)^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big] + \frac{11}{3}\Big[\frac{1}{X^2} - \frac{1}{Y^2}\Big]\ln\frac{X^2}{Y^2} \\
& - 2\frac{(x-y)^2}{X^2Y^2}\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big\}[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}] \\
& + \frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\Big[\Big(-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]} \\
& + \frac{(x-y)^4}{X^2Y'^2 - X'^2Y^2}\Big[\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2} - \frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Big) \\
& \times[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\} - \mathrm{Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\} - (z'\to z)] \\
& + \Big\{-\frac{(x-y)^4}{X^2Y'^2X'^2Y^2} + \frac{(x-y)^2}{(z-z')^2}\Big(\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big)\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_{z'}^\dagger\}\mathrm{Tr}\{U_{z'}U_y^\dagger\}\Big].
\end{aligned} \quad (101)$$

Promoting Wilson lines in the r.h.s of this equation to operators and adding the quark contribution from Ref. [9],

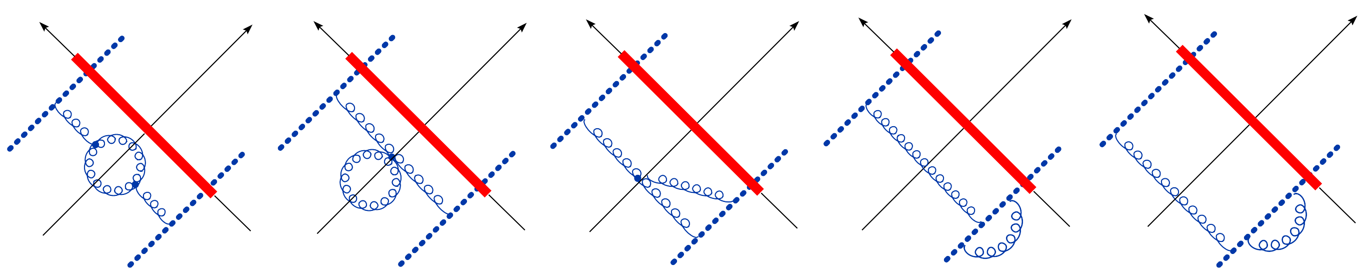

FIG. 10 (color online). Typical diagrams without the gluon-shockwave intersection.

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{quark}} = \frac{\alpha_s}{2\pi^2}\int d^2z[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - N_c\mathrm{Tr}\{U_xU_y^\dagger\}]\Big[-\frac{\alpha_s n_f}{6\pi}\frac{(x-y)^2}{X^2Y^2}\Big(\ln(x-y)^2\mu^2 + \frac{5}{3}\Big) + \frac{\alpha_s n_f}{6\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}\Big] + \frac{\alpha_s^2}{\pi^4}n_f\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}\int d^2zd^2z'\mathrm{Tr}\{t^aU_zt^bU_{z'}^\dagger - t^aU_zt^bU_z^\dagger\}\frac{1}{(z-z')^4} \times\Big\{1 - \frac{X'^2Y^2 + Y'^2X^2 - (x-y)^2(z-z')^2}{2(X'^2Y^2 - Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big\}, \tag{102}$$

we obtain the full NLO kernel cited in Eq. (5).

VI. COMPARISON TO NLO BFKL

A. Linearized forward kernel

In this section we compare our kernel to the forward NLO BFKL results [13]. The linearized equation (5) has the form

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{(x-y)^2}{X^2Y^2}\Big\{1 + \frac{\alpha_s}{4\pi}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + \Big(\frac{67}{9} - \frac{\pi^2}{3}\Big)N_c - \frac{10}{9}n_f - 2N_c\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}[\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y)] + \frac{\alpha_s^2N_c^2}{16\pi^4}\int d^2zd^2z'\Big[-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]} + \frac{(x-y)^4}{X^2Y'^2 - X'^2Y^2}\Big[\frac{1}{X^2Y'^2} + \frac{1}{Y^2X'^2}\Big] + \frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2} - \frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2} - \frac{n_f}{N_c^3}\Big\{\frac{4}{(z-z')^4} - 2\frac{X'^2Y^2 + Y'^2X^2 - (x-y)^2(z-z')^2}{(z-z')^4(X'^2Y^2 - Y'^2X^2)}\ln\frac{X'^2Y^2}{Y'^2X^2}\Big\}\Big]\hat{\mathcal{U}}(z,z'). \tag{103}$$

For the case of forward scattering, $\langle\hat{\mathcal{U}}(x,y)\rangle = \langle\hat{\mathcal{U}}(x-y)\rangle$, and the linearized equation (103) reduces to

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle = \frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{x^2}{(x-z)^2z^2}\Big\{1 + \frac{\alpha_s}{4\pi}\Big[b\ln x^2\mu^2 - b\frac{(x-z)^2 - z^2}{x^2}\ln\frac{(x-z)^2}{z^2} + \Big(\frac{67}{9} - \frac{\pi^2}{3}\Big)N_c - \frac{10}{9}n_f - 2N_c\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big\}[\langle\hat{\mathcal{U}}(x-z)\rangle + \langle\hat{\mathcal{U}}(z)\rangle - \langle\hat{\mathcal{U}}(x)\rangle] + \frac{\alpha_s^2N_c^2}{16\pi^4}\int d^2zd^2z'\Big[-\frac{4}{z^4} + \Big\{2\frac{(x-z-z')^2z'^2 + (x-z')^2(z+z')^2 - 4x^2z^2}{z^4[(x-z-z')^2z'^2 - (x-z')^2(z+z')^2]} + \frac{x^4}{(x-z-z')^2z'^2 - (x-z')^2(z+z')^2}\Big[\frac{1}{(x-z-z')^2z'^2} + \frac{1}{(x-z')^2(z+z')^2}\Big] + \frac{x^2}{z^2}\Big[\frac{1}{(x-z-z')^2z'^2} - \frac{1}{(x-z')^2(z+z')^2}\Big]\Big\}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2} - \frac{n_f}{N_c^3}\Big\{\frac{4}{z^4} - 2\frac{(x-z-z')^2z'^2 + (x-z')^2(z+z')^2 - x^2z^2}{z^4[(x-z-z')^2z'^2 - (x-z')^2(z+z')^2]}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2}\Big\}\Big]\hat{\mathcal{U}}(z)\rangle. \tag{104}$$

Using the integral J_{13} from [16], we get

$$\frac{1}{\pi}\int d^2z'\Big[\frac{x^4}{(x-z-z')^2z'^2-(x-z')^2(z+z')^2}+\frac{x^2}{z^2}\Big]\frac{1}{z'^2(x-z-z')^2}\ln\frac{(x-z-z')^2z'^2}{(x-z')^2(z+z')^2}$$
$$=\frac{2x^2}{z^2}\Big\{\frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4}+2Li_2\Big(-\frac{z^2}{x^2}\Big)-2Li_2\Big(-\frac{x^2}{z^2}\Big)\Big]-\Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\Big[\int_0^1-\int_1^\infty\Big]$$
$$\times\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Big\}, \tag{105}$$

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(x)\rangle=\frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{x^2}{(x-z)^2z^2}\Big\{1+\frac{\alpha_s}{4\pi}\Big[b\ln x^2\mu^2-b\frac{(x-z)^2-z^2}{x^2}\ln\frac{(x-z)^2}{z^2}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}n_f$$
$$-2N_c\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}\Big]\Big\}[\langle\hat{\mathcal{U}}(x-z)\rangle+\langle\hat{\mathcal{U}}(z)\rangle-\langle\hat{\mathcal{U}}(x)\rangle]+\frac{\alpha_s^2N_c^2}{4\pi^3}\int d^2z\frac{x^2}{z^2}\Big\{\Big(1+\frac{n_f}{N_c^3}\Big)\frac{3(x,z)^2-2x^2z^2}{16x^2z^2}$$
$$\times\Big(\frac{2}{x^2}+\frac{2}{z^2}+\frac{x^2-z^2}{x^2z^2}\Big)\ln\frac{x^2}{z^2}-\Big[3+\Big(1+\frac{n_f}{N_c^3}\Big)\Big(1-\frac{(x^2+z^2)^2}{8x^2z^2}+\frac{3x^4+3z^4-2x^2z^2}{16x^4z^4}(x,z)^2\Big)\Big]$$
$$\times\int_0^\infty dt\frac{1}{x^2+t^2z^2}\ln\frac{1+t}{|1-t|}+\frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4}+2Li_2\Big(-\frac{z^2}{x^2}\Big)-2Li_2\Big(-\frac{x^2}{z^2}\Big)\Big]$$
$$-\Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\Big[\int_0^1-\int_1^\infty\Big]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Big\}\mathcal{U}(z). \tag{106}$$

B. Comparison of eigenvalues

To compare the eigenvalues of Eq. (106) with NLO BFKL, we expand $\mathcal{U}(x,0)$ in eigenfunctions,

$$\langle\hat{\mathcal{U}}(x_\perp,0)\rangle=\sum_{n=-\infty}^{\infty}\int_{-(1/2)-i\infty}^{-(1/2)+i\infty}\frac{d\gamma}{2\pi i}e^{in\phi}(x_\perp^2\mu^2)^\gamma\langle\hat{\mathcal{U}}(n,\gamma)\rangle \tag{107}$$

(where ϕ is the angle in the $x_\perp$ plane), compute the evolution of $\langle\hat{\mathcal{U}}(n,\gamma)\rangle$ from Eq. (106), and compare it to the calculation based on the NLO BFKL results from [13,17]. (For the quark part of the NLO BK kernel the agreement with NLO BFKL was proven in Ref. [18]).

The relevant integrals have the form

$$\frac{1}{2\pi}\int d^2z[2(z^2/x^2)^\gamma e^{in\phi}-1]\frac{x^2}{(x-z)^2z^2}=\chi(n,\gamma),$$
$$\frac{1}{\pi}\int d^2z[2(z^2/x^2)^\gamma e^{in\phi}-1]\Big(\frac{1}{(x-z)^2}-\frac{1}{z^2}\Big)\ln\frac{(x-z)^2}{z^2}=\chi^2(n,\gamma)-\chi'(n,\gamma)-\frac{4\gamma\chi(\gamma)}{\gamma^2-\frac{n^2}{4}}, \tag{108}$$
$$\frac{1}{\pi}\int d^2z(z^2/x^2)^\gamma\frac{x^2}{(x-z)^2z^2}e^{in\phi}\ln\frac{(x-z)^2}{x^2}\ln\frac{z^2}{x^2}=\frac{1}{2}\chi''(n,\gamma)+\chi'(n,\gamma)\chi(n,\gamma).$$

Here $\chi(n,\gamma)=2\psi(1)-\psi(\gamma+\frac{n}{2})-\psi(1-\gamma+\frac{n}{2})$ [as usual, $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma function] and

$$\frac{1}{\pi}\int d^2z(z^2/x^2)^{\gamma-1}e^{in\phi}\Big\{\Big(1+\frac{n_f}{N_c^3}\Big)\frac{3(x,z)^2-2x^2z^2}{16x^2z^2}\Big(\frac{2}{x^2}+\frac{2}{z^2}+\frac{x^2-z^2}{x^2z^2}\Big)\ln\frac{x^2}{z^2}$$
$$-\Big[3+\Big(1+\frac{n_f}{N_c^3}\Big)\Big(1-\frac{(x^2+z^2)^2}{8x^2z^2}+\frac{3x^4+3z^4-2x^2z^2}{16x^4z^4}(x,z)^2\Big)\Big]\int_0^\infty dt\frac{1}{x^2+t^2z^2}\ln\frac{1+t}{|1-t|}\Big\}$$
$$=\Big\{-\Big[3+\Big(1+\frac{n_f}{N_c^3}\Big)\frac{2+3\gamma\bar{\gamma}}{(3-2\gamma)(1+2\gamma)}\Big]\delta_{0n}+\Big(1+\frac{n_f}{N_c^3}\Big)\frac{\gamma\bar{\gamma}}{2(3-2\gamma)(1+2\gamma)}\delta_{2n}\Big\}\frac{\pi^2\cos\pi\gamma}{(1-2\gamma)\sin^2\pi\gamma}\equiv F(n,\gamma), \tag{109}$$

$$\frac{1}{2\pi}\int d^2z(z^2/x^2)^{\gamma-1}e^{in\phi}\Big\{\frac{(x^2-z^2)}{(x-z)^2(x+z)^2}\Big[\ln\frac{x^2}{z^2}\ln\frac{x^2z^2(x-z)^4}{(x^2+z^2)^4}+2Li_2\Big(-\frac{z^2}{x^2}\Big)-2Li_2\Big(-\frac{x^2}{z^2}\Big)\Big]-\Big(1-\frac{(x^2-z^2)^2}{(x-z)^2(x+z)^2}\Big)\times\Big[\int_0^1-\int_1^\infty\Big]\frac{du}{(x-zu)^2}\ln\frac{u^2z^2}{x^2}\Big\}=-\Phi(n,\gamma)-\Phi(n,1-\gamma) \tag{110}$$

where [17]

$$\Phi(n,\gamma)=\int_0^1\frac{dt}{1+t}t^{\gamma-1+(n/2)}\Big\{\frac{\pi^2}{12}-\frac{1}{2}\psi'\Big(\frac{n+1}{2}\Big)-\mathrm{Li}_2(t)-\mathrm{Li}_2(-t)-\Big(\psi(n+1)-\psi(1)+\ln(1+t)+\sum_{k=1}^\infty\frac{(-t)^k}{k+n}\Big)\ln t-\sum_{k=1}^\infty\frac{t^k}{(k+n)^2}[1-(-1)^k]\Big\}. \tag{111}$$

The convenient way to calculate the integrals over the angle ϕ is to represent $\cos n\phi$ as $T_n(\cos\phi)$ and use formulas for the integration of Chebyshev polynomials from Ref. [17].

Using integrals (108)–(110) one easily obtains the evolution equation for $\mathcal{U}(n,\gamma)$ in the form

$$\frac{d}{d\eta}\langle\hat{\mathcal{U}}(n,\gamma)\rangle=\frac{\alpha_sN_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)+\frac{\alpha_sb}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma\chi(n,\gamma)}{\gamma^2-\frac{n^2}{4}}\Big]+\frac{\alpha_sN_c}{4\pi}[-\chi''(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+4\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle \tag{112}$$

where $\chi'(n,\gamma)\equiv\frac{d}{d\gamma}\chi(n,\gamma)$, etc.

Next we calculate the same thing using NLO BFKL results [13,17]. The impact factor $\Phi_A(q)$ for the color dipole $\mathcal{U}(x,y)$ is proportional to $\alpha_s(q)(e^{iqx}-e^{iqy})(e^{-iqx}-e^{-iqy})$, so one obtains the cross section of the scattering of the color dipole in the form

$$\langle\hat{\mathcal{U}}(x,0)\rangle=\frac{1}{4\pi^2}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}\alpha_s(q)(e^{iqx}-1)(e^{-iqx}-1)\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^\omega G_\omega(q,q') \tag{113}$$

where $G_\omega(q,q')$ is the partial wave of the forward Reggeized gluon scattering amplitude satisfying the equation

$$\omega G_\omega(q,q')=\delta^{(2)}(q-q')+\int d^2pK(q,p)G_\omega(p,q') \tag{114}$$

and $\Phi_B(q')$ is the target impact factor. The kernel $K(q,p)$ is symmetric with respect to $q\leftrightarrow p$ and the eigenvalues are

$$\int d^2p\Big(\frac{p^2}{q^2}\Big)^{\gamma-1}e^{in\phi}K(q,p)=\frac{\alpha_s(q)}{\pi}N_c\Big[\chi(n,\gamma)+\frac{\alpha_sN_c}{4\pi}\delta(n,\gamma)\Big],$$
$$\delta(n,\gamma)=-\frac{b}{2}[\chi'(n,\gamma)+\chi^2(n,\gamma)]+\Big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c^3}\Big)\chi(n,\gamma)+6\zeta(3)-\chi''(n,\gamma)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)\Big\}. \tag{115}$$

The corresponding expression for $\langle\hat{\mathcal{U}}(n,\gamma)\rangle$ takes the form

$$\langle\hat{\mathcal{U}}(n,\gamma)\rangle=-\frac{1}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}e^{-in\theta}\alpha_s(q)\Big(\frac{q^2}{4\mu^2}\Big)^\gamma\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^\omega G_\omega(q,q') \tag{116}$$

where θ is the angle between the $\vec{q}$ and x axes. Using Eq. (114) we obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{1}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}e^{-in\theta}\alpha_s(q)\Big(\frac{q^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{qq'}\Big)^{\omega} \times\int d^2pK(q,p)G_\omega(p,q'). \tag{117}$$

The integration over q can be performed using

$$\int d^2q\alpha_s(q)\Big(\frac{q^2}{p^2}\Big)^{\gamma-1}e^{in\phi}K(q,p) = \frac{\alpha_s^2(p)}{\pi}N_c\Big[\chi(n,\gamma)-\frac{b\alpha_s}{4\pi}\chi'(n,\gamma)+\frac{\alpha_sN_c}{4\pi}\delta(n,\gamma)\Big] \tag{118}$$

[recall that $K(q,p)=K(p,q)$ and $\alpha_s(p)=\alpha_s-\frac{b\alpha_s^2}{4\pi}\ln\frac{p^2}{\mu^2}$ with our accuracy]. The result is

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_s}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-in\varphi}\Big(\frac{p^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^{\omega}G_\omega(p,q') \times\frac{\alpha_s(p)}{\pi}N_c\Big[\chi\Big(n,\gamma-\frac{\omega}{2}\Big)-\frac{b\alpha_s}{4\pi}\chi'\Big(n,\gamma-\frac{\omega}{2}\Big)+\frac{\alpha_sN_c}{4\pi}\delta\Big(n,\gamma-\frac{\omega}{2}\Big)\Big] \tag{119}$$

where the angle φ corresponds to $\vec{p}$. Since $\omega\sim\alpha_s$ we can neglect terms $\sim\omega$ in the argument of δ and expand $\chi(n,\gamma-\frac{\omega}{2})\simeq\chi(n,\gamma)-\frac{\omega}{2}\chi'(n,\gamma)$. Using again Eq. (114) in the leading order we can replace extra ω by $\frac{\alpha_s}{\pi}N_c\chi(n,\gamma)$ and obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_s}{2\pi^2}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-in\varphi}\Big(\frac{p^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^{\omega}G_\omega(p,q') \times\frac{\alpha_s^2(p)}{\pi}N_c\Big[\chi(n,\gamma)-\frac{b\alpha_s}{4\pi}\chi'(n,\gamma)+\frac{\alpha_sN_c}{4\pi}[\delta(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)]\Big]. \tag{120}$$

Finally, expanding $\alpha_s^2(p)\simeq\alpha_s(p)(\alpha_s-\frac{b\alpha_s^2}{4\pi}\ln\frac{p^2}{\mu^2})\alpha_s(\mu)$ we obtain

$$s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle = -\frac{\alpha_sN_c}{2\pi^3}\cos\frac{\pi n}{2}\frac{\Gamma(-\gamma+\frac{n}{2})}{\Gamma(1+\gamma+\frac{n}{2})}\Big\{\chi(n,\gamma)\Big(1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}\Big)-\frac{b\alpha_s}{4\pi}\chi'(n,\gamma) +\frac{\alpha_sN_c}{4\pi}[\delta(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)]\Big\}\int\frac{d^2p}{p^2}\frac{d^2q'}{q'^2}e^{-in\varphi}\alpha_s(p)\Big(\frac{p^2}{4\mu^2}\Big)^{\gamma}\Phi_B(q') \times\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{pq'}\Big)^{\omega}G_\omega(p,q') \tag{121}$$

which can be rewritten as an evolution equation,

$$\begin{aligned}s\frac{d}{ds}\langle\hat{\mathcal{U}}(n,\gamma)\rangle &= \frac{\alpha_sN_c}{\pi}\Big\{\Big(1+\frac{b\alpha_s}{4\pi}\Big[\chi(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}-\frac{d}{d\gamma}\Big]\Big)\chi(n,\gamma)+\frac{\alpha_sN_c}{4\pi}[\delta(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)]\Big\}\langle\mathcal{U}(n,\gamma)\rangle\\ &= \frac{\alpha_sN_c}{\pi}\Big\{\Big[1-\frac{b\alpha_s}{4\pi}\frac{d}{d\gamma}+\Big(\frac{67}{9}-\frac{\pi^2}{3}\Big)N_c-\frac{10}{9}\frac{n_f}{N_c^2}\Big]\chi(n,\gamma)+\frac{\alpha_sb}{4\pi}\Big[\frac{1}{2}\chi^2(n,\gamma)-\frac{1}{2}\chi'(n,\gamma)-\frac{2\gamma}{\gamma^2-\frac{n^2}{4}}\chi(\gamma)\Big]\\ &\quad+\frac{\alpha_sN_c}{4\pi}[-\chi''(n,\gamma)-2\chi(n,\gamma)\chi'(n,\gamma)+6\zeta(3)+F(n,\gamma)-2\Phi(n,\gamma)-2\Phi(n,1-\gamma)]\Big\}\langle\hat{\mathcal{U}}(n,\gamma)\rangle.\end{aligned} \tag{122}$$

This eigenvalue coincides with Eq. (112) up to the extra term $2\zeta(3)$. It would correspond to the additional contribution to the r.h.s. of Eq. (5) in the form of $\frac{\alpha_s^2N_c^2}{4\pi^2}\zeta(3)\mathrm{Tr}U_xU_y^\dagger$, which contradicts the requirement $\frac{d}{d\eta}U_xU_y^\dagger=0$ at $x=y$. A possible reason for the disagreement is the connection between the matrix element of the color dipole with a rigid cutoff $\alpha<\sigma$ and the cutoff by energy s in Eq. (113). It is worth noting that the coefficient $6\zeta(3)$ in Eq. (122) agrees with the $j\to1$ asymptotics of the three-loop anomalous dimensions of leading-twist gluon operators [19].

It should be emphasized that the coincidence of terms with the nontrivial γ dependence proves that there is no additional $O(\alpha_s)$ correction to the vertex of the gluon-shockwave interaction coming from the small loop inside the shockwave; see Fig. 11 [In other words, all the effects coming from the small loop in the shockwave are absorbed in the renormalization of the coupling constant in the definition of the U operator (6)]. In the case of the quark loop, we proved the above statement by the comparison of our results for $\mathrm{Tr}\{U_xU_y^\dagger\}$ in the shockwave background with explicit light-cone calculation of the behavior of

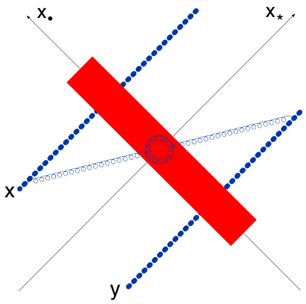

FIG. 11 (color online). Gluon loop inside the shockwave.

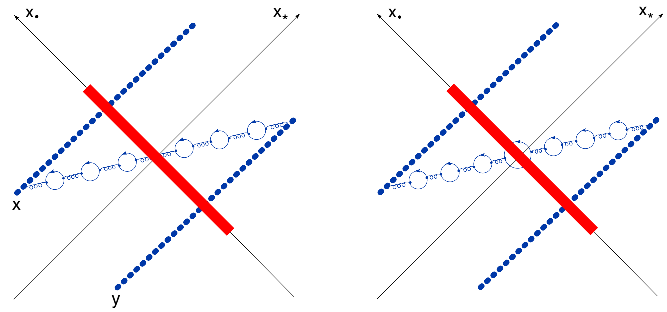

FIG. 12 (color online). Renormalon bubble chain of quark loops.

$\mathrm{Tr}\{U_x U_y^\dagger\}$ as $x \to y$ [9]. For the gluon loop, we can use the NLO BFKL results as an independent calculation. Let us repeat the arguments of Ref. [9] for this case. The characteristic transverse scale inside the shockwave is small (see the discussion in Ref. [9]), and therefore the contribution of the diagram in Fig. 11 reduces to the contribution of a certain operator *local* in the transverse space. This would bring the additional terms with nontrivial z dependence to the kernel, which translates into the nontrivial additional γ-dependent term in the eigenvalues. Such terms do not exist, and therefore the gluon interaction with the shockwave does not get an extra $O(\alpha_s)$ correction.

VII. ARGUMENT OF THE COUPLING CONSTANT IN THE BK EQUATION

In this section we briefly summarize the results of the renormalon-based analysis of the argument of the coupling constant carried in Refs. [9,10].

To get an argument of the coupling constant we can trace the quark part of the β function (proportional to n_f). In the leading log approximation $\alpha_s \ln\frac{p^2}{\mu^2} \sim 1$, $\alpha_s \ll 1$, the quark part of the β function comes from the bubble chain of quark loops in the shockwave background. We can either have no intersection of the quark loop with the shockwave [see Fig. 12(a)] or we may have one of the loops in the shockwave background [see Fig. 12(b)].

The sum of these diagrams yields

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle = 2\alpha_s \mathrm{Tr}\{t^a U_x t^b U_y^\dagger\} \times \int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}] \times [e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}] \times \frac{1}{p^2(1+\frac{\alpha_s}{6\pi}\ln\frac{\mu^2}{p^2})}\Big(1 - \frac{\alpha_s n_f}{6\pi}\ln\frac{l^2}{\mu^2}\Big) \times \partial_\perp^2 U^{ab}(l) \frac{1}{(p-l)^2(1+\frac{\alpha_s}{6\pi}\ln\frac{\mu^2}{(p-l)^2})} \tag{123}$$

where we have left only the β-function part of the quark loop. Replacing the quark part of the β function $-\frac{\alpha_s}{6\pi}n_f \ln\frac{p^2}{\mu^2}$ by the total contribution $\frac{\alpha_s}{4\pi} b \ln\frac{p^2}{\mu^2}$, we get

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle = 2\mathrm{Tr}\{t^a U_x t^b U_y^\dagger\} \times \int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 q [e^{i(p,x)_\perp} - e^{i(p,y)_\perp}] \times [e^{-i(p-l,x)_\perp} - e^{-i(p-l,y)_\perp}] \frac{\alpha_s(p^2)}{p^2} \times \alpha_s^{-1}(l^2)\partial_\perp^2 U^{ab}(q) \frac{\alpha_s((p-l)^2)}{(p-l)^2}. \tag{124}$$

In principle, one should also include the "renormalon dressing" of the double-log and conformal terms in Eq. (5). We think, however, that they form a separate contribution which has nothing to do with the argument of the BK equation.

To go to the coordinate space, we expand the coupling constants in Eq. (124) in powers of $\alpha_s = \alpha_s(\mu^2)$, i.e. return back to Eq. (123) with $\frac{\alpha_s}{6\pi}n_f \to -b\frac{\alpha_s}{4\pi}$. Unfortunately, the Fourier transformation to the coordinate space can be performed explicitly only for the two nontrivial terms of the expansion $\alpha_s(p^2) \simeq \alpha_s - \frac{b\alpha_s}{4\pi}\ln p^2/\mu^2 + (\frac{b\alpha_s}{4\pi} \times \ln p^2/\mu^2)^2$. In the first order we get the running-coupling part of the NLO BK equation (5),

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle = \frac{\alpha_s}{2\pi^2}\int d^2z [\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c \mathrm{Tr}\{U_x U_y^\dagger\}] \times \Big[\frac{(x-y)^2}{X^2Y^2}\Big(1 + b\frac{\alpha_s}{4\pi}\ln(x-y)^2\mu^2\Big) - b\frac{\alpha_s}{4\pi}\frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}\Big]. \tag{125}$$

The result of the Fourier transformation up to the second order has the form [9,10]

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle = \frac{\alpha_s}{2\pi^2}\int d^2z[\mathrm{Tr}\{U_x U_z^\dagger\}\mathrm{Tr}\{U_z U_y^\dagger\} - N_c\mathrm{Tr}\{U_x U_y^\dagger\}]\bigg\{\frac{(x-y)^2}{X^2Y^2}\bigg[1+\frac{b\alpha_s}{4\pi}\bigg(\ln(x-y)^2\mu^2+\frac{5}{3}\bigg) + \bigg(\frac{b\alpha_s}{4\pi}\bigg)^2\ln^2(x-y)^2\mu^2\bigg] + \frac{b\alpha_s}{4\pi}\frac{1}{X^2}\ln\frac{X^2}{Y^2}\bigg[1+\frac{b\alpha_s}{4\pi}\ln(x-y)^2\mu^2+\frac{b\alpha_s}{4\pi}\ln X^2\mu^2\bigg] - \frac{b\alpha_s}{4\pi}\frac{1}{Y^2}\ln\frac{X^2}{Y^2}\bigg[1+\frac{b\alpha_s}{4\pi}\ln(x-y)^2\mu^2+\frac{b\alpha_s}{4\pi}\ln Y^2\mu^2\bigg]\bigg\} + \ldots. \tag{126}$$

We extrapolate the ln+ln^2 terms in the above equation as follows:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_x \hat U_y^\dagger\} = \frac{\alpha_s((x-y)^2)}{2\pi^2}\int d^2z[\mathrm{Tr}\{\hat U_x \hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z \hat U_y^\dagger\} - N_c\mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}] \tag{127}$$

$$\times\bigg[\frac{(x-y)^2}{X^2Y^2}+\frac{1}{X^2}\bigg(\frac{\alpha_s(X^2)}{\alpha_s(Y^2)}-1\bigg)+\frac{1}{Y^2}\bigg(\frac{\alpha_s(Y^2)}{\alpha_s(X^2)}-1\bigg)\bigg] + \ldots \tag{128}$$

where dots stand for the remaining conformal terms and the ln^2 term. (Here we promoted Wilson lines in the r.h.s. to operators.)

When the sizes of the dipoles are very different, the kernel of the above equation reduces to

$$\begin{aligned} &\frac{\alpha_s((x-y)^2)}{2\pi^2}\frac{(x-y)^2}{X^2Y^2} && |x-y| \ll |x-z|, |y-z|,\\ &\frac{\alpha_s(X)^2)}{2\pi^2X^2} && |x-z| \ll |x-y|, |y-z|,\\ &\frac{\alpha_s(Y)^2)}{2\pi^2Y^2} && |y-z| \ll |x-y|, |x-z|. \end{aligned} \tag{129}$$

In the earlier paper [9] Eq. (127) was interpreted as an indication that the argument of the coupling constant is the size of the parent dipole $x-y$. We are grateful to G. Salam for pointing out that the proper interpretation is the size of the smallest dipole as follows from Eq. (129).

It is instructive to compare our result to the paper [10], where the NLO BK equation is rewritten in terms of three effective coupling constants. The authors of Ref. [10] extrapolate Eq. (126) in a different way,

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat U_x \hat U_y^\dagger\} = \frac{1}{2\pi^2}\int d^2z[\mathrm{Tr}\{\hat U_x \hat U_z^\dagger\}\mathrm{Tr}\{\hat U_z \hat U_y^\dagger\} - N_c\mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}]\bigg[\frac{1}{X^2}\alpha_s(X^2) + \frac{1}{Y^2}\alpha_s(Y^2) - \frac{2(x-z,y-z)}{X^2Y^2}\times\frac{\alpha_s(X^2)\alpha_s(Y^2)}{\alpha_s(R^2)}\bigg] \tag{130}$$

where R^2 is some scale interpolating between X^2 and Y^2 (the explicit form can be found in Ref. [10]). Theoretically, until the Fourier transformations in all orders in $\ln p^2/\mu^2$ are performed, both of these interpretations are models of the high-order behavior of the running-coupling constant. The convenience of these models can be checked by the numerical estimates of the size of the neglected term(s) in comparison to terms taken into account by the model; see the discussions in Refs. [20].

VIII. CONCLUSIONS AND OUTLOOK

We have calculated the NLO kernel for the evolution of the color dipole. It consists of three parts: the running-coupling part proportional to the β function (see diagrams shown in Fig. 8), the conformal part describing $1\to 3$ dipoles transition (diagrams in Fig. 6), and the nonconformal term coming from the diagrams in Fig. 9. The result agrees with the forward NLO BFKL kernel [13] up to a term proportional to $\alpha_s^2\zeta(3)$ times the original dipole. We think that the difference could be due to different definitions of the cutoff in the longitudinal momenta. There is not any obvious preferred definition of the cutoff in the longitudinal momenta, so it can be chosen in any way convenient for practical calculations of higher orders. (It is worth noting that all cutoffs should give the same α_s correction to the intercept of the BFKL Pomeron determined by the rightmost singularity in the complex ω plane). Our goal was to study the dipole amplitudes with the cutoff closely related to the small-x asymptotics of the anomalous dimensions of twist-2 gluon operators. It would be instructive to get the $j\to 1$ asymptotics of the anomalous dimensions of gluon operators directly from Eq. (5), without a Fourier transformation of our result to the momentum space and comparing to NLO BFKL as it is done in Sec. VI. The study is in progress.

There is a recent paper [21] where the dipole form of the nonforward NLO BFKL kernel is calculated using the nonforward NLO BFKL kernel [22]. The kernel obtained in [21] is different from our result (and not conformally invariant). We think that at least part of the difference comes from the fact that the evolution kernel (5) should be compared to the nonsymmetric "evolution" NLO BFKL kernel $K^{\mathrm{evol}}(q,p)$ rather than to the symmetric kernel $K(q,p)$ defined by Eq. (113). The kernel K^{evol} corresponds to the Green function $\tilde G_\omega$ defined by Eq. (113) with a different lower cutoff for the longitudinal integration

$$\langle\hat{\mathcal{U}}(x,0)\rangle=\frac{1}{4\pi^2}\int\frac{d^2q}{q^2}\frac{d^2q'}{q'^2}\alpha_s(q)(e^{iqx}-1)(e^{-iqx}-1)\times\Phi_B(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\Big(\frac{s}{q'^2}\Big)^{\omega}\tilde{G}_\omega(q,q'). \quad (131)$$

$\tilde{G}_\omega(q,q')$ satisfies Eq. (114) with the kernel $K^{\rm evol}$,

$$\omega\tilde{G}_\omega(q,q')=\delta^{(2)}(q-q')+\int d^2pK^{\rm evol}(q,p)\tilde{G}_\omega(p,q') \quad (132)$$

and the relation between $K^{\rm evol}(q,p)$ and $K(q,p)$ has the form (cf. Ref. [13])

$$K^{\rm evol}(q,p)=K(q,p)-\frac{1}{2}\int d^2q'K(q,q')\ln\frac{q^2}{q'^2}K(q',p). \quad (133)$$

It is easy to see that the structure (131) repeats itself after differentiation with respect to s, so it can be rewritten as an evolution equation for $\mathcal{U}(x)$ [whereas the derivative of the original formula (113) does not have the structure of the evolution equation due to an extra $\frac{1}{|q|^\omega}$]. In terms of eigenvalues, the modified kernel (133) lead to the shifts of the type $\chi(n,\gamma)\to\chi(n,\gamma-\frac{\omega}{2})$ which we saw in Sec. VI B.

It should be emphasized that the conformally invariant NLO kernel describes the evolution of the lightlike Wilson lines with the rigid cutoff in the longitudinal momenta (17). On the contrary, for dipoles with the non-lightlike slope, the sum of the diagrams in Fig. 6 is not conformally invariant (see Appendix B). The reason is that a general Wilson line is a nonlocal operator which is not conformally invariant to begin with—for example, the non-lightlike Wilson line turns into a circle under the inversion $x^\mu\to x^\mu/x^2$. With the lightlike Wilson lines, the situation is different. Formally, a Wilson line

$$[\infty p_1+x_\perp,-\infty p_1+x_\perp]=\mathrm{P}\exp\Big\{ig\int_{-\infty}^{\infty}dx^+A_+(x^+,x_\perp)\Big\} \quad (134)$$

is invariant under the inversion $x^\mu\to x^\mu/x^2$ [with respect to the point with a zero (−) component]. Indeed, $(x^+,x_\perp)^2=-x_\perp^2$, so after the inversion $x_\perp\to x_\perp/x_\perp^2$ and $x^+\to x^+/x_\perp^2$, and therefore

$$[\infty p_1+x_\perp,-\infty p_1+x_\perp]\to\mathrm{P}\exp\Big\{ig\int_{-\infty}^{\infty}d\frac{x^+}{x_\perp^2}A_+\Big(\frac{x^+}{x_\perp^2},x_\perp\Big)\Big\}=[\infty p_1+x_\perp,-\infty p_1+x_\perp]. \quad (135)$$

Thus, it is not surprising that the bulk of our NLO kernel for the lightlike dipoles is conformally invariant in the transverse space. The part proportional to the β function is not conformally invariant and should not be, but there is another term, $\sim\ln\frac{(x-y)^2}{(x-z)^2}\ln\frac{(x-y)^2}{(y-z)^2}$, which is not invariant. The reason for this is probably the cutoff $|\alpha|<\sigma$, which can be expressed as a cutoff in the longitudinal coordinate x^+, and therefore under the inversion $x^+\to x^+/x_\perp^2$ the cutoff can pick up some logs of transverse separations. It is worth noting that conformal and nonconformal terms come from graphs with different topology: the conformal terms come from the $1\to3$ dipole diagrams in Fig. 6 which describe the dipole creation, while the nonconformal double-log term comes from the $1\to2$ dipole transitions (see Fig. 9) which can be regarded as a combination of dipole creation and dipole recombination. It is possible that in the effective action language, symmetric with respect to the projectile and the target [23], the evolution kernel is conformally invariant. We hope to study this problem in a separate publication.

Finally, let us present the evolution equation for matrix elements of color dipoles for large nuclei in the leading-N_c approximation. Using the standard mean-field approximation [5]

$$\langle A|\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\}|A\rangle=\langle A|\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\}|A\rangle,$$
$$\langle A|\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\}|A\rangle=\langle A|\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat{U}_z\hat{U}_{z'}^\dagger\}|A\rangle\langle A|\mathrm{Tr}\{\hat{U}_{z'}\hat{U}_y^\dagger\}|A\rangle,$$

one easily obtains from Eq. (5) $[N(x,y)\equiv\langle A|\mathcal{U}(x,y)|A\rangle]$

$$\frac{d}{d\eta}N(x,y)=\frac{\alpha_sN_c}{2\pi^2}\int d^2z\frac{(x-y)^2}{X^2Y^2}\Big\{1+\frac{\alpha_sN_c}{4\pi}\Big[\frac{11}{3}\ln(x-y)^2\mu^2-\frac{11}{3}\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2}+\frac{67}{9}-\frac{\pi^2}{3}-2\ln\frac{X^2}{(x-y)^2}\ln\frac{Y^2}{(x-y)^2}\Big]\Big\}$$
$$\times[N(x,z)+N(z,y)-N(x,y)-N(x,z)N(z,y)]$$
$$+\frac{\alpha_s^2N_c^2}{8\pi^4}\int d^2zd^2z'\Big\{-\frac{2}{(z-z')^4}+\Big[\frac{X^2Y'^2+X'^2Y^2-4(x-y)^2(z-z')^2}{(z-z')^4(X^2Y'^2-X'^2Y^2)}+\frac{(x-y)^4}{X^2Y'^2(X^2Y'^2-X'^2Y^2)}$$
$$+\frac{(x-y)^2}{X^2Y'^2(z-z')^2}\Big]\ln\frac{X^2Y'^2}{X'^2Y^2}\Big\}[N(z,z')-N(x,z)N(z,z')-N(z,z')N(z',y)-N(x,z)N(z',y)+N(x,z)N(z,y)$$
$$+N(x,z)N(z,z')N(z',y)]. \quad (136)$$

In this closed form the NLO evolution equation (136) can be used for numerical simulations of the dipole evolution.

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APPENDIX A: UV PART OF THE ONE-TO-THREE DIPOLES KERNEL

As we mentioned above, it is convenient to separate the UV-divergent and UV-finite parts of Eq. (43) by writing down $U_z^{mm'}U_{z'}^{nn'} = (U_z^{mm'}U_{z'}^{nn'} - U_z^{mm'}U_z^{nn'}) + U_z^{mm'}U_z^{nn'}$. The contribution of the first part leads to Eq. (58), while the second UV-divergent terms have the same color structure as the leading order BK equation. After replacing $U_z^{mm'}U_{z'}^{nn'}$ by $U_z^{mm'}U_z^{nn'}$, integrating over u with the prescription (47), and changing variables to $k_2 = q_2 = k'$, $p = q_1 + q_2$, $l = q_1 - k_1$ (so that $q_1 = p - k'$, $k_1 = p - l - k'$, and $k_1 + k_2 = p - l$), Eq. (41) turns into

$$\begin{aligned}\langle \mathrm{Tr}\{\hat U_x \hat U_y^\dagger\}\rangle_{\text{Fig. 6 } z'\to z} &= \frac{g^4}{8\pi^2}\int_0^\sigma \frac{d\alpha}{\alpha}\int d^2z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big)\\
&\times\bigg[\Big\{\int d\!\!{}^-{}^{2-\varepsilon}p\, d\!\!{}^-{}^{2-\varepsilon}l F_1(p,l) + \int d\!\!{}^-{}^2p\, d\!\!{}^-{}^2l F_2(p,l)\Big\}(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})\\
&+ (e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)})\\
&\times \frac{(k',p-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\\
&+ \int d\!\!{}^-{}^2p\, d\!\!{}^-{}^2l\, d\!\!{}^-{}^2k'(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l-k',X)-i(k',Y)} - e^{-i(p-l-k',Y)-i(k',X)})\\
&\times \frac{(k',p-l-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-k')}{p^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2}\bigg]\end{aligned} \tag{A1}$$

where

$$\begin{aligned}F_1(p,l) &= \int d\!\!{}^-{}^{2-\varepsilon}k'\bigg(\frac{1-\frac{\varepsilon}{2}}{p^2(p-l)^2}\Big\{-2 - \frac{(p-k')^2 + (p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2}\ln\frac{(p-k')^2}{(p-k'-l)^2} + \frac{k'^2 + (p-k')^2}{(p-k')^2 - k'^2}\ln\frac{(p-k')^2}{k'^2}\\
&+ \frac{(p-k'-l)^2 + k'^2}{(p-k'-l)^2 - k'^2}\ln\frac{(p-k'-l)^2}{k'^2}\Big\} + \frac{2(p,p-l)}{p^2(p-l)^2}\Big\{\Big(\frac{(p-k',p-k'-l)}{(p-k')^2(p-k'-l)^2} - \frac{1}{k'^2}\Big)\ln\frac{(p-k')^2(p-k'-l)^2}{k'^4}\\
&- \Big(\frac{(p-k',p-k'-l)}{(p-k')^2} + \frac{(p-k',p-k'-l)}{(p-k'-l)^2} + 2\Big)\frac{\ln(p-k')^2/(p-k'-l)^2}{(p-k')^2 - (p-k'-l)^2}\Big\}\\
&+ \frac{2}{p^2}\frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{p^2}{k'^2} + \frac{2}{(p-l)^2}\frac{(p-l,p-k')}{(p-k')^2k'^2}\ln\frac{(p-l)^2}{k'^2}\bigg)\end{aligned} \tag{A2}$$

and

$$\begin{aligned}F_2(p,l) &= \mu^{2\epsilon}\int d\!\!{}^-{}^2k'\bigg(2l_i\Big[\Big(-\frac{p_i}{p^2} + \frac{(p-k')_i}{(p-k')^2}\Big)\frac{\ln(p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2 - (p-k'-l)^2]} + \Big(\frac{(p-l)_i}{(p-l)^2} - \frac{(p-l-k')_i}{(p-k'-l)^2}\Big)\\
&\times \frac{1/k'^2}{(p-k')^2 - (p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}\Big]\end{aligned} \tag{A3}$$

$$+\frac{2}{p^2}\bigg[\frac{2(l,p-k'-l)(p,k')/k'^2}{(p-k'-l)^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-k'-l)^2}-\frac{(p-k',p-l-k')(p,k')}{k'^2(p-k')^2(p-k'-l)^2}\ln\frac{(p-k')^2}{k'^2}$$
$$+\frac{(l,k')/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}-\frac{(p-k'-l,k')}{2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2}+\frac{(p-l,k')/k'^2}{(p-l-k')^2-k'^2}\ln\frac{(p-k'-l)^2}{k'^2}$$
$$+\frac{2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}-\frac{2\ln(p-l-k')^2/k'^2}{(p-l-k')^2-k'^2}+\frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{(p-k')^2}{p^2}\bigg]$$
$$+\frac{2}{(p-l)^2}\bigg[\frac{-2(l,p-k')(p-l,k')/k'^2}{(p-k')^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-k'-l)^2}-\frac{(p-k',p-l-k')(p-l,k')}{k'^2(p-k')^2(p-k'-l)^2}\ln\frac{(p-k'-l)^2}{k'^2}$$
$$-\frac{(l,k')/k'^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}+\frac{(p,k')/k'^2}{(p-k')^2-k'^2}\ln\frac{(p-k')^2}{k'^2}-\frac{(p-k',k')}{2k'^2(p-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$+\frac{2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}-\frac{2\ln(p-k')^2/k'^2}{(p-k')^2-k'^2}+\frac{(p-l,p-k')}{(p-k')^2k'^2}\ln\frac{(p-l-k')^2}{(p-l)^2}\bigg]\bigg). \tag{A4}$$

We need to perform the integration over k'. Let us start with the UV-divergent term $\sim F_1$. Using the integrals

$$4\pi\int d\!\!{}^-{}^dk'\frac{(l-k')_i}{k'^2(l-k')^2}\ln\frac{p^2}{k'^2}=l_i\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)}\frac{\Gamma(2-\frac{d}{2})}{l^{4-d}}\bigg[\ln\frac{p^2}{l^2}+\frac{2}{d-2}+\psi\Big(2-\frac{d}{2}\Big)+\psi(d-1)-\psi\Big(\frac{d}{2}\Big)-\psi(1)\bigg],$$

$$4\pi\int d\!\!{}^-{}^dk'\frac{(k',k'-l)}{k'^2(l-k')^2}\ln\frac{(p-k')^2}{k'^2}=\frac{1}{2}\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2},$$

$$4\pi\int d\!\!{}^-{}^dk'\frac{(p-k',p-k'-l)}{(p-k')^2(p-k'-l)^2}\ln\frac{(p-k')^2(p-k'-l)^2}{k'^4}=-\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2},$$

$$4\pi\int d\!\!{}^-{}^dk'\frac{(p,p-l-k')}{(p-l-k')^2k'^2}\ln\frac{p^2}{k'^2}=(p,p-l)\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(d-1)}\frac{\Gamma(2-\frac{d}{2})}{|p-l|^{4-d}}\times\bigg[\ln\frac{p^2}{(p-l)^2}+\frac{2}{d-2}+\psi\Big(2-\frac{d}{2}\Big)+\psi(d-1)-\psi\Big(\frac{d}{2}\Big)-\psi(1)\bigg] \tag{A5}$$

one obtains

$$F_1(p,l)=\frac{1}{4\pi}\bigg\{\frac{2(p,p-l)}{p^2(p-l)^2}\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Gamma(\varepsilon/2)\Big(-4+\frac{1-\frac{\varepsilon}{2}}{3-\varepsilon}\Big)+\frac{2(p,p-l)}{p^2(p-l)^2}\Big(\frac{11}{3}\ln\frac{l^2}{\mu^2}-\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2}-\ln^2\frac{(p-l)^2}{p^2}+\frac{\pi^2}{3}\Big)-\frac{\ln p^2/l^2}{3(p-l)^2}-\frac{\ln(p-l)^2/l^2}{3p^2}+O(\varepsilon)\bigg\}. \tag{A6}$$

Let us, at first, consider the UV-divergent contribution

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{UV}}=\frac{\alpha_s^2}{\pi}\mu^{2\varepsilon}\int d^{2-\varepsilon}z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU^\dagger_z\}\mathrm{Tr}\{U_zU^\dagger_y\}-\frac{1}{2}\mathrm{Tr}\{U_xU^\dagger_y\}\Big)\times\int d\!\!{}^-{}^{2-\varepsilon}p\,d\!\!{}^-{}^{2-\varepsilon}l(e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2}\times\bigg[\frac{\Gamma^2(1-\frac{\varepsilon}{2})}{\Gamma(2-\varepsilon)}\Gamma(\varepsilon/2)\Big(-4+\frac{1-\frac{\varepsilon}{2}}{3-\varepsilon}\Big)+\frac{11}{3}\ln\frac{l^2}{\mu^2}+O(\varepsilon)\bigg]. \tag{A7}$$

To this contribution we should add the counterterm corresponding to quark and gluon loops lying inside the shockwave. The rigorous calculation of the counterterm was performed in Ref. [9], and the result is

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{CT}} = -b\frac{\alpha_s^2}{\pi}\frac{2}{\varepsilon}\int d^{2-\varepsilon}z_\perp\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big) \times \int d\!\!{}^-{}^d p\, d\!\!{}^-{}^d l(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2} \tag{A8}$$

where we need the gluon part of $b(=\frac{11}{3}N_c)$. After subtraction of the counterterm (A8) the UV-divergent contribution (A7) reduces to

$$\frac{d}{d\sigma}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{UV\text{-}CT}} = \frac{\alpha_s^2}{\pi}\int d^2z\Big(\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big) \times \int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l(e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})\frac{(p,p-l)}{p^2(p-l)^2}\Big[\frac{11}{3}\ln\frac{l^2}{\mu^2} - \frac{67}{9}\Big] \tag{A9}$$

so one obtains the regularized F_1 in the form

$$F_1^{\mathrm{reg}}(p,l) = \frac{1}{4\pi}\Big[\frac{2(p,p-l)}{p^2(p-l)^2}\Big(\frac{11}{3}\ln\frac{l^2}{\mu^2} - \frac{67}{9} - \ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2} - \ln^2\frac{(p-l)^2}{p^2} + \frac{\pi^2}{3}\Big) - \frac{\ln p^2/l^2}{3(p-l)^2} - \frac{\ln(p-l)^2/l^2}{3p^2}\Big]. \tag{A10}$$

It is convenient to calculate first the Fourier transform with $e^{i(p,X)-i(p-l,Y)}$. Using the integrals

$$\begin{aligned}\int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}\frac{(p,p-l)}{p^2(p-l)^2}\ln\frac{l^2}{\mu^2} &= -\frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln\frac{X^2Y^2}{\Delta^2}\mu^2,\\ \int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}\frac{(p,p-l)}{p^2(p-l)^2}\ln\frac{p^2}{l^2}\ln\frac{(p-l)^2}{l^2} &= \frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2},\\ \int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}\frac{(p,p-l)}{p^2(p-l)^2}\ln^2\frac{(p-l)^2}{p^2} &= \frac{1}{4\pi^2}\frac{(X,Y)}{X^2Y^2}\ln^2\frac{X^2}{Y^2}\end{aligned} \tag{A11}$$

we get

$$\int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}F_1^{\mathrm{reg}}(p,l) = -\frac{11}{24\pi^3}\frac{(X,Y)}{X^2Y^2}\Big(\ln\frac{X^2Y^2}{\Delta^2}\mu^2 + \frac{67}{33}\Big) - \frac{1}{16\pi^3}\frac{(X,Y)}{X^2Y^2}\Big[\ln^2\frac{X^2}{\Delta^2} + \ln^2\frac{Y^2}{\Delta^2} + \ln^2\frac{X^2}{Y^2} - \frac{2\pi^2}{3}\Big] - \frac{1}{48\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2} + \frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big]. \tag{A12}$$

Hereafter we use the notation $\Delta \equiv X - Y = x - y$.

Next we calculate the F_2 contribution. We need the following Fourier integrals:

$$\int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}\int d\!\!{}^-{}^2 k' 2l_i\Big[\Big(-\frac{p_i}{p^2} + \frac{(p-k')_i}{(p-k')^2}\Big)\frac{\ln(p-k')^2/(p-l-k')^2}{k'^2[(p-k')^2 - (p-k'-l)^2]} + \Big(\frac{(p-l)_i}{(p-l)^2} - \frac{(p-l-k')_i}{(p-k'-l)^2}\Big)\frac{1/k'^2}{(p-k')^2 - (p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2}\Big] = \frac{(X,Y)}{16\pi^3X^2Y^2}\Big[\ln^2\frac{X^2}{Y^2} + \frac{2\pi^2}{3}\Big], \tag{A13}$$

$$\begin{aligned}&\int d\!\!{}^-{}^2 p\, d\!\!{}^-{}^2 l e^{i(p,\Delta)+i(l,Y)}\Big[\frac{4}{p^2}\int d\!\!{}^-k'\frac{(l,p-l-k')(p,k')}{k'^2(p-k'-l)^2((p-k')^2 - (p-k'-l)^2)}\ln\frac{(p-k')^2}{(p-k'-l)^2}\\ &\quad - \frac{4}{(p-l)^2}\int d\!\!{}^-k'\frac{(l,p-k')(p-l,k')}{k'^2(p-k')^2((p-k')^2 - (p-k'-l)^2)}\ln\frac{(p-k')^2}{(p-k'-l)^2}\Big]\\ &= \frac{1}{16\pi^3}\Big\{-\frac{\pi^2}{3}\frac{(X+Y)^2}{X^2Y^2} + \frac{2}{Y^2}\int_0^1 du\frac{\ln u}{u - \frac{X^2}{X^2-Y^2}} + \frac{2}{X^2}\int_0^1 du\frac{\ln u}{u + \frac{Y^2}{X^2-Y^2}}\\ &\quad + \frac{i\kappa}{X^2Y^2}\Big(2\int_0^1 du\Big[\frac{\ln u}{u - \frac{(X,\Delta)-i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(Y,\Delta)+i\kappa}{\Delta^2}} - \mathrm{c.c.}\Big] + \ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big)\Big\},\end{aligned} \tag{A14}$$

where $\kappa = \sqrt{X^2Y^2 - (X,Y)^2}$, and

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\Delta)+i(l,Y)} \int d\!\!\!/^2 k' \Big[-\frac{2(p-k',p-l-k')(p,k')}{p^2k'^2(p-k')^2(p-k'-l)^2} \ln\frac{(p-k')^2}{k'^2} - \frac{2(p-k',p-l-k')(p-l,k')}{(p-l)^2k'^2(p-k')^2(p-k'-l)^2} \ln\frac{(p-k'-l)^2}{k'^2} \Big]$$
$$= \frac{1}{32\pi^3X^2Y^2}\Big[X^2\ln^2\frac{X^2}{\Delta^2} + Y^2\ln^2\frac{Y^2}{\Delta^2} + 2(X,Y)\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}\Big] + \frac{i\kappa}{16\pi^3X^2Y^2}\int_0^1 du\Big[\frac{\ln u}{u - \frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.} - \frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big], \tag{A15}$$

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\Delta)+i(l,Y)} \int d\!\!\!/^2 k' \Big(\frac{2(l,k')p^{-2}}{k'^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2} + \frac{2(p-l,k')p^{-2}}{k'^2[(p-k'-l)^2-k'^2]}\ln\frac{(p-l-k')^2}{k'^2} - \frac{2(l,k')(p-l)^{-2}}{k'^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2} + \frac{2(p,k')(p-l)^{-2}}{k'^2[(p-k')^2-k'^2]}\ln\frac{(p-k')^2}{k'^2}\Big)$$
$$= -\frac{1}{16\pi^3}\Big(\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2} - \frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}\Big)\ln\frac{X^2}{Y^2} + \frac{1}{8\pi^3}\Big[\frac{\pi^2}{6X^2} + \frac{\pi^2}{6Y^2} - \frac{1}{X^2}\int_0^1 \frac{du\ln u}{u + \frac{Y^2}{X^2-Y^2}} - \frac{1}{Y^2}\int_0^1\frac{du\ln u}{u - \frac{X^2}{X^2-Y^2}}\Big], \tag{A16}$$

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\delta)+i(l,Y)} \int d\!\!\!/^2 k' \Big[-\frac{(p-k'-l,k')}{p^2k'^2(p-l-k')^2}\ln\frac{(p-k')^2}{k'^2} - \frac{(p-k',k')}{(p-l)^2k'^2(p-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\Big]$$
$$= \frac{1}{32\pi^3}\ln\frac{X^2}{Y^2}\Big[\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2} - \frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}\Big], \tag{A17}$$

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\Delta)+i(l,Y)} \int d\!\!\!/^2 k' \Big[\frac{4/p^2}{(p-k')^2-(p-k'-l)^2}\ln\frac{(p-k')^2}{(p-l-k')^2} - \frac{4\ln(p-l-k')^2/k'^2}{p^2[(p-l-k')^2-k'^2]} + \frac{4}{(p-l)^2[(p-k')^2-(p-k'-l)^2]}\ln\frac{(p-k')^2}{(p-l-k')^2} - \frac{4\ln(p-k')^2/k'^2}{(p-l)^2[(p-k')^2-k'^2]}\Big] = \frac{1}{4\pi^3Y^2}\ln\frac{X^2}{\Delta^2} + \frac{1}{4\pi^3X^2}\ln\frac{Y^2}{\Delta^2}, \tag{A18}$$

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\Delta)+i(l,Y)} \int d\!\!\!/^2 k' \Big[\frac{2(p,p-l-k')}{p^2(p-l-k')^2k'^2}\ln\frac{(p-k')^2}{p^2} + \frac{2(p-l,p-k')}{(p-k')^2k'^2(p-l)^2}\ln\frac{(p-l-k')^2}{(p-l)^2}\Big]$$
$$= \frac{i\kappa}{16\pi^3X^2Y^2}\Big\{2\int_0^1 du\Big[\frac{\ln u}{u - \frac{(X,\Delta)+i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(Y,\Delta)-i\kappa}{\Delta^2}} - \text{c.c.}\Big] - \ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\} + \frac{(X,Y)}{16\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2}. \tag{A19}$$

Adding the integrals (A13)–(A17) we obtain

$$\int d\!\!\!/^2 p d\!\!\!/^2 l e^{i(p,\Delta)+i(l,Y)} F_2(p,l) = \frac{1}{4\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2} + \frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big] + \frac{(X+Y)^2}{32\pi^3X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2} + \frac{(X,Y)}{8\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2} + \frac{i\kappa}{16\pi^3X^2Y^2}\Big\{\int_0^1 du\Big[\frac{\ln u}{u - \frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u + \frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.}\Big] - \frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\} \tag{A20}$$

and therefore

$$\int d^2p d^2l e^{i(p,\Delta)+i(l,Y)}[F_1^{\rm reg}(p,l)+F_2(p,l)] = -\frac{1}{8\pi^3}\frac{(X,Y)}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]+\frac{1}{32\pi^3}\frac{\Delta^2}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}$$
$$+\frac{11}{48\pi^3}\Big[\frac{1}{X^2}\ln\frac{Y^2}{\Delta^2}+\frac{1}{Y^2}\ln\frac{X^2}{\Delta^2}\Big]$$
$$+\frac{i\kappa}{16\pi^3X^2Y^2}\Big\{\int_0^1 du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-\text{c.c.}\Big]$$
$$-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\}. \tag{A21}$$

Note that the r.h.s. of this equation is finite as $X\to Y$ (taken separately, the contributions of F_1 and F_2 are singular in this limit):

$$\int d^2p d^2l e^{i(l,X)}[F_1^{\rm reg}(p,l)+F_2(p,l)] = -\frac{1}{8\pi^3}\frac{1}{X^2}\Big[\frac{11}{3}\ln X^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]. \tag{A22}$$

Using Eqs. (A18) and (A19) we obtain

$$\int d^2p d^2l e^{i(p,\Delta)+i(l,Y)}[F_1^{\rm reg}(p,l)+F_2(p,l)](e^{i(p,X)}-e^{i(p,Y)})(e^{-i(p-l,X)}-e^{-i(p-l,Y)})$$
$$=-\frac{1}{8\pi^3}\frac{\Delta^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]-\frac{1}{16\pi^3}\frac{\Delta^2}{X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}$$
$$-\frac{i\kappa}{8\pi^3X^2Y^2}\Big\{\int_0^1 du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-\text{c.c.}\Big]-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big\}. \tag{A23}$$

Now we turn our attention to the last two terms in Eq. (A1). Using Fourier transformation

$$\int d^2k_1 d^2k_2 e^{-i(k_1,x_1)-i(k_2,x_2)}\frac{k_{2i}}{(k_1+k_2)^2k_2^2}\ln\frac{k_1^2}{k_2^2}=\frac{i}{8\pi^2}\Big(x_{1i}-\frac{(x_1,x_{12})}{x_{12}^2}x_{12i}\Big)\frac{1}{i\kappa_{12}}\Big\{\int_0^1 du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_1^2}}-\text{c.c.}\Big]$$
$$+\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_2,x_{12})+i\kappa_{12}}{(x_2,x_{12})-i\kappa_{12}}\Big\}+\frac{ix_{12i}}{16\pi^2x_{12}^2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{x_1^2}{x_2^2} \tag{A24}$$

[where $x_{12}\equiv x_1-x_2$ and $\kappa_{12}\equiv\sqrt{x_1^2x_2^2-(x_1,x_2)^2}$], one easily obtains

$$\int d^2p d^2l d^2k'(e^{-i(p-l,X)}-e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)}-e^{i(p-k',Y)+i(k',X)})\frac{(k',p-k')}{(p-l)^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$=\frac{i\kappa}{16\pi^3X^2Y^2}\Big[\int_0^1 du\Big(\frac{\ln u}{u-\frac{(X,Y)-i\kappa}{X^2}}+\frac{\ln u}{u-\frac{(X,Y)-i\kappa}{Y^2}}-\text{c.c.}\Big)+\frac{1}{2}\ln\frac{X^2}{Y^2}\ln\frac{[(\Delta,X)+i\kappa][(\Delta,Y)+i\kappa]}{[(\Delta,X)-i\kappa][(\Delta,Y)-i\kappa]}\Big]$$
$$=\frac{i\kappa}{16\pi^3X^2Y^2}\Big[-\int_0^1 du\Big[\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}}+\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}}-\text{c.c.}\Big]-\frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big]-\frac{(X,Y)}{32\pi^3X^2Y^2}\ln^2\frac{X^2}{Y^2}. \tag{A25}$$

Similarly,

$$\int d^2k_1 d^2k_2 e^{-i(k_1,x_1)-i(k_2,x_2)}\frac{(k_1,k_2)k_{1i}}{(k_1+k_2)^2k_1^2k_2^2}\ln\frac{k_1^2}{k_2^2}$$
$$=\frac{i}{16\pi^2}\Big(x_{1i}-\frac{(x_1,x_{12})}{x_{12}^2}x_{12i}\Big)\frac{1}{i\kappa}\Big[\int_0^1 du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_1^2}}-\text{c.c.}\Big]+\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_2,x_{12})+i\kappa}{(x_2,x_{12})-i\kappa}\Big]-\frac{i}{16\pi^2}\Big(x_{2i}-\frac{(x_2,x_{12})}{x_{12}^2}x_{12i}\Big)$$
$$\times\frac{1}{i\kappa}\Big[\int_0^1 du\Big[\frac{\ln u}{u+\frac{(x_2,x_{12})-i\kappa}{x_2^2}}-\text{c.c.}\Big]+\frac{1}{2}\ln\frac{x_2^2}{x_{12}^2}\ln\frac{(x_1,x_{12})+i\kappa}{(x_1,x_{12})-i\kappa}\Big]+\frac{ix_{12i}}{32\pi^2x_{12}^2}\ln\frac{x_1^2x_2^2}{x_{12}^4}\ln\frac{x_1^2}{x_2^2}$$
$$+\frac{i}{16\pi^2}\Big(x_{12i}-\frac{(x_1,x_{12})}{x_1^2}x_{1i}\Big)\frac{1}{i\kappa}\Big[\int_0^1 du\Big[\frac{\ln u}{u-\frac{(x_1,x_{12})-i\kappa}{x_{12}^2}}-\text{c.c.}\Big]-\frac{1}{2}\ln\frac{x_1^2}{x_{12}^2}\ln\frac{(x_1,x_2)+i\kappa}{(x_1,x_2)-i\kappa}\Big]-\frac{ix_{1i}}{32\pi^2x_1^2}\ln\frac{x_1^2}{x_2^2}\ln\frac{x_2^2}{x_{12}^2} \tag{A26}$$

and therefore

$$\int d\!\!\!^-{}^2p d\!\!\!^-{}^2l d\!\!\!^-{}^2k'(e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)})\frac{2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$= \frac{i\kappa Y^{-2}}{16\pi^3X^2}\Big[\int_0^1 du\Big(-2\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}} - 2\frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}} - \text{c.c.}\Big) + \ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big] - \frac{(X,Y)}{32\pi^3X^2Y^2}\ln\frac{X^2\Delta^2}{Y^4}\ln\frac{X^2}{\Delta^2}$$
$$- \frac{(X,Y)}{32\pi^3X^2Y^2}\ln\frac{Y^2\Delta^2}{X^4}\ln\frac{Y^2}{\Delta^2} - \frac{1}{32\pi^3}\Big(\frac{1}{X^2}+\frac{1}{Y^2}\Big)\ln\frac{X^2}{Y^2}\ln\frac{Y^2}{\Delta^2}. \tag{A27}$$

Adding Eqs. (A25) and (A27) we obtain

$$\int d\!\!\!^-{}^2p d\!\!\!^-{}^2l d\!\!\!^-{}^2k'(e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)})$$
$$\times \frac{(k',p-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}$$
$$= \frac{i\kappa}{16\pi^3X^2Y^2}\Big[\int_0^1 du(\frac{\ln u}{u-\frac{(\Delta,X)+i\kappa}{\Delta^2}} + \frac{\ln u}{u+\frac{(\Delta,Y)-i\kappa}{\Delta^2}} - c.c.) - \frac{1}{2}\ln\frac{X^2Y^2}{\Delta^4}\ln\frac{(X,Y)+i\kappa}{(X,Y)-i\kappa}\Big] + \frac{\Delta^2}{32\pi^3X^2Y^2}\ln\frac{X^2}{\Delta^2}\ln\frac{Y^2}{\Delta^2}. \tag{A28}$$

It is easy to see that the contribution of the last term in Eq. (A1) is equal to (A28), so we get

$$\langle \mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle = \frac{g^4}{8\pi^2}\int_0^\sigma\frac{d\alpha}{\alpha}\int d^2z\Big[\frac{N_c}{2}\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{2}\mathrm{Tr}\{U_xU_y^\dagger\}\Big]$$
$$\times\Big[\int d\!\!\!^-{}^2p d\!\!\!^-{}^2l[F_1^{\mathrm{reg}}(p,l) + F_2(p,l)](e^{i(p,X)} - e^{i(p,Y)})(e^{-i(p-l,X)} - e^{-i(p-l,Y)})$$
$$+ 2\int d\!\!\!^-{}^2p d\!\!\!^-{}^2l d\!\!\!^-{}^2k'(e^{-i(p-l,X)} - e^{-i(p-l,Y)})(e^{i(p-k',X)+i(k',Y)} - e^{i(p-k',Y)+i(k',X)})$$
$$\times\frac{(k',p-k')(p-k')^2 - 2(p-k',p-l-k')(k',p-l-k')}{(p-l)^2(p-k')^2k'^2(p-l-k')^2}\ln\frac{(p-l-k')^2}{k'^2}\Big]$$
$$= -\frac{\alpha_s^2N_c}{8\pi^3}\int d^2z\Big[\mathrm{Tr}\{U_xU_z^\dagger\}\mathrm{Tr}\{U_zU_y^\dagger\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big]\frac{\Delta^2}{X^2Y^2}\Big[\frac{11}{3}\ln\frac{X^2Y^2}{\Delta^2}\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big]. \tag{A29}$$

Note that the dilogarithms and products of logarithms have canceled. The simplicity of the final result indicates that there should be a less tedious derivation, but we were not able to find it.

APPENDIX B: CUTOFF DEPENDENCE OF THE NLO KERNEL

We will repeat the procedure from Sec. III C, this time using the cutoff by the slope.

$$\langle K_{\mathrm{NLO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{shockwave}} = \frac{d}{d\eta}\langle\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{shockwave}} - \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat U_x\hat U_y^\dagger\}\rangle_{\mathrm{shockwave}}. \tag{B1}$$

Instead of Eq. (18) we get

$$g^4\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle$$
$$=\frac{1}{2}g^2\frac{s^2}{4}f^{anl}f^{bn'l'}\int d\!\!\!{}^-\alpha d\!\!\!{}^-\alpha_1 d\!\!\!{}^-\beta d\!\!\!{}^-\beta' d\!\!\!{}^-\beta_1 d\!\!\!{}^-\beta'_1 d\!\!\!{}^-\beta_2 d\!\!\!{}^-\beta'_2\int d^2zd^2z'\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2 e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp}$$
$$\times\frac{4\alpha_1(\alpha-\alpha_1)U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1,z)_\perp-i(q_2-k_2,z')_\perp}d_{\bullet\lambda}(\alpha p_1+\beta p_2+q_{1\perp}+k_{1\perp})d_{\lambda'\bullet}(\alpha p_1+\beta' p_2+q_{2\perp}+k_{2\perp})}{(\beta-\beta_1-\beta_2+i\epsilon)(\beta'-\beta'_1-\beta'_2+i\epsilon)(\beta+\xi\alpha-i\epsilon)(\beta'+\xi\alpha'-i\epsilon)[\alpha\beta s-(q_1+q_2)^2_\perp+i\epsilon][\alpha\beta' s-(k_1+k_2)^2_\perp+i\epsilon]}$$
$$\times\frac{d_{\mu\xi}(\alpha_1p_1+\beta_1p_2+q_{1\perp})}{\alpha_1\beta_1s-q^2_{1\perp}+i\epsilon}\frac{d^\xi_{\mu'}(\alpha_1p_1+\beta'_1p_2+k_{1\perp})}{\alpha_1\beta'_1s-k^2_{1\perp}+i\epsilon}\frac{d_{\nu\eta}((\alpha-\alpha_1)p_1+\beta_2p_2+q_{2\perp})}{(\alpha-\alpha_1)\beta_2s-q^2_{2\perp}+i\epsilon}$$
$$\times\frac{d^\eta_{\nu'}((\alpha-\alpha_1)p_1+\beta'_2p_2+k_{2\perp})}{(\alpha-\alpha_1)\beta'_2s-k^2_{2\perp}+i\epsilon}\Gamma^{\mu\nu\lambda}(\alpha p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\times\Gamma^{\mu'\nu'\lambda'}(\alpha p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})\qquad\text{(B2)}$$

where $\xi=e^{-2\eta_1}$. In this formula $\frac{1}{\beta+\xi\alpha-i\epsilon}$ comes from the integration over the u parameter in the l.h.s. and $\frac{1}{\beta'+\xi\alpha'-i\epsilon}$ from the integration over the v parameter.

Taking residues at $\beta=-\xi\alpha$, $\beta'=-\xi\alpha'$ and $\beta_2=-\beta_1$, $\beta'_2=-\beta'_1$ we obtain

$$\int_0^\infty du\int_{-\infty}^0 dv\langle\hat{A}^a_\bullet(un+x_\perp)\hat{A}^b_\bullet(vn+y_\perp)\rangle$$
$$=\frac{1}{2}g^2\frac{s^2}{4}f^{anl}f^{bn'l'}\int d\!\!\!{}^-\alpha d\!\!\!{}^-\alpha_1 d\!\!\!{}^-\beta_1 d\!\!\!{}^-\beta'_1\int d^2zd^2z'\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2 e^{i(q_1+q_2,x)_\perp-i(k_1+k_2,y)_\perp}4\frac{\alpha_1(\alpha-\alpha_1)}{\alpha^2}$$
$$\times U_z^{nn'}U_{z'}^{ll'}e^{-i(q_1-k_1)z-i(q_2-k_2)z'}\frac{(q_{1\perp}+q_{2\perp})_\lambda}{(q_1+q_2)^2_\perp+\xi\alpha^2}\frac{(k_{1\perp}+k_{2\perp})_{\lambda'}}{(k_1+k_2)^2_\perp+\xi\alpha^2}\frac{d^\xi_\mu(\alpha_1p_1+q_{1\perp})}{\alpha_1\beta_1s-q^2_{1\perp}+i\epsilon}\frac{d_{\xi\mu'}(\alpha_1p_1+k_{1\perp})}{\alpha_1\beta'_1s-k^2_{1\perp}+i\epsilon}$$
$$\times\frac{d^\eta_\eta((\alpha-\alpha_1)p_1+q_{2\perp})}{-(\alpha-\alpha_1)\beta_1s-q^2_{2\perp}+i\epsilon}\frac{d_{\eta\nu'}((\alpha-\alpha_1)p_1+k_{2\perp})}{-(\alpha-\alpha_1)\beta'_1s-k^2_{2\perp}+i\epsilon}$$
$$\times\Gamma^{\mu\nu\lambda}(\alpha_1p_1+q_{1\perp},(\alpha-\alpha_1)p_1+q_{2\perp},-\alpha p_1-q_{1\perp}-q_{2\perp})$$
$$\times\Gamma^{\mu'\nu'\lambda'}(\alpha_1p_1+k_{1\perp},(\alpha-\alpha_1)p_1+k_{2\perp},-\alpha p_1-k_{1\perp}-k_{2\perp})\qquad\text{(B3)}$$

which leads to [cf. Eq. (23)]

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle=\frac{g^4}{4\pi^2}\mathrm{Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^1du\bar{u}u\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2$$
$$\times\frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2\bar{u}+q_2^2u)(k_1^2\bar{u}+k_2^2u)}$$
$$\times\left[(q_1^2-q_2^2)\delta_{ij}-\frac{2}{u}q_{1i}(q_1+q_2)_j+\frac{2}{\bar{u}}(q_1+q_2)_iq_{2j}\right]$$
$$\times\left[(k_1^2-k_2^2)\delta_{ij}-\frac{2}{u}k_{1i}(k_1+k_2)_j+\frac{2}{\bar{u}}(k_1+k_2)_ik_{2j}\right]$$
$$=\frac{g^4}{4\pi^2}\mathrm{Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\xi\frac{d}{d\xi}\int_0^\infty d\alpha\int_0^\alpha d\alpha'\int d\!\!\!{}^-{}^2q_1 d\!\!\!{}^-{}^2q_2 d\!\!\!{}^-{}^2k_1 d\!\!\!{}^-{}^2k_2$$
$$\times\frac{(\alpha-\alpha')\alpha' e^{i(q_1,x-z)_\perp+i(q_2,x-z')_\perp-i(k_1,y-z)_\perp-i(k_2,y-z')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2(\alpha-\alpha')+q_2^2\alpha')(k_1^2(\alpha-\alpha')+k_2^2\alpha')}$$
$$\times\left[\frac{\delta_{ij}}{\alpha}(q_1^2-q_2^2)-\frac{2}{\alpha'}q_{1i}(q_1+q_2)_j+\frac{2}{\alpha-\alpha'}(q_1+q_2)_iq_{2j}\right]$$
$$\times\left[\frac{\delta_{ij}}{\alpha}(k_1^2-k_2^2)-\frac{2}{\alpha'}k_{1i}(k_1+k_2)_j+\frac{2}{\alpha-\alpha'}(k_1+k_2)_ik_{2j}\right]\qquad\text{(B4)}$$

(recall that $\frac{d}{d\eta}=-2\xi\frac{d}{d\xi}$). The contribution which is sensitive to the subtraction of $(\mathrm{LO})^2$ is

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\xi\frac{d}{d\xi}\int_0^\infty d\alpha\int_0^\alpha d\alpha'\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2 \times \frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2](q_1^2(\alpha-\alpha')+q_2^2\alpha')(k_1^2(\alpha-\alpha')+k_2^2\alpha')} \times\left[\frac{\alpha}{\alpha'}(q_1,k_1)+\frac{\alpha}{\alpha-\alpha'}(q_2,k_2)\right](q_1+q_2,k_1+k_2). \tag{B5}$$

The "+" prescription (45) leads to the subtraction

$$\frac{d}{d\eta}\langle \mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle = \xi\frac{d}{d\xi}\frac{g^4}{\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\int_0^\infty\frac{d\alpha}{\alpha}\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2(q_1+q_2,k_1+k_2) \times\frac{e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}\left\{\int_0^\alpha\frac{d\alpha'}{\alpha'}\left[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\right]+\int_0^\alpha\frac{d\alpha'}{\alpha-\alpha'}\left[\frac{\alpha^2(q_2,k_2)}{[q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_2,k_2)}{q_2^2k_2^2}\right]\right\}. \tag{B6}$$

The details of the upper cutoff in α do not matter since they correspond to changes in the impact factor which do not affect the evolution. For example,

$$\begin{aligned} &-2\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\frac{1}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}\int_0^\alpha\frac{d\alpha'}{\alpha'}\left[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\right]\\ &=\frac{1}{(q_1+q_2)^2(k_1+k_2)^2}\int_0^1\frac{du}{u}\left[\frac{(q_1,k_1)}{(q_1^2\bar{u}+q_2^2u][k_1^2\bar{u}+k_2^2u]}-\frac{(q_1,k_1)}{q_1^2k_1^2}\right]\\ &=\frac{d}{d\sigma}\int_0^\sigma\frac{d\alpha}{\alpha}\frac{1}{(q_1+q_2)^2(k_1+k_2)^2}\int_0^\alpha\frac{d\alpha'}{\alpha'}\left[\frac{\alpha^2(q_1,k_1)}{(q_1^2(\alpha-\alpha')+q_2^2\alpha'][k_1^2(\alpha-\alpha')+k_2^2\alpha']}-\frac{(q_1,k_1)}{q_1^2k_1^2}\right] \end{aligned} \tag{B7}$$

where the last line is exactly our "rigid cutoff" with "+" subtraction (45).

On the contrary, the details of the upper cutoff in α' are essential for the evolution equation (B1). The contribution to $\langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle$ corresponding to the "slope" cutoff (6) has the form

$$\langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle^{\mathrm{slope}} = -\frac{g^4}{2\pi^2}\mathrm{Tr}\{t^aU_xt^bU_y^\dagger\}f^{anl}f^{bn'l'}\int d^2zd^2z'U_z^{nn'}U_{z'}^{ll'}\int d\!\!\!{}^-{}^2q_1d\!\!\!{}^-{}^2q_2d\!\!\!{}^-{}^2k_1d\!\!\!{}^-{}^2k_2\frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2} \times e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}\int_0^\infty\frac{d\alpha'}{\alpha'}\left[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\right]$$

and therefore the difference between the subtractions in "rigid cutoff" (B6) and "slope cutoff" (B7) prescriptions can be written as

$$\begin{aligned}
&\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm rigid}-K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm slope}\rangle_{\rm Fig.\,3(a)}\\
&=\frac{g^4}{2\pi^2}{\rm Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U^{nn'}_zU^{ll'}_{z'}\int d\!\!{}^-{}^2q_1d\!\!{}^-{}^2q_2d\!\!{}^-{}^2k_1d\!\!{}^-{}^2k_2e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}\\
&\quad\times\Big\{\frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2}\int_0^\infty\frac{d\alpha'}{\alpha'}\Big[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\Big]\\
&\quad+2\xi\frac{d}{d\xi}\int_0^\infty\frac{d\alpha}{\alpha}\frac{(q_1+q_2,k_1+k_2)}{[(q_1+q_2)^2+\xi\alpha^2][(k_1+k_2)^2+\xi\alpha^2]}\Big[\frac{(q_1,k_1)}{q_1^2k_1^2}+\frac{(q_2,k_2)}{q_2^2k_2^2}\Big]\int_0^\alpha\frac{d\alpha'}{\alpha'}\Big\}\\
&=-\frac{g^4}{2\pi^2}{\rm Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2zd^2z'U^{nn'}_zU^{ll'}_{z'}\int d\!\!{}^-{}^2q_1d\!\!{}^-{}^2q_2d\!\!{}^-{}^2k_1d\!\!{}^-{}^2k_2e^{i(q_1,X)_\perp+i(q_2,X')_\perp-i(k_1,Y)_\perp-i(k_2,Y')_\perp}\\
&\quad\times\int_0^\infty\frac{d\alpha'}{\alpha'}\Big\{\frac{(q_1+q_2,k_1+k_2)}{(q_1+q_2)^2(k_1+k_2)^2}\Big[\frac{(q_1,k_1)}{(q_1^2+\xi\alpha'^2)(k_1^2+\xi\alpha'^2)}+\frac{(q_2,k_2)}{(q_2^2+\xi\alpha'^2)(k_2^2+\xi\alpha'^2)}\Big]\\
&\quad-\frac{(q_1+q_2,k_1+k_2)}{[(q_1+q_2)^2+\xi\alpha'^2][(k_1+k_2)^2+\xi\alpha'^2]}\Big[\frac{(q_1,k_1)}{q_1^2k_1^2}+\frac{(q_2,k_2)}{q_2^2k_2^2}\Big]\Big\}.
\end{aligned}$$

It is instructive to rewrite this result in Schwinger's notations,

$$\begin{aligned}
&\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm rigid}-K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm slope}\rangle_{\rm Fig.\,3(a)}\\
&=\frac{g^4}{2\pi^2}{\rm Tr}\{t^aU_xt^bU^\dagger_y\}f^{anl}f^{bn'l'}\int d^2z\int_0^\infty\frac{d\alpha'}{\alpha'}\Big[\Big(x\Big|\frac{p_i}{p^2+\xi\alpha'^2}\Big|z\Big)U^{nn'}_z\Big(z\Big|\frac{p_i}{p^2+\xi\alpha'^2}\Big|y\Big)\Big(z\Big|\frac{p_j}{p^2}U^{ll'}\frac{p_j}{p^2}\Big|y\Big)\\
&\quad-\Big(x\Big|\frac{p_i}{p^2}\Big|z\Big)U^{nn'}_z\Big(z\Big|\frac{p_i}{p^2}\Big|y\Big)\Big(z\Big|\frac{p_j}{p^2+\xi\alpha'^2}U^{ll'}\frac{p_j}{p^2+\xi\alpha'^2}\Big|y\Big)\Big].
\end{aligned}$$

We see now that the difference between the two regularizations of the longitudinal divergence is given by the difference of $(\mathrm{LO})^2$ contributions with cutoffs in α determined by the momenta on the first and on the second step of $(\mathrm{LO})^2$ evolution.

It is easy to see that for the sum of all diagrams this yields [see Eq. (53)]

$$\begin{aligned}
\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm rigid}-K_{\rm LO}{\rm Tr}\{\hat U_x\hat U^\dagger_y\}^{\rm slope}\rangle&=\alpha_s^2\int d^2zd^2z'[{\rm Tr}\{U_xU^\dagger_z\}{\rm Tr}\{U_zU^\dagger_{z'}\}{\rm Tr}\{U_{z'}U^\dagger_y\}-{\rm Tr}\{U_xU^\dagger_zU_{z'}U^\dagger_yU_zU^\dagger_{z'}\}\\
&\quad+(z\leftrightarrow z')]\int_0^\infty\frac{dt}{t}\Big\{\Big[\Big(x\Big|\frac{p_i}{p^2+t}\Big|z\Big)-\Big(y\Big|\frac{p_i}{p^2+t}\Big|z\Big)\Big]^2\\
&\quad\times\Big[\Big(z\Big|\frac{p_i}{p^2}\Big|z'\Big)-\Big(y\Big|\frac{p_i}{p^2}\Big|z'\Big)\Big]^2-\Big[\Big(x\Big|\frac{p_i}{p^2}\Big|z\Big)-\Big(y\Big|\frac{p_i}{p^2}\Big|z\Big)\Big]^2\\
&\quad\times\Big[\Big(z\Big|\frac{p_i}{p^2+t}\Big|z'\Big)-\Big(y\Big|\frac{p_i}{p^2+t}\Big|z'\Big)\Big]^2\\
&\quad+\Big[\Big(x\Big|\frac{p_i}{p^2+t}\Big|z'\Big)-\Big(y\Big|\frac{p_i}{p^2+t}\Big|z'\Big)\Big]^2\Big[\Big(z'\Big|\frac{p_i}{p^2}\Big|z\Big)-\Big(x\Big|\frac{p_i}{p^2}\Big|z\Big)\Big]^2\\
&\quad-\Big[\Big(x\Big|\frac{p_i}{p^2}\Big|z'\Big)-\Big(y\Big|\frac{p_i}{p^2}\Big|z'\Big)\Big]^2\Big[\Big(z'\Big|\frac{p_i}{p^2+t}\Big|z\Big)-\Big(x\Big|\frac{p_i}{p^2+t}\Big|z\Big)\Big]^2\Big\}.
\end{aligned}$$

Using the integral

$$\begin{aligned}
\int_0^\infty\frac{dt}{t^{1-\epsilon}}\Big(x\Big|\frac{p_i}{p^2+t}\Big|z\Big)\Big(y\Big|\frac{p_i}{p^2+t}\Big|z\Big)&=-\frac{(X,Y)}{4\pi^2}\int_0^1du\frac{\Gamma(\epsilon)\Gamma(2-\epsilon)(4\bar uu)^{-\epsilon}}{(X^2\bar u+Y^2u)^{2-\epsilon}}\\
&=\frac{(X,Y)}{4\pi^2X^2Y^2}\Big(-\frac{1}{\epsilon}-\ln4+\frac{X^2\ln Y^2-Y^2\ln X^2}{X^2-Y^2}-\ln X^2Y^2\Big)+O(\epsilon)
\end{aligned}$$

we obtain

$$\begin{aligned}\langle K_{\rm LO}{\rm Tr}\{\hat U_x\hat U_y^\dagger\}^{\rm rigid}-K_{\rm LO}{\rm Tr}\{\hat U_x\hat U_y^\dagger\}^{\rm slope}\rangle=&-\frac{\alpha_s^2}{16\pi^4}\int d^2zd^2z'\frac{1}{(z-z')^2X^2Y^2}\\&\times[{\rm Tr}\{U_xU_z^\dagger\}{\rm Tr}\{U_zU_{z'}^\dagger\}{\rm Tr}\{U_{z'}U_y^\dagger\}-{\rm Tr}\{U_xU_z^\dagger U_{z'}U_y^\dagger U_zU_{z'}^\dagger\}+z\leftrightarrow z']\\&\times\left(\frac{Y^2}{Y'^2}\left\{(X,Y)\left[\frac{X^2+Y^2}{X^2-Y^2}\ln\frac{X^2}{Y^2}+2\right]+(\Delta,Y)\ln X^2-(\Delta,X)\ln Y^2\right\}\right.\\&-\frac{\Delta^2}{Y'^2}\left\{(z-z',Y')\left[\frac{(z-z')^2+Y'^2}{(z-z')^2-Y'^2}\ln\frac{(z-z')^2}{Y'^2}+2\right]\right.\\&\left.\left.-(Y,Y')\ln(z-z')^2+(Y,z-z')\ln Y'^2\right\}+x\leftrightarrow y\right).\end{aligned}\tag{B8}$$

The NLO kernel for the evolution of color dipoles with respect to the slope is the sum of Eq. (5) and the correction (B8). Note that the correction term (B8) is not conformally invariant (cf. Ref. [24]). This is hardly surprising since the non-lightlike Wilson line turns into a circle under the inversion $x_\mu \to x_\mu/x^2$. Also, the forward kernel of the correction term will probably spoil the agreement with the NLO BFKL. We were not able to compute the corresponding eigenvalues, but it is hard to imagine how the integral of the r.h.s. of Eq. (B8) with $(z-z')^{2\gamma}$ can be a number independent of γ [recall that our discrepancy with NLO BFKL is $2\zeta(3)$].

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