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Wiener-Fliess Composition of Formal Power Series: Additive Static Feedback and Shuffle Rational Series

Subbarao Venkatesh Guggilam
Old Dominion University, gsvenky89@gmail.com

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by

Subbarao Venkatesh Guggilam
B. Tech, 2011, Amrita Vishwa Vidyapeetham, India
M. Tech, 2014, IIT Bombay, India
M. Sc, 2016, Chennai Mathematical Institute, India

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Approved by:

W. Steven Gray (Director)

Oscar R. González (Member)

Luis A. Duffaut Espinosa (Member)

Alexander Schmeding (Member)

ABSTRACT

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Subbarao Venkatesh Guggilam
Old Dominion University, 2021
Director: Dr. W. Steven Gray

The problem statement for this dissertation is two-fold. The first problem considered is when does a Chen-Fliess series in an additive static feedback connection with a formal static map yield a closed-loop system with a Chen-Fliess series expansion? This work proves that such a closed-loop system always has a Chen-Fliess series representation. Furthermore, an algorithm based on the Hopf algebras for the shuffle group and the dynamic output feedback group is designed to compute the generating series of the closed-loop system. It is proved that the additive static feedback connection preserves local convergence and relative degree, but a counterexample shows that the additive static feedback does not preserve global convergence in general. This dissertation then pivots to the second problem considered, the shuffle rationality problem. The notion of shuffle rationality and shuffle recognizability are first defined, akin to the traditional notion of rational series in bilinear systems theory. It is proved that shuffle rationality and shuffle recognizability coincide, similar to Schützenberger's theorem. An equivalent characterization of shuffle rational series is provided in terms of a canonical state space realization. Specifically, it is shown that a shuffle rational series corresponds to a realization of a nilpotent bilinear system cascaded with a static rational map.

To the memory of my vivacious and loving mom, Swapna Guggilam

&

my ever supportive and motivating dad, Subbarao Guggilam.

ACKNOWLEDGEMENTS

This dissertation might bear my name, but that does not describe the whole picture. There are a plethora of people that I need to be grateful to for helping me see the light at the end of the day. Foremost, I would like to thank my advisor Prof. W. Steven Gray. I was a lost student dwindling nowhere in academia before Prof. Gray found me, contacted me and persuaded me to land in the United States for my Ph.D. Besides technical and mathematical prowess, I learned a lot of life lessons and also the right attitude and methodology for the pursuit of academic research. His meticulous planning and uncompromising adherence in upholding the standards of any publishable research document was a huge learning curve for a rookie researcher such as myself. He always furnished me the freedom to disagree with him on technical discussions without flinching. That freedom helped me a lot in my thinking and putting out my ideas without any apprehensions. I am certainly going to miss the Friday 13:00 EST in-person meetings with him that I had during the pre-Covid era. I would like to thank my committee: Dr. Oscar R. González, Dr. Luis A. Duffaut Espinosa, and Dr. Alexander Schmeding for agreeing to be part of my committee and for their valuable suggestions on my work and possible future lines of work that could be extended from the dissertation.

Up next, I would like to thank Graduate Assistant of the Electrical and Computer Engineering Department Mrs. Lori K. Barrett and Fiscal Operation Specialist Mrs. Romina Samson for making my stay at the department a pleasant one. These two people were literally my 9-1-1 for any administrative issues that I had encountered during my Ph.D. I would like to extend my thanks to one of the student staff, Ms. Consolota Baynor, who indeed became one of my very best friends that I will cherish. The conversations with these three people always made my day. I would like to thank one of my dear friends Dr. Grace C. Rajan who took up the role of my informal academic advisor and was always there to help me with the initial jitters and confusions that I had as an international student. This long marathon of my Ph.D. would have never been possible without the group of my close friends, informally titled “Among us”. Yashwanth (a soon to be Ph.D. graduate), Prateek, Heramb, Abhilash, Srinath, Anoop, Yogaraj, Yash and Abhinav consitute the group. The conversations with this bunch of friends were always fun beyond measure, releasing any accumulation of stress. My survival in these Covid times without this group is simply beyond doubt. Although all of these people are much younger than me, they are mature beyond their age, and I have indeed learned a lot from them. Being in a group of friends that were highly but differently talented has always helped me to stay grounded and focused. I am pretty sure each of them have a bright future in their grasps.

I would like to mention my father, Mr. Subbarao Guggilam, who has unflinchingly and unfailingly tried to motivate me all my life. There were times where I had lost faith in myself, and yet, he always had a hand around my shoulder to propel me further, despite the fact that life and the world were throwing all sorts of questions to him. Without my father, I am a nobody and shall forever be indebted to him. I would like to thank my late mom, Mrs. Swapna Guggilam, for all her blessings from heaven. It is probably my irrevocable regret that I wake up with every morning that my sweet mom is not alive today to see my first baby steps into this world of research and academia. This dissertation work is hence and rightfully dedicated to my parents.

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CHAPTER 1

INTRODUCTION

The goal of this chapter is to describe the problems addressed in this dissertation and provide an outline of the document. The problem statement for the dissertation is two-fold; hence, the motivation behind each problem is explained.

1.1 MOTIVATION

1.1.1 Additive Static Feedback Problem

The interconnection of systems has been studied since the inception of the mathematical system theory [Belevitch (1968), Fliess & Bourlès (1994)]. The static feedback connection is of prime importance in the analysis of nonlinear input-affine systems [Isidori (1995)]. The concept of feedback equivalence between two state space models, arising from the existence of a static feedback law relating them, is used in the motion planning problem in robotics [Murray & Sastry (1993)]. The discretization of a continuous-time nonlinear system, which is feedback equivalent to a nilpotent system, is widely utilized in nonlinear digital control problems such as output tracking [Di Gamberardina, et al. (1996), Monaco & Normand-Cyrot (2001)].

This dissertation addresses the class of analytic nonlinear input-output operators called *Fliess operators*. The interconnection of interest is the additive static output feedback connection. A *Chen-Fliess* series is a functional series of iterated integrals of the input used to describe an input-output system [Fliess (1981)]. The iterated integrals are indexed by words X^* over a noncommutative alphabet X. The generating series of a Chen-Fliess series F_c is a noncommutative formal power series c whose coefficients provide the weights for the iterated integrals. A Chen-Fliess series is convergent if the series describes the input-output behavior on some non zero interval of time and for all integrable inputs that are sufficiently

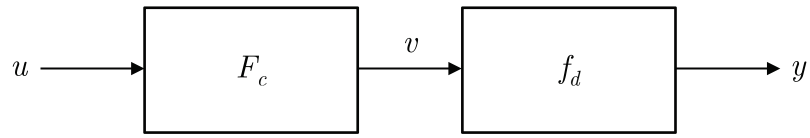


Fig. 1: Wiener-Fliess connection

small. A convergent Chen-Fliess series is called a Fliess operator. If F_c and F_d are two input-output systems represented by Chen-Fliess functional series, then it was shown in [Ferfera (1979), Gray & Li (2005)] that the feedback interconnection of two such systems always renders a closed-loop system in the same class. Its corresponding generating series, written as the *feedback product* $c@d$, can be efficiently computed in terms of a combinatorial Hopf algebra which is commutative, graded and connected [Duffaut Espinosa, et al. (2016), Foissy (2015), Gray, et al. (2014a)]. Convergence of the closed-loop system was characterized in detail by [Thitsa & Gray (2012)]. Variations of the feedback product were used to solve system inversion problems [Gray, et al. (2014b)] and trajectory generation problems [Duffaut Espinosa & Gray (2017)]. The single-input, single-output (SISO) multiplicative dynamic output feedback connection was treated in [Gray & Ebrahimi-Fard (2017)]. However, the existing framework cannot accommodate the scenario where the dynamical system F_d in the feedback path is replaced with a memoryless (static) function f_d, namely, the static feedback connection. The main problem is that the closed-loop involves the cascade connection of a Fliess operator F_c with the static map f_d, namely the *Wiener-Fliess connection* as shown in Figure 1. The presence of a static map in the configuration renders the object incompatible with the algebra used in the analysis of the dynamic feedback case.

The Wiener-Fliess composition of a Fliess operator F_c with a memoryless map f_d was defined, albeit with restrictions, in [Gray & Thitsa (2012)]. The Chen-Fliess series for a state space model is a special case of the Wiener-Fliess connection [Fliess (1981)]. The restriction in [Gray & Thitsa (2012)] was that the generating series c of the Fliess operator F_c in the forward path has to be *proper*. However, the definition is expanded in the present work for the case when c is non-proper but with the new restriction that the static map f_d should be a polynomial. An analysis of the contractive nature of the Wiener-Fliess composition product is also presented, which is essential to prove that the closed-loop system has a Chen-Fliess series representation, say $F_{c\hat{@}d}$. The effect on relative degree of the nonlinear plant F_c in closed-loop is also characterized. Next, the focus turns to actually computing the generating series $c\hat{@}d$. What is needed in this regard are *two* Hopf algebras, the output feedback Hopf algebra corresponding to the dynamic feedback case [Gray, et al. (2014b)], and the Hopf algebra corresponding to the shuffle group [Gray, et al. (2014b), Venkatesh & Gray (2020)]. The shuffle product appears naturally in nonlinear control theory when systems are interconnected in parallel (taking the product of the outputs) [Ree (1958), Fliess (1981)] and in series [Ferfera (1979), Ferfera (1980), Gray, et al. (2014a)]. To facilitate these calculations, the underlying Hopf algebra (e.g., see [Abe (2004), Sweedler (1969)]) for the group of non-proper formal power series under the shuffle product is introduced. The interplay of these two combinatorial Hopf algebra structures is used to compute what will be called the *Wiener-Fliess feedback product*, $c\hat{@}d$. It will be shown that this product has a natural interpretation as a transformation group acting on the plant, which preserves the relative degree of the plant.

The proof that the additive static feedback configuration has a Chen-Fliess series representation solves the computational aspect of the problem but opens up issues regarding convergence. In particular, does the additive static feedback configuration of a Fliess operator F_c with an analytic function f_d have a Fliess operator $F_{c\hat{@}d}$ representing the closed-loop system? An equivalent question is to find conditions under which the Chen-Fliess series $F_{c\hat{@}d}$ is convergent (locally or globally). Further questions can be asked such as does the additive static feedback preserve local convergence, or global convergence or both. These questions are answered in this dissertation. This will require an analysis of the local and global convergence of the shuffle product, mixed composition product, and the Wiener-Fliess composition product as the Wiener-Fliess feedback product involves all three of these products on formal power series. The characterization of global convergence uses the Fréchet topology on the space S_∞^m of all formal series corresponding to globally convergent Fliess operators. The Fréchet topology stems from the *filtration* of Banach spaces $S_\infty^m(R)$, $R > 0$, which is addressed in Chapter 2. With the topological structure in hand, the questions get further multiplied as one could ask whether the shuffle product and mixed composition product defined on $\mathbb{R}^m\langle\langle X\rangle\rangle$ are continuous in its arguments when restricted to the Fréchet space S_∞^m. It is important to note that the question of local convergence of the shuffle product was first addressed in [Thitsa & Gray (2012)], where it was proved that the shuffle product preserves local convergence. However, the present work dissects the question completely to answer all possible cases: the shuffle product of two locally convergent series, the shuffle product on the space S_∞^m, the shuffle product between a locally convergent series and a series in S_∞^m. The work of [Thitsa & Gray (2012)] also addressed the global convergence of the shuffle product but only under restricted conditions. The global convergence of the shuffle product was expanded in [Winter-Arboleda (2019)]. However, the proof was based on the property that shuffle product is closed in each of the Banach spaces $S_\infty^m(R)$. This is proved false in this dissertation with a counterexample, and the global convergence of the shuffle product is completely reworked.

In summary, this part of the dissertation was primarily focused on computing the Chen-Fliess series representation of the additive static feedback connection and providing sufficient conditions for the convergence of the series representation of the closed-loop system.

1.1.2 Shuffle Rational Series

Let X be a finite set of noncommuting symbols. Let $\mathbb{R}\langle X\rangle$ and $\mathbb{R}\langle\langle X\rangle\rangle$ denote, respectively, the set of polynomials and formal power series over X with real coefficients. Each set forms an $\mathbb{R}$-vector space and an associative $\mathbb{R}$-algebra under the catenation (Cauchy) product. The smallest subset of $\mathbb{R}\langle\langle X\rangle\rangle$ containing $\mathbb{R}\langle X\rangle$ which is closed under addition, scalar multiplication, the Cauchy product, and inversion (in the Cauchy product sense), that is, the *rational closure* of $\mathbb{R}\langle X\rangle$, constitutes the set of *rational series* [Berstel & Reutenauer (1988)]. Let F_c denote the Chen-Fliess series having $c \in \mathbb{R}\langle\langle X\rangle\rangle$ as its generating series. The following statements are known to be equivalent:

1. Series c is rational.

2. Series c is recognizable (Schützenberger's theorem [Schützenberger (1961)]).

3. Series c has a Hankel matrix with finite rank [Fliess (1974)].

4. Operator $y = F_c[u]$ has a bilinear state space realization [Fliess (1983)].

5. Operator $y = F_c[u]$ satisfies a linear ordinary differential equation in y of order equal to its Hankel rank and having coefficients which are rational functions of $\{u, \dot{u}, \ddot{u}, \ldots\}$ [Fliess & Reutenauer (1982), Fliess & Reutenauer (1983)](see also [Wang & Sontag (1992), Wang & Sontag (1995)]).

Bilinear systems play a special role in the theory of nonlinear control systems [Isidori (1995), Elliot (2009), Nijmeijer & Van der Schaft (1990)]. The definition of rationality given above, however, is simply one instance of a more general concept. Namely, given any associative algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ and an arbitrary subalgebra $\mathscr{F}$, the corresponding set of rational series is defined as those series in the rational closure of $\mathscr{F}$. Therefore, it is natural to ask whether any other notion of rationality has utility in the context of nonlinear control theory. If so, are there equivalences analogous to those given above in this alternative setting? The objective of this part of the dissertation is to partially answer this question in the affirmative by providing one specific example, namely, by replacing the Cauchy product on $\mathbb{R}\langle\langle X\rangle\rangle$ with the *shuffle product*. The latter forms a commutative algebra and is in some sense the adjoint of the Cauchy product [Reutenauer (1993)]. This alternative product leads directly to the notion of *shuffle-rationality*. The next objective is to provide two equivalent characterizations of shuffle-rationality, namely, the analogues of statements 2 and 4 above. The concept

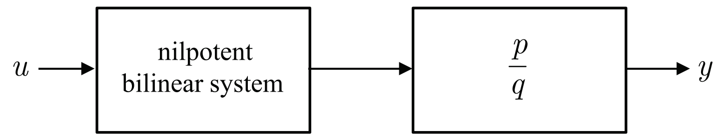


Fig. 2: Wiener-Fliess system comprised of a nilpotent bilinear system and a static rational function

of shuffle-recognizability will be introduced and then a shuffle version of Schützenberger's theorem is proved. Next, it is shown that there is a correspondence between shuffle-rational series and a class of state space realizations which are bilinear in the state and nilpotent but have rational output functions as shown in Figure 2. A common theme in all of the analysis is the evaluation of rational functions on formal power series. The computations are facilitated by the Hopf algebra corresponding to the shuffle group. Finally, as an application, it is shown how to model bilinear systems with output saturation in this context. It should be stated that the question of whether there exist analogous versions of statements 3 and 5 in this setting is an open question at present.

1.1.3 The Intersection Point: Wiener-Fliess Composition

The research problems outlined in Sections 1.1.1 and 1.1.2 might appear to be disjointed. However, their point of tangency is the Wiener-Fliess composition as in Figure 1. The additive static feedback problem requires the definition and contractive properties of the Wiener-Fliess composition product in the ultrametric topology to prove the existence of a generating series for the closed-loop system. Furthermore, the computational algorithm to compute the Wiener-Fliess composition, designed based on the Hopf algebra corresponding to the shuffle group, is utilized for computing the generating series for the additive static feedback connection. The convergence problem pertaining to the additive static feedback product could be attempted only after identifying the convergence conditions for the Wiener-Fliess composition product. On the other hand, it is proved in the dissertation that a Fliess operator with a shuffle rational series as the generating series has a nilpotent bilinear state space realization with a Wiener-Fliess composition of static rational function as in Figure 2. Hence, the branching point of the dissertation is the Wiener-Fliess composition.

1.2 PROBLEM STATEMENT

The primary objectives of the dissertation are to:

1. Characterize the Wiener-Fliess composition product in the ultrametric topology and to develop an algorithm to compute it based on the Hopf algebra corresponding to the shuffle group.

2. Prove that a Chen-Fliess series F_c in an additive static output feedback connection with a formal function f_d has a Chen-Fliess series representation and provide an algorithm to compute the additive static feedback product.

3. Prove that the shuffle product and the mixed composition product preserve both local and global convergence.

4. Characterize the convergence of the Wiener-Fliess composition product.

5. Prove that the additive static output feedback configuration preserves local convergence but not necessarily global convergence.

6. Define the shuffle rationality and shuffle rational series.

7. Introduce the notion of shuffle recognizability and prove that a series is shuffle rational if and only if it is shuffle recognizable.

8. Describe and prove a canonical structure for a state space realization corresponding to a Fliess operator generated by a shuffle rational series.

1.3 THESIS OUTLINE

The remainder of this dissertation is organized as follows. Chapter 2 is a self-contained introduction to the concepts of Fliess operators, Hopf algebras and the Fréchet topology on formal power series. Following this, Chapter 3 introduces the definition of the Wiener-Fliess composition product, develops its properties and renders a closed expression for computing the generating series of the additive static feedback product. The chapter also includes an algorithm designed using the Hopf algebras corresponding to the shuffle group and the dynamic output feedback group for the computation of the additive static feedback product. Chapter 4 is devoted to answering the convergence properties of the additive static feedback product. The chapter begins by addressing local convergence for the mixed composition and the Wiener-Fliess composition products. The chapter then proceeds to treat the global convergence of the shuffle, mixed composition and Wiener-Fliess composition products. The chapter concludes by applying this analysis to the local and global convergence problems for the additive static feedback product. The dissertation then pivots in Chapter 5 to address shuffle rationality and its equivalent characterizations. Chapter 6 summarizes the main contributions of this dissertation, and introduces some potential new problems and conjectures for future work, albeit with some partially worked out results and insights.

CHAPTER 2

PRELIMINARIES

The goal of this chapter is to provide the mathematical definitions and results that are essential for presenting the main contributions of this work. This chapter aims to provide a brief introduction to the noncommutative formal power series and, hence, to Chen-Fliess series and their interconnections. Prior to that, there is a section on preliminaries about Hopf algebras and Fréchet topology. The chapter presumes the reader has been exposed to the basics of point-set topology, functional analysis (to the definition of normed spaces and seminormed spaces) and abstract algebra (to the level of modules and algebras over commutative rings) [Dugundji (1966), Conway (1985), Dummit & Foote (2004)].

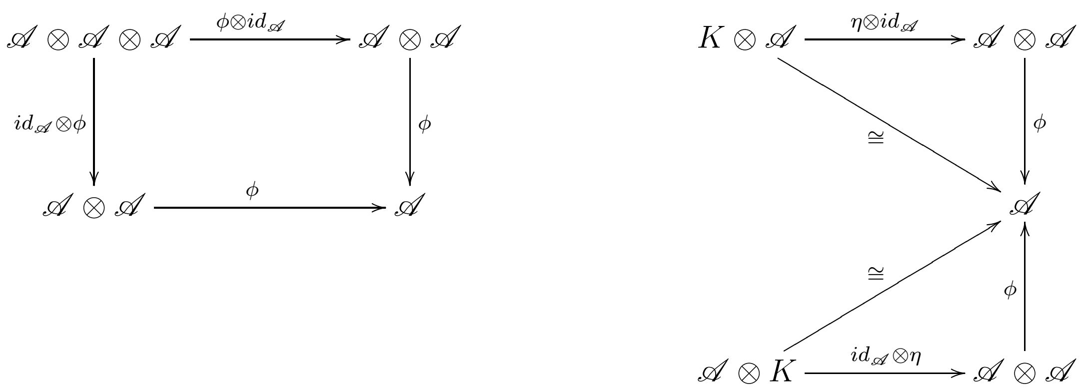
2.1 ALGEBRAIC PRELIMINARIES

The goal of this section is to provide the definitions of necessary algebraic objects with respect to the dissertation. The treatment only considers unital associative algebras; hence, the properties are implicitly imbibed into the definition. The sections describe the algebraic structures of an algebra, coalgebra, bialgebra and Hopf algebra. [Abe (2004),Sweedler (1969)].

2.1.1 Algebra

The definition of an algebra can be facilitated through the category of modules. This notion allows one to define the concept of a coalgebra (the dual notion) with ease. Let K be a commutative ring with identity.

Definition 2.1.1. An algebra over K is a K-module $\mathscr{A}$ along with the morphisms of K-modules $\phi : \mathscr{A} \otimes \mathscr{A} \longrightarrow A$, called the multiplication or product map, and $\eta : K \longrightarrow A$, called the unit map, such that the following arrow diagrams are commutative.



(2.1.1)

The tuple $(\mathscr{A}, \phi, \eta)$ is called a K-algebra.

The commutative diagram (2.1.1) means that a K-algebra $\mathscr{A}$ must satisfy the following properties:

1. The product map ϕ must be associative.
2. The scalar multiplication through the η map must have a unit.

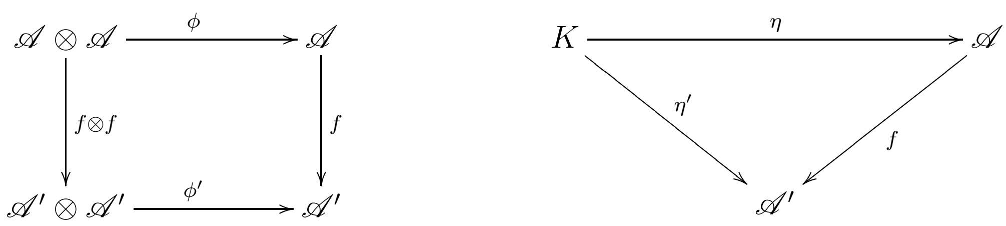
Example 2.1.1. The following are examples of algebras.

1. The set of smooth functions over the reals denoted by $\mathscr{C}^{\infty}(\mathbb{R})$ forms an $\mathbb{R}$-algebra. The product map is the pointwise product of the functions, and the unit map is the ring isomorphism from $\mathbb{R}$ to the constant functions.
2. A commutative unital ring R forms an R-algebra. The product map is just the product defined on the ring structure of R, and the unit map is the identity map.

□

The concept of a K-algebra morphism is defined next.

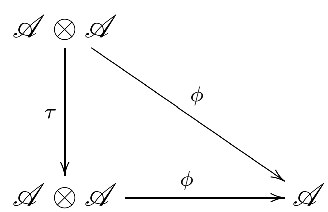
Definition 2.1.2. Let $(\mathscr{A}, \phi, \eta)$, $(\mathscr{A}', \phi', \eta')$ be K-algebras. A map $f : \mathscr{A} \longrightarrow \mathscr{A}'$ is called a K-algebra morphism provided the following diagrams commute.



Definition 2.1.3. Let P and Q be modules over K. The twisting morphism τ of K-modules is $\tau : P \otimes Q \longrightarrow Q \otimes P$ with

$$\tau(p \otimes q) = q \otimes p \quad \forall\, q \in Q, p \in P.$$

A K-algebra $\mathscr{A}$ is commutative if and only if the following diagram commutes.

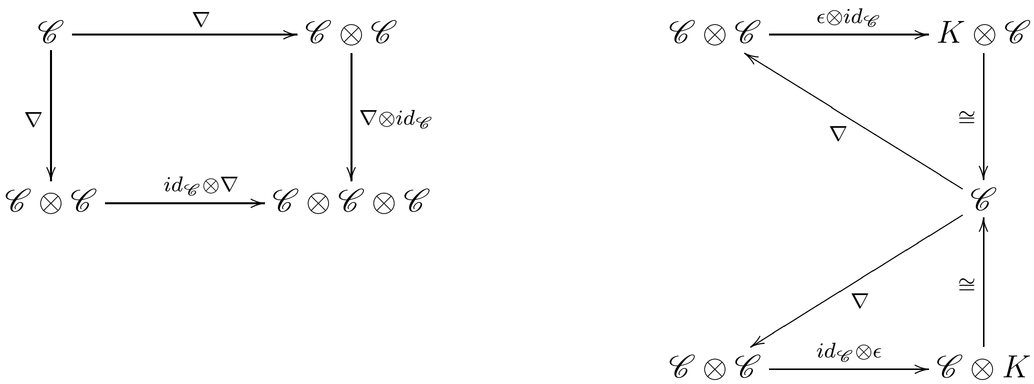


As an example, the $\mathbb{R}$-algebra $\mathscr{C}^{\infty}(\mathbb{R})$ is a commutative algebra. The K-algebra $\mathscr{A}$ is a graded algebra if the underlying K-module structure is graded viz. $\mathscr{A} = \bigoplus_{n\in\mathbb{N}_0} \mathscr{A}_n$ where $\mathscr{A}_n$ is a K-module for all $n \in \mathbb{N}_0$ such that $\phi\left(\mathscr{A}_m \otimes \mathscr{A}_n\right) \subseteq \mathscr{A}_{m+n}\ \forall\, m, n \in \mathbb{N}_0$. The graded K-algebra is connected if $\eta : K \longrightarrow \mathscr{A}_0$ is a K-algebra isomorphism.

2.1.2 Coalgebra

The notion of a K-coalgebra is a categorical structure dual to that of a K-algebra.

Definition 2.1.4. A K-coalgebra $\mathscr{C}$ is a K-module with the K-module morphisms $\nabla : \mathscr{C} \longrightarrow \mathscr{C} \otimes \mathscr{C}$, called the comultiplication or coproduct map, and $\epsilon : \mathscr{C} \longrightarrow K$, called the counit map, such that the following diagrams commute.



(2.1.2)

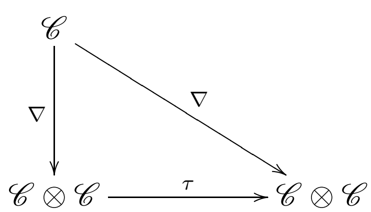
The tuple $(\mathscr{C}, \nabla, \epsilon)$ is called a K-coalgebra.

The commutative diagram (2.1.1) means that a K-algebra $\mathscr{A}$ must satisfy the following properties:

1. The coproduct map ∇ must be coassociative.

2. The counit map is the categorical dual to the unit map for a K-algebra.

The coalgebra $\mathscr{C}$ is called cocommutative if the following diagram commutes,



where τ is the twisting morphism of K-modules as in Definition 2.1.3. The Sweedler notation is very useful in representing the coproduct map and is used in the dissertation in Chapters 3 and 5.

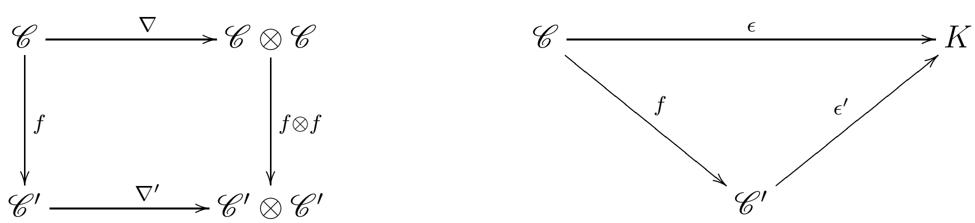
Definition 2.1.5. [Sweedler (1969)]. Given the K-coalgebra tuple $(\mathscr{C}, \nabla, \epsilon)$ and $c \in \mathscr{C}$, then the Sweedler notation to denote the coproduct of c is

$$\nabla(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

where $c_{(1)}, c_{(2)} \in \mathscr{C}$ are the components of the tensors resulting from the coproduct of c.

Next, the definition of a K-coalgebra morphism is given.

Definition 2.1.6. Let $(\mathscr{C}, \nabla, \epsilon)$, $(\mathscr{C}', \nabla', \epsilon')$ be K-coalgebras. A map $f : \mathscr{C} \longrightarrow \mathscr{C}'$ is called a K-coalgebra morphism provided the following diagrams commute.



Example 2.1.2. Let $(P, \leq)$ be a locally finite poset. For $x, y \in P$ denote the interval as $[x, y] = \{z \in P : x \leq z \leq y\}$. Let $\mathscr{P}$ be the free $\mathbb{R}$-module formed by the collection of finite posets formed from P or finite intervals from P. $\mathscr{P}$ can be endowed with an $\mathbb{R}$-coalgebra structure. Define the coproduct map ∇ and the counit map ϵ as

$$\nabla([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y] \ \forall x \leq y \in P$$

$$\epsilon([x, y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

The tuple $(\mathscr{P}, \nabla, \epsilon)$ is known as the *incidence coalgebra* with respect to the poset P. □

2.1.3 Bialgebra

The bialgebra structure over a commutative ring is fundamental for defining a Hopf algebra. A bialgebra is an amalgamation of the algebra and coalgebra structures such that both are compatible with each other.

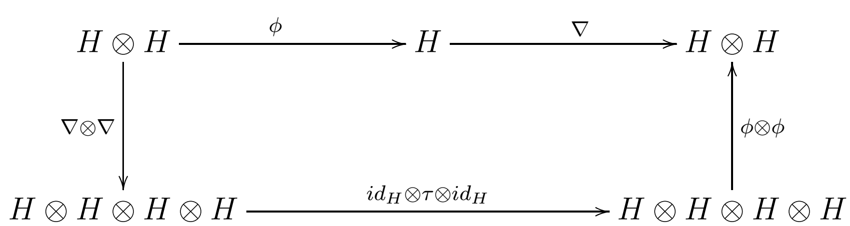
Definition 2.1.7. A bialgebra H over K is a tuple $(H, \phi, \eta, \nabla, \epsilon)$ such that

1. H is a K-module.

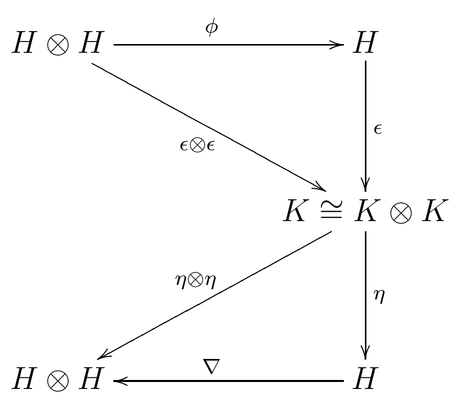
2. (H, ϕ, η) is a K-algebra, where ϕ and η are the product and unit maps respectively.

3. (H, ∇, ϵ) is a K-coalgebra, where ∇ and ϵ are the coproduct and counit maps respectively.

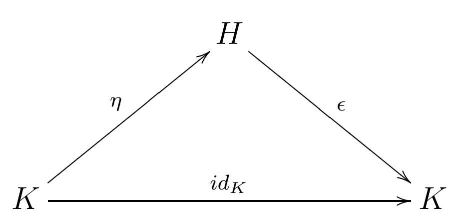
such that the following arrow diagrams commute.



(2.1.3)



(2.1.4)



(2.1.5)

The diagrams (2.1.3) and (2.1.4) describe that the product map ϕ and the unit map η are K-coalgebra morphisms, while the coproduct map ∇ and the counit map ϵ are K-algebra morphisms. The diagram (2.1.5) describes that the unit map η is a section of the counit map ϵ in the category of K-modules.

Example 2.1.3. Let G be a finite group. Denote $\mathbb{R}[G]$ as the free $\mathbb{R}$-module generated over the group G viz. an element $c \in \mathbb{R}[G]$ is given as

$$c = \sum_{x \in G} \alpha_x x.$$

The product map on $\mathbb{R}[G]$ can be defined using the group product on G as

$$\left(\sum_{x \in G} \alpha_x x \right) \left(\sum_{yinG} \alpha_y y \right) = \sum_{z \in G} \left(\sum_{xy=z} \alpha_x \alpha_y \right) z.$$

The unit element is just the identity element of the group G. Hence, $\mathbb{R}[G]$ is an $\mathbb{R}$-algebra called the *group algebra*. The group algebra can be extended to a bialgebra structure by defining the coproduct map ∇ and the counit map ϵ as:

$$\nabla(g) = g \otimes g$$
$$\epsilon(g) = 1$$

$\forall g \in G$ and then extending linearly over $\mathbb{R}[G]$. It can be verified that $\mathbb{R}[G]$ is an $\mathbb{R}$-bialgebra under this coproduct and counit map definition [Sweedler (1969)]. □

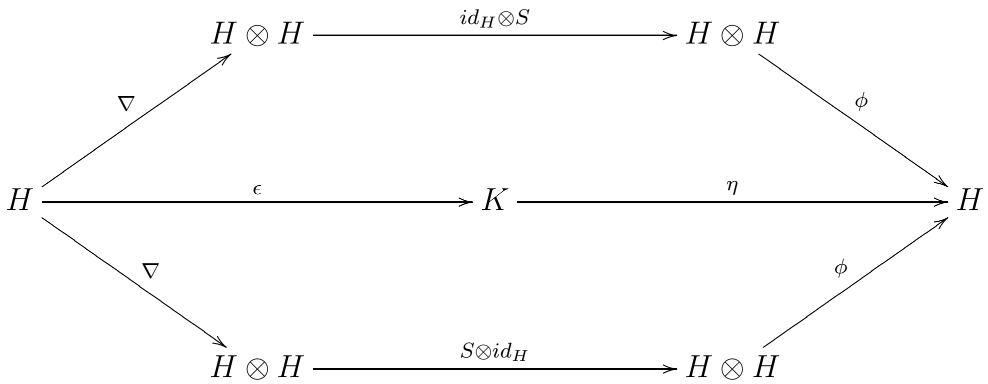
2.1.4 Hopf Algebra

Hopf algebras are an important class of bialgebras. A Hopf algebra is a bialgebra equipped with a K-linear map called an antipode.

Definition 2.1.8. A Hopf algebra H over K is a tuple $(H, \phi, \eta, \nabla, \epsilon, S)$ such that the following conditions are satisfied:

1. $(H, \phi, \eta, \nabla, \epsilon)$ is a K-bialgebra.

2. $S : H \longrightarrow H$ is a K-linear map such that the following diagram commutes.



(2.1.6)

Using Sweedler notation, the arrow diagram (2.1.6) implies that $\forall c \in H$,

$$\sum_{(c)} S\left(c_{(1)}\right) c_{(2)} = \sum_{(c)} c_{(1)} S\left(c_{(2)}\right) = \epsilon\,(c)\, 1_H\ ,$$

where 1_H is the multiplicative unit of the Hopf algebra H. The computation of the antipode of an element c becomes slightly easier when the algebra structure of H is graded and connected and is discussed in Section 3.2.

Example 2.1.4. Consider the $\mathbb{R}$-bialgebra $\mathbb{R}[G]$ described in Example 2.1.3. The bialgebra can be extended to a Hopf algebra structure where the antipode map S is given by the group inverse operation. Hence, $S(g) = g^{-1}$ for all $g \in G$ and then extending linearly over $\mathbb{R}[G]$. □

2.2 INVERSE SYSTEM OF LOCALLY CONVEX SPACES AND FRÉCHET SPACE

The goal of this section is to furnish the reader with the definition of the inverse limit in the category of locally convex spaces. Prior to that, the definitions of a topological vector space and a locally convex space are provided. For the rest of this section, the field $\mathbb{R}$ is equipped with the standard topology.

Definition 2.2.1. [Schaefer & Wolff (1999)] Let E be an $\mathbb{R}$-vector space together with a Hausdorff topology τ on E such that the vector addition map $+ : E \times E \longrightarrow E$ and the scalar multiplication map $\cdot : \mathbb{R} \times E \longrightarrow E$ are continuous. Then, the topological space (E, τ) (or just E) is called a topological vector space.

Example 2.2.1. The following are examples of a topological vector space.

1. Every normed space $(E, \|\cdot\|)$ on $\mathbb{R}$ is a topological vector space on $\mathbb{R}$.

2. The space of real valued sequences $\mathbb{R}^\omega$ is a topological vector space on $\mathbb{R}$, where the topology endowed is the product topology. Thus, $\mathbb{R}^\omega$ is isomorphic to the product of countable copies of $\mathbb{R}$.

□

The definition of a locally convex space is given next.

Definition 2.2.2. [Schaefer & Wolff (1999)] A topological vector space (E, τ) (on $\mathbb{R}$) is called a locally convex space if there is a family of continuous seminorms $\{p_i : E \longrightarrow [0, \infty[\}_{i \in I}$ over some index set I such that

1. τ is the initial topology with respect to the family of canonical projections $\{q_i : E \longrightarrow E/\ker(p_i)\}_{i \in I}$.

2. $\bigcap_{i \in I} \ker(p_i) = \{0\}$.

The family of seminorms $\{p_i\}_{i \in I}$ is called a generating family of seminorms.

Example 2.2.2. The following are examples of a locally convex space.

1. Every normed space $(E, \|\cdot\|)$ on $\mathbb{R}$ is a locally convex space where the generating family of seminorms is a singleton consisting of the norm $\|\cdot\|$.

2. The space of real-valued smooth functions on the closed unit interval denoted by $\mathscr{C}^\infty([0,1],\mathbb{R})$ is a locally convex space where the generating family of seminorms is the countable family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ and $\forall n \in \mathbb{N}$,

$$\|f\|_n = \sup_{x\in[0,1]} |f^{(n)}(x)|,$$

where $f \in \mathscr{C}^\infty([0,1],\mathbb{R})$ and $f^{(n)}$ is the n^{th}-derivative of f.

□

Theorem 2.2.1. *[Schaefer & Wolff (1999)] Let X and Y be locally convex spaces and let $\mathscr{P}$ be the generating family of seminorms for the topology on X. A linear map $L : X \longrightarrow Y$ is continuous if and only if for each continuous seminorm q on Y, there exists a finite subset $\{p_i : i = 1, 2, \ldots, n\}$ of $\mathscr{P}$ and a constant $c > 0$ such that $q \circ L \leq c \sup_{i=1,\ldots,n} p_i$.*

Definition 2.2.3. [Schaefer & Wolff (1999)] Let $(I, \leq)$ be a directed set. The inverse system of locally convex spaces is a collection of locally convex spaces $(X_i)_{i\in I}$ such that there exists continuous linear maps $\mu_{ij} : X_i \longrightarrow X_j \;\forall\, i \leq j$ satisfying the following conditions:

1. $\mu_{ij} \circ \mu_{jk} = \mu_{ik} \;\forall\, i \leq j \leq k$.

2. $\mu_{ii} : X_i \longrightarrow X_i$ is the identity morphism.

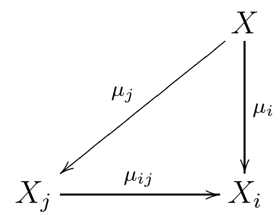
Then the tuple $\left((X_i)_{i\in I}, (\mu_{ij})_{\substack{i,j\in I\\ i\leq j}}\right)$ forms an inverse system or projective system of the collection of locally convex spaces.

The following is an example of the inverse system of a collection of locally convex spaces.

Example 2.2.3. Consider the set directed set $\mathbb{N}$ with the usual ordering $\leq$. Let space $\mathbb{R}^n$ be equipped with the norm topology under the canonical norm $\|\cdot\|_2$ on $\mathbb{R}^n$. It is easy to verify that the canonical projection map $\pi_{mn} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a continuous linear map whenever $m < n$. Hence, the tuple $\left((\mathbb{R}^n)_{n\in\mathbb{N}}, (\pi_{mn})_{\substack{m,n\in\mathbb{N}\\ m\leq n}}\right)$ forms an inverse system of this collection of locally convex spaces.

□

Definition 2.2.4. [Schaefer & Wolff (1999)] Let $\left((X_i)_{i\in I}, (\mu_{ij})_{\substack{i,j\in I\\ i\leq j}}\right)$ be an inverse system of a collection of locally convex spaces. The inverse or projective limit of the system is defined to be the tuple $\left(X, (\mu_i)_{i\in I}\right)$, where X is a locally convex space equipped with a family of continuous linear maps $\mu_i : X \longrightarrow X_i$ such that $\forall i, j \in I$ the following diagram commutes.



The topology endowed on the limit space X is the initial topology with respect to the family of maps $(\mu_i)_{i\in I}$. The inverse limit is denoted as $X = \varprojlim_{i\in I} X_i$.

The inverse limit of locally convex spaces exists and is unique up to a linear homeomorphism as a consequence of the universal property of the inverse limit in the category of locally convex spaces.

Example 2.2.4. Consider Example 2.2.3. The projective limit of the inverse system $\left((\mathbb{R}^n)_{n\in\mathbb{N}}, (\pi_{mn})_{\substack{m,n\in\mathbb{N}\\ m\leq n}}\right)$ is the locally convex space $\mathbb{R}^\omega$ which is the space of all real valued sequences along with the family of continuous projection maps $\pi_n : \mathbb{R}^\omega \longrightarrow \mathbb{R}^n\ \forall n \in \mathbb{N}$. The limit topological space $\mathbb{R}^\omega$ is then isomorphic to $\mathbb{R}^\omega$ (as in maps from $\mathbb{N}$ to $\mathbb{R}$) endowed with the topology of pointwise convergence. □

The concept of a Fréchet space is introduced next. It is central to the understanding of the dissertation and is used to define the convergence of Chen-Fliess series as described in Section 2.4.

Definition 2.2.5. [Schaefer & Wolff (1999)] A Fréchet space X is defined to be a locally convex metrizable complete topological vector space.

A more practical definition of a Fréchet space is as follows.

Definition 2.2.6. [Schaefer & Wolff (1999)] A complete topological space X is a Fréchet space if its topology is homeomorphic to the topology induced by a countable separated family of seminorms.

The following theorem provides a more lucid way of picturing the Fréchet space as a projective limit.

Theorem 2.2.2. *[Schaefer & Wolff (1999)] A topological space X is a Fréchet space if and only if it is a projective limit of a sequence of Banach spaces.*

More properties of Fréchet space are presented in Section 2.4. The following section introduces the notions of formal power series and Chen-Fliess series.

2.3 FORMAL POWER SERIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. Its *length* is $|\eta| = k$. In particular, $|\eta|_{x_i}$ is the number of times the letter $x_i \in X$ appears in η. The set of all words including the empty word, $\emptyset$, is denoted by X^*, and $X^+ := X^* \backslash \emptyset$. The set X^* forms a monoid under catenation. The set of all words with prefix η is written as ηX^*. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is denoted by (c, η) and called the *coefficient* of η in c. A series c is *proper* when $(c, \emptyset) = 0$. The *support* of c, $\text{supp}(c)$, is the set of all words having nonzero coefficients. The *order* of c, $\text{ord}(c)$, is the length of the minimal length word in its support. Normally, c is written as a formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X \rangle\rangle$. The set $\mathbb{R}\langle\langle X \rangle\rangle$ is equipped with the partial ordering $\leq$ defined as : $c \leq d$ if and only $|(c, \eta)| \leq |(d, \eta)|$ $\forall \eta \in X^*$. A formal power series c is a polynomial when the inverse image of $\mathbb{R}^\ell \setminus \{0\}$ is a finite set. The set of all noncommutative polynomials with coefficients in $\mathbb{R}^\ell$ is denoted by $\mathbb{R}^\ell\langle X\rangle$. Alternatively,

$$\mathbb{R}^\ell\langle X\rangle = \bigoplus_{\eta\in X^*} \mathbb{R}^\ell$$
$$\mathbb{R}^\ell\langle\langle X\rangle\rangle = \prod_{\eta\in X^*} \mathbb{R}^\ell.$$

The set $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is an ultrametric space with the ultrametric

$$\kappa(c,d) = \sigma^{\mathrm{ord}(c-d)},$$

where $c,d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $\sigma \in]0,1[$. For brevity, $\kappa(c,0)$ is written as $\kappa(c)$, and $\kappa(c,d) = \kappa(c-d)$. The ultrametric space $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \kappa)$ is known to be Cauchy complete [Berstel & Reutenauer (1988)]. The following notions of *strong* and *weak* contraction maps will be used.

Definition 2.3.1. Given metric spaces (E,d) and (E',d'), a map $f : E \longrightarrow E'$ is said to be a strong contraction map if $\forall s,t \in E$, it satisfies the condition $d'(f(s),f(t)) \leq \alpha d(s,t)$, where $\alpha \in [0,1[$. If $\alpha = 1$, then the map f is said to be a weak contraction map or a non-expansive map.

In the event that the letters of X commute, the set of all corresponding formal power series is denoted by $\mathbb{R}^\ell\,[[X]]$. For any series $c \in \mathbb{R}^\ell\,[[X]]$, the natural number $\varpi(c)$ corresponds to the order of its proper part, namely $c - (c,\emptyset)$.

2.3.1 Fliess Operators

Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0,t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : \; 1 \leq i \leq m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0,t_1]$. Let $L^m_\mathfrak{p}[t_0,t_1]$ denote the set of all measurable functions defined on $[t_0,t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0,t_1] := \{u \in L^m_\mathfrak{p}[t_0,t_1] : \|u\|_\mathfrak{p} \leq R\}$. Assume $C[t_0,t_1]$ is the subset of continuous functions in $L^m_1[t_0,t_1]$. Given any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the corresponding *Chen-Fliess series* is

$$F_c[u](t) = \sum_{\eta\in X^*} (c,\eta)\, E_\eta[u](t,t_0), \tag{2.3.1}$$

where $E_\emptyset[u] = 1$ and

$$E_{x_i\bar{\eta}}[u](t,t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau,t_0)\, d\tau$$

with $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$ [Fliess (1981)]. If there exist constants $K, M > 0$ such that

$$|(c_i,\eta)| \leq KM^{|\eta|}|\eta|!, \;\; \forall\eta \in X^*, \; \forall i = 1,\ldots,\ell \tag{2.3.2}$$

then F_c constitutes a well-defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0+T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0+T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1,\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [Gray & Wang (2002)]. This map is referred to as a *Fliess operator*. A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ obeying the growth condition in (2.3.2) is called a *locally convergent* generating series. The set of all locally convergent generating series is denoted by $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. The supremum of the set of all $\max\{R,T\}$ for which a Fliess operator F_c is a well-defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0+T]$ into $B_{\mathfrak{a}}^\ell(S)[t_0,\, t_0+T]$ is called the *radius of convergence* of the Fliess operator F_c and is denoted by $\rho\,(F_c)$. A Fliess operator F_c is called *locally convergent* if $\rho\,(F_c) > 0$. If there exist constants $K, M > 0$ and $\gamma \in [0,1[$ such that

$$|(c_i,\eta)| \leq KM^{|\eta|}\left(|\eta|!\right)^\gamma, \;\; \forall\eta \in X^*, \; \forall i = 1,\ldots,\ell \tag{2.3.3}$$

then F_c constitutes a well defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,\, t_0+T]$ into $B_{\mathfrak{q}}^\ell(S)[t_0,\, t_0+T]$ for all $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1,\infty]$ are conjugate exponents [Winter-Arboleda (2019)]. The infimum of all the $\gamma \in [0,1[$ such that (2.3.3) is satisfied for a series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is called the *Gevrey order* of the series c. A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ obeying the growth condition in (2.3.3) is called a *globally convergent* series. The set of all globally convergent series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is denoted as $\mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$. A Fliess operator F_c is *globally convergent* if and only if there exists no real number $M > 0$ such that $\rho(F_c) < M$. Observe that a noncommutative polynomial $\mathbb{R}\langle X\rangle$ is a globally convergent series with Gevrey degree 0. A series $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$ is only a sufficient condition for the corresponding Fliess operator F_c to be globally convergent [Winter-Arboleda (2019), Winter-Arboleda (2015)]. Necessary conditions are presented in Subsection 2.4. In the absence of any convergence criterion, (2.3.1) only defines an operator in a formal sense.

2.3.2 Interconnections of Fliess Operators

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \sqcup\!\sqcup\, d}$, respectively [Fliess (1981)]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X'\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, where $|X'| = m + 1$, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* [Ferfera (1980)] of c and d is given by

$$c \circ d = \sum_{\eta \in X'^*} (c, \eta)\, \psi_d(\eta)(\mathbf{1}). \tag{2.3.4}$$

Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X'\rangle\rangle$ to the algebra of $\mathbb{R}$-linear endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}\,(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x'_i\eta) = \psi_d(x'_i) \circ \psi_d(\eta)$ with $\psi_d(x'_i)(e) = x_0(d_i \sqcup\!\sqcup\, e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. The composition product is linear in its left argument.

Theorem 2.3.1. *If $c, c' \in \mathbb{R}^{\ell}\langle\langle X'\rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$, then $(c + c') \circ d = c \circ d + c' \circ d$.*

When two Fliess operators F_c and F_d are interconnected to form a feedback system with F_c in the forward path and F_d in the feedback path, the generating series of the closed-loop system is denoted by the *feedback product* $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* [Gray, et al. (2014a)]. For example, in the SISO case where $X = \{x_0, x_1\}$, define the set of *unital* Fliess operators $\mathscr{F}_\delta = \{I + F_c \;:\; c \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\delta + \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group of operators under the composition product.

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + c \circledcirc d$, $c \circledcirc d := d + c \,\tilde{\circ}\, d_\delta$, and $\tilde{\circ}$ denotes the *mixed* composition product [Gray & Li (2005)]. The *mixed* composition product definition is induced by the identity $F_{c \,\tilde{\circ}\, d_\delta} = F_c \circ F_{d_\delta}$ so that

$$c \,\tilde{\circ}\, d_\delta = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(\mathbf{1}),$$

where $c \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle, d_\delta \in \mathbb{R}^m\langle\langle X_\delta\rangle\rangle$ with $|X'| = m + 1$ and ϕ_d is analogous to ψ_d in (2.3.4) except here $\phi_d(x_i)(e) = x_i e + x_0(d_i \sqcup\!\sqcup e)$ with $d_0 := 0$. Equivalently, $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ forms a group. The mixed composition product is also linear in its left argument.

Theorem 2.3.2. *If $c, c' \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then $(c + c') \,\tilde{\circ}\, d_\delta = c \,\tilde{\circ}\, d_\delta + c' \,\tilde{\circ}\, d_\delta$.*

The following theorem states that the mixed composition can be viewed as a right group action of $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ on $\mathbb{R}^\ell\langle\langle X'\rangle\rangle$.

Theorem 2.3.3. *[Gray & Duffaut Espinosa (2013)] If $c \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle$ and $d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then $(c \,\tilde{\circ}\, d_\delta) \,\tilde{\circ}\, e_\delta = c \,\tilde{\circ}\, (d_\delta \circ e_\delta)$.*

The next lemma states that the mixed composition product distributes on the left over the shuffle product.

Lemma 2.3.1. *[Gray & Li (2005)] If $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ with $e \in \mathbb{R}^m\langle\langle X'\rangle\rangle$ such that $|X| =$*

$m+1$, *then*

$$(c \sqcup\!\sqcup d)\,\tilde{\circ}\, e_\delta = (c\,\tilde{\circ}\, e_\delta) \sqcup\!\sqcup (d\,\tilde{\circ}\, e_\delta).$$

For the group of unital Fliess operators, the coordinate maps for the corresponding Hopf algebra H have the form

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}^\ell : c \mapsto (c,\eta),$$

where $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, $\eta \in X^*$. The commutative product is taken to be the Hadamard product in $\mathbb{R}^\ell$,

$$\boldsymbol{m} : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}$ is defined to map every c to $\mathbb{1} = [11\cdots 1] \in \mathbb{R}^\ell$. If the *degree* of a_η is defined as $\deg(a_\eta) = 2|\eta|_{x_0} + |\eta|_{x_1} + 1$, then H is a graded and connected $\mathbb{R}$-algebra with $H = \bigoplus_{k\geq 0} H_k$, where H_k is the set of all elements of degree k and $H_0 = \mathbb{R}\mathbf{1}$ [Foissy (2015)]. The coproduct Δ is defined so that the formal power series product $c \circledcirc d$ for the group $\mathscr{F}_\delta$ satisfies

$$\Delta a_\eta(c,d) = a_\eta(c \circledcirc d) = (c \circledcirc d, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{\circ -1} := \delta + c^{\circ -1}$ is computed.

Lemma 2.3.2. *[Gray, et al. (2014a)] The Hopf algebra $(H, \boldsymbol{m}, \Delta)$ has an antipode S satisfying $a_\eta(c^{\circ -1}) = (Sa_\eta)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$.*

With this concept, the generating series for the feedback connection, $c@d$, can be computed explicitly.

Theorem 2.3.4. *[Gray, et al. (2014a)] For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X'\rangle\rangle$, where $|X| = m+1$ and $|X'| = \ell+1$, it follows that $c@d = c\,\tilde{\circ}\,(-d\circ c)_\delta^{\circ -1}$.*

TABLE 1: Relative degrees for interconnections of SISO nonlinear systems

r_{c+d}	$\min(r_c, r_d)$ if $r_c \neq r_d$
$r_{c \sqcup\!\sqcup\, d}$	$r = \min(r_c, r_d)$ if $r_c \neq r_d$ and the series with relative degree not equal to r is non-proper
$r_{c \circ d}$	$r_c + r_d$
$r_{c\,\tilde{\circ}\,d_\delta}$	r_c
$r_{c \circledcirc d}$	$\min(r_c, r_d)$ if $r_c \neq r_d$
$r_{c^{\circ -1}}$	r_c
$r_{c@d}$	r_c

2.3.3 Relative Degree of Chen-Fliess Series

Let $X = \{x_0, x_1\}$ and F_c be a Chen-Fliess series such that $c \in \mathbb{R}\langle\langle X \rangle\rangle$. The concept of relative degree for F_c is defined via the notion of relative degree of its generating series c.

Definition 2.3.2. [Gray & Venkatesh (2019)] A generating series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r if and only if there exists some $e \in \mathbb{R}\langle\langle X \rangle\rangle$ with $x_1 \notin \operatorname{supp}(e)$ such that $c = c_N + K x_0^{r-1} x_1 + x_0^{r-1} e$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ is called the *natural part* of the series and $K \neq 0$. The relative degree of the series c is denoted as r_c.

If $c \in \mathbb{R}\langle\langle X \rangle\rangle$, the *forced part* of the series is denoted as c_F and is given by $c_F = c - c_N$. Table 1 consolidates the effect of various operations on the relative degree of formal series arising from the interconnection of Chen-Fliess series. For more detail, the reader is referred to [Gray & Venkatesh (2019)].

The following lemma is used in determining the relative degree of the Wiener-Fliess composition product as defined in Chapter 3.

Lemma 2.3.3. *[Gray & Venkatesh (2019)] If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r_c, then* $\operatorname{supp}((c^{\sqcup\!\sqcup\, k})_F) \subseteq x_0^{r_c - 1} X^+$ *for all $k \in \mathbb{N}$. If, in addition, c is also proper, then $x_0^{r_c-1} x_1 \notin$* $\operatorname{supp}((c^{\sqcup\!\sqcup\, k})_F)$ *for all $k > 1$.*

2.4 FRÉCHET TOPOLOGY FOR GLOBAL CONVERGENCE

The ultrametric topology on $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ provides a framework to prove the existence of a well-defined feedback product via fixed point theorems as described in Section 3.3. However, a convergent sequence of series in the ultrametric space $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, each of which has a well-defined Fliess operator, need not have a well-defined Fliess operator corresponding to the limit. This is demonstrated by the following example.

Example 2.4.1. Let $(c_i)_{i\in\mathbb{N}_0}$ be a sequence of series in $\mathbb{R}\langle\langle X\rangle\rangle$. Let

$$c_i = \sum_{k=0}^{i} (k!)^{1+\epsilon}\, x_1^k,$$

where $\epsilon > 0$. Observe that each c_i is a polynomial; hence, $c_i \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. It is evident that the sequence $(c_i)_{i\in\mathbb{N}_0}$ is Cauchy in the ultrametric topology. The sequence $c_i \longrightarrow c$, where c is defined as

$$c = \sum_{k=0}^{\infty} (k!)^{1+\epsilon}\, x_1^k.$$

Since $\epsilon > 0$, there exist no constants $K, M > 0$ such that $|(c, x_1^n)| \leq KM^n n!\ \forall n \in \mathbb{N}_0$. Therefore, $c \notin \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. □

This subsection describes the construction of a topology called the *Fréchet* or *seminorm* topology under which global convergence of Fliess operators is preserved in the limit.

Definition 2.4.1. Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$. Then, for any positive real number $R > 0$, define the map $||.||_{\infty,R} : \mathbb{R}\langle\langle X\rangle\rangle \mapsto \overline{\mathbb{R}_+}$ as

$$||c||_{\infty,R} = \sup_{\eta\in X^*} \left\{ |(c,\eta)| \frac{R^{|\eta|}}{|\eta|!} \right\},$$

where $\overline{\mathbb{R}_+}$ is the closure of the non-negative real line with $+\infty$. For all positive real $R > 0$, define the normed space

$$S^m_\infty(R) = \left\{ c \in \mathbb{R}^m\langle\langle X \rangle\rangle \ : ||c_i||_{\infty,R} < \infty \ \forall i = 1,\ldots,m \right\}.$$

When the series c is one-dimensional, the superscript $m = 1$ is omitted. Note that $S_\infty(R)$ is isometrically isomorphic to the Banach space $\ell^\infty(X^*)$, the space of all bounded functions from X^* to $\mathbb{R}$. Hence, the tuple $\left(S_\infty(R), +, \cdot, ||.||_{\infty,R}\right)$ forms an infinite dimensional Banach space, where $+$ and $\cdot$ represent series addition and scalar multiplication, respectively. The following theorem states that a formal series c is *locally convergent* as in (2.3.2) if and only if c belongs to $S_\infty(R)$ for some $R > 0$.

Theorem 2.4.1. *[Winter-Arboleda (2019)]* $\mathbb{R}_{LC}\langle\langle X \rangle\rangle = \bigcup_{R>0} S_\infty(R)$.

The locally convex space $S_\infty(R')$ is an infinite dimensional Banach space, the standard Bolzano-Wierstrass theorem fails to hold. Hence, not every bounded sequence in $S_\infty(R')$ has a convergent subsequence in $S_\infty(R')$ as shown in the following example.

Example 2.4.2. Consider the sequence of the series $(c_i)_{i\in\mathbb{N}_0} \in S_\infty(R)$ such that

$$c_n = \sum_{\eta \in X^n} \left(\frac{1}{R}\right)^n n!\,\eta.$$

It is evident that the sequence $(c_i)_{i\in\mathbb{N}_0}$ is bounded as $||c_n||_{\infty,R} = 1 \ \forall n \in \mathbb{N}_0$. However, note that $\forall\, m, n \in \mathbb{N}_0$ where $m \neq n$,

$$||c_m - c_n||_{\infty,R} = 1.$$

Hence, the bounded sequence $(c_i)_{i\in\mathbb{N}_0}$ has no convergent subsequence. □

Moreover, the space $S_\infty(R')$ is not a separable space viz. the Banach space does not have a countable dense topological subspace [Dahmen, et al. (2020)]. The space $S^m_\infty(R')$, which is a direct product of m Banach spaces, is provided a Banach space structure by the norm

$$||d||_{\infty,R} = \max_{i=1,2,\ldots,m} ||d_i||_{\infty,R}.$$

Let $R', R \in \mathbb{R}$ such that $0 < R < R'$. Observe that $S_\infty(R') \subset S_\infty(R)$ as vector spaces. In addition, the topology on $S_\infty(R')$ induced by norm $||.||_{\infty,R'}$ is finer than the subspace topology induced from $S_\infty(R)$. Hence, this inclusion of vector spaces is not a topological embedding. In fact, the inclusion map $i : S_\infty(R') \longrightarrow S_\infty(R)$ is a compact operator, viz. every bounded sequence in $S_\infty(R')$ has a convergent subsequence in $S_\infty(R)$ [Dahmen, et al. (2020)].

Consider the directed set $(\mathbb{R}_{>0}, \leq)$ with the usual ordering. Then $S_* = \{(S_\infty(R))_{R\in\mathbb{R}_{>0}}\}$ forms a projective system of locally convex topological vector spaces with the family of inclusion maps $i_{R'R} : S_\infty(R') \longrightarrow S_\infty(R)\ \forall 0 < R < R'$. The projective limit of the system $\left(S_*, (i_{R'R})_{0<R<R'}\right)$ is a locally convex topological vector space S_∞ defined as

$$S_\infty = \bigcap_{R\in\mathbb{R}_{>0}} S_\infty(R).$$

The limit space S_∞ is equipped with the initial topology determined by the family of canonical injections $i_R : S_\infty \longrightarrow S_\infty(R)\ \forall R > 0$. Thus,

$$c \in S_\infty \Leftrightarrow ||c||_{\infty,R} < \infty\ \forall R > 0.$$

Since the set $\mathbb{N} \subset \mathbb{R}_{>0}$ is cofinal, it is sufficient to consider the space S_∞ as the projective limit of the spaces $S_\infty(N)$ where $N \in \mathbb{N}$. Hence, the space S_∞ is the sequential projective limit of the Banach spaces $(S_\infty(N))_{N\in\mathbb{N}}$ and can be endowed with the Fréchet topology by Theorem 2.2.2. The ordered set of countable seminorms

$$||\cdot||_{\infty,1} \leq ||\cdot||_{\infty,2} \leq \cdots \leq ||\cdot||_{\infty,k} \leq \cdots$$

is called *a fundamental system of seminorms* for the Fréchet space. The Fréchet spaces are completely metrizable locally convex topological vector spaces. Hence,

$$(c_i)_{i\in\mathbb{N}_0} \to c \in S_\infty \Leftrightarrow \lim_{i\to\infty} ||c_i - c||_{\infty,R} = 0 \quad \forall R > 0.$$

Since the inclusion maps i_{MN} where $0 < N < M$ with $M, N \in \mathbb{N}$ are compact operators, the projective limit S_∞ becomes a *Fréchet-Schwartz* space [Carreras & Bonet (1987), Komatsu (1967)]. Thus, the space S_∞ is separable. A countable dense topological subspace is constructed in [Winter-Arboleda (2019)]. Hence, the limit space S_∞ is better behaved than the spaces $S_\infty(R)$ from which it is constructed. In particular, the space S_∞ satisfies a Bolzano-Wierstrass like theorem. The space $S_\infty^m \subset \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$c \in S_\infty^m \Leftrightarrow c_i \in S_\infty \ \forall i = 1, \ldots, m$$

and is endowed with the *product* topology whenever $m > 1$. The construction of the space S_∞^m is pivotal in regards to the radius of convergence of Fliess operators.

Theorem 2.4.2. *[Winter-Arboleda (2019)] A series $c \in S_\infty^m(R)$ for some $R > 0$ if and only if the corresponding Fliess operator F_c is locally convergent.*

Observe that if $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, then $||c||_{\infty,R} < \infty \ \forall R > 0$. Hence, $\mathbb{R}_{GC}^m\langle\langle X\rangle\rangle \subset S_\infty^m(R) \ \forall R > 0$ implying that $\mathbb{R}_{GC}^m\langle\langle X\rangle\rangle \subset S_\infty^m$.

Theorem 2.4.3. *[Winter-Arboleda (2019)] $S_\infty^m = \overline{\mathbb{R}_{GC}^m\langle\langle X\rangle\rangle}$, where the closure is taken in the Fréchet topology.*

Theorem 2.4.4. *[Winter-Arboleda (2019)] A series $c \in S_\infty^m$ if and only if the corresponding Fliess operator F_c is globally convergent.*

Theorem 2.4.4 asserts that it is necessary and sufficient for a series $c \in S_\infty^m$ in order for the corresponding Fliess operator F_c to describe a well-defined mapping from $B_{\mathfrak{p}}^m(R)[t_0,$

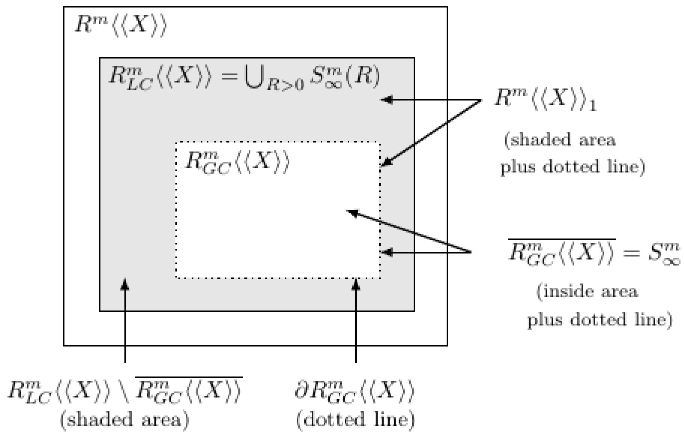


Fig. 3: The hierarchy of topological vector spaces for convergence.

$t_0+T]$ into $B^\ell_{\mathfrak{q}}(S)[t_0,\, t_0+T]$ for all $R, T > 0$. Observe that in Example 2.4.1 the sequence of polynomials is not even Cauchy in $S_\infty(R)$ space $\forall R > 0$. Hence, the sequence does not converge in the S^m_∞ space. Define $\partial\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle = S^m_\infty \setminus \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ in the Fréchet topology. The following example shows that the boundary $\partial\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is not empty.

Example 2.4.3. [Winter-Arboleda (2019)] Let $X = \{x_0, x_1\}$. The Ferfera series $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ is given by $c = x_1^* = \sum_{n=0}^\infty x_1^n$. It is evident that $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with Gevrey degree 0. Consider the series $d = c \circ c$ which describes the cascade connection of two Ferfera systems. It is known that d has Gevrey degree 1, but the corresponding Fliess operator F_d has a well-defined mapping from $B^m_{\mathfrak{p}}(R)[t_0,\, t_0+T]$ into $B^\ell_{\mathfrak{q}}(S)[t_0,\, t_0+T]$ for all $R, T > 0$. Hence, there exists a series $c \in \partial\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ with the Gevrey degree 1 such that the corresponding Fliess operator has a well-defined mapping globally. □

Define $\mathbb{R}^m\langle\langle X\rangle\rangle_1$ as the set of series with Gevrey degree 1. The hierarchy of the spaces $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle, S^m_\infty, \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ and $\mathbb{R}^m\langle\langle X\rangle\rangle_1$ are depicted in Figure 3. Such a partition of the space of formal series $\mathbb{R}^m\langle\langle X\rangle\rangle$ leads to important questions about closure and continuity of addition, various composition products and the shuffle product in each of the $S^m_\infty(R)$ spaces and in $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle, \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$ and S^m_∞. The space $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ can be viewed as the inductive limit of the topological spaces $S_\infty^m(R)$ where $R > 0$ and, hence, can be endowed with the *Silva topology* [Dahmen, et al. (2020)]. This work does not address the continuity of the mixed composition product and shuffle product in the $\mathbb{R}_{LC}^m\langle\langle X\rangle\rangle$ space and is treated in [Gray, et al. (2021)]. However, the continuity of the mixed composition product and shuffle product in the S_∞^m space will be treated in Section 4.2. The following theorem describes the closure of local and global convergence of series under addition.

Theorem 2.4.5. *[Winter-Arboleda (2019)] The following statements are true:*

1. *If $c, d \in \mathbb{R}_{LC}^m\langle\langle X\rangle\rangle$, then $c + d \in \mathbb{R}_{LC}^m\langle\langle X\rangle\rangle$.*

2. *If $c, d \in \mathbb{R}_{GC}^m\langle\langle X\rangle\rangle$, then $c + d \in \mathbb{R}_{GC}^m\langle\langle X\rangle\rangle$.*

3. *If $c, d \in S_\infty^m$, then $c + d \in S_\infty^m$.*

Theorem 2.4.5 is a direct consequence of the triangle inequality for the norm $||.||_{\infty,R}$ in each of the $S_\infty^m(R)$ spaces. The definition for the continuity of a multilinear operator on a Fréchet space is described next. Let E be a real Fréchet space with a fundamental system of seminorms $\{||.||_k\}_{k\in\mathbb{N}}$ such that $||.||_k \leq ||.||_{k+1}$.

Theorem 2.4.6. *[Meise & Vogt (1997)] Consider an $\mathbb{R}$-multilinear operator of rank n on the Fréchet space E such that*

$$T : E \times E \times \cdots \times E \longrightarrow E$$

$$(c_1, c_2, \ldots, c_n) \longmapsto T(c_1, c_2, \ldots, c_n).$$

The operator T is continuous if there exists constant $C > 0$ and a family of maps $\alpha_i : \mathbb{N} \longrightarrow \mathbb{N}$ for $i = 1, 2, \ldots, n$ such that $\forall k \in \mathbb{N}$,

$$||T(c_1, c_2, \ldots, c_n)||_k \leq C||c_1||_{\alpha_1(k)} \cdots ||c_n||_{\alpha_n(k)}.$$

Theorem 2.4.6 is used in Sections 4.2.1 and 4.2.2 where the continuity of shuffle and mixed composition product in the space S_∞^m is addressed.

2.5 FORMAL STATIC MAPS AND CONVERGENCE

Static feedback has both Chen-Fliess series and formal static maps playing key roles in its computation and convergence. This subsection provides a brief discussion on the formal static maps. Let $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$ and $d \in \mathbb{R}^k\,[[\tilde{X}]]$. A formal static function $f_d : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ around the point $z = 0$ is defined as

$$f_d\,(z) = \sum_{\eta \in \tilde{X}^*} (d, \eta)\, z^\eta,$$

where $z \in \mathbb{R}^m$, and $z^{\tilde{x}_i \eta} = z_i z^\eta \;\forall \tilde{x}_i \in \tilde{X}, \eta \in \tilde{X}^*$. The base case is taken to be $z^\emptyset = 1$. The series $d \in \mathbb{R}^k\,[[\tilde{X}]]$ is called the generating series of the static map f_d. A series $d \in \mathbb{R}\,[[\tilde{X}]]$ is said to be *locally convergent* if there exist constants $K_d, M_d > 0$ such that $|(d,\eta)| \leq K_d M_d^{|\eta|}$, $\forall \eta \in \tilde{X}^*$. A series $d \in \mathbb{R}^k\,[[\tilde{X}]]$ is said to be locally convergent if and only if each component d_i is locally convergent for $i = 1, \ldots, m$. The subset of all locally convergent series in $\mathbb{R}^k\,[[\tilde{X}]]$ is denoted as $\mathbb{R}^k_{LC}\,[[\tilde{X}]]$. The following theorem explains the significance of the definition of local convergence in the present context.

Theorem 2.5.1. *If $d \in \mathbb{R}_{LC}\,[[\tilde{X}]]$ with growth constants $K_d, M_d > 0$, then the formal static function $f_d : \mathbb{R}^m \longrightarrow \mathbb{R}$ has a finite radius of convergence $1/M_d$.*

Proof: Let $z \in \mathbb{R}^m$. From the triangle inequality on $\mathbb{R}$,

$$\begin{aligned} |f_d\,(z)| &\leq \sum_{\eta \in \tilde{X}^*} |(d,\eta)||z^\eta| \\ &\leq \sum_{\eta \in \tilde{X}^*} K_d M_d^{|\eta|} |z^\eta| \\ &= K_d \left(\sum_{n=0}^{\infty} \left(M_d |z^{\tilde{x}_1}| \right)^n \right) \cdots \left(\sum_{n=0}^{\infty} \left(M_d |z^{\tilde{x}_m}| \right)^n \right) \end{aligned}$$

$$= K_d \prod_{i=1}^{m} \left(\sum_{n=0}^{\infty} (M_d|z_i|)^n \right).$$

Observe that

$$\sum_{n=0}^{\infty} (M_d|z_i|)^n = \left(\frac{1}{1 - M_d|z_i|} \right)$$

for $|z_i| \leq 1/M_d$. Hence,

$$|f_d(z)| \leq K_d \prod_{i=1}^{m} \left(\frac{1}{1 - M_d|z_i|} \right)$$

for $\max_{i=1,\ldots,m} |z_i| \leq 1/M_d$. ■

Therefore, $d \in \mathbb{R}_{LC}[[\tilde{X}]]$ implies that the corresponding static function f_d is bounded pointwise in absolute value by a real analytic map with a finite radius of convergence. As a consequence of *Cauchy's integral formula* on polydiscs in $\mathbb{C}^m$ [Hormander (1973)], $d \in \mathbb{R}_{LC}[[\tilde{X}]]$ is a necessary and sufficient condition for the corresponding static map f_d to be locally analytic around $z = 0$. The following lemma is essential in the proof of global convergence of static maps and composition products of Chen-Fliess series.

Lemma 2.5.1. *Given $x \in [0, \infty[$ and $r \in]0, 1]$, the following inequality holds:*

$$K_r M_r^x (\Gamma(x+1))^r \leq \Gamma(rx+1) \leq \tilde{K}_r 2^x (\Gamma(x+1))^r,$$

where

$$K_r = \left(\left(\frac{2\pi}{\exp(2)} \right)^{1-r} r \right)^{\frac{1}{2}}, \quad \tilde{K}_r = 2 \left(\left(\frac{2\pi}{\exp(2)} \right)^{1-r} 4 \right)^{\frac{1}{2}} \text{ and } M_r = r^r.$$

Proof: The Stirling series for $\Gamma(z)$ where $z \in \mathbb{C}$ is given by [Abramowitz & Stegun (1988)]

$$\ln \Gamma(z) = \frac{1}{2} \ln 2\pi + \left(z - \frac{1}{2} \right) \ln z - z + \mathcal{O}\left(\frac{1}{z} \right), \qquad \Re(z) > 0,$$

where $\Re(z)$ is the real part of z. Substituting $rz+1$ for z gives

$$\ln\Gamma(rz+1) = \frac{1}{2}\ln 2\pi + \left(rz+\frac{1}{2}\right)\ln(rz+1) - (rz+1) + \mathcal{O}\left(\frac{1}{rz+1}\right), \qquad \Re(z) > 0.$$

Therefore, for $x \in \mathbb{R}_+$,

$$\Gamma(rx+1) = \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)(rx+1)^{\left(rx+\frac{1}{2}\right)}\exp(-rx)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right). \tag{2.5.1}$$

Since $r \in]0,1]$,

$$\begin{aligned}\Gamma(rx+1) &= \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)\left(r\left(x+\frac{1}{r}\right)\right)^{rx}\left(r\left(x+\frac{1}{r}\right)\right)^{\frac{1}{2}}\exp(-rx)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right)\\ &= \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)(r^r)^x r^{\frac{1}{2}}\left(x+\frac{1}{r}\right)^{rx}\left(x+\frac{1}{r}\right)^{\frac{1}{2}}\exp(-rx)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right).\end{aligned} \tag{2.5.2}$$

In addition, for $x \in \mathbb{R}_+$,

$$r\ln\Gamma(x+1) = \frac{r}{2}\ln 2\pi + r\left(x+\frac{1}{2}\right)\ln(x+1) - r(x+1) + r\mathcal{O}\left(\frac{1}{x+1}\right).$$

Therefore,

$$(\Gamma(x+1))^r = \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)^r (x+1)^{r\left(x+\frac{1}{2}\right)}\exp(-rx)\left(1+r\mathcal{O}\left(\frac{1}{x+1}\right)\right). \tag{2.5.3}$$

Hence,

$$\begin{aligned}\left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} r\right)^{\frac{1}{2}}(r^r)^x(\Gamma(x+1))^r &= \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)(r^r)^x r^{\frac{1}{2}}(x+1)^{rx}(x+1)^{\frac{r}{2}}\\ &\quad \exp(-rx)\left(1+r\mathcal{O}\left(\frac{1}{x+1}\right)\right).\end{aligned}$$

Observe that $r \in]0,1]$ implies that $1/r \geq 1$ and $r/2 \leq 1/2$. Thus,

$$\left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} r\right)^{\frac{1}{2}} (r^r)^x \left(\Gamma\left(x+1\right)\right)^r \leq \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right) (r^r)^x r^{\frac{1}{2}} \left(x+\frac{1}{r}\right)^{rx} \left(x+\frac{1}{r}\right)^{\frac{1}{2}} \exp\left(-rx\right)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right).$$

Applying (2.5.2),

$$\left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} r\right)^{\frac{1}{2}} (r^r)^x \left(\Gamma\left(x+1\right)\right)^r \leq \Gamma(rx+1).$$

Therefore,

$$K_r M_r^x (\Gamma(x+1))^r \leq \Gamma(rx+1),$$

where K_r and M_r are defined as above. This establishes the inequality on the left-hand side.

For the inequality on the right-hand side, observe from (2.5.3) that

$$\left(\Gamma\left(x+1\right)\right)^r = \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)^r (x+1)^{r\left(x+\frac{1}{2}\right)} \exp(-rx)\left(1+r\mathcal{O}\left(\frac{1}{x+1}\right)\right).$$

Since $\left(1+r\mathcal{O}\left(\frac{1}{x+1}\right)\right) \geq 1$,

$$(x+1)^{r\left(x+\frac{1}{2}\right)} \leq \frac{\left(\Gamma\left(x+1\right)\right)^r \exp\left(rx\right)}{\left(\left(\frac{2\pi}{\exp(2)}\right)^{\frac{1}{2}}\right)^r}. \tag{2.5.4}$$

From (2.5.1) it follows that

$$\begin{aligned}\Gamma(rx+1) &= \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)(rx+1)^{\left(rx+\frac{1}{2}\right)}\exp\left(-rx\right)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right)\\ &= \left(\frac{(2\pi)^{\frac{1}{2}}}{\exp(1)}\right)(rx+1)^{r\left(x+\frac{1}{2}\right)}(rx+1)^{\left(\frac{1-r}{2}\right)}\exp\left(-rx\right)\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right).\end{aligned}$$

Applying (2.5.4),

$$\Gamma(rx+1) \leq \left(\left(\frac{2\pi}{\exp(2)}\right)^{\frac{1}{2}}\right) \frac{(\Gamma(x+1))^r \exp(rx)}{\left(\left(\frac{2\pi}{\exp(2)}\right)^{\frac{1}{2}}\right)^r} (rx+1)^{\left(\frac{1-r}{2}\right)} \exp(-rx)$$
$$\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right).$$

Since $r \in]0,1]$ implies that $\left(\frac{1-r}{2}\right) < 1/2$, it follows that,

$$\Gamma(rx+1) \leq \left(\left(\frac{2\pi}{\exp(2)}\right)^{\frac{1}{2}}\right) (\Gamma(x+1))^r (rx+1)^{\frac{1}{2}} \left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right).$$

Observe $(rx+1)^{\frac{1}{2}} \leq 2^x$ and $\left(1+\mathcal{O}\left(\frac{1}{rx+1}\right)\right) \leq 2$. Hence,

$$\Gamma(rx+1) \leq \left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} 4\right)^{\frac{1}{2}} 2^x (\Gamma(x+1))^r.$$

Therefore,

$$\Gamma(rx+1) \leq \tilde{K}_r 2^x (\Gamma(x+1))^r,$$

where $\tilde{K}_r$ is defined as above. ■

Lemma 2.5.1 states that the ratio of a fractional power of the factorial of a non-negative real number (in the sense of analytic continuation) and the factorial of a fraction of a real number has an exponential growth analytically. Therefore, $\forall r \in]0,1]$ and $x \in \mathbb{R}_{>0}$

$$\Gamma(x+1)^r \approx \Gamma(rx+1)\Theta(c^x),$$

where $\Theta(.)$ is the standard *Bachman-Landau* notation, and c is some constant.

The *Gevrey order* of a series $d \in \mathbb{R}\,[[\tilde{X}]]$ is defined as

$$s = \inf\{t \geq 0 : |(d,\eta)| \leq K_d M_d^{|\eta|}\,(|\eta|!)^t\,,\ \forall \eta \in \tilde{X}^*\},$$

where $K_d, M_d > 0$ are constants and are not fixed apriori. A series $d \in \mathbb{R}\,[[\tilde{X}]]$ is said to be *globally convergent* if there exist constants $K_d, M_d > 0$ and $s \in [0,1[$ such that

$$|(d,\eta)| \leq K_d M_d^{|\eta|}\,(|\eta|!)^{-1+s}\,, \quad \forall \eta \in \tilde{X}^*.$$

Hence, a series $d \in \mathbb{R}_{GC}\,[[\tilde{X}]]$ has a Gevrey order $(-1+s)$ with $s \in [0,1[$ while a series $d \in \mathbb{R}_{LC}\,[[\tilde{X}]]$ has a Gevrey order of 0. A series $d \in \mathbb{R}^k\,[[\tilde{X}]]$ is said to be globally convergent if and only if each component d_i is globally convergent for $i = 1,\ldots,m$. The subset of all globally convergent series in $\mathbb{R}^k\,[[\tilde{X}]]$ is denoted as $\mathbb{R}^k_{GC}\,[[\tilde{X}]]$. The following theorem explains the significance of the definition of global convergence of a series with respect to its corresponding static function.

Theorem 2.5.2. *If $d \in \mathbb{R}_{GC}\,[[\tilde{X}]]$ with growth constants $K_d, M_d > 0$ and Gevrey order $(-1+s)$ with $s \in [0,1[$, then the formal static function $f_d : \mathbb{R}^m \longrightarrow \mathbb{R}$ converges over the entire domain $\mathbb{R}^m$.*

Proof: Let $z \in \mathbb{R}^m$. From the triangle inequality on $\mathbb{R}$,

$$\begin{aligned} |f_d\,(z)| &\leq \sum_{\eta \in \tilde{X}^*} |(d,\eta)||z^\eta| \\ &\leq \sum_{\eta \in \tilde{X}^*} K_d M_d^{|\eta|}\,(|\eta|!)^{-1+s}\,|z^\eta| \\ &\leq K_d \left(\sum_{n=0}^{\infty} \frac{\left(M_d |z^{\tilde{x}_1}|\right)^n}{n!^{(1-s)}} \right) \cdots \left(\sum_{n=0}^{\infty} \frac{\left(M_d |z^{\tilde{x}_m}|\right)^n}{n!^{(1-s)}} \right) \\ &= K_d \prod_{i=1}^{m} \left(\sum_{n=0}^{\infty} \frac{\left(M_d |z_i|\right)^n}{n!^{(1-s)}} \right). \end{aligned}$$

Since $n! = \Gamma(n+1)$, then by Lemma 2.5.1,

$$\left(\sum_{n=0}^{\infty} \frac{(M_d|z_i|)^n}{n!^{(1-s)}}\right) \leq \tilde{K}_r \left(\sum_{n=0}^{\infty} \frac{(2M_d|z_i|)^n}{\Gamma((1-s)\,n+1)}\right)$$
$$= \mathbb{E}_{(1-s),1}(2M_d|z_i|),$$

where $\mathbb{E}_{(1-s),1}(.)$ is the Mittag-Leffler function. Hence,

$$|f_d(z)| \leq K_d \prod_{i=1}^{m} \mathbb{E}_{(1-s),1}(2M_d|z_i|).$$

■

Observe that $s \in [0,1[$ if and only if $(1-s) \in]0,1]$. Hence, $\mathbb{E}_{(1-s),1}(\cdot)$ is an entire function on $\mathbb{C}$. Therefore, $d \in \mathbb{R}_{GC}[[\tilde{X}]]$ implies that the corresponding static map f_d is bounded pointwise in absolute value by a real analytic map which is convergent everywhere on $\mathbb{R}^m$. Observe that a commutative polynomial $d \in \mathbb{R}[\tilde{X}]$ is globally convergent with Gevrey order -1. Like in the case of Chen-Fliess series, a commutative series $d \in \mathbb{R}^m_{GC}[[\tilde{X}]]$ is only a sufficient condition for the corresponding formal static map to be convergent everywhere on $\mathbb{R}^m$. The derivation of a necessary condition for a real analytic function that is convergent everywhere on $\mathbb{R}^m$ requires more careful attention. An analytic function that is analytic everywhere on $\mathbb{R}^m$ need not extend into an entire function upon complexification of the domain.

Example 2.5.1. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = 1/(x^2+1)$. The function f is analytic everywhere on $\mathbb{R}$. The complexification of $f : \mathbb{C} \longrightarrow \mathbb{C}$ given by $f(z) = 1/(z^2+1)$ is not an entire function on $\mathbb{C}$ as the complex map f has poles at $z = \pm i$. □

A locally real analytic function always extends to a locally analytic complex function, but a function that is analytic over the entire real line does not necessarily extend to an entire function. Hence, the complexification approach does not yield to find the necessary growth condition for a real analytic function that is analytic everywhere on $\mathbb{R}^m$. The derivation of the necessary condition is deferred for future work.

2.6 SHUFFLE GROUP

This subsection presents the *shuffle group*. The computations and algorithms described in Section 3.2 are based on the Hopf algebra of the coordinate maps defined on the shuffle group. The following theorem describes the *shuffle group*.

Theorem 2.6.1. *[Gray, et al. (2014b)] The set of non-proper series in* $\mathbb{R}\langle\langle X\rangle\rangle$ *is a group under the shuffle product. In particular, the shuffle inverse of any such series* c *is*

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1-c'))^{\sqcup\!\sqcup -1} = (c,\emptyset)^{-1}(c')^{\sqcup\!\sqcup *},$$

where $c' := 1 - c/(c,\emptyset)$ *is proper and* $(c')^{\sqcup\!\sqcup *} := \sum_{k\geq 0}(c')^{\sqcup\!\sqcup k}$ *and the identity element is the constant* 1.

More generally, if $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, then the shuffle inverse is defined componentwise viz. $(c^{\sqcup\!\sqcup -1})_i = c_i^{\sqcup\!\sqcup -1}$, where $i = 1, 2, \ldots \ell$. Hence, in general, $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \sqcup\!\sqcup)$ possesses a group structure with the identity element $\mathbb{1} \triangleq [1\ 1 \cdots 1]^T \in \mathbb{R}^\ell$.

Example 2.6.1. Let $c = 1 - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$. Observe that, $c' = x_1$, and hence, $c^{\sqcup\!\sqcup -1} = x_1^{\sqcup\!\sqcup *} = \sum_{k\geq 0} k!\, x_1^k$. □

CHAPTER 3

ADDITIVE STATIC FEEDBACK PRODUCT: DEFINITION AND COMPUTATION

Consider the additive static feedback connection of a Chen-Fliess series F_c with a static map f_d in the feedback. The objective of the chapter is to prove the existence and uniqueness of a Chen-Fliess series for the closed-loop system and to compute the generating series for the closed-loop system. The chapter also develops the Hopf algebra corresponding to the shuffle group, which aids in building the algorithmic tools for computing the feedback product. Before attempting the feedback problem, the cascade connection of Chen-Fliess F_c with the static map f_d must be treated in detail.

3.1 WIENER-FLIESS CONNECTIONS

This section describes the cascade connection shown in Figure 4 of a Chen-Fliess series F_c generated by a series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and a formal static map $f_d : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$ defined without loss of generality at $z = 0$. Such configurations are called *Wiener-Fliess* connections. The connection is known to generate another well defined formal Fliess operator, and its generating series is computed through the *Wiener-Fliess composition product*. The product is well defined formally due to the local finiteness property in the following cases:

1. The Fliess operator F_c is defined by a proper series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$.

2. The formal static function $f_d : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$ is a vector of k polynomials.

The definition addressing the first case appears in [Gray & Thitsa (2012)]. However, the definition of the *Wiener-Fliess* product remains the same for both cases and is given in the following theorem.

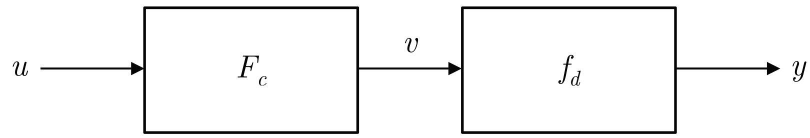


Fig. 4: Wiener-Fliess connection

Theorem 3.1.1. *Let* $X = \{x_0, x_1, \ldots, x_m\}$ *and* $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_\ell\}$*. Given a formal Fliess operator* F_c *with* $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ *and formal function* $f_d : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$ *with a generating series* $d \in \mathbb{R}^k\,[[\tilde{X}]]$ *at* $z = 0$*, viz.*

$$f_d(z) = \sum_{\tilde{\eta} \in \tilde{X}^*} (d, \tilde{\eta}) z^{\tilde{\eta}},$$

the composition $f_d \circ F_c$ *has a generating series in* $\mathbb{R}^k\langle\langle X\rangle\rangle$ *provided either of the following holds:*

1. $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ *is proper.*

2. $d \in \mathbb{R}^k\,[\tilde{X}]$.

The generating series of $f_d \circ F_c$ *is then given by the Wiener-Fliess composition product*

$$d \,\hat{\circ}\, c = \sum_{\tilde{\eta} \in \tilde{X}^*} (d, \tilde{\eta}) c^{\,\sqcup\!\sqcup\, \tilde{\eta}}, \tag{3.1.1}$$

where $c^{\,\sqcup\!\sqcup\, \tilde{x}_i\tilde{\eta}} := c_i \sqcup\!\sqcup\, c^{\,\sqcup\!\sqcup\, \tilde{\eta}}\ \forall \tilde{x}_i \in \tilde{X},\ \forall \tilde{\eta} \in \tilde{X}^*$*, and* $c^{\,\sqcup\!\sqcup\, \phi} = 1$.

The following theorem shows that the Wiener-Fliess composition product is left $\mathbb{R}$-linear.

Theorem 3.1.2. *If either of the following conditions hold,*

1. $c \in \mathbb{R}^\ell_p\,\langle\langle X\rangle\rangle$.

2. $d, e \in \mathbb{R}^k\,[\tilde{X}]$.

then $(\alpha d + e)\,\hat{\circ}\, c = \alpha(d\,\hat{\circ}\, c) + (e\,\hat{\circ}\, c)$*, where* $\alpha \in \mathbb{R}$*.*

Proof: Observe

$$\begin{aligned}
(\alpha d + e)\,\hat{\circ}\, c &= \sum_{\tilde{\eta}\in\tilde{X}^*} ((\alpha d) + e, \tilde{\eta}) c^{\sqcup\!\sqcup\,\tilde{\eta}} \\
&= \sum_{\tilde{\eta}\in\tilde{X}^*} (\alpha d, \tilde{\eta}) c^{\sqcup\!\sqcup\,\tilde{\eta}} + \sum_{\tilde{\eta}\in\tilde{X}^*} (e, \tilde{\eta}) c^{\sqcup\!\sqcup\,\tilde{\eta}} \\
&= \alpha \sum_{\tilde{\eta}\in\tilde{X}^*} (d, \tilde{\eta}) c^{\sqcup\!\sqcup\,\tilde{\eta}} + \sum_{\tilde{\eta}\in\tilde{X}^*} (e, \tilde{\eta}) c^{\sqcup\!\sqcup\,\tilde{\eta}} \\
&= \alpha(d\,\hat{\circ}\, c) + (e\,\hat{\circ}\, c).
\end{aligned}$$

■

The next lemma will be used to show that the Wiener-Fliess composition product has certain contractive properties in the ultrametric space.

Lemma 3.1.1. *Let* $X = \{x_0, x_1, \ldots, x_m\}$ *and* $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_\ell\}$*. Assume* $\eta \in \tilde{X}^+$*.*

1. *If* $c,\ \tilde{c} \in \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$*, then* $\kappa(\eta\,\hat{\circ}\, c, \eta\,\hat{\circ}\,\tilde{c}) \leq \max\{\kappa(c), \kappa(\tilde{c})\}^{(|\eta|-1)}\kappa(c,\tilde{c})$*.*

2. *If* $c,\ \tilde{c} \in \mathbb{R}^\ell\langle\langle X\rangle\rangle \setminus \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$*, then* $\kappa(\eta\,\hat{\circ}\, c, \eta\,\hat{\circ}\,\tilde{c}) \leq \kappa(c,\tilde{c})$*.*

Proof: The proof is by induction on the length of η. If $\eta = \tilde{x}_i$ for $i = 1, 2, \ldots \ell$, then

$$\begin{aligned}
\kappa(\tilde{x}_i\,\hat{\circ}\, c, \tilde{x}_i\,\hat{\circ}\,\tilde{c}) &= \kappa(c^{\sqcup\!\sqcup\,\tilde{x}_i}, \tilde{c}^{\sqcup\!\sqcup\,\tilde{x}_i}) \\
&= \kappa(c_i, \tilde{c}_i) \\
&\leq \kappa(c, \tilde{c}).
\end{aligned}$$

Hence, the base case is proved. Now assume the hypothesis is true for $|\eta| = k \geq 1$. Let $\hat{\eta} = \tilde{x}_j\eta$, where $\tilde{x}_j \in \tilde{X}$ and $\eta \in \tilde{X}^k$. Then

$$\kappa(\hat{\eta}\,\hat{\circ}\, c, \hat{\eta}\,\hat{\circ}\,\tilde{c}) = \kappa(c^{\sqcup\!\sqcup\,\tilde{x}_j\eta}, \tilde{c}^{\sqcup\!\sqcup\,\tilde{x}_j\eta})$$

$$= \kappa(c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta}, \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta})$$

$$= \kappa(c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta} - \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta})$$

$$= \kappa((c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta} - c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}) + (c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta} - \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}))$$

$$\leq \max\{\kappa(c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta} - c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}), \kappa(c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta} - \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta})\}$$

$$= \max\{\kappa(c_j \sqcup\!\sqcup\, (c^{\sqcup\!\sqcup\, \eta} - \tilde{c}^{\sqcup\!\sqcup\, \eta})), \kappa((c_j - \tilde{c}_j) \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta})\}.$$

By the triangle inequality on the ultrametric and the induction hypothesis,

$$\kappa(\hat{\eta} \,\hat{\circ}\, c, \hat{\eta} \,\hat{\circ}\, \tilde{c}) \leq \max\left\{\kappa(c) \max\{\kappa(c), \kappa(\tilde{c})\}^{(|\eta|-1)} \kappa(c, \tilde{c}), \kappa(\tilde{c})^{|\eta|} \kappa(c, \tilde{c})\right\}$$

$$= \max\{\kappa(c), \kappa(\tilde{c})\}^{|\eta|} \kappa(c, \tilde{c}),$$

which proves the claim when c is proper. If c is not proper, then $\kappa(c) = \kappa(\tilde{c}) = 1$. Therefore,

$$\kappa(\hat{\eta} \,\hat{\circ}\, c, \hat{\eta} \,\hat{\circ}\, \tilde{c}) \leq \kappa(c, \tilde{c})$$

as desired. ■

For a fixed $d \in \mathbb{R}^k[[\tilde{X}]]$ define the map $d_{\hat{\circ}} : \mathbb{R}_p^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^k\langle\langle X\rangle\rangle : c \mapsto d \,\hat{\circ}\, c$, and for a fixed $\tilde{d} \in \mathbb{R}^k[\tilde{X}]$ define the map $\tilde{d}_{\hat{\circ}} : \mathbb{R}^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^k\langle\langle X\rangle\rangle : \bar{c} \mapsto d \,\hat{\circ}\, \bar{c}$. The following theorems describe the contractive properties of $d_{\hat{\circ}}$ and $\tilde{d}_{\hat{\circ}}$.

Theorem 3.1.3. *The map $d_{\hat{\circ}}$ is a weak contraction map when $\overline{\omega}(d) = 1$ and a strong contraction map when $\overline{\omega}(d) > 1$.*

Proof: Let $c, c' \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$. Observe,

$$\kappa(d_{\hat{\circ}}(c), d_{\hat{\circ}}(c')) = \kappa(d \,\hat{\circ}\, c, d \,\hat{\circ}\, c')$$

$$= \kappa\left(\sum_{\eta \in \tilde{X}^*} (d, \eta)(c^{\sqcup\!\sqcup\, \eta} - c'^{\,\sqcup\!\sqcup\, \eta})\right)$$

$$\leq \sup_{\eta\in\tilde{X}^+} \kappa\big((d,\eta)(c^{\sqcup\!\sqcup\,\eta} - c'^{\,\sqcup\!\sqcup\,\eta})\big)$$

$$= \sup_{k\geq\,\overline{\omega}(d)} \sup_{\eta\in\tilde{X}^k} \kappa(c^{\sqcup\!\sqcup\,\eta}, c'^{\,\sqcup\!\sqcup\,\eta}).$$

Applying Lemma 3.1.1 gives

$$\kappa(d_{\,\hat{\circ}}(c), d_{\,\hat{\circ}}(c')) \leq \sup_{k\geq\,\overline{\omega}(d)} \max\{\kappa(c),\kappa(c')\}^{k-1}\kappa(c,c')$$

$$\leq \max\{\kappa(c),\kappa(c')\}^{\overline{\omega}(d)-1}\kappa(c,c').$$

■

Theorem 3.1.4. *The map $\tilde{d}_{\,\hat{\circ}}$ is a weak contraction map.*

Proof: Let $c,c'\in\mathbb{R}^\ell\langle\langle X\rangle\rangle$. Observe,

$$\kappa(\tilde{d}_{\,\hat{\circ}}(c),\tilde{d}_{\,\hat{\circ}}(c')) = \kappa(\tilde{d}\,\hat{\circ}\,c, \tilde{d}\,\hat{\circ}\,c')$$

$$= \kappa\left(\sum_{\eta\in\operatorname{supp}(\tilde{d})} (\tilde{d},\eta)(c^{\sqcup\!\sqcup\,\eta} - c'^{\,\sqcup\!\sqcup\,\eta})\right)$$

$$\leq \sup_{\eta\in\operatorname{supp}(\tilde{d})} \kappa\big((\tilde{d},\eta)(c^{\sqcup\!\sqcup\,\eta} - c'^{\,\sqcup\!\sqcup\,\eta})\big).$$

By the definition of ultrametric κ,

$$\kappa(\tilde{d}_{\,\hat{\circ}}(c),\tilde{d}_{\,\hat{\circ}}(c')) \leq \sup_{\eta\in\operatorname{supp}(\tilde{d})} \kappa\left(c^{\sqcup\!\sqcup\,\eta} - c'^{\,\sqcup\!\sqcup\,\eta}\right)$$

$$= \sup_{\eta\in\operatorname{supp}(\tilde{d})} \kappa\left(c^{\sqcup\!\sqcup\,\eta}, c'^{\,\sqcup\!\sqcup\,\eta}\right).$$

Applying Lemma 3.1.1 gives

$$\kappa(\tilde{d}_{\,\hat{\circ}}(c),\tilde{d}_{\,\hat{\circ}}(c')) \leq \sup_{\eta\in\operatorname{supp}(\tilde{d})} \kappa\left(c,c'\right)$$

$$= \kappa\,(c, c')\,.$$

■

The following theorem states the mixed associativity property involving the mixed composition product and the Wiener-Fliess composition product. This identity plays a key role in determining the generating series of the static feedback connection in Section 3.3.

Theorem 3.1.5. *If either of the following conditions hold,*

1. $c \in \mathbb{R}_p^{\ell}\,\langle\langle X\rangle\rangle$
2. $d \in \mathbb{R}^k\,[\tilde{X}]$,

with $e \in \mathbb{R}^m\langle\langle X'\rangle\rangle$ *such that* $|\tilde{X}| = \ell$ *and* $|X| = m+1$, *then* $d\,\hat{\circ}\,(c\,\tilde{\circ}\,e_\delta) = (d\,\hat{\circ}\,c)\,\tilde{\circ}\,e_\delta$.

Proof: The proof is obtained directly from the definition of the Wiener-Fliess composition product in Theorem 3.1.1 by linearly extending the identity given in Lemma 2.3.1.

■

The final theorem of the section states necessary and sufficient conditions for which relative degree is preserved under the Wiener-Fliess composition product.

Theorem 3.1.6. *Let* $X = \{x_0, x_1\}$ *and* $c \in \mathbb{R}_p\,\langle\langle X\rangle\rangle$ *with relative degree* r_c. *Assume* $d \in \mathbb{R}\,[[\tilde{x}_1]]$. *The relative degree of* $d\hat{\circ}\,c$ *is well-defined and equal to* r_c *if and only if* $(d, \tilde{x}_1) \neq 0$.

Proof: The proof follows from the formula in Theorem 3.1.1 and Lemma 2.3.3 ■

For the case when c is non-proper and $d \in \mathbb{R}\,[\tilde{x}_1]$, the relative degree of $d\,\hat{\circ}\,c$ requires caution and is hard to characterize in general. The following example demonstrates a case when the Chen-Fliess series c is non-proper but $d\,\hat{\circ}\,c$ has relative degree.

Example 3.1.1. Let $X = \{x_0, x_1\}$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $c = 1 + x_1$. Observe that c has relative degree 1. Given $d \in \mathbb{R}\,[\tilde{x}_1]$ such that $d = \tilde{x}_1^2$, then

$$d\,\hat{\circ}\,c = \tilde{x}_1^2\,\hat{\circ}\,(1 + x_1)$$

$$= (1 + x_1)^{\sqcup\!\sqcup\, 2}$$
$$= 1 + 2x_1 + 2x_1^2.$$

Hence, the relative degree of $d \,\hat{\circ}\, c$ exists and is 1. □

The following is an example when Chen-Fliess series c is non-proper but $d \,\hat{\circ}\, c$ does not have relative degree.

Example 3.1.2. Let $X = \{x_0, x_1\}$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $c = 1 + x_1$. Observe that c has relative degree 1. Given $d \in \mathbb{R}\,[\tilde{x}_1]$ such that $d = \tilde{x}_1^2 - 2\tilde{x}_1$, then

$$\begin{aligned} d \,\hat{\circ}\, c &= \tilde{x}_1^2 - 2\tilde{x}_1 \,\hat{\circ}\, (1 + x_1) \\ &= (1 + x_1)^{\sqcup\!\sqcup\, 2} - 2\,(1 + x_1) \\ &= -1 + 2x_1^2. \end{aligned}$$

Hence, the relative degree of $d \,\hat{\circ}\, c$ is not well-defined. □

3.2 HOPF ALGEBRA OF THE SHUFFLE GROUP

The goal of this section is to describe the Hopf algebra of the shuffle group as defined in Theorem 2.6.1. It is utilized subsequently to develop an algorithm to compute the Wiener-Fliess composition product. Define the set of formal power series

$$M = \{\,\mathbb{1} + d\ : d \in \mathbb{R}_p^n\,\langle\langle X\rangle\rangle\},$$

where $\mathbb{1} = [1 \cdots 1\,1]^T \in \mathbb{R}^n$. In light of Theorem 2.6.1, $(M, \sqcup\!\sqcup)$ forms an Abelian group, where the shuffle inverse of $c \in M$ is defined componentwise viz. $(c^{\sqcup\!\sqcup\, -1})_i = (c_i)^{\sqcup\!\sqcup\, -1}$. The identity element of the group M is $\mathbb{1}$. Let the set of all maps from M to $\mathbb{R}^n$ be denoted as

$\mathrm{Hom}_{\mathbf{set}}(M,\mathbb{R}^n)$. The subset $H \subset \mathrm{Hom}_{\mathbf{set}}(M,\mathbb{R}^n)$ of coordinate maps defined on group M is

$$H = \{a_\eta \ : a_\eta(c) = (c,\eta) \ : \ \eta \in X^*\}.$$

H has an $\mathbb{R}$-algebra structure with addition, scalar multiplication and product defined, respectively, as

$$\begin{aligned}(a_\eta + a_\zeta)(c) &= a_\eta(c) + a_\zeta(c)\\ (ka_\eta)(c) &= k(a_\eta(c))\\ \boldsymbol{m}(a_\eta, a_\zeta)(c) &= a_\eta(c) \odot a_\zeta(c),\end{aligned}$$

where $\eta, \zeta \in X^*, k \in \mathbb{R}$, and $\odot$ denotes the Hadamard product on $\mathbb{R}^n$. The unit for the product is given by $a_\emptyset$ with $a_\emptyset(c) = \mathbb{1}$, $\forall c \in M$. Define the coproduct $\Delta : H \longrightarrow H \bigotimes H$ as $\Delta a_\eta(c,d) = a_\eta(c ⧢ d)$, where $c, d \in M$ and $\eta \in X^*$. The counit map ϵ is defined as

$$\epsilon(a_\eta) = \begin{cases} 1 \ : \ \eta = \emptyset \\ 0 \ : \ \text{otherwise.}\end{cases}$$

It is simple to check that $(H, \boldsymbol{m}, a_\emptyset, \Delta, \epsilon)$ forms a commutative and cocommutative bialgebra. The bialgebra is graded based on word length viz. $H = \bigoplus_{k\in\mathbb{N}_0} H_k$ with $a_\eta \in H_k$ if and only if $|\eta| = k$. Since $\mathbb{R} \cong H_0$ in the category of algebras with ϵ acting as the isomorphism, H is a connected and graded bialgebra. The reduced coproduct Δ' is defined as $\Delta'(a_\eta) = \Delta(a_\eta) - a_\eta \otimes \mathbf{1} - \mathbf{1} \otimes a_\eta$ if $\eta \neq \emptyset$. Here, $\mathbf{1}$ stands for the constant map which maps c to $\mathbb{1}$ for all $c \in M$. For the case of the empty word, $\Delta'(a_\emptyset) = 0$. If $c, d \in \mathbb{R}_p^n\langle\langle X\rangle\rangle$, then their corresponding elements in the shuffle group M are $\mathbb{1} + c$ and $\mathbb{1} + d$, respectively. The shuffle product of two proper series is computed by the reduced coproduct of the corresponding elements in the shuffle group M. For all proper series $c, d \in \mathbb{R}_p^n\langle\langle X\rangle\rangle$ and $\eta \in X^*$, it follows that $(c ⧢ d, \eta) = \Delta'(a_\eta)(\mathbb{1} + c, \mathbb{1} + d)$. The antipode map $S : H \longrightarrow H$ is given by

$S(a_\eta)(c) = a_\eta(c^{\,ш\,-1})$. Since the Hopf algebra is graded and connected the antipode can be computed for any $a \in H^+$ (where $H^+ := \bigoplus_{k\geq 1} H_k$) as [Figueroa & Gracia-Bondía (2005)]

$$S(a) = -a - \sum a'_{(1)} \odot S(a'_{(2)}),$$

where the summation is taken over all components of the reduced coproduct $\Delta'(a)$ written in the Sweedler notation [Sweedler (1969), Abe (2004)]. Therefore, the tuple $(H, \boldsymbol{m}, a_\emptyset, \Delta, \epsilon, S)$ forms a commutative, cocommutative, connected and graded unital Hopf algebra.

Example 3.2.1. Reconsider Example 2.6.1, where $c = 1 - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$ so that $c^{\,ш\,-1} = \sum_{k\geq 0} k!\, x_1^k$. The goal is to determine $(c^{\,ш\,-1}, x_1^2)$ directly without computing the entire shuffle inverse. Observe

$$a_{x_1^2}(c^{\,ш\,-1}) = S(a_{x_1^2})(c),$$

and the reduced coproduct of $a_{x_1^2}$ is

$$\Delta'(a_{x_1^2}) = 2(a_{x_1} \otimes a_{x_1}).$$

Since $\Delta'(a_{x_1}) = 0$, it follows that

$$S(a_{x_1}) = -a_{x_1}.$$

Hence,

$$\begin{aligned} S(a_{x_1^2})(c) &= -a_{x_1^2}(c) - 2\big(a_{x_1}(c)(-a_{x_1}(c))\big) \\ &= 0 - 2(1(-1)) = 2. \end{aligned}$$

Therefore, $(c^{\,ш\,-1}, x_1^2) = 2$, as expected. □

An inductive algorithm is presented next to compute the coproduct Δ on H. A key feature of the algorithm is a recursively defined *partition* map $\mu: X^* \longrightarrow X^* \otimes X^*$, where $x_j\eta \mapsto (x_j \otimes \emptyset + \emptyset \otimes x_j)\mu(\eta)$ with $\eta \in X^*$, $x_j \in X$, and $\mu(\emptyset) := (\emptyset \otimes \emptyset)$. The definition of the map μ is exactly dual to the definition of the *deshuffle* coproduct $\Delta_{\sqcup\sqcup}$ described in [Foissy (2015)]. The deshuffle coproduct is described on the coordinate maps a_η for all $\eta \in X^*$ and involves the splitting of the coordinate maps. However, from an algorithmic perspective, it is more natural to split the underlying words as described in the following algorithm.

Algorithm 3.2.1. For all $\eta \in X^*$ and $c, d \in M$, the coproduct $\Delta a_\eta(c,d)$ can be computed as:

1. $\mu(\eta) = \sum \eta_{(1)} \otimes \eta_{(2)}$.

2. $\Delta a_\eta(c,d) = \sum a_{\eta_{(1)}}(c) \odot a_{\eta_{(2)}}(d)$.

This algorithm can be trivially extended to compute the reduced coproduct.

Algorithm 3.2.2. For all $\eta \in X^*$ and $c, d \in M$, the reduced coproduct $\Delta' a_\eta(c,d)$ can be computed as:

1. If $\eta = \emptyset$, then $\Delta' a_\eta(c,d) = 0$.

2. Else, $\Delta' a_\eta(c,d) = \Delta a_\eta(c,d) - a_\eta(c) \odot \mathbb{1} - \mathbb{1} \odot a_\eta(d)$.

Let Φ_c be an $\mathbb{R}$-linear homomorphism of algebras defined as $\Phi_c : H \longrightarrow \mathbb{R}^n : a_\eta \mapsto a_\eta(c)$, where $\mathbb{R}^n$ is an $\mathbb{R}$-algebra under the Hadamard product. The maps Φ_c are usually called the *characters* of the Hopf algebra H and form a group under the Hopf convolution product $\star$ defined as

$$\begin{aligned}(\Phi_c \star \Phi_d)(a_\eta) &= \boldsymbol{m} \circ (\Phi_c \otimes \Phi_d) \circ \Delta(a_\eta) \\ &= \sum \Phi_c(a_{\eta_{(1)}}) \odot \Phi_d(a_{\eta_{(2)}}) \\ &= \sum a_{\eta_{(1)}}(c) \otimes a_{\eta_{(2)}}(d)\end{aligned}$$

$$= \Delta a_\eta(c,d) = (c \sqcup\!\sqcup d, \eta).$$

Hence, alternatively, the coproduct can be realized as the Hopf convolution product of the characters of the Hopf algebra H. The group inverse for any character Φ_c is defined as $\Phi_c^{\star -1} = \Phi_{c^{\sqcup\!\sqcup -1}} = \Phi_c \circ S$. It is not hard to see that the group of characters of the Hopf algebra H and the shuffle group M are isomorphic.

Example 3.2.2. Suppose $X = \{x_1, x_2\}$. Let $c = 1 - x_1$ and $d = 1 + x_1x_2 \in \mathbb{R}\langle\langle X\rangle\rangle$. The shuffle product $c \sqcup\!\sqcup d$ is computed directly as $c \sqcup\!\sqcup d = 1 + x_1x_2 - 2x_1^2x_2 - x_1x_2x_1$. The objective is to find only $(c \sqcup\!\sqcup d, x_1x_2x_1) = \Delta a_{x_1x_2x_1}(c,d)$ using Algorithm 3.2.1.

(1) Recursively apply the map μ to compute the partition of the word $x_1x_2x_1$:

$$\begin{aligned}
\mu(x_1x_2x_1) &= (x_1 \otimes \emptyset + \emptyset \otimes x_1)\mu(x_2x_1) \\
&= (x_1 \otimes \emptyset + \emptyset \otimes x_1)(x_2 \otimes \emptyset + \emptyset \otimes x_2)\mu(x_1) \\
&= (x_1 \otimes \emptyset + \emptyset \otimes x_1)(x_2 \otimes \emptyset + \emptyset \otimes x_2)(x_1 \otimes \emptyset + \emptyset \otimes x_1) \\
&= (x_1 \otimes \emptyset + \emptyset \otimes x_1)(x_2x_1 \otimes \emptyset + x_2 \otimes x_1 + x_1 \otimes x_2 + \emptyset \otimes x_2x_1) \\
&= x_1x_2x_1 \otimes \emptyset + x_1x_2 \otimes x_1 + x_1^2 \otimes x_2 + x_1 \otimes x_2x_1 + x_2x_1 \otimes x_1 + x_2 \otimes x_1^2 + \\
&\quad x_1 \otimes x_1x_2 + \emptyset \otimes x_1x_2x_1.
\end{aligned}$$

(2) Compute the coproduct:

$$\begin{aligned}
\Delta a_{x_1x_2x_1}(c,d) &= (c, x_1x_2x_1)(d,\emptyset) + (c, x_1x_2)(d, x_1) + (c, x_1^2)(d, x_2) + (c, x_1)(d, x_2x_1) + \\
&\quad (c, x_2x_1)(d, x_1) + (c, x_2)(d, x_1^2) + (c, x_1)(d, x_1x_2) + (c, \emptyset)(d, x_1x_2x_1) \\
&= (0)(1) + (0)(0) + (0)(0) + (-1)(0) + (0)(0) + (0)(0) + (-1)(1) + (1)(0) \\
&= -1.
\end{aligned}$$

Therefore, $(c \sqcup\!\sqcup d, x_1x_2x_1) = -1$ as computed from the direct shuffle product calculation. □

A key observation is that Algorithm 3.2.2 can be utilized to compute the Wiener-Fliess composition product (3.1.1). Specifically, if $\bar{c} \in \mathbb{R}^{\ell}_{p}\langle\langle X\rangle\rangle$, define the corresponding group element in M as $c \triangleq \mathbb{1} + \bar{c}$. If $\tilde{\eta} = \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_s} \in \tilde{X}^*$ and $\zeta \in X^*$, then

$$(\bar{c}^{\,\sqcup\!\sqcup\,\tilde{\eta}}, \zeta) = (\Delta'^{\circ(s-1)}a_{\zeta})(c_{i_1}, c_{i_2}, \ldots, c_{i_s}),$$

where $(\Delta'^{\circ(s-1)}a_{\zeta})$ denotes the composition of the reduced coproduct map with itself $s-1$ times and then applied to the coordinate map a_{ζ}. Computationally, this boils down to splitting the word ζ into all possible s subwords, say $\zeta = \alpha_1\alpha_2\cdots\alpha_s$, where $\alpha_i \in X^+$, and then finding the Hadamard product of the coefficients corresponding to each subword with respect to the proper part of the series in the argument. That is,

$$(\bar{c}^{\,\sqcup\!\sqcup\,\tilde{\eta}}, \zeta) = \sum_{\substack{\alpha_1,\ldots,\alpha_s \in X^+ \\ \zeta \in \alpha_1 \,\sqcup\!\sqcup\, \cdots \,\sqcup\!\sqcup\, \alpha_s}} (\bar{c}_{i_1}, \alpha_1) \odot (\bar{c}_{i_2}, \alpha_2) \odot \cdots \odot (\bar{c}_{i_k}, \alpha_s).$$

The extension of this framework to the computation of Wiener-Fliess composition product is described next. Let $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_{\ell}\}$ be the commuting alphabet and $\tilde{X}^*$ the set of commuting words with $\tilde{X}^+ = \tilde{X}^* \setminus \{\emptyset\}$. Hence, $\forall \tilde{\eta} = \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_s} \in \tilde{X}^*$ define the computational operators on the Hopf algebra H as $\chi_{\tilde{\eta}} : H \longrightarrow H$ such that $a_{\eta} \mapsto \chi_{\tilde{\eta}}(a_{\eta})$ (where $\eta \in X^*$) and

$$\chi_{\tilde{\eta}}a_{\eta}(c) = \Delta'^{\circ(s-1)}a_{\eta}(c_{i_1}, c_{i_2}, \ldots, c_{i_k}) = \left(c^{\,\sqcup\!\sqcup\,\tilde{\eta}}, \eta\right) = (\tilde{\eta} \,\hat{\circ}\, \bar{c}, \eta),$$

where $c \in M, \bar{c} \in \mathbb{R}^{\ell}_{p}\langle\langle X\rangle\rangle$ and $c = \mathbb{1} + \bar{c}$. If $d \in \mathbb{R}^{k}[[\tilde{X}]]$, then the Wiener-Fliess composition $d \,\hat{\circ}\, \bar{c}$ can be computed as

$$(d \,\hat{\circ}\, \bar{c}, \eta) = \left(\left[(d, \emptyset)\,\epsilon + \sum_{\tilde{\eta} \in \tilde{X}^+} (d, \tilde{\eta})\,\chi_{\tilde{\eta}}\right] a_{\eta}\right)(\mathbb{1} + \bar{c}).$$

The framework for computing the Wiener-Fliess composition product for the non-proper case when $\bar{c} \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^k[\tilde{X}]$ requires more careful attention. Consider the case where $\bar{c}$ is non-proper such that $(\bar{c}_i, \emptyset) = r_i \neq 0 \ \forall i = 1, \ldots, \ell$. The corresponding group element of $\bar{c}$ in M is c where $c_i = \left(\frac{1}{r_i}\right)\bar{c}_i \ \forall i = 1, \ldots, \ell$. If $\tilde{\eta} = \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_s} \in \tilde{X}^*$ and $\zeta \in X^*$, then

$$(\bar{c}^{\,\sqcup\!\sqcup\, \tilde{\eta}}, \zeta) = \left(\prod_{j=1}^{s} r_{i_j}\right)\left(\Delta^{\circ(s-1)}a_{\zeta}\right)(c_{i_1}, c_{i_2}, \ldots, c_{i_s}),$$

where $(\Delta^{\circ(s-1)}a_{\zeta})$ denotes the composition of the coproduct map with itself $s-1$ times and then applied to the coordinate map a_{ζ}. Hence, $\forall \tilde{\eta} = \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_s} \in \tilde{X}^*$, define the computational operator $\hat{\chi}_{\tilde{\eta}}$ on the Hopf algebra H viz. $\hat{\chi}_{\tilde{\eta}} : H \longrightarrow H$ such that $a_{\eta} \mapsto \hat{\chi}_{\tilde{\eta}}(a_{\eta})$ (where $\eta \in X^*$) and

$$\hat{\chi}_{\tilde{\eta}} a_{\eta}(c) = \left(\prod_{j=1}^{s}(\bar{c}_{i_j}, \emptyset)\right)\left(\Delta^{\circ(s-1)}a_{\zeta}\right)(c_{i_1}, c_{i_2}, \ldots, c_{i_s}) = \left(c^{\,\sqcup\!\sqcup\, \tilde{\eta}}, \eta\right) = (\tilde{\eta} \,\hat{\circ}\, \bar{c}, \eta),$$

where $\bar{c} \in \mathbb{R}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ such that $(\bar{c}_i, \emptyset) \neq 0 \ \forall i = 1, \cdots, \ell$, and $c \in M$ is the corresponding group element of $\bar{c}$. Therefore, let $\bar{c} \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^k[\tilde{X}]$. The Wiener-Fliess composition $d \,\hat{\circ}\, \bar{c}$ can be computed as

$$(d \,\hat{\circ}\, \bar{c}, \eta) = \left(\left[(d, \emptyset)\,\epsilon + \sum_{\tilde{\eta} \in \mathrm{supp}(d)} (d, \tilde{\eta})\,\hat{\chi}_{\tilde{\eta}}\right] a_{\eta}\right)(c).$$

Note that this framework based on Hopf algebra H has limitations when $d \in \mathbb{R}^k[\tilde{X}]$ and $\bar{c} \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ is a non-proper series such that $\exists j \in \{1, \ldots, \ell\} : (c_j, \emptyset) = 0$. Observe that this is only a possibility if $\ell > 1$.

3.3 FLIESS OPERATORS UNDER STATIC OUTPUT FEEDBACK

Assume $|X| = m + 1$ and $|\tilde{X}| = \ell$. Let F_c be a Chen-Fliess series with a generating series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$. Assume it is interconnected with a static formal map f_d with generating series

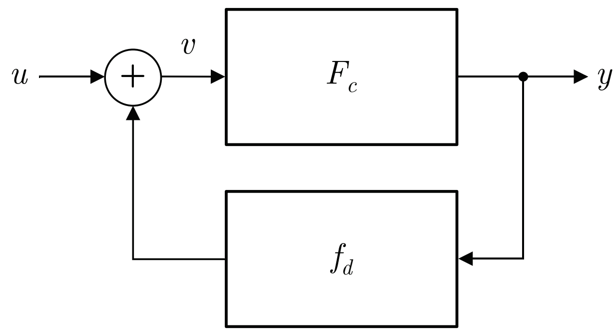


Fig. 5: Fliess operator F_c with static output feedback f_d.

$d \in \mathbb{R}^m [[\tilde{X}]]$ in the additive output feedback configuration shown in Figure 5 satisfying either of the following conditions:

1. The series c is proper.
2. d is only a polynomial.

The first objective of this section is to show that the closed-loop system always has a Chen-Fliess series representation, say $y = F_e[u]$, where $e \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$.

If this is the case, then necessarily

$$\begin{aligned} F_e[u] &= y \\ &= F_c[u + f_d(y)] \\ &= F_c[u + f_d \circ F_e[u]] \\ &= F_{c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta}[u] \end{aligned}$$

for any admissible u. From the uniqueness of generating series, the series e has to satisfy the fixed point equation

$$e = c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta. \tag{3.3.1}$$

Observe that e must be a proper series whenever c is proper. It follows directly from the definition of the mixed composition product that for all $w \in \mathbb{R}^k\langle\langle X'\rangle\rangle$, the series $c\,\tilde{\circ}\,w_\delta \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle$ is proper if and only if $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$, where $k = |X| - 1$. The following lemmas will be used to show that (3.3.1) always has a unique fixed point in both cases.

Lemma 3.3.1. *If $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$, then the map $Q_{c,d} : \mathbb{R}_p^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}_p^\ell\langle\langle X\rangle\rangle : e \mapsto c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta$ is a strong contraction map in the ultrametric topology on the space $\mathbb{R}_p^\ell\langle\langle X\rangle\rangle$.*

Proof: First observe that $\kappa(h_\delta) = \kappa(h)$, $\forall h \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$. Now define two maps, $d_{\hat{\circ},\delta} : e \mapsto (d\,\hat{\circ}\,e)_\delta$ and $c_{\tilde{\circ}} : f \mapsto c\,\tilde{\circ}\,f_\delta$, where $f \in \mathbb{R}^m\langle\langle X\rangle\rangle$. Note that $Q_{c,d}(e) = (c_{\tilde{\circ}} \circ d_{\hat{\circ},\delta})(e)$. It is known that $c_{\tilde{\circ}}$ is a strong contraction map in the ultrametric topology [Gray & Li (2005)], so it only needs to be shown that $d_{\hat{\circ},\delta}$ is at least a non-expansive map.

Consider first the case where $\overline{\omega}(d) = 1$. By Theorem 3.1.3, $\kappa(d_{\hat{\circ},\delta}(e)) \leq \kappa(e)$. Therefore, $d_{\hat{\circ},\delta}$ is a weak contraction map.

Consider next the case where $\overline{\omega}(d) > 1$. Since $e \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ then $\mathrm{ord}(e) \geq 1$. Therefore, $\kappa(e) \leq \sigma$ with $\sigma \in]0,1[$. By Theorem 3.1.3, $\kappa(d_{\hat{\circ},\delta}(e)) \leq \sigma\kappa(e)$. Hence, $d_{\hat{\circ},\delta}$ is a strong contraction map. ■

The counterpart of Lemma 3.3.1 for the case when d is a polynomial but c is allowed to be an arbitrary formal series not necessarily proper is proven next.

Lemma 3.3.2. *If $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[\tilde{X}]$, then the map $\tilde{Q}_{c,d} : \mathbb{R}^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^\ell\langle\langle X\rangle\rangle : e \mapsto c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta$ is a strong contraction map in the ultrametric topology on the space $\mathbb{R}^\ell\langle\langle X\rangle\rangle$.*

Proof: Define the maps, $\tilde{d}_{\hat{\circ},\delta} : e \mapsto (d\,\hat{\circ}\,e)_\delta$ and $c_{\tilde{\circ}} : f \mapsto c\,\tilde{\circ}\,f_\delta$, where $f \in \mathbb{R}^m\langle\langle X\rangle\rangle$. Note that $\tilde{Q}_{c,d}(e) = (c_{\tilde{\circ}} \circ d_{\hat{\circ},\delta})(e)$. Since $\kappa(h_\delta) = \kappa(h)$, $\forall h \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, from Theorem 3.1.4, $d_{\hat{\circ},\delta}$ is a weak contraction map in ultrametric topology. As seen in Lemma 3.3.1, the map $c_{\tilde{\circ}}$ is a strong contraction map in ultrametric space. Therefore, $\tilde{Q}_{c,d}(e) = (c_{\tilde{\circ}} \circ d_{\hat{\circ},\delta})(e)$ is a strong contraction map in the ultrametric topology on the space $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. ■

The following fixed point theorem establishes the first main result of the section, which follows subsequently.

Theorem 3.3.1. *Let X be a noncommutative alphabet and $\tilde{X}$ a commutative alphabet such that $|X| = m+1$ and $|\tilde{X}| = \ell$. The following statements are true:*

1. *Given a proper series $c \in \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$, the series $c\tilde{\circ}(-d\hat{\circ}c)_{\delta}^{-1} \in \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ is a unique fixed point of the map $Q_{c,d}$ defined in the Lemma 3.3.1.*

2. *Given a non-proper series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[\tilde{X}]$, the series $c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1} \in \mathbb{R}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ is a unique fixed point of the map $\tilde{Q}_{c,d}$ defined in the Lemma 3.3.2.*

Proof: Let $c \in \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$. If $e := c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1}$, then

$$\begin{aligned} Q_{c,d}(e) &= c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_{\delta} \\ &= c\,\tilde{\circ}\,(d\,\hat{\circ}\,(c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1}))_{\delta}. \end{aligned}$$

Applying Theorem 3.1.5 yields

$$\begin{aligned} Q_{c,d}(e) &= c\,\tilde{\circ}\,((d\,\hat{\circ}\,c)\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1})_{\delta} \\ &= c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1} = e. \end{aligned}$$

Therefore, $c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{-1}$ is the unique fixed point of $Q_{c,d}$. Note that the uniqueness is guaranteed as the ultrametric spaces are Hausdorff spaces. The proof for the case when $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[\tilde{X}]$ is similar. ■

Theorem 3.3.2. *Let $|X| = m+1$ and $|\tilde{X}| = \ell$. Let either of the following be given:*

1. *A proper series $c \in \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$.*

2. *A non-proper series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[\tilde{X}]$.*

Then the generating series for the closed-loop system in Figure 5 *is the* Wiener-Fliess feedback product $c\hat{@}d := c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_\delta^{-1}$.

The computation of $(-d\,\hat{\circ}\,c)$ can be performed via the coproduct of the Hopf algebra of the shuffle group as described in Section 3.2. The group inverse $(-d\,\hat{\circ}\,c)_\delta^{-1}$ can be computed via the antipode of the Faá di Bruno type Hopf algebra corresponding to the group $(\mathbb{R}^\ell\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. (A particularly efficient algorithm appears in [Ebrahimi-Fard & Gray (2017)].) Hence, the calculation of the generating series for the static feedback case is an interplay between these two very distinct Hopf algebras.

The notion that feedback can be described mathematically as a transformation group acting on the plant is well established in control theory [Brockett (1978)]. The following theorem describes the situation in the present context.

Theorem 3.3.3. *The Wiener-Fliess feedback product is a*

1. *right group action by the additive group* $\left(\mathbb{R}^m\,[[\tilde{X}]], +, 0\right)$ *on the set* $\mathbb{R}_p^\ell\,\langle\langle X\rangle\rangle$ *when the Chen-Fliess series is proper;*

2. *right group action by the additive group* $\left(\mathbb{R}^m\,[\tilde{X}], +, 0\right)$ *on the set* $\mathbb{R}^\ell\langle\langle X\rangle\rangle \setminus \mathbb{R}_p^\ell\,\langle\langle X\rangle\rangle$ *when the Chen-Fliess series is non-proper,*

where $|X| = m+1$ *and* $|\tilde{X}| = \ell$.

Proof: Let $d_1, d_2 \in \mathbb{R}^m\,[[\tilde{X}]]$ and $c \in \mathbb{R}_p^\ell\,\langle\langle X\rangle\rangle$. It needs to be proved that

$$(c\hat{@}d_1)\hat{@}d_2 = c\hat{@}(d_1 + d_2).$$

From Theorem 3.3.2 observe that

$$\begin{aligned}(c\hat{@}d_1)\hat{@}d_2 &= (c\hat{@}d_1)\,\tilde{\circ}\,(-d_2\,\hat{\circ}\,(c\hat{@}d_1))_\delta^{-1}\\ &= (c\,\tilde{\circ}\,(-d_1\,\hat{\circ}\,c)_\delta^{-1})\,\tilde{\circ}\,(-d_2\,\hat{\circ}\,(c\hat{@}d_1))_\delta^{-1}.\end{aligned}$$

Applying Theorem 2.3.3 and then Theorem 3.3.2 gives

$$
\begin{aligned}
(c\hat{@}d_1)\hat{@}d_2 &= c\,\tilde{\circ}\,\left[(-d_1\,\hat{\circ}\,c)_\delta^{-1}\circ(-d_2\,\hat{\circ}\,(c\hat{@}d_1))_\delta^{-1}\right] \\
&= c\,\tilde{\circ}\,\left[(-d_2\,\hat{\circ}\,(c\hat{@}d_1))_\delta\circ(-d_1\,\hat{\circ}\,c)_\delta\right]^{-1} \\
&= c\,\tilde{\circ}\,\left[(-d_2\,\hat{\circ}\,(c\,\tilde{\circ}\,(-d_1\,\hat{\circ}\,c)_\delta^{-1}))_\delta\circ(-d_1\,\hat{\circ}\,c)_\delta\right]^{-1}.
\end{aligned}
$$

In light of Theorem 3.1.5,

$$
(c\hat{@}d_1)\hat{@}d_2 = c\,\tilde{\circ}\left[\Big((-d_2\,\hat{\circ}\,c)\,\tilde{\circ}\,(-d_1\,\hat{\circ}\,c)_\delta^{-1}\Big)_\delta\circ\Big(-d_1\,\hat{\circ}\,c\Big)_\delta\right]^{-1}.
$$

Expanding the group product of $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle,\circ)$, it follows that

$$
(c\hat{@}d_1)\hat{@}d_2 = c\,\tilde{\circ}\left[\left(\left(\Big((-d_2\,\hat{\circ}\,c)\,\tilde{\circ}\,(-d_1\,\hat{\circ}\,c)_\delta^{-1}\Big)\,\tilde{\circ}\,\Big(-d_1\,\hat{\circ}\,c\Big)_\delta\right)+\Big(-d_1\,\hat{\circ}\,c\Big)\right)_\delta\right]^{-1}.
$$

Finally, from Theorem 2.3.3,

$$
(c\hat{@}d_1)\hat{@}d_2 = c\,\tilde{\circ}\,[-d_1\,\hat{\circ}\,c+(-d_2\,\hat{\circ}\,c)]_\delta^{-1},
$$

so that via the left linearity of Wiener-Fliess composition,

$$
(c\hat{@}d_1)\hat{@}d_2 = c\,\tilde{\circ}\,[-(d_1+d_2)\,\hat{\circ}\,c]_\delta^{-1} = c\hat{@}(d_1+d_2).
$$

The proof is analogous for the case when c is non-proper and $d_1,d_2\in\mathbb{R}^m\,[\tilde{X}]$. ■

It is worth noting that for dynamic output feedback the transformation group is $(\mathbb{R}^m\langle\langle X\rangle\rangle,+)$, while here it is $(\mathbb{R}^m\,[[\tilde{X}]],+)$ (or $(\mathbb{R}^m\,[\tilde{X}],+)$) that plays this role. The final theorem states that a SISO nonlinear input-output system with relative degree has its relative degree left invariant under static output feedback.

Theorem 3.3.4. *Let $X=\{x_0,x_1\}$ and $c\in\mathbb{R}\langle\langle X\rangle\rangle$ have relative degree. If either of the* *following conditions hold:*

1. *c is proper and $d \in \mathbb{R}\,[[\tilde{x}_1]]$;*

2. *c is non-proper and $d \in \mathbb{R}\,[\tilde{x}_1]$,*

then $c\hat{@}d$ has relative degree equal to that of c.

Proof: The proof follows from the formula in Theorem 3.3.2 and the relative degree properties summarized in Table 1. ■

Observe that in the SISO case, the Wiener-Fliess composition product of a non-proper Chen-Fliess series c with relative r_c and a commutative polynomial $d \in \mathbb{R}\,[\tilde{X}]$ can fail to have a well-defined relative degree as demonstrated in Example 3.1.2. However, the static feedback configuration of the non-proper series c with the commutative polynomial d always has well defined relative degree and is r_c as proven in Theorem 3.3.4.

Example 3.3.1. Consider a normalized forced pendulum equation

$$\ddot{\theta} + \sin\theta = u \tag{3.3.2}$$

with input u, angular displacement θ, and output $y = \theta$. Under the feedback law $u = v+\sin\theta$, the system is transformed into a double integrator $\ddot{\theta} = v$. For example, with $\theta(0) = 0$ and $\dot{\theta}(0) = 1$, the closed-loop system is described by

$$y(t) = t + \int_0^t \int_0^{\tau_2} v(\tau_1)\, d\tau_1\, d\tau_2,$$

or equivalently, $y = F_{c\hat{@}d}[v]$ with $c\hat{@}d = x_0 + x_0x_1$. Clearly, the series has relative degree two. The same result can be established via Theorem 3.3.2. The following computations were all done via Mathematica. It is easily checked that the open-loop system $y = F_c[u]$ has the generating series

$$c = x_0 + x_0x_1 - x_0^3 - x_0^3x_1 + 2x_0^5 + 4x_0^5x_1 + 2x_0^4x_1x_0 + x_0^3x_1x_0^2 + \cdots$$

and has relative degree 2 as expected. The sinusoidal static output feedback map has generating series $d \in \mathbb{R}\,[[\tilde{x}_1]]$ given by

$$d = \tilde{x}_1 - \frac{1}{3!}\tilde{x}_1^3 + \frac{1}{5!}\tilde{x}_1^5 - \frac{1}{7!}\tilde{x}_1^7 + \cdots$$

Using the computational methods described above and computing the composition antipode for words up to length four, it is found that

$$c\hat{@}d \approx x_0 + x_0x_1 + \mathcal{O}(x_0^6).$$

The terms $\mathcal{O}(x_0^6)$ are the error terms due to the need to truncate all the underlying series at each step of the calculation in the Wiener-Fliess feedback product formula. The order of these error terms can be increased but at a significant computational cost. $\square$

CHAPTER 4

CONVERGENCE OF ADDITIVE STATIC FEEDBACK PRODUCT

The goal of this chapter is to answer the following questions. Does the additive static feedback connection preserve local convergence? That is, does the additive feedback connection of a locally convergent Fliess operator with a locally convergent analytic function have a locally convergent Fliess operator representing the closed-loop system? Taking a step further, does the additive static feedback connection preserve global convergence? The strategy is to characterize the convergence under individual products appearing in the Wiener-Fliess feedback product derived in Chapter 3.

4.1 LOCAL CONVERGENCE OF THE MIXED COMPOSITION AND WIENER-FLIESS COMPOSITION

The goal of this section is to prove the closure of $\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ under the mixed composition product and Wiener-Fliess composition product. These results are vital in proving that the Wiener-Fliess feedback product preserves local convergence. The following lemma is essential.

Lemma 4.1.1. *Let $e \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ be a proper series such that $|(e,\zeta)| \leq K_e M_e^{|\zeta|-j}|\zeta|!\ \forall \zeta \in X^+$ where $j \geq 0$. Then $e^{\sqcup\!\sqcup\, n}$ is a proper and locally convergent series $\forall n \geq 1$ such that $|(e^{\sqcup\!\sqcup\, n},\zeta)| \leq K_e^n M_e^{|\zeta|-nj}|\zeta|!\binom{|\zeta|-1}{n-1}\ \forall \zeta \in X^+$.*

Proof: Observe $\forall n \geq 1$ and $\forall \zeta \in X^+$ that

$$|(e^{\sqcup\!\sqcup\, n},\zeta)| \leq \sum_{\substack{i_1,\ldots,i_n>0\\ i_i+\cdots+i_n=|\zeta|}} (K_e M_e^{i_1-j} i_1!)\cdots(K_e M_e^{i_n-j} i_n!)$$

$$
\begin{aligned}
&= K_e^n M_e^{|\zeta|-nj}|\zeta|! \sum_{\substack{i_1,\ldots,i_n>0 \\ i_i+\cdots+i_n=|\zeta|}} 1 \\
&= K_e^n M_e^{|\zeta|-nj}|\zeta|! \binom{|\zeta|-1}{n-1}.
\end{aligned}
$$

■

4.1.1 Local Convergence of Mixed Composition Product

This subsection addresses the question of whether the mixed composition product of two locally convergent Chen-Fliess series has a well defined Fliess operator representation. The following lemma is used in the proof of the main result.

Lemma 4.1.2. *Let* $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ *such that* $|(d_i,\zeta)| \leq K_{d_i} M_{d_i}^{|\zeta|}|\zeta|!$, *where* $\zeta \in X^*$, *and* d_i *is the* i^{th} *component of the series* d, *then the proper series* $e = \sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i$ *is also locally convergent. Specifically,*

$$
|(e,\zeta)| \leq (1+mK_d)M_d^{|\zeta|-1}|\zeta|!, \quad \forall\, \zeta \in X^*,
$$

where $K_d = \max\limits_{i=1,\ldots,m} K_{d_i}$ *and* $M_d = \max\limits_{i=1,\ldots,m} M_{d_i}$.

Proof: Let $\zeta \in \tilde{X}^*$. By triangle inequality,

$$
\begin{aligned}
\left|\left(\sum_{i=1}^m d_i, \zeta\right)\right| &\leq \sum_{i=1}^m |(d_i,\zeta)| \\
&\leq \sum_{i=1}^m K_{d_i} M_{d_i}^{|\zeta|}|\zeta|! \\
&\leq m \max_{i=1,\ldots,m} K_{d_i} M_{d_i}^{|\zeta|}|\zeta|!.
\end{aligned}
$$

Define $K_d = \max\limits_{i=1,\ldots,m} K_{d_i}$ and $M_d = \max\limits_{i=1,\ldots,m} M_{d_i}$. Then,

$$
\left|\left(\sum_{i=1}^m d_i, \zeta\right)\right| \leq mK_d M_d^{|\zeta|}|\zeta|!.
$$

Hence,

$$\left(\sum_{j=0}^{m}\tilde{x}_j+\tilde{x}_0\sum_{j=1}^{m}d_j,\zeta\right)=\sum_{j=0}^{m}(\tilde{x}_j,\zeta)+\sum_{j=1}^{m}\left(d_j,\tilde{x}_0^{-1}(\zeta)\right)$$
$$\leq\begin{cases}1 & :\ \zeta\in\tilde{X}\setminus\{\tilde{x}_0\}\\ 1+mK_d & :\ \zeta=\tilde{x}_0\\ mK_dM_d^{|\zeta|-1}(|\zeta|-1)! & :\ \zeta=\tilde{x}_0\tilde{X}^+\\ 0 & :\ \text{otherwise.}\end{cases}$$

Note that $(1+mK_d)M_d^{|\zeta|-1}|\zeta|!$ bounds the right-hand side on all cases. Therefore,

$$|(e,\zeta)|\leq(1+mK_d)M_d^{|\zeta|-1}|\zeta|!\quad\forall\,\zeta\in\tilde{X}^*.$$

■

The following theorem is the core result of this subsection. It proves that the mixed composition product preserves local convergence.

Theorem 4.1.1. *Let* $c\in\mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *such that* $|(c,\eta)|\leq K_cM_c^{|\eta|}|\eta|!\ \forall\eta\in X^*$. *Assume* $d\in\mathbb{R}^m_{LC}\langle\langle\tilde{X}\rangle\rangle$ *such that* $|(d_i,\zeta)|\leq K_{d_i}M_{d_i}^{|\zeta|}|\zeta|!$, *where* $\zeta\in\tilde{X}^*$, *and* d_i *is the* i^{th} *component of the series* d. *Then* $c\,\tilde{\circ}\,d_\delta\in\mathbb{R}_{LC}\langle\langle\tilde{X}\rangle\rangle$ *such that*

$$|(c\,\tilde{\circ}\,d_\delta,\eta)|\leq\begin{cases}\left(\dfrac{K_cM_c(1+mK_d)}{(1+mK_d)M_c+M_d}\right)\left[(1+mK_d)M_c+M_d\right]^{|\eta|}|\eta|! & \textit{if } \eta\neq\emptyset\\ K_c & \textit{if } \eta=\emptyset,\end{cases}$$

where $K_d=\max\limits_{i=1,\ldots,m}K_{d_i}$ *and* $M_d=\max\limits_{i=1,\ldots,m}M_{d_i}$.

Proof: Observe

$$c\,\tilde{\circ}\,d_\delta=\sum_{\eta\in X^*}(c,\eta)\eta\,\tilde{\circ}\,d_\delta$$

$$
\begin{aligned}
&\leq \sum_{\eta\in X^*} K_c M_c^{|\eta|}|\eta|!\; \eta \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\eta\in X^n} \eta \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} (x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m}) \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} \left(\frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{x_m^{\sqcup\!\sqcup\, r_m}}{r_m!}\right) \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} \binom{n}{r_0\cdots r_m} (x_0 \,\tilde{\circ}\, d_\delta)^{\sqcup\!\sqcup\, r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (x_m \,\tilde{\circ}\, d_\delta)^{\sqcup\!\sqcup\, r_m} \\
&= K_c \sum_{n=0}^{\infty} \left[M_c \left(\sum_{j=0}^{m} x_j \,\tilde{\circ}\, d_\delta \right)\right]^{\sqcup\!\sqcup\, n} \\
&= K_c \sum_{n=0}^{\infty} \left[M_c \left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right)\right]^{\sqcup\!\sqcup\, n}.
\end{aligned}
$$

Observe that $M_c\left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i\right)$ is a proper series. Hence, $|(c \,\tilde{\circ}\, d_\delta, \emptyset)| \leq K_c$. Now let $\eta \in \tilde{X}^\alpha$ where $\alpha \in \mathbb{N}$. Then,

$$
\begin{aligned}
(c \,\tilde{\circ}\, d_\delta, \eta) &\leq K_c \left(\sum_{n=1}^{\alpha} \left[M_c \left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right)\right]^{\sqcup\!\sqcup\, n}, \eta \right) \\
&= K_c \sum_{n=1}^{\alpha} M_c^n \left(\left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right)^{\sqcup\!\sqcup\, n}, \eta \right).
\end{aligned}
$$

Using Lemma 4.1.1, Lemma 4.1.2 and the triangle inequality,

$$
\begin{aligned}
|(c \,\tilde{\circ}\, d_\delta, \eta)| &\leq K_c \sum_{n=1}^{\alpha} M_c^n (1+mK_d)^n M_d^{\alpha-n} \alpha! \binom{\alpha-1}{n-1} \\
&= K_c M_c (1+mK_d) \left\{ \sum_{n=0}^{\alpha-1} \binom{\alpha-1}{n} \left[M_c(1+mK_d)\right]^n M_d^{(\alpha-1)-n}] \right\} \alpha! \\
&= K_c M_c (1+mK_d) \left[(1+mK_d)M_c + M_d\right]^{\alpha-1} \alpha! \\
&= \left(\frac{K_c M_c (1+mK_d)}{(1+mK_d)M_c + M_d} \right) \left[(1+mK_d)M_c + M_d\right]^{\alpha} \alpha!
\end{aligned}
$$

$$= \left(\frac{K_c M_c (1 + mK_d)}{(1 + mK_d)M_c + M_d} \right) [(1 + mK_d)M_c + M_d]^{|\eta|} \, |\eta|!.$$

Therefore, $c \,\tilde{\circ}\, d_\delta \in \mathbb{R}_{LC}\langle\langle \tilde{X} \rangle\rangle$. ■

Theorem 4.1.1 has proved that the mixed composition product is closed under local convergence. The following theorem reinterprets the result in terms of the $S_\infty(R)$ spaces.

Theorem 4.1.2. *If $c \in S_\infty(R)$ and $d \in S_\infty^m(R)$, then*

$$||c \,\tilde{\circ}\, d_\delta||_{\infty,\bar{R}} \leq ||c||_{\infty,R} \,,$$

where $\bar{R} = \left(\frac{R}{2 + m||d||_{\infty,R}} \right)$ *and* $||d||_{\infty,R} = \max_{i=1,\ldots,m} ||d_i||_{\infty,R}$. *Hence,* $c \,\tilde{\circ}\, d_\delta \in S_\infty(\bar{R})$.

Proof: Recall that $c \in S_\infty(R)$ if and only if $||c||_{\infty,R} < \infty$. Thus,

$$|(c, \eta)| \leq ||c||_{\infty,R} \left(\frac{1}{R} \right)^{|\eta|} |\eta|!, \quad \forall \eta \in X^*.$$

Similarly, for $i = 1, \ldots, m$,

$$|(d_i, \eta)| \leq ||d||_{\infty,R} \left(\frac{1}{R} \right)^{|\eta|} |\eta|!, \quad \forall \eta \in X^*,$$

where $||d||_{\infty,R} = \max_{i=1,\ldots,m} ||d_i||_{\infty,R}$. Hence, using Theorem 4.1.1,

$$|(c \,\tilde{\circ}\, d_\delta, \eta)| \leq \begin{cases} \left(\dfrac{||c||_{\infty,R} \left(1 + m||d||_{\infty,R}\right)}{2 + m||d||_{\infty,R}} \right) \left[\dfrac{2 + m||d||_{\infty,R}}{R} \right]^{|\eta|} |\eta|! & \text{if } \eta \neq \emptyset \\ ||c||_{\infty,R} & \text{if } \eta = \emptyset. \end{cases}$$

Observe that

$$\frac{||c||_{\infty,R} \left(1 + m||d||_{\infty,R}\right)}{2 + m||d||_{\infty,R}} \leq ||c||_{\infty,R}.$$

If $\bar{R} = \left(\dfrac{R}{2+m||d||_{\infty,R}}\right)$, then

$$||c \,\tilde{\circ}\, d_\delta||_{\infty,\bar{R}} \leq ||c||_{\infty,R}.$$

■

Theorem 4.1.2 has provided a description of the convergence of the mixed composition product when both arguments are locally convergent. The following theorem characterizes the convergence when the right argument is in the Fréchet space S_∞^m.

Theorem 4.1.3. *If $c \in S_\infty(R)$ and $d \in S_\infty^m$, then $c \,\tilde{\circ}\, d_\delta \in S_\infty(\bar{R})$, where $\bar{R} = \left(\dfrac{R}{2+m||d||_{\infty,R}}\right)$.*

Proof: Recall that $d \in S_\infty^m$ if and only if $d \in S_\infty^m(R')$ $\forall R' > 0$. Therefore, $c \in S_\infty(R)$ and $d \in S_\infty^m(R)$. Applying Theorem 4.1.2,

$$||c \,\tilde{\circ}\, d_\delta||_{\infty,\bar{R}} \leq ||c||_{\infty,R} < \infty,$$

where $\bar{R} = \left(\dfrac{R}{2+m||d||_{\infty,R}}\right)$. ■

The following theorem states that the mixed composition product is a continuous linear operator on a Banach space with respect to its left argument.

Theorem 4.1.4. *Given $d \in S_\infty^m$, define a map $\phi_{d,R}$ between Banach spaces*

$$\phi_d : S_\infty(R) \longrightarrow S_\infty(\bar{R})$$
$$c \longmapsto c \,\tilde{\circ}\, d_\delta,$$

where $\bar{R} = \left(\dfrac{R}{2+m||d||_{\infty,R}}\right)$. Then, $\phi_{d,R}$ is a bounded linear operator with $||\phi_{d,R}|| \leq 1$ under the operator norm. Hence, the linear map $\phi_{d,R}$ is Lipschitz with Lipschitz constant less than or equal to 1.

Proof: The map $\phi_{d,R}$ is linear by Theorem 2.3.2. If $c \in S_\infty(R)$, then $\phi_{d,R}(c) \in S_\infty(\bar{R})$ by Theorem 4.1.3, where $\bar{R} = \left(\dfrac{R}{2+m||d||_{\infty,R}}\right)$. Observe that

$$d \in S_\infty^m \Leftrightarrow d \in S_\infty^m(R)\ \forall R > 0.$$

Hence, by Theorem 4.1.2

$$||\phi_{d,R}(c)||_{\infty,\bar{R}} = ||c \,\tilde{\circ}\, d_\delta||_{\infty,\bar{R}} \leq ||c||_{\infty,R}.$$

Therefore, $||\phi_{d,R}|| \leq 1$. ■

4.1.2 Local Convergence of Wiener-Fliess Composition Product

This subsection addresses the preservation of local convergence under the Wiener-Fliess composition product. The case where c is a noncommutative formal proper series, as defined in Theorem 3.1.1, is only considered here. The case where d is a commutative polynomial is in Section 4.2.3. The following theorem in spirit has appeared in [Gray & Thitsa (2012)], and it was using exponential generating functions. In this work, an alternate proof is provided using only elementary combinatorics.

Theorem 4.1.5. *Let* $d \in \mathbb{R}_{LC}[[\tilde{X}]]$ *with* $|\tilde{X}| = k$ *and* $|(d,\eta)| \leq K_d M_d^{|\eta|}\ \forall \eta \in \tilde{X}^*$. *If* $c \in \mathbb{R}^k_{p,LC}\langle\langle X\rangle\rangle$ *such that* $|(c_i,\zeta)| \leq K_{c_i} M_{c_i}^{|\zeta|}|\zeta|!\ \forall \zeta \in X^*$, *then* $d \,\hat{\circ}\, c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ *with*

$$|(d \,\hat{\circ}\, c,\eta)| \leq \begin{cases} \left(\frac{kK_dK_cM_d}{1+kK_cM_d}\right)\left[M_c\left(1+kK_cM_d\right)\right]^{|\eta|}|\eta|! & \text{if } \eta \neq \emptyset \\ K_d & \text{if } \eta = \emptyset, \end{cases}$$

where $K_c = \max\limits_{i=1,\ldots,k} K_{c_i}$ *and* $M_c = \max\limits_{i=1,\ldots,k} M_{c_i}$.

Proof: Observe

$$
\begin{aligned}
d \,\hat{\circ}\, c &= \sum_{\eta \in \tilde{X}^*} (d,\eta) c^{\,\sqcup\!\sqcup\, \eta} \\
&\leq \sum_{\eta \in \tilde{X}^*} K_d M_d^{|\eta|} c^{\,\sqcup\!\sqcup\, \eta} \\
&= \sum_{n=0}^{\infty} K_d M_d^n \sum_{\eta \in \tilde{X}^n} c^{\,\sqcup\!\sqcup\, \eta} \\
&= \sum_{n=0}^{\infty} K_d M_d^n \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} \tilde{x}_1^{i_1} \cdots \tilde{x}_k^{i_k} \,\hat{\circ}\, c \\
&= \sum_{n=0}^{\infty} K_d M_d^n \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} c_1^{\,\sqcup\!\sqcup\, i_1} \,\sqcup\!\sqcup\, c_2^{\,\sqcup\!\sqcup\, i_2} \,\sqcup\!\sqcup\, \cdots \,\sqcup\!\sqcup\, c_k^{\,\sqcup\!\sqcup\, i_k}.
\end{aligned}
$$

Since $\binom{n}{i_1 \cdots i_k} \geq 1$,

$$
\begin{aligned}
d \,\hat{\circ}\, c &\leq \sum_{n=0}^{\infty} K_d M_d^n \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} \binom{n}{i_1 \cdots i_k} c_1^{\,\sqcup\!\sqcup\, i_1} \,\sqcup\!\sqcup\, c_2^{\,\sqcup\!\sqcup\, i_2} \,\sqcup\!\sqcup\, \cdots \,\sqcup\!\sqcup\, c_k^{\,\sqcup\!\sqcup\, i_k} \\
&= \sum_{n=0}^{\infty} K_d M_d^n \left(\sum_{j=1}^{k} c_j \right)^{\sqcup\!\sqcup\, n} \\
&= K_d \sum_{n=0}^{\infty} \left(M_d \sum_{j=1}^{k} c_j \right)^{\sqcup\!\sqcup\, n}.
\end{aligned}
$$

Note that $M_d \sum_{j=1}^{k} c_j$ is a proper series. Hence, $|(d \,\hat{\circ}\, c, \emptyset)| \leq K_d$.

Now let $\eta \in X^\alpha$ where $\alpha \in \mathbb{N}$. Then,

$$
\begin{aligned}
(d \,\hat{\circ}\, c, \eta) &\leq K_d \left(\left[\sum_{n=1}^{\alpha} \left(M_d \sum_{j=1}^{k} c_j \right) \right]^{\sqcup\!\sqcup\, n}, \eta \right) \\
&= K_d \sum_{n=1}^{\alpha} M_d^n \left(\left(\sum_{j=1}^{k} c_j \right)^{\sqcup\!\sqcup\, n}, \eta \right).
\end{aligned}
$$

Observe that $\left(\sum_{j=1}^{k} c_j, \zeta \right) \leq k K_c M_c^{|\zeta|} |\zeta|!$, where $K_c = \max\limits_{i=1,\ldots,k} K_{c_i}$ and $M_c = \max\limits_{i=1,\ldots,k} M_{c_i}$.

Using Lemma 4.1.1 and the triangle inequality,

$$\begin{aligned}
|(d\,\hat{\circ}\,c,\eta)| &\leq K_d\left[\sum_{n=1}^{\alpha} k^n K_c^n M_d^n M_c^{\alpha}\alpha!\binom{\alpha-1}{n-1}\right] \\
&= K_d M_c^{\alpha}\alpha!\left[\sum_{n=1}^{\alpha} k^n K_c^n M_d^n\binom{\alpha-1}{n-1}\right] \\
&= K_d\left(kK_cM_d\right)M_c^{\alpha}\left[\sum_{n=0}^{\alpha-1}\left(kK_cM_d\right)^n\binom{\alpha-1}{n}\right]\alpha! \\
&= K_d\left(kK_cM_d\right)M_c^{\alpha}\left(1+kK_cM_d\right)^{\alpha-1}\alpha! \\
&= \left(\frac{kK_dK_cM_d}{1+kK_cM_d}\right)\left[M_c\left(1+kK_cM_d\right)\right]^{\alpha}\alpha! \\
&= \left(\frac{kK_dK_cM_d}{1+kK_cM_d}\right)\left[M_c\left(1+kK_cM_d\right)\right]^{|\eta|}|\eta|!.
\end{aligned}$$

Therefore, $d\,\hat{\circ}\,c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. ■

4.2 GLOBAL CONVERGENCE OF SHUFFLE, MIXED COMPOSITION AND WIENER-FLIESS COMPOSITION PRODUCTS

This section addresses the preservation of global convergence under the shuffle product, mixed composition and the Wiener-Fliess composition product. The proofs of these convergence theorems need a few preliminary results. In particular, the proofs of global convergence involve the use of fractional powers of multinomial coefficients. Recall that the gamma function, the $\Gamma(\cdot)$ restricted to $\mathbb{R}_+$ is the analytic continuation of the factorial map on the non-negative integers. Hence, the analytic continuation of the multinomial coefficient is defined in the following way.

Definition 4.2.1. If $\alpha \in \mathbb{R}_{\geq 0}$ and $i_i, i_2, \ldots, i_s \in \mathbb{R}_{\geq 0}$ such that $\sum_{j=1}^{s} i_j = \alpha$, then

$$\binom{\alpha}{i_1\; i_2 \cdots i_s} = \frac{\Gamma(\alpha+1)}{\Gamma(i_1+1)\;\Gamma(i_2+1)\cdots\Gamma(i_s+1)}.$$

The following lemma is central to proving that S_∞^m is closed under the shuffle product.

Lemma 4.2.1. *Let* $\alpha \in \mathbb{R}_{\geq 0}$ *and* $i_1, i_2, \ldots, i_v \in \mathbb{R}_{\geq 0}$ *such that* $\sum_{j=1}^{v} i_j = \alpha$. *If* $r \in]0, 1]$, *then*

$$\binom{\alpha}{i_1\ i_2 \cdots i_v}^r \leq \left(\frac{K_r^v}{\tilde{K}_r}\right)\left(\frac{M_r}{2}\right)^{\alpha}\binom{r\alpha}{ri_1\ ri_2 \cdots ri_v},$$

where $K_r = \left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} r\right)^{\frac{1}{2}}$, $\tilde{K}_r = 2\left(\left(\frac{2\pi}{\exp(2)}\right)^{1-r} 4\right)^{\frac{1}{2}}$, *and* $M_r = r^r$.

Proof: Observe

$$\binom{\alpha}{i_1\ i_2 \cdots i_v}^r = \frac{(\Gamma(\alpha+1))^r}{(\Gamma(i_1+1))^r\ (\Gamma(i_2+1))^r \cdots (\Gamma(i_s+1))^r}.$$

Using the Lemma 2.5.1,

$$\begin{aligned}\binom{\alpha}{i_1\ i_2 \cdots i_v}^r &\leq \frac{2^{-r}\tilde{K}_r^{-1}\Gamma(r\alpha+1)}{K_r^{-1}M_r^{-i_1}\Gamma(ri_1+1)\,K_r^{-1}M_r^{-i_2}\Gamma(ri_2+1)\cdots K_r^{-1}M_r^{-i_v}\Gamma(ri_v+1)} \\ &= \left(\frac{K_r^v}{\tilde{K}_r}\right)\left(\frac{M_r}{2}\right)^{\alpha}\binom{r\alpha}{ri_1\ ri_2 \cdots ri_v}.\end{aligned}$$

■

The following theorem is known as the *Neoclassical Inequality* and is an extension of the multinomial theorem extended to arbitrary positive fractional powers of non-negative reals.

Theorem 4.2.1. *[Lyons & Qian (2002)] Let* $r \in]0, 1]$ *and* $m \in \mathbb{N}$. *If* $n \in \mathbb{N}_0$ *and* $x_1, x_2, \ldots, x_m \geq 0$, *then*

$$\sum_{\substack{i_1,\ldots,i_m \in \mathbb{N}_0 \\ i_1+i\cdots+i_m=n}} \binom{rn}{ri_1 \cdots ri_m} x_1^{ri_1} \cdots x_m^{ri_m} \leq \left(\frac{1}{r}\right)^{2(m-1)} (x_1+\cdots+x_m)^n.$$

If $r = 1$ above, then the inequality becomes an equality and reduces to the well known multinomial theorem albeit restricted to positive reals.

4.2.1 Global Convergence of Shuffle Product

The goal of this subsection is to prove that the S_∞^m is closed under the shuffle product. This problem was first addressed in Theorem 4.1.4 of [Winter-Arboleda (2019)]. However, the proof was built on the assertion that the norm $||.||_{\infty,R}$ is submultiplicative under shuffle product on the $S_\infty^m(R)$ space, which is untrue. In this subsection, it is proved that $S_\infty^m(R)$ is not closed under the shuffle product. Hence, the global convergence claim is proved on the basis of new results proved in this subsection.The following theorem is a slightly restrictive case that the shuffle product is closed in the space $\mathbb{R}_{GC}^m\langle\langle X\rangle\rangle$. This result also has appeared in Lemma 4.1.4 of [Winter-Arboleda (2019)], but the growth constants of the shuffle product were incorrectly bounded. The corrected result appears here.

Theorem 4.2.2. *Let $c_1, c_2, \ldots, c_k$ be a finite nonempty collection of formal power series such that $c_i \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ $\forall i = 1,2,\ldots,k$. Let $\{K_i\}_{i=1}^k, \{M_i\}_{i=1}^k$, and $\{\gamma_i\}_{i=1}^k$ be a collection of constants such that $K_i, M_i > 0$ and $\gamma_i \in [0,1[$ $\forall i = 1,2,\ldots,k$ with $|(c_i,\eta)| \leq K_i M_i^{|\eta|} (|\eta|!)^{\gamma_i}$ $\forall \eta \in X^*$. Then, $c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ such that $|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq \bar{K}\bar{M}^{|\eta|} (|\eta|!)^{\bar{\gamma}}$, where*

$$\bar{K} = \left(\frac{\left(\prod_{i=1}^k K_i\right)(1-\bar{\gamma})^2}{\tilde{K}_{1-\bar{\gamma}}}\right)\left(\frac{K_{1-\bar{\gamma}}}{(1-\bar{\gamma})^2}\right)^k$$
$$\bar{M} = \left[\frac{M_{1-\bar{\gamma}}\left(M_1^{1-\bar{\gamma}} + \cdots + M_k^{1-\bar{\gamma}}\right)}{2}\right]$$
$$\bar{\gamma} = \max_{i=1,\ldots,k} \gamma_i,$$

and the constants $K_{1-\bar{\gamma}}, \tilde{K}_{1-\bar{\gamma}}$, and $M_{1-\bar{\gamma}}$ are defined as

$$K_{1-\bar{\gamma}} = \left(\left(\frac{2\pi}{\exp(2)}\right)^{\bar{\gamma}}(1-\bar{\gamma})\right)^{\frac{1}{2}}, \quad \tilde{K}_{1-\bar{\gamma}} = 2\left(\left(\frac{2\pi}{\exp(2)}\right)^{\bar{\gamma}} 4\right)^{\frac{1}{2}}$$
$$M_{1-\bar{\gamma}} = (1-\bar{\gamma})^{(1-\bar{\gamma})}.$$

Proof: Observe that $\forall \eta \in X^n$ where $n \in \mathbb{N}_0$,

$$(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta) = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} (c_1, \eta_1)(c_2, \eta_2) \cdots (c_k, \eta_k)$$

$$(\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta).$$

Using triangle inequality on $\mathbb{R}$,

$$|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} |(c_1, \eta_1)||(c_2, \eta_2)| \cdots |(c_k, \eta_k)|$$

$$(\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta)$$

$$\leq \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} K_1 M_1^{i_1} (i_1!)^{\gamma_1} \cdots K_k M_k^{i_k} (i_k)^{\gamma_k}$$

$$(\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta).$$

Observe,

$$\sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} (\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta) = \binom{n}{i_1 \cdots i_k} \; \forall \eta \in X^n.$$

Therefore,

$$|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq K_1 \cdots K_k \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} M_1^{i_1} \cdots M_k^{i_k}$$

$$(i_1)^{\gamma_1} \cdots (i_k)^{\gamma_k}.$$

If $\bar{\gamma} \triangleq \max_{i=1,\ldots,k} \gamma_i$, then

$$
\begin{aligned}
|(c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k, \eta)| &\leq \left(\prod_{i=1}^{k} K_i\right) \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} M_1^{i_1} \cdots M_k^{i_k} \\
&\qquad (i_1)^{\bar{\gamma}} \cdots (i_k)^{\bar{\gamma}} \\
&= \left(\prod_{i=1}^{k} K_i\right) \left[\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k}^{1-\bar{\gamma}} M_1^{i_1} \cdots M_k^{i_k} \right] n!^{\bar{\gamma}} \\
&= \left(\prod_{i=1}^{k} K_i\right) \left[\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k}^{1-\bar{\gamma}} \left(M_1^{\frac{1}{1-\bar{\gamma}}}\right)^{(1-\bar{\gamma})i_1} \right. \\
&\qquad \left. \cdots \left(M_k^{\frac{1}{1-\bar{\gamma}}}\right)^{(1-\bar{\gamma})i_k} \right] n!^{\bar{\gamma}}.
\end{aligned}
$$

Observe $\bar{\gamma} \in [0,1[$ if and only if $(1-\bar{\gamma}) \in]0,1]$. Hence, using Lemma 4.2.1,

$$
\begin{aligned}
|(c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k, \eta)| &\leq \left(\prod_{i=1}^{k} K_i\right) \left(\frac{K_{1-\bar{\gamma}}^k}{\tilde{K}_{1-\bar{\gamma}}}\right) \left(\frac{M_{1-\bar{\gamma}}}{2}\right)^n \left[\sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \right. \\
&\qquad \binom{(1-\bar{\gamma})\, n}{(1-\bar{\gamma})\, i_1 \cdots (1-\bar{\gamma})\, i_k} \left(M_1^{\frac{1}{1-\bar{\gamma}}}\right)^{(1-\bar{\gamma})i_1} \\
&\qquad \left. \cdots \left(M_k^{\frac{1}{1-\bar{\gamma}}}\right)^{(1-\bar{\gamma})i_k} \right] n!^{\bar{\gamma}}.
\end{aligned}
$$

Finally, applying the neoclassical inequality from Theorem 4.2.1 gives

$$
\begin{aligned}
|(c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k, \eta)| &\leq \left(\frac{\prod_{i=1}^{k} K_i}{\tilde{K}_{1-\bar{\gamma}}}\right) (K_{1-\bar{\gamma}})^k \left(\frac{1}{1-\bar{\gamma}}\right)^{2(k-1)} \\
&\qquad \left[\frac{M_{1-\bar{\gamma}} \left(M_1^{1-\bar{\gamma}} + \cdots + M_k^{1-\bar{\gamma}}\right)}{2}\right]^n (n!)^{\bar{\gamma}} \\
&= \left(\frac{\left(\prod_{i=1}^{k} K_i\right)(1-\bar{\gamma})^2}{\tilde{K}_{1-\bar{\gamma}}}\right) \left(\frac{K_{1-\bar{\gamma}}}{(1-\bar{\gamma})^2}\right)^k
\end{aligned}
$$

$$\left[\frac{M_{1-\bar{\gamma}}\left(M_1^{1-\bar{\gamma}}+\cdots+M_k^{1-\bar{\gamma}}\right)}{2}\right]^n (n!)^{\bar{\gamma}}.$$

■

The following corollary is an immediate result of Theorem 4.2.2.

Corollary 4.2.1. *If* $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$*, then* $c^{\sqcup\!\sqcup\, n} \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle \ \forall n \in \mathbb{N}_0$*.*

The following example demonstrates that the Banach space $S_\infty^m(R)$ is not closed under the shuffle product, where $R > 0$, thus disproving the assertion made in Theorem 4.1.4 of [Winter-Arboleda (2019)].

Example 4.2.1. Let $c \in \mathbb{R}\langle\langle X\rangle\rangle$ be defined as

$$c = \sum_{\eta \in X^*} K \left(\frac{1}{R}\right)^{|\eta|} |\eta|!\, \eta.$$

Observe that $||c||_{\infty,R} = K < \infty$. Therefore, $c \in S_\infty(R)$. Then, $\forall \eta \in X^n$

$$\begin{aligned}
\left(c^{\sqcup\!\sqcup\, 2}, \eta\right) &= \sum_{i=0}^{n} \sum_{\substack{\eta_1 \in X^i \\ \eta_2 \in X^{n-i}}} (c, \eta_1)\,(c, \eta_2)\,(\eta_1 \sqcup\!\sqcup\, \eta_2, \eta) \\
&= \sum_{i=0}^{n} \left(K\left(\frac{1}{R}\right)^i i!\right)\left(K\left(\frac{1}{R}\right)^{n-i}(n-i)!\right) \sum_{\substack{\eta_1 \in X^i \\ \eta_2 \in X^{n-i}}} (\eta_1 \sqcup\!\sqcup\, \eta_2, \eta) \\
&= K^2\left(\frac{1}{R}\right)^n \sum_{i=0}^{n} i!\,(n-i)! \sum_{\substack{\eta_1 \in X^i \\ \eta_2 \in X^{n-i}}} (\eta_1 \sqcup\!\sqcup\, \eta_2, \eta)\,.
\end{aligned}$$

Observe for $0 \leq i \leq n$,

$$\sum_{\substack{\eta_1 \in X^i \\ \eta_2 \in X^{n-i}}} (\eta_1 \sqcup\!\sqcup\, \eta_2, \eta) = \binom{n}{i}.$$

Hence, $\forall \eta \in X^n$

$$\begin{aligned}
\left(c^{\sqcup\!\sqcup\, 2}, \eta\right) &= K^2 \left(\frac{1}{R}\right)^n \sum_{i=0}^{n} i!\,(n-i)!\binom{n}{i} \\
&= K^2 \left(\frac{1}{R}\right)^n n! \sum_{i=0}^{n} 1 \\
&= K^2 \left(\frac{1}{R}\right)^n n!\,(n+1) \\
&= K^2 \left(\frac{1}{R}\right)^n (n+1)!.
\end{aligned}$$

Therefore, $c^{\sqcup\!\sqcup\, 2} = \sum_{\eta \in X^*} K^2 \left(\frac{1}{R}\right)^{|\eta|} (|\eta|+1)!\eta$. Clearly,

$$\left|\left|c^{\sqcup\!\sqcup\, 2}\right|\right|_{\infty,R} = \sup_{\eta \in X^*} \{K^2\,(|\eta|+1)\} \to \infty.$$

Hence, $c \in S_\infty(R)$ but $c^{\sqcup\!\sqcup\, 2} = c \sqcup\!\sqcup\, c \notin S_\infty(R)$. The conclusion is that the Banach space $S_\infty^m(R)$ is not closed under the shuffle product. □

The following theorem characterizes a kind of *almost* submultiplicative property of $||\cdot||_{\infty,R}$ with respect to the shuffle product of formal power series.

Theorem 4.2.3. *Let $c_1, c_2, \ldots, c_k$ be a finite nonempty collection of formal power series such that $c_i \in S_\infty(R)$ $\forall i = 1, 2, \ldots, k$. Then, $c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k \in S_\infty(R')$ $\forall R' = \epsilon R$, where $\epsilon \in]0,1[$ and*

$$||c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k||_{\infty,R'} \leq \frac{1}{(1-\epsilon)^k} ||c_1||_{\infty,R} ||c_2||_{\infty,R} \cdots ||c_k||_{\infty,R}.$$

Proof: Observe $\forall \eta \in X^n$, where $n \in \mathbb{N}$,

$$(c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k, \eta) = \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1 + \cdots + i_k = n}} \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} (c_1, \eta_1)(c_2, \eta_2) \cdots (c_k, \eta_k)$$

$$(\eta_1 \sqcup\!\sqcup\, \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, \eta_k, \eta).$$

Using triangle inequality on $\mathbb{R}$,

$$|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1+\cdots+i_k=n}} \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} |(c_1, \eta_1)||(c_2, \eta_2)| \cdots |(c_k, \eta_k)| \\ (\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta)$$

Hence,

$$|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1+\cdots+i_k=n}} \left(\max_{\zeta \in X^{i_1}} |(c_1, \zeta)|\right) \cdots \left(\max_{\zeta \in X^{i_k}} |(c_k, \zeta)|\right) \\ \sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} (\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta).$$

Applying the identity

$$\sum_{\substack{j=1 \\ \eta_j \in X^{i_j}}}^{k} (\eta_1 \sqcup\!\sqcup \eta_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup \eta_k, \eta) = \binom{n}{i_1 \cdots i_k} \; \forall \eta \in X^n$$

gives

$$|(c_1 \sqcup\!\sqcup c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup c_k, \eta)| \leq \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} \left(\max_{\zeta \in X^{i_1}} |(c_1, \zeta)|\right) \cdots \left(\max_{\zeta \in X^{i_k}} |(c_k, \zeta)|\right).$$

Since $c_i \in S_\infty(R) \; \forall i = 1 \cdots k$,

$$\max_{\zeta \in X^j} |(c_i, \zeta)| \leq \frac{||c_i||_{\infty,R} (j)!}{R^j} \quad \forall j \in \mathbb{N}.$$

Therefore,

$$|(c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k, \eta)| \leq \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} \left(||c_1||_{\infty,R}\frac{i_1!}{R^{i_1}}\right) \cdots \left(||c_k||_{\infty,R}\frac{i_k!}{R^{i_k}}\right)$$
$$= \frac{\left(||c_1||_{\infty,R} \cdots ||c_k||_{\infty,R}\right) n!}{R^n} \sum_{\substack{i_1 \ldots i_k \geq 0 \\ i_1+\cdots+i_k=n}} 1$$
$$= \frac{\left(||c_1||_{\infty,R} \cdots ||c_k||_{\infty,R}\right) n!}{R^n} \binom{n+k-1}{k-1}, \quad \forall \eta \in X^n.$$

Hence,

$$||c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k||_{\infty,R'} = \sup_{\eta \in X^*} \left\{ |(c_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k.\eta)| \frac{R'^{|\eta|}}{|\eta|!} \right\}$$
$$\leq \sup_{n \in \mathbb{N}_0} \left\{ \left(||c_1||_{\infty,R} \cdots ||c_k||_{\infty,R}\right) \binom{n+k-1}{k-1} \left(\frac{R'}{R}\right)^n \right\}$$
$$= \left(||c_1||_{\infty,R} \cdots ||c_k||_{\infty,R}\right) \sup_{n \in \mathbb{N}_0} \left\{ \epsilon^n \binom{n+k-1}{k-1} \right\}$$
$$\leq \left(||c_1||_{\infty,R} \cdots ||c_k||_{\infty,R}\right) \sum_{n=0}^{\infty} \left\{ \epsilon^n \binom{n+k-1}{k-1} \right\}.$$

Noting that $\epsilon \in]0, 1[$,

$$\sum_{n=0}^{\infty} \left\{ \epsilon^n \binom{n+k-1}{k-1} \right\} = \left(\sum_{n=0}^{\infty} \epsilon^n \right)^k = \frac{1}{(1-\epsilon)^k}.$$

Therefore,

$$||c_1 \sqcup\!\sqcup\, c_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k||_{\infty,R'} \leq \frac{1}{(1-\epsilon)^k} ||c_1||_{\infty,R} ||c_2||_{\infty,R} \cdots ||c_k||_{\infty,R} < \infty.$$

■

The shuffle product is a bilinear map and is bounded in norm according to Theorem 4.2.3. Hence, its continuity is a consequence of this property which is given in the following corollary.

TABLE 2: Summary of convergence results for the shuffle product.

c	d	$c \sqcup\!\sqcup d$	Theorem
$\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$	$\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$	$\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$	Theorem 4.2.2
$S^m_\infty(R)$	$S^m_\infty(R)$	$S^m_\infty(R')$ where $R' = \epsilon R \; \forall \epsilon \in]0,1[$.	Theorem 4.2.3
$S^m_\infty(R)$	S^m_∞	$S^m_\infty(R')$ where $R' = \epsilon R \; \forall \epsilon \in]0,1[$.	Theorem 4.2.4
S^m_∞	S^m_∞	S^m_∞	Theorem 4.2.5

Corollary 4.2.2. *Define a bilinear map,*

$$\hat{\sqcup\!\sqcup} : S^m_\infty(R) \times S^m_\infty(R) \longrightarrow S^m_\infty(R')$$

$$c \times d \longmapsto c \sqcup\!\sqcup d \; ,$$

where $R' = \epsilon R$ for a fixed $\epsilon \in]0,1[$. Then the operator $\hat{\sqcup\!\sqcup}$ is bounded with $|| \hat{\sqcup\!\sqcup} || \leq \left(\frac{1}{1-\epsilon}\right)^2$ under the operator norm. Hence, the operator $\hat{\sqcup\!\sqcup}$ is continuous.

The following theorem characterizes convergence of the shuffle product when one of its arguments is in the Fréchet space S^m_∞. Note that the shuffle product is symmetric; hence, it is of no consequence considering either the first argument or the second argument in S^m_∞. The convergence results for the shuffle product are summarized and tabulated in Table 2.

Theorem 4.2.4. *Let $c \in S^m_\infty(R)$ and $d \in S^m_\infty$. Then, $c \sqcup\!\sqcup d \in S^m_\infty(R') \; \forall R' = \epsilon R$, where $\epsilon \in]0,1[$.*

Proof: Recall that

$$d \in S^m_\infty \Leftrightarrow d \in S^m_\infty(\hat{R}) \; \forall \hat{R} > 0.$$

Therefore, $c \in S^m_\infty(R)$ and $d \in S^m_\infty(R)$. Applying Theorem 4.2.3,

$$||c \,\tilde{\circ}\, d_\delta||_{\infty,R'} \leq \left(\frac{1}{1-\epsilon}\right)^2 ||c||_{\infty,R} ||d||_{\infty,R} < \infty,$$

$\forall R' = \epsilon R$, where $\epsilon \in]0,1[$. ■

The following theorem is the main result of this subsection. It states that the Fréchet space S_∞^m is closed under the shuffle product.

Theorem 4.2.5. *If $c, d \in S_\infty^m$, then $c \sqcup\!\sqcup\, d \in S_\infty^m$.*

Proof: Observe

$$c, d \in S_\infty^m \Leftrightarrow c_i, d_i \in S_\infty(R) \quad \forall R > 0,\ \forall i = 1, \ldots, m.$$

Fix $\epsilon \in]0,1[$ and $\forall R' > 0$ define $R = \left(\frac{1}{\epsilon}\right) R'$. Using Theorem 4.2.3,

$$||(c \sqcup\!\sqcup\, d)_i||_{\infty,R'} = ||c_i \sqcup\!\sqcup\, d_i||_{\infty,R'} \leq \frac{||c_i||_{\infty,R}||d_i||_{\infty,R}}{(1-\epsilon)^2} \quad \forall i = 1, \ldots, m$$
$$< \infty.$$

Therefore,

$$(c \sqcup\!\sqcup\, d)_i \in S_\infty(R') \quad \forall R' > 0,\ \forall i = 1, \ldots, m \Longleftrightarrow (c \sqcup\!\sqcup\, d) \in S_\infty^m(R') \quad \forall R' > 0.$$

Hence, $c \sqcup\!\sqcup\, d \in S_\infty^m$. ■

The following corollary is a consequence of Theorem 4.2.3 and is used in this work repeatedly in Subsections 4.2.2 and 4.2.3.

Corollary 4.2.3. *If $c \in S_\infty(R)$, then $\forall R' = \epsilon R$, where $\epsilon \in]0,1[$, it follows that*

$$\left|\left|c^{\sqcup\!\sqcup\, k}\right|\right|_{\infty,R'} \leq \frac{||c||_{\infty,R}^k}{(1-\epsilon)^k}.$$

Theorem 4.2.5 has shown that S_∞^m is closed under shuffle product. The following theorem states that the bilinear shuffle product is a continuous bilinear map on S_∞^m.

Theorem 4.2.6. *Define the bilinear map on the Fréchet space* S_∞^m

$$\bar{\sqcup\!\sqcup} : S_\infty^m \times S_\infty^m \longrightarrow S_\infty^m$$
$$c \times d \longmapsto c \sqcup\!\sqcup d.$$

Then the operator $\bar{\sqcup\!\sqcup}$ *is a continuous bilinear map.*

Proof: Observe that the set of norms $\{||.||_{\infty,k}\}_{k\in\mathbb{N}}$ are linearly ordered as

$$||.||_{\infty,1} \leq ||.||_{\infty,2} \leq \cdots \leq ||.||_{\infty,n} \leq ||.||_{\infty,n+1} \leq \cdots ,$$

and thus form a fundamental system of seminorms for the Fréchet space S_∞^m. Fix an $\epsilon \in]0,1[$ and define the map α_ϵ such that

$$\alpha_\epsilon : \mathbb{N} \longrightarrow \mathbb{N}$$
$$k \longmapsto \lceil \epsilon k \rceil ,$$

where $\lceil . \rceil$ is the ceiling function. Observe that, $c \in S_\infty^m$ if and only if $c \in S_\infty^m(R)\ \forall R > 0$. Applying Theorem 4.2.3,

$$|| \bar{\sqcup\!\sqcup} (c,d)||_{\infty,k} \leq \left(\frac{1}{1-\epsilon}\right)^2 ||c||_{\infty,\alpha_\epsilon(k)}||d||_{\infty,\alpha_\epsilon(k)}$$

$\forall k \in \mathbb{N}$ and $\forall c,d \in S_\infty^m$. Hence, the bilinear map $\bar{\sqcup\!\sqcup}$ is continuous by Theorem 2.4.6. ■

Using Theorem 2.4.2, Theorem 2.4.4 and Table 2, the following propositions must hold. The parallel product configuration of two locally convergent Fliess operators F_c, F_d is represented by a locally convergent Fliess operator $F_{c \sqcup\!\sqcup d}$. The parallel product configuration of two globally convergent Fliess operators F_c, F_d is represented by a globally convergent Fliess operator $F_{c \sqcup\!\sqcup d}$.

4.2.2 Global Convergence of Mixed Composition Product

The goal of this subsection is to prove the Fréchet space S_∞^m is closed under the mixed composition product. The mixed composition product is a noncommutative product; hence, there are four possible cases based on where the argument series of the composition product can lie in the hierarchy of spaces as shown in Figure 3. The continuity of the mixed composition product with respect to its left argument in S_∞ space is also proven in this subsection. The following theorem describes a particular condition under which the mixed composition product $c\,\tilde{\circ}\,d_\delta$ belongs to $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Theorem 4.2.7. *If $c \in \mathbb{R}\langle X\rangle$ and $d \in \mathbb{R}^m_{GC}\langle\langle \bar{X}\rangle\rangle$, then $c\,\tilde{\circ}\,d_\delta \in \mathbb{R}_{GC}\langle\langle \bar{X}\rangle\rangle$.*

Proof: Without loss of generality assume c is a polynomial of degree $N \in \mathbb{N}_0$. Since $c \in \mathbb{R}\langle X\rangle$ implies $\exists$ constants $K_c, M_c > 0$ such that $|(c,\eta)| \leq K_c M_c^{|\eta|}$, $\forall \eta \in \cup_{i=0}^N X^i$, then

$$
\begin{aligned}
c\,\tilde{\circ}\,d_\delta &= \sum_{\eta\in\mathrm{supp}(c)} (c,\eta)\eta\,\tilde{\circ}\,d_\delta \\
&\leq \sum_{\eta\in\mathrm{supp}(c)} K_c M_c^{|\eta|}\eta\,\tilde{\circ}\,d_\delta \\
&\leq \sum_{n=0}^{N} K_c M_c^n \sum_{\eta\in X^n} \eta\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{N} K_c M_c^n \sum_{\substack{r_0\ldots r_m\geq 0\\ r_0+\cdots+r_m=n}} (x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m})\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{N} K_c M_c^n \sum_{\substack{r_0\ldots r_m\geq 0\\ r_0+\cdots+r_m=n}} \left(\frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{x_m^{\sqcup\!\sqcup\, r_m}}{r_m!}\right)\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{N} \frac{K_c M_c^n}{n!} \sum_{\substack{r_0\ldots r_m\geq 0\\ r_0+\cdots+r_m=n}} \binom{n}{r_0\cdots r_m} (x_0\,\tilde{\circ}\,d_\delta)^{\sqcup\!\sqcup\, r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (x_m\,\tilde{\circ}\,d_\delta)^{\sqcup\!\sqcup\, r_m} \\
&= K_c \sum_{n=0}^{N} \frac{\left[M_c\left(\sum_{j=0}^m x_j\,\tilde{\circ}\,d_\delta\right)\right]^{\sqcup\!\sqcup\, n}}{n!} \\
&= K_c \sum_{n=0}^{N} \frac{\left[M_c\left(\sum_{j=0}^m \bar{x}_j + \bar{x}_0 \sum_{i=1}^m d_i\right)\right]^{\sqcup\!\sqcup\, n}}{n!}
\end{aligned}
$$

$$\leq K_c \sum_{n=0}^{N} \left[M_c \left(\sum_{j=0}^{m} \bar{x}_j + \bar{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n}.$$

Observe that $\sum_{j=0}^{m} \bar{x}_j \in \mathbb{R}\langle X \rangle \subset \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle$. Since $d \in \mathbb{R}^m_{GC}\langle\langle \bar{X} \rangle\rangle$ implies $d_i \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle$ $\forall i = 1, \ldots, m$, then $\bar{x}_0 \sum_{i=1}^{m} d_i \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle$. Therefore, by Theroem 2.4.5,

$$\sum_{j=0}^{m} \bar{x}_j + \bar{x}_0 \sum_{i=1}^{m} d_i \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle.$$

Applying Corollary 4.2.1, $\left[M_c \left(\sum_{j=0}^{m} \bar{x}_j + \bar{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n} \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle$ $\forall n \leq N$. Hence, by virtue of Theorem 2.4.5,

$$\sum_{n=0}^{N} \left[M_c \left(\sum_{j=0}^{m} \bar{x}_j + \bar{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n} \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle.$$

Therefore,

$$c \,\tilde{\circ}\, d_\delta \leq K_c \sum_{n=0}^{N} \left[M_c \left(\sum_{j=0}^{m} \bar{x}_j + \bar{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n} \in \mathbb{R}_{GC}\langle\langle \bar{X} \rangle\rangle.$$

■

The main assertion that the Fréchet space S^m_∞ is closed under mixed composition product requires some preliminary results. As an initial step, the following theorem characterizes the convergence of mixed composition product $c \,\tilde{\circ}\, d_\delta$ when $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ and $d \in S^m_\infty(R)$.

Theorem 4.2.8. *Let $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ such that $|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!^\gamma$ $\forall \eta \in X^*$, and $\gamma \in [0, 1[$. If $d \in S^m_\infty(R)$, then $c \,\tilde{\circ}\, d_\delta \in S_\infty(R')$, $\forall R' = \epsilon R$, where $\epsilon \in]0, 1[$.*

Proof: Observe

$$\begin{aligned} c \,\tilde{\circ}\, d_\delta &= \sum_{\eta \in X^*} (c, \eta) \eta \,\tilde{\circ}\, d_\delta \\ &\leq \sum_{\eta \in X^*} K_c M_c^{|\eta|} |\eta|!^\gamma \, \eta \,\tilde{\circ}\, d_\delta \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} K_c M_c^n n!^{\gamma} \sum_{\eta \in X^n} \eta \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n!^{\gamma} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=n}} (x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m}) \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n!^{\gamma} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=n}} \left(\frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{x_m^{\sqcup\!\sqcup\, r_m}}{r_m!} \right) \,\tilde{\circ}\, d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n!^{(\gamma-1)} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=n}} \binom{n}{r_0 \cdots r_m} (x_0 \,\tilde{\circ}\, d_\delta)^{\sqcup\!\sqcup\, r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (x_m \,\tilde{\circ}\, d_\delta)^{\sqcup\!\sqcup\, r_m} \\
&= K_c \sum_{n=0}^{\infty} \frac{\left[M_c \left(\sum_{j=0}^m x_j \,\tilde{\circ}\, d_\delta \right) \right]^{\sqcup\!\sqcup\, n}}{n!^{(1-\gamma)}} \\
&= K_c \sum_{n=0}^{\infty} \frac{\left[M_c \left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i \right) \right]^{\sqcup\!\sqcup\, n}}{n!^{(1-\gamma)}}.
\end{aligned}
$$

From Lemma 2.5.1, $(\Gamma(n+1))^{(1-\gamma)} = n!^{(1-\gamma)} \geq \dfrac{1}{\tilde{K}_{1-\gamma}} 2^{-n}\Gamma((1-\gamma)n+1)$, where $\tilde{K}_{1-\gamma} = 2\left(\left(\dfrac{2\pi}{\exp(2)}\right)^{\gamma} 4\right)^{\frac{1}{2}}$. Hence,

$$
c \,\tilde{\circ}\, d_\delta \leq K_c \tilde{K}_{1-\gamma} \sum_{n=0}^{\infty} \frac{\left[2M_c \left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i \right) \right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\gamma)n+1\right)}.
$$

Define $\tilde{K}_c \triangleq K_c \tilde{K}_{1-\gamma}$ and $\tilde{M}_c \triangleq 2M_c$. Then,

$$
c \,\tilde{\circ}\, d_\delta \leq \tilde{K}_c \sum_{n=0}^{\infty} \frac{\left[\tilde{M}_c \left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i \right) \right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\gamma)n+1\right)}.
$$

Therefore,

$$
||c \,\tilde{\circ}\, d_\delta||_{\infty,R'} \leq \tilde{K}_c \left\| \sum_{n=0}^{\infty} \frac{\left[\tilde{M}_c \left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i \right) \right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\gamma)n+1\right)} \right\|_{\infty,R'}.
$$

Observe that $S_\infty(R')$ is a Banach space with topology defined by $||.||_{\infty,R'}$. Hence, by the triangle inequality

$$||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} \le \tilde{K}_c \sum_{n=0}^{\infty} \frac{\tilde{M}_c{}^n \left|\left|\left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i\right)^{\sqcup n}\right|\right|_{\infty,R'}}{\Gamma\left((1-\gamma)n+1\right)}.$$

Next, observe that $\left|\left|\sum_{j=0}^m x_j\right|\right|_{\infty,R} = R$ and $||\tilde{x}_0 \sum_{i=1}^m d_i||_{\infty,R} \le mR||d||_{\infty,R}$, where $||d||_{\infty,R} = \max_{i=1\cdots m}\left\{||d_i||_{\infty,R}\right\}$. Hence, again by the triangle inequality,

$$\begin{aligned} \left|\left|\left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i\right)\right|\right|_{\infty,R} &\le \left|\left|\sum_{j=0}^m \tilde{x}_j\right|\right|_{\infty,R} + \left|\left|\tilde{x}_0 \sum_{i=1}^m d_i\right|\right|_{\infty,R} \\ &\le R + mR||d||_{\infty,R} < \infty. \end{aligned}$$

Therefore, $\left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i\right) \in S_\infty(R)$. Hence, $\forall\ R' = \epsilon R$ where $\epsilon \in]0,1[$, by virtue of Corollary 4.2.3, it follows that

$$\begin{aligned} ||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} &\le \tilde{K}_c \sum_{n=0}^{\infty} \frac{\left(\frac{\tilde{M}_c}{1-\epsilon}\right)^n \left|\left|\left(\sum_{j=0}^m \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^m d_i\right)\right|\right|_{\infty,R}^n}{\Gamma\left((1-\gamma)n+1\right)} \\ &\le \tilde{K}_c \sum_{n=0}^{\infty} \frac{\left(\frac{\tilde{M}_c}{1-\epsilon}\right)^n \left(R + mR||d||_{\infty,R}\right)^n}{\Gamma\left((1-\gamma)n+1\right)} \\ &= \tilde{K}_c \sum_{n=0}^{\infty} \frac{\left[\frac{\tilde{M}_c R}{1-\epsilon}\left(1 + m||d||_{\infty,R}\right)\right]^n}{\Gamma\left((1-\gamma)n+1\right)} \\ &= \tilde{K}_c \mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}_c R\left(1 + m||d||_{\infty,R}\right)}{1-\epsilon}\right). \end{aligned}$$

Since $\gamma \in [0,1[\Leftrightarrow (1-\gamma) \in]0,1]$, the Mittag-Leffler function $\mathbb{E}_{(1-\gamma),1}(\cdot)$ is an entire function. Note that $d \in S_\infty^m(R)$ and $\epsilon \in]0,1[$ imply that $\left(\frac{\tilde{M}_c R\left(1 + m||d||_{\infty,R}\right)}{1-\epsilon}\right) < \infty$.

Therefore,

$$||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} \leq \tilde{K}_c \mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}_c R\left(1+m||d||_{\infty,R}\right)}{1-\epsilon}\right) < \infty.$$

Therefore, $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and $d \in S_\infty^m(R)$ implies that $c\,\tilde{\circ}\,d_\delta \in S_\infty(R')\ \forall R' = \epsilon R$, where $\epsilon \in]0,1[$. ■

Theorem 4.2.8 describes the convergence of $c\,\tilde{\circ}\,d_\delta$ when its left argument $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ but the right argument d lies in $S_\infty^m(R)$ for some $R > 0$. The mixed composition product does not belong to $S_\infty(R)$ space *but arbitrarily close* to it. The subsequent step is to let the right argument d be in the limit space S_∞^m and characterize the convergence of $c\,\tilde{\circ}\,d_\delta$.

Theorem 4.2.9. *If $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and $d \in S_\infty^m$, then $c\,\tilde{\circ}\,d_\delta \in S_\infty$.*

Proof: Recall that

$$d \in S_\infty^m \Longleftrightarrow d \in S_\infty^m(R)\ \forall R > 0.$$

Fix $\epsilon \in]0,1[$ and $\forall R' > 0$ define $R = \left(\frac{1}{\epsilon}\right) R'$. Since $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, applying Theorem 4.2.8 gives $c\,\tilde{\circ}\,d_\delta \in S_\infty^m(R')$. Therefore,

$$c\,\tilde{\circ}\,d_\delta \in S_\infty(R')\ \forall R' > 0 \Longleftrightarrow c\,\tilde{\circ}\,d_\delta \in S_\infty.$$

■

Theorems 4.2.8 and 4.2.9 state that when the left argument c in the mixed composition product $c\,\tilde{\circ}\,d_\delta$ is in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$, the convergence of $c\,\tilde{\circ}\,d_\delta$ is limited by the convergence of the series d. The next step is to check when the left-hand argument c is in the closure of the $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$, the S_∞ space. Prior to that, the following lemma is needed to proceed.

Lemma 4.2.2. *Let $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ such that $|(c,\eta)| \leq K_c M_c^{|\eta|}|\eta|!\ \forall \eta \in X^*$. If $d \in S_\infty^m(R)$,* *then* $\forall\ R' = \epsilon R$, *where* $\epsilon \in]0,1[$, *it follows that*

$$||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} \leq K_c \sum_{n=0}^{\infty} \left(\frac{M_c R \left(1 + m||d||_{\infty,R}\right)}{1-\epsilon} \right)^n,$$

where $||d||_{\infty,R} = \max_{i=1,\cdots m} ||d_i||_{\infty,R}$.

Proof: Observe

$$\begin{aligned}
c\,\tilde{\circ}\,d_\delta &= \sum_{\eta\in X^*} (c,\eta)\eta\,\tilde{\circ}\,d_\delta \\
&\leq \sum_{\eta\in X^*} K_c M_c^{|\eta|} |\eta|!\,\eta\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\eta\in X^n} \eta\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} (x_0^{r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_m^{r_m})\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n n! \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} \left(\frac{x_0^{\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \frac{x_m^{\sqcup\!\sqcup\, r_m}}{r_m!} \right)\,\tilde{\circ}\,d_\delta \\
&= \sum_{n=0}^{\infty} K_c M_c^n \sum_{\substack{r_0,\ldots,r_m\geq 0\\ r_0+\cdots+r_m=n}} \binom{n}{r_0 \cdots r_m} (x_0\,\tilde{\circ}\,d_\delta)^{\sqcup\!\sqcup\, r_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup (x_m\,\tilde{\circ}\,d_\delta)^{\sqcup\!\sqcup\, r_m} \\
&= K_c \sum_{n=0}^{\infty} \left[M_c \left(\sum_{j=0}^{m} x_j\,\tilde{\circ}\,d_\delta \right) \right]^{\sqcup\!\sqcup\, n} \\
&= K_c \sum_{n=0}^{\infty} \left[M_c \left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} &\leq K_c \left\| \sum_{n=0}^{\infty} \left[M_c \left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right) \right]^{\sqcup\!\sqcup\, n} \right\|_{\infty,R'} \\
&\leq K_c \sum_{n=0}^{\infty} {M_c}^n \left\| \left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i \right)^{\sqcup\!\sqcup\, n} \right\|_{\infty,R'}.
\end{aligned}$$

Observe that $\left|\left|\sum_{j=0}^{m} x_j\right|\right|_{\infty,R} = R$ and $||\tilde{x}_0 \sum_{i=1}^{m} d_i||_{\infty,R} \leq mR||d||_{\infty,R}$, where $||d||_{\infty,R} = \max_{i=1\cdots m}\left\{||d_i||_{\infty,R}\right\}$. Hence,

$$\left|\left|\left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i\right)\right|\right|_{\infty,R} \leq \left|\left|\sum_{j=0}^{m} \tilde{x}_j\right|\right|_{\infty,R} + \left|\left|\tilde{x}_0 \sum_{i=1}^{m} d_i\right|\right|_{\infty,R}$$
$$\leq R + mR||d||_{\infty,R} < \infty.$$

Therefore, $\left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i\right) \in S_\infty(R)$. Hence, $\forall\ R' = \epsilon R$, where $\epsilon \in]0,1[$, by virtue of Corollary 4.2.3,

$$\begin{aligned} ||c \,\tilde{\circ}\, d_\delta||_{\infty,R'} &\leq K_c \sum_{n=0}^{\infty} \left(\frac{M_c}{1-\epsilon}\right)^n \left|\left|\left(\sum_{j=0}^{m} \tilde{x}_j + \tilde{x}_0 \sum_{i=1}^{m} d_i\right)\right|\right|_{\infty,R}^n \\ &\leq K_c \sum_{n=0}^{\infty} \left(\frac{M_c}{1-\epsilon}\right)^n \left(R + mR||d||_{\infty,R}\right)^n \\ &= K_c \sum_{n=0}^{\infty} \left(\frac{M_c R\left(1 + m||d||_{\infty,R}\right)}{1-\epsilon}\right)^n. \end{aligned}$$

■

The following theorem describes the convergence of the mixed composition product $c \,\tilde{\circ}\, d_\delta$ when the left argument c is in Fréchet space S_∞ but the right argument is locally convergent viz. $d \in S_\infty^m(R)$ for some $R > 0$.

Theorem 4.2.10. *If $c \in S_\infty$ and $d \in S_\infty^m(R)$, then $c \tilde{\circ} d_\delta \in S_\infty(R')$ $\forall R' = \epsilon R$, where $\epsilon \in]0,1[$.*

Proof: Observe that $c \in S_\infty$ implies there exists a Cauchy sequence $\{c_i\}_{i\in\mathbb{N}}$ such that $c_i \to c$ in the Fréchet topology, where $c_i \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ $\forall i \in \mathbb{N}$ viz.

$$c_i \to c \in S_\infty \Leftrightarrow ||c - c_i||_{\infty,\bar{R}} \to 0 \quad \forall \bar{R} > 0.$$

Choose $\bar{R} = \dfrac{2R\left(1 + m||d||_{\infty,R}\right)}{1-\epsilon} < \infty$ for a fixed $\epsilon \in]0,1[$. Then,

$$||c - c_i||_{\infty,\bar{R}} \to 0 \Leftrightarrow \ \exists N : \forall n > N; ||c - c_n||_{\infty,\bar{R}} < \frac{\epsilon'}{2}. \tag{4.2.1}$$

By virtue of (4.2.1),

$$||c - c_n||_{\infty,\bar{R}} < \frac{\epsilon'}{2} \Leftrightarrow |(c - c_n, \eta)| \leq \frac{\epsilon' |\eta|!}{2\bar{R}^{|\eta|}} \quad \forall \eta \in X^*.$$

Using the triangle inequality on $\mathbb{R}$,

$$|(c,\eta)| - |(c_n,\eta)| \leq |(c - c_n, \eta)| \leq \frac{\epsilon' |\eta|!}{2\bar{R}^{|\eta|}} \quad \forall \eta \in X^*.$$

Therefore, $\forall \eta \in X^*$:

$$|(c,\eta)| \leq |(c_n,\eta)| + \frac{\epsilon' |\eta|!}{2\bar{R}^{|\eta|}}.$$

Since $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$, there exist constants $K, M > 0$ and $\gamma \in [0, 1[$ such that $|(c_n, \eta)| \leq KM^{|\eta|}\left(|\eta|!\right)^{\gamma}$. Hence,

$$|(c,\eta)| \leq KM^{|\eta|}\left(|\eta|!\right)^{\gamma} + \frac{\epsilon' |\eta|!}{2\bar{R}^{|\eta|}} \quad \forall \eta \in X^*.$$

Therefore,

$$\begin{aligned} c \,\tilde{\circ}\, d_\delta &= \sum_{\eta \in X^*} (c,\eta)\, \eta \,\tilde{\circ}\, d_\delta \\ &\leq \sum_{\eta \in X^*} \left\{ KM^{|\eta|}\left(|\eta|!\right)^{\gamma} + \frac{\epsilon' |\eta|!}{2\bar{R}^{|\eta|}} \right\} \eta \,\tilde{\circ}\, d_\delta \\ &= \sum_{n=0}^{\infty} \left\{ KM^{|\eta|}\left(|\eta|!\right)^{\gamma} \sum_{\eta \in X^n} \eta \,\tilde{\circ}\, d_\delta \right\} + \frac{\epsilon'}{2} \sum_{n=0}^{\infty} \left\{ \frac{n!}{\bar{R}^n} \sum_{\eta \in X^n} \eta \,\tilde{\circ}\, d_\delta \right\}. \end{aligned}$$

Hence,

$$||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} \leq \left|\left|\sum_{n=0}^{\infty}\left\{KM^{|\eta|}\left(|\eta|!\right)^{\gamma}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}+\frac{\epsilon'}{2}\sum_{n=0}^{\infty}\left\{\frac{n!}{\bar{R}^n}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}\right|\right|_{\infty,R'}.$$

Using the triangle inequality on $S_\infty(R')$,

$$||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} \leq \left|\left|\sum_{n=0}^{\infty}\left\{KM^{|\eta|}\left(|\eta|!\right)^{\gamma}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}\right|\right|_{\infty,R'}+\frac{\epsilon'}{2}\left|\left|\sum_{n=0}^{\infty}\left\{\frac{n!}{\bar{R}^n}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}\right|\right|_{\infty,R'}.$$

By Theorem 4.2.8,

$$\left|\left|\sum_{n=0}^{\infty}\left\{KM^{|\eta|}\left(|\eta|!\right)^{\gamma}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}\right|\right|_{\infty,R'} \leq \tilde{K}\mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}}{(1-\epsilon)}R\left(1+m||d||_{\infty,R}\right)\right)<\infty,$$

where $\tilde{K}=K\tilde{K}_{1-\gamma}$ with $\tilde{K}_{1-\gamma}=\left(\left(\frac{2\pi}{\exp(2)}\right)^{\gamma}4\right)^{\frac{1}{2}}$, $\tilde{M}=2M$, and $\mathbb{E}_{(1-\gamma),1}(\cdot)$ is the Mittag-Leffler function, which is an entire function as $(1-\gamma)\in]0,1]$. By Lemma 4.2.2,

$$\left|\left|\sum_{n=0}^{\infty}\left\{\frac{n!}{\bar{R}^n}\sum_{\eta\in X^n}\eta\,\tilde{\circ}\,d_\delta\right\}\right|\right|_{\infty,R'} \leq \sum_{n=0}^{\infty}\left(\frac{R\left(1+m||d||_{\infty,R}\right)}{\bar{R}\,(1-\epsilon)}\right)^n.$$

Therefore,

$$\begin{aligned}||c\,\tilde{\circ}\,d_\delta||_{\infty,R'} &\leq \tilde{K}\mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}}{(1-\epsilon)}R\left(1+m||d||_{\infty,R}\right)\right)+\frac{\epsilon'}{2}\sum_{n=0}^{\infty}\left(\frac{R\left(1+m||d||_{\infty,R}\right)}{\bar{R}\,(1-\epsilon)}\right)^n\\ &\leq \tilde{K}\mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}}{(1-\epsilon)}R\left(1+m||d||_{\infty,R}\right)\right)+\frac{\epsilon'}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\\ &\leq \tilde{K}\mathbb{E}_{(1-\gamma),1}\left(\frac{\tilde{M}}{(1-\epsilon)}R\left(1+m||d||_{\infty,R}\right)\right)+\epsilon'<\infty.\end{aligned}$$

■

Theorem 4.2.10 has asserted that the convergence of the mixed composition product

$c\,\tilde{\circ}\,d_\delta$ is limited by the convergence of the right argument d when the left argument c is in the Fréchet space S_∞. The following theorem states that the Fréchet space S_∞ is closed under the mixed composition product, which is the ultimate goal of this subsection.

Theorem 4.2.11. *If $c \in S_\infty$ and $d \in S_\infty^m$, then $c\,\tilde{\circ}\,d_\delta \in S_\infty$.*

Proof: Recall

$$d \in S_\infty^m \Longleftrightarrow d \in S_\infty^m(R)\ \forall R > 0.$$

Fix $\epsilon \in]0,1[$ and $\forall R' > 0$ define $R = \left(\frac{1}{\epsilon}\right) R'$. Since $c \in S_\infty$, using Theorem 4.2.10 gives $c\,\tilde{\circ}\,d_\delta \in S_\infty(R')$. Therefore, the theorem is proved as

$$c\,\tilde{\circ}\,d_\delta \in S_\infty(R')\ \forall R' > 0 \Longleftrightarrow c\,\tilde{\circ}\,d_\delta \in S_\infty.$$

■

The following theorem goes a step further than Theorem 4.2.11 stating that the mixed composition product is not just closed in S_∞ space but is indeed continuous on S_∞ in its left argument.

Theorem 4.2.12. *Given $d \in S_\infty^m$, define the linear operator λ_d on the Fréchet space S_∞ such that,*

$$\begin{aligned} \lambda_d : S_\infty &\longrightarrow S_\infty \\ c &\longmapsto c\,\tilde{\circ}\,d_\delta. \end{aligned}$$

Then, λ_d is continuous.

Proof: Define a map $\alpha : \mathbb{N} \longrightarrow \mathbb{N}$ such that $k \longmapsto \left\lceil \left(2 + m||d||_{\infty,k}\right) k \right\rceil$. By Theorem 4.1.2,

TABLE 3: Summary of convergence results for the mixed composition product.

c	d	$c\,\tilde{\circ}\,d_\delta$	Theorem
$\mathbb{R}\langle X\rangle$	$\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$	$\mathbb{R}_{GC}\langle\langle X\rangle\rangle$	Theorem 4.2.7
$S_\infty(R)$	$S^m_\infty(R)$	$S_\infty(\bar{R})$ where $\bar{R} = \left(\frac{R}{2+m\|\|d\|\|_{\infty,R}}\right)$.	Theorem 4.1.2
$S_\infty(R)$	S^m_∞	$S_\infty(\bar{R})$ where $\bar{R} = \left(\frac{R}{2+m\|\|d\|\|_{\infty,R}}\right)$.	Theorem 4.1.3
S_∞	$S^m_\infty(R)$	$S^m_\infty(R')$ where $R' = \epsilon R\;\forall\epsilon \in]0,1[$.	Theorem 4.2.10
S_∞	S^m_∞	S_∞	Theorem 4.2.11

$\forall k \in \mathbb{N}$

$$\begin{aligned} ||c\,\tilde{\circ}\,d_\delta||_{\infty,k} &\leq ||c||_{\infty,\left(2+m||d||_{\infty,k}\right)k} \\ &\leq ||c||_{\infty,\alpha(k)}. \end{aligned}$$

Therefore, applying Theorem 2.4.6, the linear operator λ_d is continuous on S_∞. ■

The results regarding the convergence of the mixed composition product are summarized in Table 3. Observe that from Theorem 2.4.2, Theorem 2.4.4 and Table 3, the following statements can be inferred. The mixed cascade configuration of two locally convergent Fliess operators F_c, F_d is represented by a locally convergent Fliess operator $F_{c\,\tilde{\circ}\,d_\delta}$. The mixed cascade configuration of two globally convergent Fliess operators F_c, F_d is represented by a globally convergent Fliess operator $F_{c\,\tilde{\circ}\,d_\delta}$.

4.2.3 Global Convergence of Wiener-Fliess Product

The goal of this section is to show that the Wiener-Fliess composition product preserves global convergence. That is, the Wiener-Fliess composition of a globally convergent commutative series d and a noncommutative series c in the Fréchet space S_∞, $d\,\hat{\circ}\,c$, lies in the Fréchet space. Recall from Theorem 3.1.1, the definition is two-fold. This section considers both these cases in detail. The following theorem is a particular case where the Wiener-Fliess composition yields a series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Theorem 4.2.13. *If $d \in \mathbb{R}[\tilde{X}]$ with $|\tilde{X}| = m$ and $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$, then $d \,\hat{\circ}\, c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$.*

Proof: Assume d is a polynomial of degree $N \in \mathbb{N}_0$. Since $d \in \mathbb{R}[\tilde{X}]$, there exist constants $K_d, M_d > 0$ such that $|(d,\eta)| \leq \dfrac{K_d M_d^{|\eta|}}{|\eta|!}$ $\forall \eta \in \cup_{i=0}^N \tilde{X}^i$. Observe

$$
\begin{aligned}
d \,\hat{\circ}\, c &= \sum_{\eta \in \mathrm{supp}(d)} (d,\eta) c^{\sqcup\!\sqcup\, \eta} \\
&\leq \sum_{\eta \in \mathrm{supp}(d)} \frac{K_d M_d^{|\eta|}}{|\eta|!} c^{\sqcup\!\sqcup\, \eta} \\
&= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\eta \in \tilde{X}^n} c^{\sqcup\!\sqcup\, \eta} \\
&= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} \tilde{x}_1^{i_1} \cdots \tilde{x}_k^{i_k} \,\hat{\circ}\, c \\
&= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} c_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k}.
\end{aligned}
$$

Since $\binom{n}{i_1 \cdots i_k} \geq 1$,

$$
\begin{aligned}
d \,\hat{\circ}\, c &\leq \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} \binom{n}{i_1 \cdots i_k} c_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k} \\
&= \sum_{n=0}^{N} \frac{K_d M_d^n \left(\sum_{j=1}^m c_j\right)^{\sqcup\!\sqcup\, n}}{n!} \\
&= K_d \sum_{n=0}^{N} \frac{\left[M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{n!} \\
&\leq K_d \sum_{n=0}^{N} \left[M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}.
\end{aligned}
$$

Since $c \in \mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$, then $c_j \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ $\forall j = 1,\ldots,m$. Hence, $\sum_{j=1}^m c_i \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ by Theorem 2.4.5. Using Corollary 4.2.1, $\left[M_d\left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n} \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ $\forall n \leq N$. Hence, by virtue of Theorem 2.4.5,

$$\sum_{n=0}^{N}\left[M_d\left(\sum_{j=1}^{m}c_j\right)\right]^{\sqcup\!\sqcup\, n}\in\mathbb{R}_{GC}\langle\langle X\rangle\rangle.$$

Therefore,

$$d\,\hat{\circ}\,c\leq K_d\sum_{n=0}^{N}\left[M_d\left(\sum_{j=1}^{m}c_j\right)\right]^{\sqcup\!\sqcup\, n}\in\mathbb{R}_{GC}\langle\langle X\rangle\rangle.$$

■

The following theorem addresses the convergence for the Wiener-Fliess composition of a locally convergent noncommutative proper series with a globally convergent commutative series d .

Theorem 4.2.14. *Let $d\in\mathbb{R}_{GC}[[\tilde{X}]]$ with $|\tilde{X}|=m$, growth constants $K_d, M_d>0$, and Gevrey order $(-1+\bar{s})$ with $\bar{s}\in[0,1[$ such that $|(d,\tilde{\eta})|\leq K_d M_d^{|\tilde{\eta}|}(|\tilde{\eta}|!)^{-1+\bar{s}}$ $\forall\tilde{\eta}\in\tilde{X}^*$. If c is a proper series such that $c\in S_\infty^m(R)\cap\mathbb{R}_p^m\langle\langle X\rangle\rangle$, then $d\,\hat{\circ}\,c\in S_\infty(R')$ $\forall R'=\epsilon R$, where $\epsilon\in]0,1[$.*

Proof: Observe

$$\begin{aligned}
d\,\hat{\circ}\,c&=\sum_{\eta\in\tilde{X}^*}(d,\eta)c^{\sqcup\!\sqcup\,\eta}\\
&\leq\sum_{\eta\in\tilde{X}^*}K_dM_d^{|\eta|}(|\eta|!)^{-1+\bar{s}}c^{\sqcup\!\sqcup\,\eta}\\
&=\sum_{n=0}^{\infty}K_dM_d^{n}(n!)^{-1+\bar{s}}\sum_{\eta\in\tilde{X}^n}c^{\sqcup\!\sqcup\,\eta}\\
&=\sum_{n=0}^{\infty}\frac{K_dM_d^n}{(n!)^{1-\bar{s}}}\sum_{\substack{i_1\cdots i_k\geq 0\\ i_1+\cdots+i_k=n}}\tilde{x}_1^{i_1}\cdots\tilde{x}_k^{i_k}\;\hat{\circ}\,c\\
&=\sum_{n=0}^{\infty}\frac{K_dM_d^n}{(n!)^{1-\bar{s}}}\sum_{\substack{i_1\cdots i_k\geq 0\\ i_1+\cdots+i_k=n}}c_1^{\sqcup\!\sqcup\, i_1}\sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2}\sqcup\!\sqcup\,\cdots\sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k}.
\end{aligned}$$

Since $\binom{n}{i_1 \cdots i_k} \geq 1$,

$$d \,\hat{\circ}\, c \leq \sum_{n=0}^{\infty} \frac{K_d M_d^n}{(n!)^{1-\bar{s}}} \sum_{\substack{i_1 \cdots i_k \geq 0 \\ i_1 + \cdots + i_k = n}} \binom{n}{i_1 \cdots i_k} c_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k}$$
$$= \sum_{n=0}^{\infty} \frac{K_d M_d^n \left(\sum_{j=1}^m c_j\right)^{\sqcup\!\sqcup\, n}}{(n!)^{1-\bar{s}}}$$
$$= K_d \sum_{n=0}^{\infty} \frac{\left[M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{(n!)^{1-\bar{s}}}.$$

Lemma 2.5.1 implies that

$$(\Gamma(n+1))^{(1-\bar{s})} = n!^{(1-\bar{s})} \geq \frac{1}{\tilde{K}_{1-\bar{s}}} 2^{-n} \Gamma((1-\bar{s})n+1),$$

where $\tilde{K}_{1-\bar{s}} = 2\left(\left(\frac{2\pi}{\exp(2)}\right)^{\bar{s}} 4\right)^{\frac{1}{2}}$. Hence,

$$d \,\hat{\circ}\, c \leq K_d \tilde{K}_{1-\bar{s}} \sum_{n=0}^{\infty} \frac{\left[2M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\bar{s})n+1\right)}.$$

Define $\tilde{K}_d \triangleq K_d \tilde{K}_{1-\bar{s}}$ and $\tilde{M}_d \triangleq 2M_d$. Therefore,

$$d \,\hat{\circ}\, c \leq \tilde{K}_d \sum_{n=0}^{\infty} \frac{\left[\tilde{M}_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\bar{s})n+1\right)},$$

and consequently,

$$||d \,\hat{\circ}\, c||_{\infty,R'} \leq \tilde{K}_d \left\| \sum_{n=0}^{\infty} \frac{\left[\tilde{M}_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{\Gamma\left((1-\bar{s})n+1\right)} \right\|_{\infty,R'}.$$

Applying the triangle inequality,

$$||d\,\hat{\circ}\,c||_{\infty,R'} \leq \tilde{K}_d \sum_{n=0}^{\infty} \frac{\tilde{M}_d^{\;n} \left|\left|\left(\sum_{j=1}^{m} c_j\right)^{\sqcup\!\sqcup\, n}\right|\right|_{\infty,R'}}{\Gamma\left((1-\bar{s})n+1\right)}.$$

In addition, $\left|\left|\left(\sum_{j=1}^{m} c_j\right)\right|\right|_{\infty,R} \leq \sum_{j=1}^{m} ||c_j||_{\infty,R} \leq m||c||_{\infty,R} < \infty$. Therefore, $\sum_{j=1}^{m} c_j \in S_\infty(R)$. Hence, $\forall\ R' = \epsilon R$, where $\epsilon \in]0,1[$, by virtue of Corollary 4.2.3,

$$\begin{aligned}
||d\,\hat{\circ}\,c||_{\infty,R'} &\leq \tilde{K}_d \sum_{n=0}^{\infty} \frac{\left(\frac{\tilde{M}_d}{1-\epsilon}\right)^n \left|\left|\left(\sum_{j=1}^{k} c_j\right)\right|\right|_{\infty,R}^n}{\Gamma\left((1-\bar{s})n+1\right)} \\
&\leq \tilde{K}_d \sum_{n=0}^{\infty} \frac{\left(\frac{\tilde{M}_d}{1-\epsilon}\right)^n \left(m||c||_{\infty,R}\right)^n}{\Gamma\left((1-\bar{s})n+1\right)} \\
&= \tilde{K}_d \sum_{n=0}^{\infty} \frac{\left[\frac{\tilde{M}_d}{1-\epsilon}\left(m||c||_{\infty,R}\right)\right]^n}{\Gamma\left((1-\bar{s})n+1\right)} \\
&= \tilde{K}_d\, \mathbb{E}_{(1-\bar{s}),1}\left(\frac{\tilde{M}_d m||c||_{\infty,R}}{1-\epsilon}\right).
\end{aligned}$$

Since $\bar{s} \in [0,1[$, the Mittag-Leffler function $\mathbb{E}_{(1-\bar{s}),1}(\cdot)$ is an entire function. Note that $c \in S_\infty^m(R)$ and $\epsilon \in]0,1[$ imply $\left(\frac{\tilde{M}_c m||c||_{\infty,R}}{1-\epsilon}\right) < \infty$. Therefore,

$$||d\,\hat{\circ}\,c||_{\infty,R'} \leq \tilde{K}_d\, \mathbb{E}_{(1-\bar{s}),1}\left(\frac{\tilde{M}_d m||c||_{\infty,R}}{1-\epsilon}\right) < \infty.$$

Hence, $d \in \mathbb{R}_{GC}\,[[\tilde{X}]]$ and $c \in S_\infty^m(R) \cap \mathbb{R}_p^m\,\langle\langle X\rangle\rangle$ imply that $d\,\hat{\circ}\,c\ \in S_\infty(R')\ \forall R' = \epsilon R$, where $\epsilon \in]0,1[$. ■

Theorem 4.2.14 establishes that the convergence of the Wiener-Fliess composition product $d\,\hat{\circ}\,c$ is limited by the convergence of the proper noncommutative series c, when the commutative series d is globally convergent. The following addresses the convergence of the Wiener-Fliess composition product when the series c lies in the Fréchet space.

Theorem 4.2.15. *If $d \in \mathbb{R}_{GC}\,[[\tilde{X}]]$ with $|\tilde{X}| = m$ and $c \in S_\infty^m \cap \mathbb{R}_p^m\,\langle\langle X\rangle\rangle$, then $d\,\hat{\circ}\,c \in S_\infty$.*

Proof: Recall

$$c \in S_\infty^m \cap \mathbb{R}_p^m \langle\langle X \rangle\rangle \Longleftrightarrow c \in S_\infty^m(R) \cap \mathbb{R}_p^m \langle\langle X \rangle\rangle \ \forall R > 0.$$

Fix $\epsilon \in]0,1[$ and $\forall R' > 0$ define $R = \left(\frac{1}{\epsilon}\right) R'$. Since $d \in \mathbb{R}_{GC}[[\tilde{X}]]$, using Theorem 4.2.14 gives $d \,\hat{\circ}\, c \in S_\infty(R')$. Therefore,

$$d \,\hat{\circ}\, c \in S_\infty(R') \ \forall R' > 0 \Longleftrightarrow d \,\hat{\circ}\, c \in S_\infty.$$

■

Theorem 4.2.15 proved that Wiener-Fliess composition preserves global convergence when the noncommutative series in the composition is proper. The next step is to revisit the question for the case of the Wiener-Fliess composition when the commutative series is restricted to being a polynomial.

Theorem 4.2.16. *If $d \in \mathbb{R}[\tilde{X}]$ and $c \in S_\infty^m(R)$, then $d \,\hat{\circ}\, c \in S_\infty(R')$ $\forall R' = \epsilon R$, where $\epsilon \in]0,1[$.*

Proof: Assume d is a polynomial of degree $N \in \mathbb{N}_0$. Since $d \in \mathbb{R}[\tilde{X}]$, there exist constants $K_d, M_d > 0$ such that $|(d,\eta)| \leq \dfrac{K_d M_d^{|\eta|}}{|\eta|!}$, $\forall \eta \in \cup_{i=0}^N \tilde{X}^i$. Observe

$$\begin{aligned} d \,\hat{\circ}\, c &= \sum_{\eta \in \mathrm{supp}(d)} (d,\eta) c^{\sqcup\!\sqcup\, \eta} \\ &\leq \sum_{\eta \in \mathrm{supp}(d)} \frac{K_d M_d^{|\eta|}}{|\eta|!} c^{\sqcup\!\sqcup\, \eta} \\ &= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\eta \in \tilde{X}^n} c^{\sqcup\!\sqcup\, \eta} \\ &= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k = n}} \tilde{x}_1^{i_1} \cdots \tilde{x}_k^{i_k} \,\hat{\circ}\, c \end{aligned}$$

$$= \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} c_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k}.$$

Since $\binom{n}{i_1 \cdots i_k} \geq 1$,

$$\begin{aligned} d \,\hat{\circ}\, c &\leq \sum_{n=0}^{N} \frac{K_d M_d^n}{n!} \sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \binom{n}{i_1 \cdots i_k} c_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup\, c_2^{\sqcup\!\sqcup\, i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_k^{\sqcup\!\sqcup\, i_k} \\ &= \sum_{n=0}^{N} \frac{K_d M_d^n \left(\sum_{j=1}^m c_j\right)^{\sqcup\!\sqcup\, n}}{n!} \\ &= K_d \sum_{n=0}^{N} \frac{\left[M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} ||d \,\hat{\circ}\, c||_{\infty,R'} &\leq K_d \left|\left| \sum_{n=0}^{N} \frac{\left[M_d \left(\sum_{j=1}^m c_j\right)\right]^{\sqcup\!\sqcup\, n}}{n!} \right|\right|_{\infty,R'} \\ &\leq K_d \sum_{n=0}^{N} \frac{{M_d}^n \left|\left| \left(\sum_{j=1}^m c_j\right)^{\sqcup\!\sqcup\, n} \right|\right|_{\infty,R'}}{n!}. \end{aligned}$$

In addition, $\left|\left|\left(\sum_{j=1}^m c_j\right)\right|\right|_{\infty,R} \leq \sum_{j=1}^m ||c_j||_{\infty,R} \leq m||c||_{\infty,R} < \infty$. Therefore, $\sum_{j=1}^m c_j \in S_\infty(R)$. Hence, $\forall\ R' = \epsilon R$, where $\epsilon \in]0,1[$, by virtue of Corollary 4.2.3,

$$\begin{aligned} ||d \,\hat{\circ}\, c||_{\infty,R'} &\leq K_d \sum_{n=0}^{N} \frac{\left(\frac{M_d}{1-\epsilon}\right)^n \left|\left|\left(\sum_{j=1}^k c_j\right)\right|\right|_{\infty,R}^n}{n!} \\ &\leq K_d \sum_{n=0}^{N} \frac{\left(\frac{M_d}{1-\epsilon}\right)^n \left(m||c||_{\infty,R}\right)^n}{n!} \\ &= K_d \sum_{n=0}^{N} \frac{\left[\frac{M_d}{1-\epsilon}\left(m||c||_{\infty,R}\right)\right]^n}{n!} \\ &\leq K_d \sum_{n=0}^{\infty} \frac{\left(\frac{M_d}{1-\epsilon}\right)^n \left|\left|\left(\sum_{j=1}^k c_j\right)\right|\right|_{\infty,R}^n}{n!} \end{aligned}$$

$$= K_d \exp\left(\frac{M_d m ||c||_{\infty,R}}{1-\epsilon}\right).$$

Observe that $c \in S_\infty^m(R)$ and $\epsilon \in]0,1[$ imply that $\left(\frac{\tilde{M}_c m ||c||_{\infty,R}}{1-\epsilon}\right) < \infty$. Therefore,

$$||d \,\hat{\circ}\, c||_{\infty,R'} \leq K_d \exp\left(\frac{M_d m ||c||_{\infty,R}}{1-\epsilon}\right) < \infty.$$

Hence, $d \in \mathbb{R}\,[\tilde{X}]$ and $c \in S_\infty^m(R)$ imply that $d \,\hat{\circ}\, c \ \in S_\infty(R')\ \forall R' = \epsilon R$, where $\epsilon \in]0,1[$. ■

Theorem 4.2.16 proves that the convergence of the Wiener-Fliess composition product $d \,\hat{\circ}\, c$ is limited by the convergence of the noncommutative series c, when d is restricted to being a polynomial. The result is analogous to the case of Wiener-Fliess composition when c is restricted to being proper as stated in Theorem 4.2.14. The following theorem asserts that Wiener-Fliess composition preserves global convergence when the commutative series d is restricted to being a polynomial.

Theorem 4.2.17. *Let $d \in \mathbb{R}\,[\tilde{X}]$ with $|\tilde{X}| = m$ and $c \in S_\infty^m$, then $d \,\hat{\circ}\, c \in S_\infty$.*

Proof: Recall

$$c \in S_\infty^m \Longleftrightarrow c \in S_\infty^m(R)\ \forall R > 0.$$

Fix $\epsilon \in]0,1[$ and $\forall R' > 0$ define $R = \left(\frac{1}{\epsilon}\right) R'$. Since $d \in \mathbb{R}\,[\tilde{X}]$, applying Theorem 4.2.16 gives $d \,\hat{\circ}\, c \in S_\infty(R')$. Therefore,

$$d \,\hat{\circ}\, c \in S_\infty(R')\ \forall R' > 0 \Longleftrightarrow d \,\hat{\circ}\, c \in S_\infty.$$

■

Using Theorem 2.4.2, Theorem 2.4.4 and Table 4, the following statements can be asserted. Under the assumptions stated in Theorem 3.1.1, the cascade connection of a locally

TABLE 4: Summary of convergence results for the Wiener-Fliess composition product.

d	c	$d \,\hat{\circ}\, c$	Theorem
$\mathbb{R}_{LC}[[\tilde{X}]]$	$\mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$	$\mathbb{R}_{LC}\langle\langle X\rangle\rangle$	Theorem 4.1.5
$\mathbb{R}_{GC}[[\tilde{X}]]$	$S^m_\infty(R) \cap \mathbb{R}^m_p\langle\langle X\rangle\rangle$	$S_\infty(R')$ where $R' = \epsilon R\ \forall \epsilon \in]0,1[$.	Theorem 4.2.14
$\mathbb{R}_{GC}[[\tilde{X}]]$	$S^m_\infty \cap \mathbb{R}^m_p\langle\langle X\rangle\rangle$	S_∞	Theorem 4.2.15
$\mathbb{R}[\tilde{X}]$	$\mathbb{R}^m_{GC}\langle\langle X\rangle\rangle$	$\mathbb{R}_{GC}\langle\langle X\rangle\rangle$	Theorem 4.2.13
$\mathbb{R}[\tilde{X}]$	$S^m_\infty(R)$	$S_\infty(R')$ where $R' = \epsilon R\ \forall \epsilon \in]0,1[$.	Theorem 4.2.16
$\mathbb{R}[\tilde{X}]$	S^m_∞	S_∞	Theorem 4.2.17

convergent Fliess operator F_c with a locally convergent real analytic function f_d is represented by a locally convergent Fliess operator $F_{d\,\hat{\circ}\,c}$. Similarly, the cascade connection of a globally convergent Fliess operator F_c with a globally convergent real analytic function f_d (as characterized in Theorem 2.5.2) has a globally convergent Fliess operator representation given by $F_{d\,\hat{\circ}\,c}$. This subsection and subsection 4.1.2 have shown that the Wiener-Fliess composition product preserves both local and global convergence. This section has characterized the convergence for the mixed composition product and Wiener-Fliess composition product, both of which are utilized in the computation of the Wiener-Fliess feedback as described in Theorem 3.3.2. Next, the convergence results obtained in this section is used to prove the local convergence of Wiener-Fliess feedback product.

4.3 LOCAL CONVERGENCE OF WIENER-FLIESS FEEDBACK

The objective of this section is to prove that additive static feedback preserves local convergence. It translates as, a locally convergent Fliess operator F_c in additive static feedback with a locally convergent analytic map f_d is represented by a locally convergent Fliess operator $F_{c\hat{@}d}$. The following result is essential to produce the desired result. It states that the antipode of the Hopf algebra corresponding to dynamic output feedback group preserves local convergence.

Theorem 4.3.1. *[Gray, et al. (2014a)] For any $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants*

$K_c, M_c > 0$ it follows that

$$|(c^{\circ -1}, \eta)| \leq K(\mathscr{A}(K_c)M_c)^{|\eta|}|\eta|! \quad \forall \eta \in X^*$$

for some $K > 0$ and

$$\mathscr{A}(K_c) = \frac{1}{1 - mK_c \ln(1 + \frac{1}{mK_c})}.$$

Theorem 4.3.1 is essential as the additive static feedback product involves the antipode operation from the dynamic output feedback group. The following states that the local convergence is preserved by the Wiener-Fliess feedback product.

Theorem 4.3.2. *Given a series $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}[[\tilde{X}]]$ with $|X| = m+1$ and $|\tilde{X}| = \ell$, if either of the following conditions hold:*

1. *The series c is proper,*

2. *The commutative series d is a polynomial,*

then $c\hat{@}d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$. Specifically, if c is proper, then $c\hat{@}d \in \mathbb{R}^{\ell}_{p,LC}\langle\langle X\rangle\rangle$.

Proof: Consider the case of c being a proper series. Clearly, $d \in \mathbb{R}^{m}_{LC}[[\tilde{X}]]$ if and only if $-d \in \mathbb{R}^{m}_{LC}[[\tilde{X}]]$. Since $c \in \mathbb{R}^{\ell}_{p,LC}\langle\langle X\rangle\rangle$, then by Theorem 4.1.5

$$(-d \,\hat{\circ}\, c) \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle.$$

Hence, applying Theorem 4.3.1 yields

$$(-d \,\hat{\circ}\, c)^{\circ -1} \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle \Leftrightarrow \delta + (-d \,\hat{\circ}\, c)^{\circ -1} = (-d \,\hat{\circ}\, c)^{\circ -1}_{\delta} \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle.$$

Therefore, by Theorem 4.1.1

$$c\hat{@}d = c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)^{\circ -1}_{\delta} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle.$$

Since c is proper, then by definition of mixed composition product,

$$c\hat{@}d = c\,\tilde{\circ}\,(-d\,\hat{\circ}\,c)_{\delta}^{\circ-1} \in \mathbb{R}^{\ell}_{p,LC}\langle\langle X\rangle\rangle.$$

Now consider the case of $d \in \mathbb{R}^m[\tilde{X}]$. Since $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$, then by Theorem 4.2.16, $(-d\,\hat{\circ}\,c) \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$. The rest of the proof is exactly analogous to the previous case.

■

The following example demonstrates that Wiener-Fliess feedback product does not preserve global convergence.

Example 4.3.1. [Thitsa & Gray (2012)] Let $X = \{x_0, x_1\}$. Define $c \in \mathbb{R}\langle\langle X\rangle\rangle$ as $c = x_1^* = \sum_{k=0}^{\infty} x_1^k$. Observe that $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. The Fliess operator F_c describes the input-output behavior of the state space model

$$\dot{z} = zu, \quad z(0) = 1,$$
$$y = z,$$

where $z(t), u(t) \in \mathbb{R}$. Define $d \in \mathbb{R}[[w]]$ as the monomial $d = w$. Note that the monomial $d \in \mathbb{R}_{GC}[[w]]$. The Fliess operator $F_{c\hat{@}d}$ describes the closed-loop system of F_c under unity feedback. The zero-input dynamics of the closed-loop system are given by the solution of the following differential equation

$$\dot{z} = z^2, \quad z(0) = 1.$$

Specifically, $z(t) = \left(\dfrac{1}{1-t}\right) = 1 + t + t^2 + \cdots$ for $t < 1$. Recall that $E_{x_0^n}[u] = \dfrac{t^n}{n!}$. The zero-input response corresponds to the $F_{\left(c\hat{@}d\right)_N}$, where $\left(c\hat{@}d\right)_N$ is the natural part of the

Wiener-Fliess feedback product given by

$$\left(c\hat{@}d\right)_N = \sum_{k=0}^{\infty} k!x_0^k.$$

Observe that the subseries $\left(c\hat{@}d\right)_N$ is only locally convergent. Hence, the Wiener-Fliess feedback product of $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}_{GC}[[w]]$, $c\hat{@}d$, is only locally convergent. □

Finally, under the assumptions stated in Theorem 4.3.2 and applying Theorem 2.4.2 the following statement can be asserted. The additive static feedback connection of a locally convergent Fliess operator F_c with a locally convergent real analytic function f_d is represented by a locally convergent Fliess operator $F_{c\hat{@}d}$.

CHAPTER 5

SHUFFLE RATIONAL FORMAL SERIES

The dissertation shifts the focus onto the second of the two problems being considered. *Rational series* in the literature are those in the rational closure of polynomials with respect to the Cauchy product [Berstel & Reutenauer (1988)]. This chapter describes the abstract notion of *rationality* and considers the rational closure of polynomials with respect to the shuffle product and its equivalent characterizations. Prior to that, the evaluation of static rational function of formal series is addressed.

5.1 RATIONAL FUNCTIONS OF FORMAL SERIES

This section describes the evaluation of commutative polynomial maps and rational functions over noncommutative formal power series. First observe that the set of non-proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ constitutes a group under the shuffle product [Gray, et al. (2014b)]. The shuffle inverse in this case is taken to be

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1-c'))^{\sqcup\!\sqcup -1} = (c,\emptyset)^{-1}(c')^{\sqcup\!\sqcup *}, \tag{5.1.1}$$

where $c' \triangleq 1 - c/(c,\emptyset)$ is proper, and $(c')^{\sqcup\!\sqcup *} \triangleq \sum_{k\in\mathbb{N}_0}(c')^{\sqcup\!\sqcup k}$. Here $(c')^{\sqcup\!\sqcup k} \triangleq c' \sqcup\!\sqcup (c')^{\sqcup\!\sqcup k-1}$ with $(c')^{\sqcup\!\sqcup 0} = 1$.

Example 5.1.1. Let $c = 1 - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$ so that $c' = x_1$. Then $c^{\sqcup\!\sqcup -1} = x_1^{\sqcup\!\sqcup *} = \sum_{k\in\mathbb{N}_0} k!\, x_1^k$. □

Since $\mathbb{R}\langle\langle X\rangle\rangle$ under the shuffle product is a commutative and associative $\mathbb{R}$-algebra, for any $k \in \mathbb{N}$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$ one can write $c^{\sqcup\!\sqcup k} = y^k(c)$, where $y^k \in \mathbb{R}[y]$. Similarly, if d is a proper series, then

$$(1-d)^{\sqcup\!\sqcup -1} = d^{\sqcup\!\sqcup *}$$

$$= 1 + d + d^{\sqcup\!\sqcup\, 2} + d^{\sqcup\!\sqcup\, 3} + \cdots$$
$$= (1 + y + y^2 + y^3 + \cdots)(d)$$
$$= \left(\frac{1}{1-y}\right)(d).$$

Therefore, the shuffle inverse of a series can be written as a rational function of the proper part of the series. The notions of a polynomial map and a rational function of a noncommutative formal power series are formalized by the following definition.

Definition 5.1.1. Let $p, q \in \mathbb{R}[y]$ and $c \in \mathbb{R}\langle\langle X \rangle\rangle$. Assume $p(y) = \sum_{i=0}^{k} a_i y^i$, where $k \in \mathbb{N}_0$, and $q((c, \emptyset)) \neq 0$. The composition of p and c is defined as

$$p(c) = \sum_{i=0}^{k} a_i c^{\sqcup\!\sqcup\, i}.$$

Extending the definition to rational functions gives

$$\frac{p}{q}(c) = p(c) \sqcup\!\sqcup\, q(c)^{\sqcup\!\sqcup\, -1}.$$

These definitions can be generalized to functions on k-tuples of series by first observing that $\mathbb{R}\langle\langle X \rangle\rangle$ is a commutative ring with the shuffle product. Therefore, $\mathbb{R}\langle\langle X \rangle\rangle$ is an $\mathbb{R}\langle\langle X \rangle\rangle$-module, where scalar multiplication of the ring with a module element is also defined by the shuffle product. In other words, $\mathbb{R}\langle\langle X \rangle\rangle \bigotimes_{\mathbb{R}\langle\langle X \rangle\rangle} \mathbb{R}\langle\langle X \rangle\rangle$ is isomorphic to $\mathbb{R}\langle\langle X \rangle\rangle$ in the category of commutative $\mathbb{R}\langle\langle X \rangle\rangle$-modules. Let A, B be $\mathbb{R}$-modules and denote the set of all $\mathbb{R}$-linear morphisms by $\mathrm{Hom}(A, B)$. Recall that $\mathrm{Hom}(A, B)$ forms an $\mathbb{R}$-module by itself. Let $\Gamma \in \mathbb{R}$-module $\mathrm{Hom}\Big(A, \mathrm{Hom}\big(B, C\big)\Big)$, where the modules $A \triangleq \bigoplus_{k\in\mathbb{N}} \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]$, $B \triangleq \bigoplus_{k\in\mathbb{N}} \prod_{i=1}^{k} \mathbb{R}\langle\langle X \rangle\rangle$, and $C \triangleq \bigoplus_{k\in\mathbb{N}} \bigotimes_{\mathbb{R}\langle\langle X \rangle\rangle}{}_{i=1}^{k} \mathbb{R}\langle\langle X \rangle\rangle$. The morphism Γ is defined as

$$\Gamma\left(\bigotimes_{\mathbb{R}}{}_{i=1}^{k} p_i\right)(c_1, c_2, \ldots, c_k) = \bigotimes_{\mathbb{R}\langle\langle X \rangle\rangle}{}_{i=1}^{k} p_i(c_i),$$

where $p_i \in \mathbb{R}[y_i]$ and $c_i \in \mathbb{R}\langle\langle X\rangle\rangle$, $i = 1, 2, \ldots k$. The right-hand side is expanded using the shuffle product as

$$\bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle\, i=1}^{k} p_i(c_i) = p_1(c_1) ⧢ p_2(c_2) ⧢ \cdots ⧢ p_k(c_k).$$

The image of Γ, denoted by $\mathrm{Im}(\Gamma)$, is an $\mathbb{R}$-module. In fact, it possesses an $\mathbb{R}$-algebra structure as $\bigotimes_{\mathbb{R}\langle\langle X\rangle\rangle\, i=1}^{k} \mathbb{R}\langle\langle X\rangle\rangle$ is an $\mathbb{R}$-algebra. If $p, p' \in \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$, then

$$(\Gamma(p)\Gamma(p'))(c_1, \ldots, c_k) = (\Gamma(p))(c_1, \ldots, c_k) \; ⧢ \; (\Gamma(p'))(c_1, \ldots, c_k).$$

As the shuffle product has no zero divisors, it is simple to check that the underlying ring structure on $\mathbb{R}$-algebra $\mathrm{Im}(\Gamma)$ is an integral domain. Hence, the quotient field $\mathrm{Im}(\Gamma)$ is the set of rational functions

$$\frac{\Gamma(p)}{\Gamma(p')}(c_1, \ldots, c_k) = \Gamma(p)(c_1, \ldots, c_k) \; ⧢ \; (\Gamma(p')(c_1, \ldots, c_k))^{⧢\, -1},$$

where $p, p' \in \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$. The symbol Γ is henceforth suppressed for brevity so that given $p \otimes p' \in \mathbb{R}[y] \otimes \mathbb{R}[y']$,

$$p \otimes p'(c, c') = \big(\Gamma(p \otimes p')\big)(c, c') = p(c) ⧢ p'(c').$$

Likewise, if $p, q \in \bigotimes_{\mathbb{R}\, i=1}^{k} \mathbb{R}[y_i]$, then

$$\frac{p}{q}(c_1, \ldots, c_k) = \frac{\Gamma(p)}{\Gamma(q)}(c_1, \ldots, c_k).$$

The evaluation of rational functions over a noncommutative formal power series requires one to compute shuffle powers and the shuffle inverse of series. Such computations can be implemented algorithmically with the aid of the Hopf algebra corresponding to the shuffle group as described in Chapter 3 and summarized below.

Consider the set

$$M \triangleq \{1 + c : c \in \mathbb{R}\langle\langle X\rangle\rangle,\ (c, \emptyset) = 0\} \subset \mathbb{R}\langle\langle X\rangle\rangle.$$

M is an Abelian group under the shuffle product with 1 as the identity element. The shuffle inverse is defined as in (5.1.1). The set of coordinate maps on $\mathbb{R}\langle\langle X\rangle\rangle$ is taken to be

$$H = \{a_\eta : M \longrightarrow \mathbb{R},\ \eta \in X^*\},$$

where $a_\eta(c) = (c, \eta)$. H constitutes a commutative $\mathbb{R}$-algebra with addition, scalar multiplication and product defined, respectively, as

$$\begin{aligned} (a_\eta + a_\zeta)(c) &= a_\eta(c) + a_\zeta(c) \\ (k a_\eta)(c) &= k(a_\eta(c)) \\ \boldsymbol{m}(a_\eta, a_\zeta)(c) &= a_\eta(c) a_\zeta(c), \end{aligned}$$

where $\eta, \zeta \in X^*, k \in \mathbb{R}$. The unit for the product is $\mathbf{1} \sim a_\emptyset$ so that $\mathbf{1}(c) = 1, \forall c \in M$. Define the coproduct $\Delta : H \longrightarrow H \bigotimes H$ as $\Delta a_\eta(c, d) = a_\eta(c \sqcup\!\sqcup d)$, where $c, d \in M$ and $\eta \in X^*$. It can be computed inductively as

$$\begin{aligned} \Delta \mathbf{1} &= \mathbf{1} \otimes \mathbf{1} \\ \Delta \circ \theta_i &= (\theta_i \otimes \mathbf{1} + \mathbf{1} \otimes \theta_i) \circ \Delta, \end{aligned}$$

where θ_i denotes the vector space endomorphism on H specified by $\theta_i a_\eta = a_{x_i \eta}$, $i = 0, 1, \ldots, m$. The counit map ϵ is defined as

$$\epsilon(a_\eta) = \begin{cases} k : & a_\eta = k a_\emptyset \\ 0 : & \text{otherwise.} \end{cases}$$

It is simple to check that $(H, \boldsymbol{m}, \mathbf{1}, \Delta, \epsilon)$ forms a commutative and cocommutative bialgebra structure. The bialgebra is graded based on word length. Hence, $H = \bigoplus_{k\in\mathbb{N}_0} H_k$ with $a_\eta \in H_k$ if and only if $|\eta| = k$. Since $\mathbb{R} \cong H_0$ in the category of algebras with ϵ acting as the isomorphism, H is a connected and graded bialgebra, and thus a Hopf algebra [Figueroa & Gracia-Bondía (2005)]. The reduced coproduct Δ' is defined as

$$\Delta'(a_\eta) = \begin{cases} \Delta(a_\eta) - a_\eta \otimes \mathbf{1} - \mathbf{1} \otimes a_\eta \ : a_\eta \neq a_\emptyset \\ 0 \ : a_\eta = a_\emptyset. \end{cases}$$

Using Sweedler's notation, the coproduct can be written as

$$\Delta(a_\eta) = \sum a_{(\eta_1)} \otimes a_{(\eta_2)},$$

where the sum is over all words η_1, η_2 such that $\eta_1 \sqcup\!\sqcup\, \eta_2 = \eta$ [Sweedler (1969)]. The antipode map $S : H \longrightarrow H$ is given by $S(a_\eta)(c) = a_\eta(c^{\sqcup\!\sqcup\, -1})$. It can be computed inductively for any $a \in H^+$ (where $H^+ \triangleq \bigoplus_{k\geq 1} H_k$) by

$$S(a_\eta) = -a_\eta - \sum a'_{(\eta_1)} S(a'_{(\eta_2)}),$$

where the summation is taken over all the components of the reduced coproduct $\Delta'(a_\eta)$.

The coproduct Δ is useful for computing shuffle powers of formal power series. For example, if $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is non-proper, then $c = (c, \emptyset)c'$, where $c' \in M$. For any $\eta \in X^*$, it follows that

$$\begin{aligned}(c^{\sqcup\!\sqcup\, 2}, \eta) &= (c, \emptyset)^2 (c'^{\sqcup\!\sqcup\, 2}, \eta) \\ &= (c, \emptyset)^2 a_\eta(c'^{\sqcup\!\sqcup\, 2}) \\ &= (c, \emptyset)^2 \Delta a_\eta(c', c').\end{aligned}$$

In the case where c is proper, one can use the corresponding group element $(1 + c)$ and compute the reduced coproduct since $(c^{\sqcup\!\sqcup\, 2}, \eta) = \Delta' a_\eta((1 + c), (1 + c))$. The shuffle inverse of a non-proper series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be computed directly using the antipode S as

$$\begin{aligned}(c^{\sqcup\!\sqcup\, -1}, \eta) &= (c, \emptyset)^{-1}(c'^{\,\sqcup\!\sqcup\, -1}, \eta) \\ &= (c, \emptyset)^{-1} S(a_\eta)(c').\end{aligned}$$

The coproduct can be linearly extended to computing the polynomial map of arbitrary formal power series. Let $\Delta^{\circ k}$ denote the composition of the coproduct Δ with itself k times where $k \geq 1$. If $c \in M$ and $\eta \in X^*$, for brevity $\Delta^{\circ k} a_\eta(c, c, \ldots, c)$ with the argument c repeated $(k+1)$ times is written as $\Delta^{\circ k} a_\eta(c)$. Suppose $p \in \mathbb{R}[x]$ is written as $p(x) = \sum_{i=0}^{m} a_i x^i$. Then for $c \in \mathbb{R}\langle\langle X \rangle\rangle$ observe

$$\begin{aligned}(p(c), \eta) &= a_\eta(p(c)) \\ &= \left\{a_0 \epsilon + a_1 + a_2 \Delta + \cdots + a_{m-1} \Delta^{\circ(m-2)} + a_m \Delta^{\circ(m-1)}\right\}(a_\eta)(c).\end{aligned}$$

Now assume $p(x), q(x) \in \mathbb{R}[x]$ such that

$$\frac{p}{q}(x) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{j=0}^{n} b_j x^j},$$

and $q(1) = 1$ without loss of generality. The computation of $(p/q)(c)$ is done as follows

$$\left(\frac{p}{q}(c), \eta\right) = \big(P(\epsilon, \Delta) \otimes Q(\epsilon, \Delta, S)\big) \circ \Delta a_\eta(c),$$

where

$$P(\epsilon, \Delta) = a_0 \epsilon + a_1 + a_2 \Delta + \cdots + a_{m-1} \Delta^{\circ(m-2)} + a_m \Delta^{\circ(m-1)}. \tag{5.1.2a}$$

$$Q(\epsilon, \Delta, S) = b_0(\epsilon \circ S) + b_1 S + b_2(\Delta \circ S) + \cdots + b_{n-1}(\Delta^{\circ(n-2)} \circ S) + b_n(\Delta^{\circ(n-1)} \circ S). \tag{5.1.2b}$$

Here $P(\epsilon, \Delta)$ and $Q(\epsilon, \Delta, S)$ are the operator polynomials corresponding to the rational function p/q. This computation is abbreviated as

$$\left(\frac{p}{q}(c), \eta\right) = \Upsilon(\epsilon, \Delta, S)(a_\eta)(c),$$

where $\Upsilon(\epsilon, \Delta, S) \triangleq \big(P(\epsilon, \Delta) \bigotimes Q(\epsilon, \Delta, S)\big) \circ \Delta$. The operator Υ is viewed as the computational block for the rational function p/q. The computation of a rational function of a series in M is naturally extended when $p, q \in \bigotimes_{\mathbb{R}i=1}^{k} \mathbb{R}[y_i]$. Let $p = p_1 \otimes p_2 \otimes \cdots \otimes p_k$ and $q = q_1 \otimes q_2 \otimes \cdots \otimes q_k$, where $p_i, q_i \in \mathbb{R}[y_i]$. Let $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_k$ be the corresponding computational blocks. Therefore,

$$\left(\frac{p}{q}(c_1, \ldots, c_k), \eta\right) = \left(\bigotimes_{i=1}^{k} \Upsilon_i\right) \circ \Delta^{\circ(k-1)} a_\eta(c_1, \ldots, c_k), \tag{5.1.3}$$

where $c_1, c_2, \ldots, c_k \in M$.

The computational framework above can be further extended to rational functions of arbitrary non-proper formal power series. Let $c \in \mathbb{R}\langle\langle X \rangle\rangle$ be non-proper with $(c, \emptyset) = \alpha \neq 0$ and $c = \alpha c'$. Fix $p(x), q(x) \in \mathbb{R}[x]$ such that

$$\frac{p}{q}(x) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{j=0}^{n} b_j x^j},$$

and $q(\alpha) = 1$ without loss of generality. The computation of $(p/q)(c)$ is done as

$$\left(\frac{p}{q}(c), \eta\right) = \left(\big(\alpha P(\alpha^{-1}\epsilon, (\alpha\Delta))\big) \otimes \big(\alpha Q(\alpha^{-1}\epsilon, (\alpha\Delta), S)\big) \circ \Delta\right)(a_\eta)(c'),$$

where the operator polynomials P and Q are defined as in (5.1.2). The computation is abbreviated as the computational block $\overline{\Upsilon}$ so that

$$\left(\frac{p}{q}(c), \eta\right) = \overline{\Upsilon}(\epsilon, \Delta, S)(a_\eta)(c'),$$

where $\overline{\Upsilon}(\epsilon, \Delta, S) \triangleq \left(\alpha P(\alpha^{-1}\epsilon, (\alpha\Delta)) \bigotimes \alpha Q(\alpha^{-1}\epsilon, (\alpha\Delta), S)\right) \circ \Delta$. The computation of a rational function of a non-proper series is naturally extended when $p, q \in \bigotimes_{\mathbb{R}i=1}^{k} \mathbb{R}[y_i]$. Let $p = p_1 \otimes p_2 \otimes \cdots \otimes p_k$ and $q = q_1 \otimes q_2 \otimes \cdots \otimes q_k$, where $p_i, q_i \in \mathbb{R}[y_i]$. Let $\overline{\Upsilon}_1, \overline{\Upsilon}_2, \ldots, \overline{\Upsilon}_k$ be the corresponding computational blocks and $c_1, c_2, \ldots, c_k$ be all non-proper series such that $(c_i, \emptyset) = \alpha_i \neq 0$, $i = 1, 2, \ldots, k$. Let $c_i' \in M$ be the corresponding group element of c_i defined as $c_i = \alpha_i c_i'$. In which case,

$$\left(\frac{p}{q}(c_1, \ldots, c_k), \eta\right) = \left(\bigotimes_{i=1}^{k} \overline{\Upsilon}_i\right) \circ \Delta^{\circ(k-1)} a_\eta(c_1', \ldots, c_k').$$

The framework for the computation of a rational function of a proper series c such that $q(0) = 1$ is extended from (5.1.2) similarly except that the coproduct Δ in the operator polynomials is replaced with the reduced coproduct Δ'. The extension to computation of rational function of ordered collection of proper series $c_1, c_2, \ldots, c_k$ is immediate with respect to (5.1.3) except that the coproduct is replaced with the reduced coproduct. The case where $c_1, c_2, \ldots, c_k$ is a *mixture* of proper and non-proper series is difficult and does not fit well in the current scheme.

5.2 RATIONAL SERIES

This section describes the concept of rationally closed subalgebras and, hence, the notion of rational series in the broadest sense. First, the classical example is briefly reviewed followed by the notion of a shuffle-rational series.

Definition 5.2.1. [Fliess (1974)] An $\mathbb{R}$-subalgebra $\mathscr{F}$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ is said to be *rationally closed* if and only if the inverse of all invertible elements of $\mathscr{F}$ belongs to $\mathscr{F}$. The *rational closure* of an $\mathbb{R}$-subalgebra $\mathscr{F}'$ of an $\mathbb{R}$-algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ is the smallest rationally closed subalgebra $\mathscr{F}$ containing $\mathscr{F}'$.

Classically, *rational series* are defined to be those in the rational closure of the $\mathbb{R}$-subalgebra of polynomials $\mathbb{R}\langle X\rangle$, where the $\mathbb{R}$-algebra structure on $\mathbb{R}\langle\langle X\rangle\rangle$ is under the

Cauchy product [Berstel & Reutenauer (1988)]. This noncommutative algebra of rational series is denoted by $\mathbb{R}\langle(X)\rangle$. Since $\mathbb{R}\langle\langle X\rangle\rangle$ also forms a commutative $\mathbb{R}$-algebra under the shuffle product, a corresponding notion of rationality is possible as described next.

Definition 5.2.2. The rational closure of the $\mathbb{R}$-subalgebra $\mathbb{R}\langle X\rangle$ of the shuffle algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ is called the algebra of *shuffle-rational series* and denoted by $\mathbb{R}^{\sqcup\sqcup}\langle(X)\rangle$.

In other words, $\mathbb{R}^{\sqcup\sqcup}\langle(X)\rangle$ is the smallest rationally closed subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$ under the shuffle product that contains $\mathbb{R}\langle X\rangle$. The next example establishes that $\mathbb{R}\langle(X)\rangle \not\subset \mathbb{R}^{\sqcup\sqcup}\langle(X)\rangle$, while the subsequent example shows that $\mathbb{R}^{\sqcup\sqcup}\langle(X)\rangle \not\subset \mathbb{R}\langle(X)\rangle$.

Example 5.2.1. Let c be the rational series

$$\begin{aligned} c &= (1-x_1)^{-1} \\ &= 1 + x_1 + x_1^2 + x_1^3 + \cdots \\ &= 1 + \frac{x_1}{1!} + \frac{x_1^{\sqcup\sqcup 2}}{2!} + \frac{x_1^{\sqcup\sqcup 3}}{3!} + \cdots \\ &\triangleq \exp(x_1^{\sqcup\sqcup}). \end{aligned}$$

Observe that c cannot be represented by a finite number of shuffle products as the exponential map is an entire function and cannot be represented by a finite number of terms or as a rational function. Hence, it is not shuffle-rational. □

Example 5.2.2. Let c be the shuffle-rational series

$$\begin{aligned} c &= (1-x_1)^{\sqcup\sqcup -1} \\ &= 1 + x_1 + x_1^{\sqcup\sqcup 2} + x_1^{\sqcup\sqcup 3} + \cdots \\ &= 1 + x_1 + 2!x_1^2 + 3!x_1^3 + \cdots \end{aligned}$$

Clearly, $c \notin \mathbb{R}\langle(X)\rangle$ since all rational series have Gevrey order $s = 0$ [Berstel & Reutenauer (1988)]. □

In the case of a single indeterminate $X = \{x\}$, it is simple to verify that

$$\mathbb{R}\langle(X)\rangle = \left\{\frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{R}[X]\right\}$$
$$\mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle = \left\{p(x) \sqcup\!\sqcup\, q(x)^{\sqcup\!\sqcup\, -1} : p(x), q(x) \in \mathbb{R}[X]\right\},$$

where $q(x) \neq 0$ in either case.

5.3 RECOGNIZABLE SERIES

In this section, the classical definition of a recognizable series is first introduced. Schützenberger showed in [Schützenberger (1961)] that a series is rational under the Cauchy product if and only if it is recognizable. Next, the shuffle analogue of Schützenberger's theorem is stated and proved.

Definition 5.3.1. [Berstel & Reutenauer (1988)] A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is said to be *recognizable* if $\exists N \in \mathbb{N}$, a monoid morphism $\mu : X^* \longrightarrow \mathbb{R}^{N\times N}$, and vectors $\lambda, \gamma \in \mathbb{R}^N$ such that $(c, w) = \lambda^T \mu(w)\gamma$, $\forall w \in X^*$. Note that $\mathbb{R}^{N\times N}$ is considered to be a multiplicative monoid. The tuple (λ, μ, γ) is called a *representation* of c with dimension N. The set of all recognizable series is denoted by $\mathbb{R}^{rec}\langle\langle X\rangle\rangle$.

The following lemma will be useful in the work that follows.

Lemma 5.3.1. *A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is a polynomial if and only if it has a representation (λ, μ, γ) with $\mu(x_i)$ being a strictly upper triangular matrix $\forall x_i \in X$.*

Proof: If c has a representation with $\mu(x_i)$ strictly upper triangular $\forall x_i \in X$, then $\exists k \in \mathbb{N}$ such that $\mu(w) = 0$ when $|w| \geq k$, as strictly upper triangular matrices are always nilpotent. Hence, c is a polynomial. Conversely, if c is a polynomial, then the underlying vector fields of any realization of F_c form a nilpotent distribution [Kawski (1992)]. Since $\mathbb{R}\langle X\rangle \subset \mathbb{R}\langle(X)\rangle$, and the underlying vector fields associated with any generating series in $\mathbb{R}\langle(X)\rangle$ comes from the Lie algebra $\mathfrak{gl}(\mathbb{R}^N)$, the fact that the subalgebra of strictly upper triangular matrices is a nilpotent Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathbb{R}^N)$ completes the proof [Humphreys (1973)].

■

Example 5.3.1. It is easily checked that $c = x_0 x_1$ has the representation

$$\mu(x_0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mu(x_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$\lambda = e_1 = [1\ 0\ 0]^T$, and $\gamma = e_3 = [0\ 0\ 1]^T$. Note that $\mu(x_0)$ and $\mu(x_1)$ are strictly upper triangular matrices and nilpotent of index 2. □

The notion of shuffle recognizability is presented next.

Definition 5.3.2. Let $k \in \mathbb{N}$ and $\{N_1, N_2, \ldots N_k\}$ be a multiset of k positive integers. Let $\{\lambda_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k$ be ordered collections of k vectors such that $\lambda_i, \gamma_i \in \mathbb{R}^{N_i}$. Assume $\{\mu_i\}_{i=1}^k$ is an ordered collection of k monoid morphisms $\mu_i : X^* \longrightarrow \mathbb{R}^{N_i \times N_i}$ such that $\mu_i(x_j)$ is a strictly upper triangular matrix $\forall x_j \in X$, $i = 1, \ldots, k$. Define two polynomials $p, q \in \bigotimes_{\mathbb{R}\, i=1}^{\ k} \mathbb{R}[y_i]$ such that $q(\lambda_i^T \gamma_i) \neq 0$, $i = 1, \ldots, k$. A series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ is said to be shuffle-recognizable if $c = p/q\left(\sum_{w \in X^*} \lambda^T \mu(w) \gamma\, w \right)$, where $\lambda^T = (\lambda_1^T \times \lambda_2^T \times \cdots \times \lambda_k^T)$, $\mu = (\mu_1 \times \mu_2 \times \cdots \times \mu_k)$, and $\gamma = (\gamma_1 \times \gamma_2 \times \cdots \times \gamma_k)$. The tuple $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ is called a k order *shuffle-representation* of c. The set of all such shuffle-recognizable series is denoted by $\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X \rangle\rangle$.

Let $c \in \mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X \rangle\rangle$ with a shuffle-representation $(p, q, \{\lambda\}_{i=1}^k, \{\mu\}_{i=1}^k, \{\gamma\}_{i=1}^k)$. The computation of (c, η), $\eta \in X^*$ can be made algorithmic using the Hopf algebra corresponding to the shuffle group. By Lemma 5.3.1, observe that the expression $\sum_{w \in X^*} \lambda^T \mu(w) \gamma$ is a Cartesian product of k polynomials, say $d_1, d_2, \cdots d_k$. Hence, for all $\eta \in X^*$

$$(c, \eta) = \left(\frac{p}{q}(d_1, d_2, \ldots, d_k), \eta \right),$$

which can be computed directly using (5.1.3). In addition,

$$\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle = \Big\{ p(c_1, c_2, \ldots, c_k) \sqcup\!\sqcup\, q(c_1, c_2, \ldots, c_k)^{\sqcup\!\sqcup\, -1} \,:\, p, q \in \bigotimes_{\mathbb{R}\ i=1}^{k} \mathbb{R}[y_i] \Big\}, \tag{5.3.1}$$

where $c_1, \ldots, c_k \in \mathbb{R}\langle X\rangle$, $(q(c_1, \ldots, c_k), \emptyset) \neq 0$.

Example 5.3.2. Suppose

$$\begin{aligned} c &= 1 + x_1 + x_1^{\sqcup\!\sqcup\, 2} + \cdots + x_1^{\sqcup\!\sqcup\, k} + \cdots \\ &= (1 - x_1)^{\sqcup\!\sqcup\, -1} \\ &= (\frac{1}{1-y})(x_1). \end{aligned}$$

Note that $\mu : X^* \longrightarrow \mathbb{R}^{2\times 2}$, where $\mu(x_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\gamma = e_2$, and $\lambda = e_1$ give a representation of x_1, that is, $x_1 = \sum_{w\in X^*} \lambda^T \mu(w)\gamma\, w$. Hence, $c = x_1^{\sqcup\!\sqcup\, *}$ is a shuffle-recognizable series with shuffle-representation $(1, 1-y, \{e_1\}, \{\mu\}, \{e_2\})$. $\square$

Equation (5.3.1) states that the set of shuffle-recognizable series are generated by finite shuffle products of polynomials and their shuffle inverses. Hence, $\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle \subseteq \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ in the category of sets. This leads to the central question of whether $\mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle = \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$. The following theorem states that this is the case.

Theorem 5.3.1. *A series is shuffle-rational if and only if it is shuffle-recognizable.*

Proof: It is sufficient to prove that $\mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle \subseteq \mathbb{R}^{\sqcup\!\sqcup\, rec}\langle\langle X\rangle\rangle$. First it is shown that all polynomials and the shuffle inverses of shuffle-invertible polynomials are shuffle-recognizable. As every shuffle-rational series is in the rational closure of the polynomials, it only remains to be shown that shuffle-recognizability is preserved under the remaining shuffle-rational operations: scalar multiplication, addition, and the shuffle product.

Let $c \in \mathbb{R}\langle X\rangle$. From Lemma 5.3.1 there exists a nilpotent representation (λ, μ, γ) such that

$$c = \sum_{w \in X^*} \lambda^T \mu(w) \gamma \, w \quad = \frac{y}{1} \left(\sum_{w \in X^*} \lambda^T \mu(w) \gamma \, w \right).$$

Therefore, trivially, c is shuffle-recognizable with tuple $(y, 1, \{\lambda\}, \{\mu\}, \{\gamma\})$. It is equally clear that if c is non-proper, and $c' = 1 - (c/(c, \emptyset))$ has a representation $(\lambda', \mu', \gamma')$, then $d = c^{\sqcup\!\sqcup\, -1}$ is $\sqcup\!\sqcup$-recognizable with tuple $((c, \emptyset)^{-1}, 1 - y, \{\lambda'\}, \{\mu'\}, \{\gamma'\})$.

Shuffle-recognizability is also preserved by scalar multiplication. If $\alpha \in \mathbb{R}$ and d is shuffle-recognizable with representation $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$, then αd is shuffle-recognizable with representation $(\alpha p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$.

It is next shown that shuffle-recognizability is closed under addition. Let $d, d' \in \mathbb{R}^{\sqcup\!\sqcup} \langle (X) \rangle$ with representations $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ and $(p', q', \{\lambda'_i\}_{i=1}^n, \{\mu'_i\}_{i=1}^n, \{\gamma'_i\}_{i=1}^n)$, respectively. Define $\lambda^T = \prod_{i=1}^k \lambda_i^T$, $\gamma = \prod_{i=1}^n \gamma_i$, $\mu = \prod_{i=1}^k \mu_i$, and likewise for λ'^T, γ' and μ'. In which case,

$$\begin{aligned} d + d' &= \frac{p}{q}\,(c_1) + \frac{p'}{q'}\,(c_2) \\ &= (p(c_1) \sqcup\!\sqcup\, q(c_1)^{\sqcup\!\sqcup\, -1}) + (p'(c_2) \sqcup\!\sqcup\, q'(c_2)^{\sqcup\!\sqcup\, -1}) \end{aligned}$$

with $c_1 = \sum_{w \in X^*} \lambda^T \mu(w) \gamma \, w$ and $c_2 = \sum_{w \in X^*} \lambda'^T \mu'(w) \gamma' \ w$. Multiplying by $q(c_1) \sqcup\!\sqcup\, q(c_1)^{\sqcup\!\sqcup\, -1} \sqcup\!\sqcup\, q'(c_2) \sqcup\!\sqcup\, q'(c_2)^{\sqcup\!\sqcup\, -1}$ on both sides of the expression on the right gives

$$\begin{aligned} d + d' &= [(p(c_1) \sqcup\!\sqcup\, q'(c_2)) + (p'(c_2) \sqcup\!\sqcup\, q(c_1))] \sqcup\!\sqcup \\ &\qquad [q(c_1) \sqcup\!\sqcup\, q'(c_2)]^{\sqcup\!\sqcup\, -1} \\ &= \{(p \otimes_{\mathbb{R}} q') + (q \otimes_{\mathbb{R}} p')\}(c_1, c_2) \sqcup\!\sqcup \\ &\qquad \big(\{q \otimes_{\mathbb{R}} q'\}(c_1, c_2)\big)^{\sqcup\!\sqcup\, -1}. \end{aligned}$$

Hence, $d + d'$ is shuffle-recognizable with representation $\big(((p \otimes_{\mathbb{R}} q') + (q \otimes_{\mathbb{R}} p')), q \otimes_{\mathbb{R}} q', \{\Lambda_i\}_{i=1}^{k+n}, \{\Psi_i\}_{i=1}^{k+n}, \{\Gamma_i\}_{i=1}^{k+n}\big)$, where $\Lambda_i = \lambda_i$, $\Psi_i = \mu_i$, and $\Gamma_i = \gamma_i$ if $1 \leq i \leq k$, and $\Lambda_i = \lambda'_{i-k}$, $\Psi_i = \mu'_{i-k}$, and $\Gamma_i = \gamma'_{i-k}$ if $(k+1) \leq i \leq (k+n)$.

Finally, the case of the shuffle product is addressed. Using the same notation as in the previous case, observe

$$\begin{aligned} d \sqcup\!\sqcup d' &= \frac{p}{q}(c_1) \sqcup\!\sqcup \frac{p'}{q'}(c_2) \\ &= p(c_1) \sqcup\!\sqcup q(c_1)^{\sqcup\!\sqcup -1} \sqcup\!\sqcup p'(c_2) \sqcup\!\sqcup q'(c_2)^{\sqcup\!\sqcup -1} \\ &= (p(c_1) \sqcup\!\sqcup p'(c_2)) \sqcup\!\sqcup (q(c_1) \sqcup\!\sqcup q'(c_2))^{\sqcup\!\sqcup -1} \\ &= (p \otimes_{\mathbb{R}} p')(c_1, c_2) \sqcup\!\sqcup ((q \otimes_{\mathbb{R}} q')(c_1, c_2))^{\sqcup\!\sqcup -1} \\ &= (p \otimes_{\mathbb{R}} p') \left(\sum_{w \in X^*} (\lambda^T \times \lambda'^T)(\mu \times \mu')(w)(\gamma \times \gamma')w \right). \end{aligned}$$

Hence, $d \sqcup\!\sqcup d'$ is shuffle-recognizable with representation $\big(p \otimes_{\mathbb{R}} p', q \otimes_{\mathbb{R}} q', \{\Lambda_i\}_{i=1}^{k+n}, \{\Psi_i\}_{i=1}^{k+n}, \{\Gamma_i\}_{i=1}^{k+n}\big)$. ■

The final theorem is useful in the next section where shuffle-rational series are used as generating series for Fliess operators.

Theorem 5.3.2. *A series $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ is globally convergent if $c = p(c_1, c_2, \ldots, c_k) \sqcup\!\sqcup q(c_1, c_2, \ldots, c_k)^{-1}$ with $\deg(q) = 0$ and only locally convergent otherwise.*

Proof: If $\deg(q) = 0$, then c is a polynomial, which is always globally convergent. It is known that if $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ then $c^{\sqcup\!\sqcup -1} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ [Gray, et al. (2014b)]. Hence, if $\deg(q) \neq 0$ then, $q(c)^{\sqcup\!\sqcup -1}$ is locally convergent. Therefore, $p(c) \sqcup\!\sqcup q(c)^{\sqcup\!\sqcup -1} \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. ■

5.4 STATE SPACE REALIZATIONS

This section presents a realization theory for Fliess operators with shuffle-rational generating series. The classical result is given first for rational series as a point of comparison. Then it is shown that a series is shuffle-rational if and only if its corresponding Fliess operator has a certain *Wiener-Fliess* realization as shown in Figure 6 [Venkatesh & Gray (2020)]. As an application, the concept is applied to bilinear systems with output saturation.

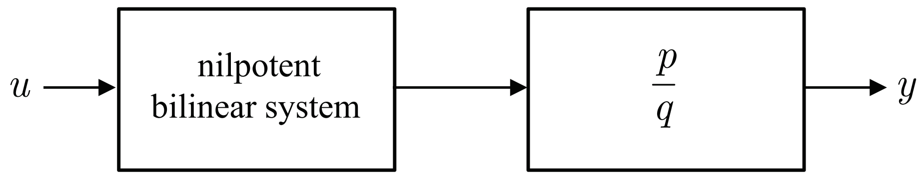


Fig. 6: Wiener-Fliess system comprised of a nilpotent bilinear system and a static rational function

Theorem 5.4.1. *[Fliess (1974), Fliess (1981)] A series $c \in \mathbb{R}\langle(X)\rangle$ if and only if the Fliess operator $y = F_c[u]$ has a bilinear state space realization*

$$\dot{z}(t) = \left(A_0 + \sum_{i=1}^{m} A_i u_i(t)\right) z(t), \quad z(0) = \gamma$$
$$y(t) = \lambda^T z(t),$$

where $A_i \in \mathbb{R}^{N\times N}$ for $i = 0, 1 \ldots, m$, and $\gamma, \lambda \in \mathbb{R}^N$.

The following is the shuffle-rational analogue of this theorem. Note that convergence of the underlying operator is ensured by Theorem 5.3.2.

Theorem 5.4.2. *A series $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ if and only if the Fliess operator $y = F_c[u]$ has a state space realization consisting of a nilpotent bilinear system followed by a static rational function.*

Proof: If $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$, then there exists a k order shuffle-representation $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ of c. From Lemma 5.3.1 and Theorem 5.4.1 it follows that each tuple $(\lambda_i, \mu_i, \gamma_i)$ corresponds to a nilpotent bilinear realization

$$\dot{z}_i(t) = \left(A_{i,0} + \sum_{j=1}^{m} A_{i,j} u_j(t)\right) z_i(t), \quad z_i(0) = \gamma_i$$
$$y_i(t) = \lambda_i^T z_i(t),$$

where $z_i(t) \in \mathbb{R}^{N_i}$, and the matrices $A_{i,j}$ are strictly upper triangular matrices.

Let $d_i = \sum_{w\in X^*} \lambda_i^T \mu_i(w)\gamma_i\, w$ denote the corresponding generating polynomials so that

$$c = \prod_{i=1}^{k} p_i(d_i) \sqcup\!\sqcup\, q_i(d_i)^{\sqcup\!\sqcup\, -1}.$$

It then follows directly that

$$F_c[u] = \prod_{i=1}^{k} \frac{p_i}{q_i}\big(F_{d_i}[u]\big),$$

so that $y = F_c[u]$ has the desired realization with state $z = [z_1^T\ z_2^T \cdots\ z_k^T]^T$, and $A_i =$ blkdiag$(A_{1,i}, \ldots, A_{k,i})$, $i = 0, 1, \ldots, m$.

Conversely, consider an arbitrary nilpotent system

$$\dot{z}(t) = \left(A_0 + \sum_{i=1}^{m} A_i u_i\right) z(t), \quad z(0) = z_0 \tag{5.4.1a}$$

$$y(t) = \frac{p}{q}(z(t)), \tag{5.4.1b}$$

where each $A_i \in \mathbb{R}^{N\times N}$ is strictly upper triangular, and $p, q \in \mathbb{R}[z_1, z_2, \ldots, z_N]$ such that $q(z) \neq 0$ on a neighborhood of z_0. First observe that the following set of N nilpotent bilinear systems

$$\dot{z}(t) = \left(A_0 + \sum_{i=1}^{m} A_i u_i(t)\right) z(t), \quad z(0) = z_0$$

$$y_i(t) = z_i(t),$$

has a corresponding representation (e_i, μ, γ) with respect to Lemma 5.3.1, where $\mu(x_i) = A_i$, $i = 1, 2, \ldots, N$, and $\gamma = z_0$. If $d_i \triangleq \sum_{w\in X^*} e_i^T \mu(w)\gamma$, then $F_{d_i}[u] = z_i$, $i = 1, 2, \ldots, N$. As the tensor algebras $\mathbb{R}[z_1, z_2, \ldots, z_N]$ and $\bigotimes_{i=1}^{N} \mathbb{R}[z_i]$ are isomorphic, there exists $p', q' \in \bigotimes_{i=1}^{N} \mathbb{R}[z_i]$ such that

$$\frac{p}{q}(z) = \frac{p'}{q'}(z_1, z_2, \ldots, z_N).$$

Therefore, the input-output behavior of system (5.4.1) is described by a Fliess operator F_d, where

$$d = \frac{p'}{q'}(d_1, d_2, \ldots, d_N).$$

Hence, $d \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ since it has the shuffle-representation $\left(p', q', \{e_i\}_{i=1}^N,\ \{\mu_i\}_{i=1}^N, \{\gamma_i\}_{i=1}^N\right)$, where $\mu_i = \mu$ and $\gamma_i = \gamma$ for $i = 1, 2, \ldots, N$. ■

The following example illustrates how a system with a shuffle-rational generating series can appear in practice.

Example 5.4.1. Consider a double integrator system with zero initial conditions followed by a saturation nonlinearity

$$\Gamma(x) = \begin{cases} \min(x, 1) & : x \geq 0 \\ \max(x, -1) & : x < 0. \end{cases}$$

As shown in Figure 7, Γ is well approximated by the hyperbolic tangent function as

$$\tanh(1.15x) = \frac{\exp(1.15x) - \exp(-1.15x)}{\exp(1.15x) + \exp(-1.15x)}.$$

Using a Taylor series approximation of the exponential functions up to degree N gives

$$\tanh(1.15x) \approx f_n(x) \triangleq \frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{n} \frac{(1.5x)^{2k+1}}{(2k+1)!}}{\sum_{k=0}^{n} \frac{(1.5x)^{2k}}{(2k)!}},$$

where $n = \lceil \frac{N}{2} \rceil$. The quality of the approximation of $\Gamma(x)$ for a few values of n is shown in Figure 7. The input-output behavior of the overall system is given by

$$y(t) = \Gamma(F_c[u])(t) \approx f_n(F_c[u])(t) = \frac{p}{q}\big(F_c[u]\big)(t) = F_d[u](t) = \hat{y}(t),$$

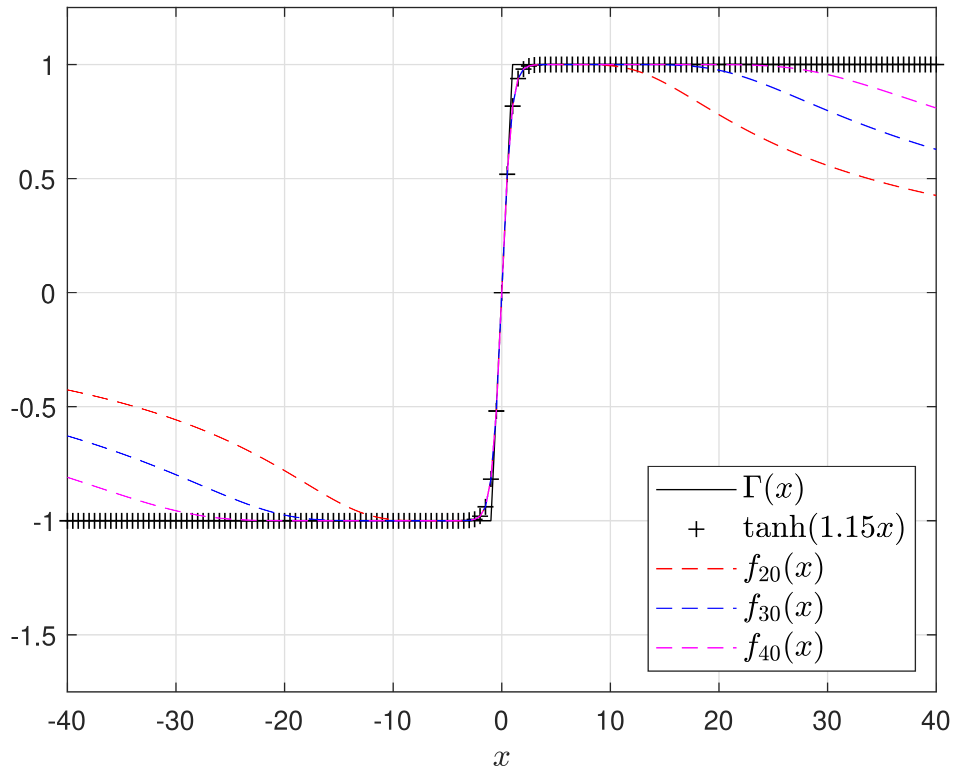


Fig. 7: Saturation function $\Gamma(x)$ and its approximations

where $c = x_0 x_1$ and

$$d = \left(\sum_{k=0}^{n} \frac{(1.5c)^{\shuffle\,(2k+1)}}{(2k+1)!} \right) \shuffle \left(\sum_{k=0}^{n} \frac{(1.5c)^{\shuffle\,(2k)}}{(2k)!} \right)^{\shuffle\,-1}.$$

The output response $y(t)$ of the given system and its shuffle-rational approximation $\hat{y}(t)$ when $n = 100$ are shown in Figure 8 for the applied input $u(t) = \cos(t)$. □

In light of Example 5.4.1, the following statement can be asserted. Shuffle-rational series can approximate nilpotent bilinear systems with hard non-linearities such as the saturation non-linearity. Nilpotent bilinear systems appear in the modeling of robotic systems [Murray & Sastry (1993)]. Hence, the shuffle-rational series potentially have a great applicability in the modeling of engineering systems where the associated hard nonlinearities are difficult to

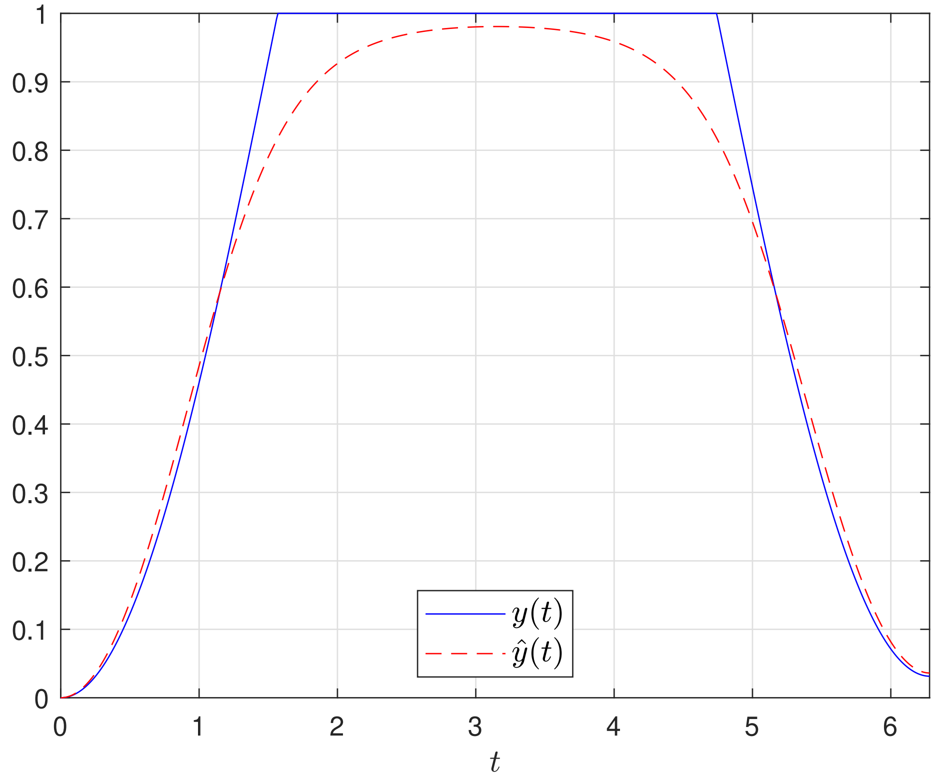


Fig. 8: Output response $y(t)$ and its shuffle-rational approximation $\hat{y}(t)$ in Example 5.4.1

neglect.

CHAPTER 6

CONCLUSIONS, CONJECTURES AND FUTURE WORK

The objectives of this chapter are to draw the conclusions from the dissertation and provide a few insights into the potential future lines of research extending from the dissertation. The discussion of future topics is meant only to serve as an initial viewpoint, and the reader is cautioned against taking it as a final word.

6.1 CONCLUSIONS

In this dissertation it was proved that when a Chen-Fliess series F_c is in additive static feedback connection with a static map f_d, the closed-loop system has a Chen-Fliess series representation, and the closed form expression of the generating series is developed. It is also proved that the additive static feedback connection preserves local convergence and does not preserve global convergence in general with the aid of a counterexample. The dissertation also characterized the convergence of the shuffle product, mixed composition product and Wiener-Fliess composition product in the course of proving the preservation of local convergence under static feedback connection. All these products were shown to preserve both the local and global convergence of Fliess operators. Next, the notions of shuffle rationality and shuffle recognizability were developed akin to the traditional notion of rational series and recognizability. An analogue of Schützenberger's theorem for the shuffle-rational series showed that shuffle rationality and shuffle recognizability are equivalent. Another equivalent characteristic of shuffle rational series was given in terms of canonical state space realizations. It is known that a Fliess operator F_c, where c is a rational series, has a canonical bilinear state space realization. Likewise, it was shown that a Fliess operator F_c, where c is shuffle-rational, has a canonical bilinear state space realization as a Wiener-Fliess composition of a nilpotent bilinear system with a rational static map.

6.2 REMARKS ON FUTURE WORK

The dissertation has completely answered the questions that were posed. However, the answers suggest new directions for research. For example, regarding the static feedback problem which was addressed in Chapters 3 and 4, the same questions can be asked for a multiplicative static feedback connection instead of the additive feedback connection. If a Chen-Fliess series F_c is in a multiplicative feedback connection with a static map f_d, does the closed-loop system have a Chen-Fliess series representation? If so, can a closed form expression for the generating series of the closed-loop system be derived and computed? Does the closed-loop system preserve local convergence? Does the closed-loop system preserve global convergence? The remainder of this chapter provides some preliminary observations regarding the shuffle rationality problem.

6.2.1 Hankel Rank of a Series

An equivalent notion of rational series is described by the finiteness of its Hankel rank. The Hankel matrix for a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$, denoted by $\mathscr{H}(c)$, is given by an infinite tableau of entries with rows and columns indexed by the words, arranged by some ordering. An element in $\eta\zeta$ (η-row, ζ-column) position viz. $\mathscr{H}(c)_{\eta\zeta} = (c, \eta\zeta)$. A bilinear realization exists for a series $c \in \mathbb{R}\langle\langle X\rangle\rangle$, equivalently $c \in \mathbb{R}\langle(X)\rangle$ iff rank of $\mathscr{H}(c)$ is finite. The dimension of the minimal bilinear state space realization for the generating series c is given by the rank of $\mathscr{H}(c)$ [Fliess (1974)]. The notions of Hankel matrix and Hankel rank evidently cannot be extended for the $\sqcup\!\sqcup$-rational series, since for $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$, the series no longer embraces the bilinear structure if $\deg(q) \neq 0$, as seen from the previous subsection. This can be further illustrated by revisiting Example 5.2.2.

Example 6.2.1. Let $X = \{x_1\}$ and $c = x_1^{\sqcup\!\sqcup\, *} \in \mathbb{R}\langle\langle X\rangle\rangle$. Then, $c = \sum_{k\in\mathbb{N}_0} k!x_1^k$. It is simple to observe that rank of $\mathscr{H}(c)$ is infinite. □

Remark: For the rest of this section, the shuffle rational series is often abbreviated as $\sqcup\!\sqcup$-rational series for brevity.

Given that $\sqcup\!\sqcup$-rational series describe nilpotent systems with *Wiener-Fliess* composition of a rational function, mathematical tools more sophisticated than linear algebra are required to have similar equivalent criteria due to the non-linearities in the output equation. The underlying picture becomes clearer in the next subsection, where the structure of input-output differential equations satisfied by Fliess operators F_c where $c \in \mathbb{R}\langle(X)\rangle$ and $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ are investigated.

6.2.2 Structure of the Input-Output Differential Equation

An equivalent characterization of a rational series is given by the structure of the differential equation characterizing its input-output behaviour. The fact that a rational series c should have a finite Hankel rank provides a view of the structure of the differential equation in discussion. It is first important to understand the concept of differential field before further discussion [Kaplansky (1957)].

Definition 6.2.1. A differential field L is a commutative field with the additive endomorphism D on L satisfying the Leibnitz rule

$$D(a.b) = D(a).b + a.D(b) \ \forall a, b \in L.$$

The constants of a differential field L, denoted by $\Omega(L)$, is the subfield given by the kernel of the endomorphism D viz.

$$\Omega(L) = \{v \in L : D(v) = 0\}.$$

Given a formal input u, let $\dot{u}, \ddot{u}, \ldots, u^{(n)}, \ldots$ denote the successive formal derivatives of u with respect to time. Define $\mathbb{R}\{u\}$ to be the $\mathbb{R}$-algebra of polynomials in the countable number of commutative indeterminates $u, \dot{u}, \ddot{u}, \ldots$. Since this $\mathbb{R}$-algebra (polynomial ring) is an integral domain, consider its field of fractions (quotient field), denoted by $\mathbb{R}(\{u\})$. It is easy to verify that $\mathbb{R}(\{u\})$ is a differential field, and it is known that $\Omega(\mathbb{R}(\{u\})) \cong \mathbb{R}$ in the category of fields [Fliess & Reutenauer (1982), Fliess & Reutenauer (1983)]. A rational series $c \in \mathbb{R}\langle(X)\rangle$, say of Hankel rank N, satisfies a differential equation of order N with coefficients from $\mathbb{R}(\{u\})$. The following result is a precise description of the preceding fact.

Theorem 6.2.1. *A series $d \in \mathbb{R}\langle(X)\rangle$ has a Hankel matrix of rank N if and only if $y, \dot{y}, \ddot{y}, \ldots, y^{(N-1)}$ are $\mathbb{R}(\{u\})$-linearly independent where $y = F_d[u]$ and hence, d satisfies a differential equation of the form*

$$y^{(N)} + c_1 y^{(N-1)} + c_2 y^{(N-2)} + \cdots + c_N y = 0,$$

where $c_1, c_2, \ldots, c_N \in \mathbb{R}(\{u\})$.

The above result proved in [Fliess & Reutenauer (1983)] provides the structure of a rational system which is a homogeneous linear differential equation with the coefficients that are rational functions of the input and their successive derivatives with respect to time. The following example is an illustration of this fact.

Example 6.2.2. Let $X = \{x_1, x_2\}$ and consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ given by $c = (x_1x_2)^*$. It is simple to check that the series c is rational and its Hankel rank is two. The series has the following bilinear realization with states z_1, z_2; inputs u_1, u_2; and the output y.

$$\dot{z}_1 = z_2 u_1$$

$$\dot{z}_2 = z_1 u_2$$

$$y = z_1.$$

The above realization satisfies the following input-output linear homogeneous differential equation

$$\ddot{y} - \left(\frac{\dot{u}_1}{u_1}\right)\dot{y} - u_1 u_2 y = 0.$$

□

To better understand what follows, it is necessary to understand the extension of a differential field. Let (A, d_A) and (B, d_B) be two differential algebras. Then $h : A \longrightarrow B$ is called a differential algebra homomorphism if h is an algebra homomorphism and h commutes with the derivation viz. $h \circ d_A = d_B \circ h$. A differential field L' is said to be an extension of a differential field L if and only if the inclusion map $i : L \longrightarrow L'$ is an L-differential algebra homomorphism. The differential field $\mathbb{R}(\{u\})$ in the preceding theorem is an extension of the trivial differential field $\mathbb{R}$. Consider the $\mathbb{R}$-algebra $\mathbb{R}\{u, y\}$ defined to be the algebra of polynomials in the countable number of commutative indeterminates $u, \dot{u}, \ddot{u}, \cdots$ and $y, \dot{y}, \ddot{y}, \cdots$. Since it is an integral domain, consider its quotient field, denoted by $\mathbb{R}(\{u, y\})$. This is algebra of fractions of polynomials in the input u and output y and their successive derivatives with respect to time. Hence, $\mathbb{R}(\{u, y\})$ is a differential field and is traditionally called the *observation algebra*. It is the equivalent of an *algebraically closed field* for differential polynomials underlying system theory. The following arrow diagram precisely illustrates the field extensions.

$$\mathbb{R} \quad \longrightarrow \quad \mathbb{R}(\{u\}) \quad \longrightarrow \quad \mathbb{R}(\{u, y\})$$

Say a field L' is an extension of L. Then $[L' : L]$ denotes the dimension of the vector space L' over the field L. If $[L' : L]$ is finite, then L' is said to be a finite extension of L. For example, in the case of algebraic fields, the field $\mathbb{C}$ is a finite extension of $\mathbb{R}$ as $[\mathbb{C} : \mathbb{R}] = 2$.

Definition 6.2.2. An input-output differential equation is called algebraic of order k, if the differential equation has the form $p(u, \dot{u}, \ddot{u}, \ldots, u^{(k)}, y, \dot{y}, \ddot{y}, \ldots, y^{(k)}) = 0$ where p is a polynomial in $\mathbb{R}\{u,y\}$.

The following theorem states a necessary condition for the existence of an algebraic input-output differential equation for a given input-ouput system [Wang (1990)].

Theorem 6.2.2. *If a formal dynamical system $\Sigma : u \mapsto y$ satisfies an algebraic input-output differential equation, then $\left[\mathbb{R}(\{u,y\}) : \mathbb{R}(\{u\})\right]$ is finite.*

If the formal system Σ can be described by a Fliess operator with generating series $c \in \mathbb{R}\langle\langle X\rangle\rangle$, then the degree of extension is alternatively called the *Lie rank* of the series. The literature normally approaches this topic through the Lie algebra of the formal vector fields and distributions [Fliess (1981)]. Note, when a series c is rational viz. $c \in \mathbb{R}\langle(X)\rangle$, the Hankel rank of c is greater than or equal to the degree of the extension $\left[\mathbb{R}(\{u,y\}) : \mathbb{R}(\{u\})\right]$. However, $\sqcup\!\sqcup$-rational series do not necessarily satisfy an input-output algebraic differential equation. The following is an instance, where the series $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ satisfies an input-output algebraic differential equation.

Example 6.2.3. Let $X = \{x_0, x_1\}$ and say $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ is described by $c = (x_0x_1)^{\sqcup\!\sqcup *}$. Note, the underlying polynomial d (viz $c = d^{\sqcup\!\sqcup *}$) is described by $d = x_0x_1$, and the Fliess operator F_d satisfies the input-output algebraic differential equation of minimal order two, equivalently, $[\mathbb{R}\{u,y\} : \mathbb{R}\{u\}] = 2$ with respect to the series d. Let $y_c = F_c[u]$ denote the output of the system described by the generating series c and likewise y_d denotes the output of the system described by the generating series (polynomial) d. The input-output differential equation satisfied by y_d is given by $\ddot{y_d} = u$. Then,

$$y_c = \sum_{k\in\mathbb{N}_0} y_d^k = (1-y_d)^{-1} = \left(\frac{1}{1-z}\right)(y_d)$$
$$\dot{y_c} = \left(\frac{1}{(1-z)^2}\right)(y_d)\dot{y_d} = y_c^2\dot{y_d}.$$

Observe that $y_c \neq 0$. Hence,

$$\dot{y_d} = \frac{\dot{y_c}}{y_c^2}\,.$$

Therefore,

$$\ddot{y}_c = 2y_c\dot{y}_c\dot{y}_d + y_c^2\ddot{y}_d = \frac{2\dot{y}_c^{\,2}}{y_c} + y_c^2 u.$$

Thus,

$$y_c\ddot{y}_c - 2\dot{y}_c^{\,2} - y_c^3 u = 0.$$

Hence, the system describing the series c satisfies an algebraic input-output differential equation of order two locally given by $y_c\ddot{y}_c - 2\dot{y}_c^{\,2} - y_c^3 u = 0$. Note that the shuffle closure did not change the degree of the extension from the underlying polynomial. □

The following example is an instance where the shuffle closure of the polynomial does not satisfy an algebraic input-output differential equation.

Example 6.2.4. Let $X = \{x_1\}$ and say $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle\langle X\rangle\rangle$ is described by $c = x_1 \sqcup\!\sqcup (1 + x_1^{\sqcup\!\sqcup 2})^{\sqcup\!\sqcup -1}$. Note, the underlying polynomial d is x_1, and the Fliess operator F_d satisfies the input-output differential equation of order one, given by $\dot{y}_d = u$. By following the previous example, the input-output differential equation locally satisfied by the Fliess operator F_c is

$$\dot{y}_c - \frac{2y_c^2\sqrt{1-4y_c^2}}{1-\sqrt{1-4y_c^2}}u = 0.$$

Hence, the system describing the series c does not satisfy an algebraic input-output differential equation due to presence of the radical $\sqrt{1-4y_c^2}$. However, note that the order of the differential equation is invariant from the underlying polynomial. □

The following counterexample demonstrates that not all Fliess operators described by a $\sqcup\!\sqcup$-rational series can even be described by an input-output differential equation.

Example 6.2.5. Let $X = \{x_0, x_1\}$ and consider $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle\langle X\rangle\rangle$ described by $c = (x_1)^{\sqcup\!\sqcup *} \sqcup\!\sqcup x_0x_1$. The series $(x_1)^{\sqcup\!\sqcup *}$ satisfies the differential equation $\dot{y}_1 - y_1^2 u = 0$, where $y_1 = F_{(x_1)^{\sqcup\!\sqcup *}}[u]$, and the polynomial x_0x_1 satisfies the input-output differential equation $\ddot{y}_2 = u$. Let $y = F_c[u]$, then $y = y_1y_2$. The time derivative of y is not invertible; hence, y_1, y_2

and their successive derivatives cannot be expressed merely in terms of y and its derivatives. Therefore, an input-output differential equation is not possible. □

The above examples inspire the following theorem about the existence and nature of the input-output differential equations for $\sqcup\!\sqcup$-rational series. Only a sketch of the proof is provided.

Theorem 6.2.3. *Let* $c \in \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$ *with the representation tuple* $(p, q, \lambda, \mu, \gamma)$*, where* $p, q \in \bigotimes_{\mathbb{R}\, i=1}^{\;k} \mathbb{R}[y_i]$*. Let* $d_1, d_2, \cdots d_k$ *be the underlying polynomials, where* $c = \frac{p}{q}(d_1, \cdots d_k)$*. Then the following statements are true:*

1. *Let* $k = 1$ *and* $[\mathbb{R}\{u, y\} : \mathbb{R}\{u\}] = N$ *with respect to* d_1*. If* $\left(\frac{\Gamma(p)}{\Gamma(q)}\right)'(\lambda^T\gamma) \neq 0$*, then the Fliess operator* F_c *satisfies locally an input-output differential equation of minimal order* N*, not necessarily algebraic. The structure involves radicals of the polynomials in the output variable* y*.*

2. *If* $k > 1$ *and* $\exists i \in \{1, \cdots, k\}$ *such that* $\forall j \neq i$*, polynomial* d_j *lies in the shuffle closure* d_i*. Then the series* c *can be represented as just shuffle closure* d_i*, and reverts to the case* 1 *for the existence of input-output differential equation.*

Proof: (**Sketch**) If $\left(\frac{\Gamma(p)}{\Gamma(q)}\right)'(\lambda^T\gamma) \neq 0$, then by the inverse mapping theorem, there is a formal local diffeomorphism mapping the derivatives of c to the derivatives of the underlying polynomial series around the initial condition. ■

The study of the rational closure of polynomials under the shuffle product can be connected to the study of rational maps. It is evident that the computation of the input-output differential equation relies typically on solving the roots of polynomial equations. This might pave the way for using the Galois theory of polynomials, known for studying the solvability of polynomials, making it a potential future topic. The final section is the study of shuffle closure of rational series for which most of the results can be extended from the results of the shuffle-rational series due to the generality in the definition of the map Γ.

6.2.3 Shuffle Rationality of Rational Series

The $\mathbb{R}$-algebra of rational series with the catenation product, $\mathbb{R}\langle(X)\rangle$, is closed under the shuffle product. In other words, the underlying $\mathbb{R}$-module of rational series adjoined with the bilinear shuffle product forms a commutative $\mathbb{R}$-algebra $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$ and a $\mathbb{R}$-subalgebra of $\mathbb{R}\langle\langle X\rangle\rangle$. Hence, it is a valid notion to consider the rational closure of $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$. The rational closure of $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$ in $\mathbb{R}\langle\langle X\rangle\rangle$ is denoted by $\mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$. By definition, it is evident that $\mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle} \supseteq \mathbb{R}\langle(X)\rangle \cup \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$. The following example proves that $\mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle} \neq \mathbb{R}\langle(X)\rangle \cup \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$.

Example 6.2.6. Let the alphabet $X = \{x_1, x_2\}$. Consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ given by $c = ((x_1x_2)^*)^{\sqcup\!\sqcup\, -1}$. Observe that c is the shuffle inverse of not a polynomial, hence $c \notin \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$. A direct calculation gives

$$c = 1 - x_1x_2 + 4x_1^2x_2^2 + x_1x_2x_1x_2 + \cdots$$

Note that

$$\begin{aligned}
(x_1)^0(c) &= c = 1 - x_1x_2 + 4x_1^2x_2^2 + x_1x_2x_1x_2 + \cdots \\
(x_1)^{-1}(c) &= -x_2 + 4x_1x_2^2 + x_2x_1x_2 + \cdots \\
(x_1)^{-2}(c) &= 4x_2^2 + \cdots
\end{aligned}$$

The leading term of the higher left-shift powers of x_1 on c increase in powers of x_2. The leading term of $x_1^{-k}(c)$ is unique to the series in the collection $\{x_1^{-k}(c)\}_{k\in\mathbb{N}_0}$. Hence, the collection of series $\{x_1^{-k}(c)\}_{k\in\mathbb{N}_0}$ are $\mathbb{R}$-linearly independent. Thus rank of $\mathscr{H}(c)$ is infinite. Therefore, $c \notin \mathbb{R}\langle(X)\rangle$; hence, $c \notin \mathbb{R}\langle(X)\rangle \cup \mathbb{R}^{\sqcup\!\sqcup}\langle(X)\rangle$. □

Hence, $\mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$ is not a disjoint union of $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$ and $\mathbb{R}\langle(X)\rangle$. A series $c \in \mathbb{R}\langle(X)\rangle$ implies $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ and from [Gray, et al. (2014b)], the shuffle closure of a globally convergent series c is locally convergent. Hence, if a series $c \in \mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$, then $c \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$. The following example proves that $\mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle} \neq \mathbb{R}\langle\langle X\rangle\rangle$.

Example 6.2.7. Let $X = \{x_1\}$. Consider the series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ described as $c = \sum_{n\in\mathbb{N}_0} 2^{2^n} x_1^n$. Using Landau's notation, since $2^{2^n} = \omega(n!)$, the series $c \notin \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ and hence, $c \notin \mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$. □

The notion of a series c being completely-recognizable is presented next.

Definition 6.2.3. Let $k \in \mathbb{N}$ and $\{N_1, N_2, \ldots N_k\}$ be a multiset of k positive integers. Let $\{\lambda_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k$ be ordered collections of k vectors such that $\lambda_i, \gamma_i \in \mathbb{R}^{N_i}$. Assume $\{\mu_i\}_{i=1}^k$ is an ordered collection of k monoid morphisms $\mu_i : X^* \longrightarrow \mathbb{R}^{N_i \times N_i}$. Define two polynomials $p, q \in \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]$ such that $q(\lambda_i^T \gamma_i) \neq 0$, $i = 1, \ldots, k$. A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is said to be completely-recognizable if $c = p/q\left(\sum_{w\in X^*} \lambda^T \mu(w)\gamma\, w\right)$, where $\lambda^T = (\lambda_1^T \times \lambda_2^T \times \cdots \times \lambda_k^T)$, $\mu = (\mu_1 \times \mu_2 \times \cdots \times \mu_k)$, and $\gamma = (\gamma_1 \times \gamma_2 \times \cdots \times \gamma_k)$. The tuple $(p, q, \{\lambda_i\}_{i=1}^k, \{\mu_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k)$ is called a k order *complete-representation* of c. The set of all such completely-recognizable series is denoted by $\mathbb{R}_{\sqcup\!\sqcup}^{rec}\overline{\langle(X)\rangle}$.

The expression $\left(\sum_{w\in X^*} \lambda^T \mu(w)\gamma w\right)$ defines a rational series in the traditional sense, as per Schützenberger's theorem. Hence, the set of $\mathbb{R}_{\sqcup\!\sqcup}^{rec}\overline{\langle(X)\rangle}$ consists of results of shuffle product of rational series and the shuffle inverses of invertible (non-proper) rational series. More precisely,

$$\mathbb{R}_{\sqcup\!\sqcup}^{rec}\overline{\langle(X)\rangle} = \left\{p(c_1, .., c_k) \sqcup\!\sqcup\, q(c_1, .., c_k)^{\sqcup\!\sqcup\, -1} : p, q \in \bigotimes_{\mathbb{R}}{}_{i=1}^{k} \mathbb{R}[y_i]\right\}, \tag{6.2.1}$$

where $q(c_1, ..c_k)$ is not proper and $c_1, \cdots, c_k \in \mathbb{R}\langle(X)\rangle$.

Example 6.2.8. Let $X = x_1$ and consider $c \in \mathbb{R}\langle\langle X\rangle\rangle$ described by $c = (x_1^*)^{\sqcup\!\sqcup\, -1}$. Let

$c' \triangleq 1 - x_1^*$. Then,

$$\begin{aligned} c &= (1-c')^{\sqcup\!\sqcup\,-1} = (c')^{\sqcup\!\sqcup\,*} \\ &= 1 + c' + (c')^{\sqcup\!\sqcup\,2} + \cdots + (c')^{\sqcup\!\sqcup\,k} + \cdots \\ &= \left(\frac{1}{1-y}\right)(c'). \end{aligned}$$

Note that the rational series c' has the representation tuple (λ, μ, γ), where $\mu : X^* \longrightarrow \mathbb{R}^{2\times 2}$ with $\mu(x_1) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, and $\lambda, \gamma \in \mathbb{R}^2$ with $\lambda = e_1, \gamma = e_2$. Hence, $c = (x_1^*)^{\sqcup\!\sqcup\,-1}$. Therefore, $c \in \mathbb{R}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$ has the representation $(1, 1-y, \lambda, \mu, \gamma)$ which implies $c \in \mathbb{R}^{rec}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$.

□

Note that from equation (6.2.1), it is evident that $\mathbb{R}^{rec}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle} \subseteq \mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle$. The above example inspires an analogue of Schützenberger's theorem for the set of completely-recognizable series, which is addressed in the following theorem.

Theorem 6.2.4. $\mathbb{R}_{\sqcup\!\sqcup}\langle(X)\rangle = \mathbb{R}^{rec}_{\sqcup\!\sqcup}\overline{\langle(X)\rangle}$ *are isomorphic as sets. Equivalently, a series is in the shuffle closure of rational series if and only if it is completely recognizable.*

Proof: The proof is similar to Theorem 5.3.1. ■

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VITA

Subbarao Venkatesh Guggilam

Department of Electrical and Computer Engineering

Old Dominion University

Norfolk, Virginia 23529-0246

Education

- M. Sc Computer Science, Chennai Mathematical Institute, Chennai, India, 2016
- M. Tech Systems and Control Engineering, IIT Bombay, Mumbai, India, 2014
- B. Tech Electronics and Instrumentation Engineering, Amrita Vishwa Vidyapeetham, Coimbatore, India, 2011.

Ph.D. Dissertation

Wiener-Fliess Composition of Formal Power Series: Additive Static Feedback and Shuffle Rational Series, Old Dominion University, August 2021.

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