Shape Sensitivity Analysis and Optimization of Skeletal Structures and Geometrically Nonlinear Solids

Ching-Hung Chuang
Old Dominion University

Follow this and additional works at: https://digitalcommons.odu.edu/mae_etds

Part of the Automotive Engineering Commons, Mechanical Engineering Commons, and the Structures and Materials Commons

Recommended Citation
https://digitalcommons.odu.edu/mae_etds/221

This Dissertation is brought to you for free and open access by the Mechanical & Aerospace Engineering at ODU Digital Commons. It has been accepted for inclusion in Mechanical & Aerospace Engineering Theses & Dissertations by an authorized administrator of ODU Digital Commons. For more information, please contact digitalcommons@odu.edu.
SHAPE SENSITIVITY ANALYSIS AND OPTIMIZATION OF SKELETAL STRUCTURES AND GEOMETRICALLY NONLINEAR SOLIDS

by

Ching-Hung Chuang

B.S., June 1981, Chung-Yuan Christian University, Taiwan, R.O.C.
M.E., June 1987, National Tsing Hua University, Taiwan, R.O.C.

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

ENGINEERING MECHANICS

OLD DOMINION UNIVERSITY

May 1992

Approved by:

Gene J. W. Hou (Director)

Chuh Mei

Nahil A. Sobh

Hideaki Kaneko

Duc T. Nguyen

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
ABSTRACT

SHAPE SENSITIVITY ANALYSIS AND OPTIMIZATION OF SKELETAL STRUCTURES AND GEOMETRICALLY NONLINEAR SOLIDS

Ching-Hung Chuang
Old Dominion University, 1992
Director: Dr. Gene J. W. Hou

Formulations and computational schemes for shape design sensitivity analysis and optimization have been developed for both skeletal structures and geometrically nonlinear elastic solids. The continuum approach, which is based on the weak variational form of the governing differential equation and the concept of the material derivative, plays a central role in such a development.

In the first part of this work, the eigenvalue and eigenvector sensitivity equations for skeletal structures are derived with respect to configuration variables of joint and support locations. This derivation is done by the domain method as well as the boundary method. The discrete approach for the eigenvalue and eigenvector sensitivity analysis is also presented for the purpose of numerical comparison. The resultant sensitivity equations are first validated by a cantilever beam for eigenvalue sensitivity analysis and a simply-supported beam for eigenvector sensitivity analysis. The analytical solutions can be easily obtained for both examples. Moreover, the investigation of numerical accuracy and computational efficiency of these sensitivity equations is done with examples of several skeletal structures. The results show that the domain method has an advantage to be both computationally accurate and efficient. Finally, a design optimization of a vibrating beam is presented to investigate the effects of including the support locations and the support
stiffness constants as design variables on the design. It is concluded that the support locations and the support stiffness constants are important to improve the quality of design.

The second part of this thesis explores the possibility using the Eulerian formulation as the foundation for shape sensitivity analysis and optimization of a new class of design problems in which the performance criteria are defined in the deformed configuration of a geometrically nonlinear elastic solid. The displacement and rotation of this nonlinear elastic solid are assumed to be large while its strain is assumed to be small. Shape sensitivity equations are derived based upon the Eulerian formulation as well as the total Lagrangian formulation for a general functional. A prismatic bar is evaluated analytically to validate these sensitivity equations. A design optimization scheme is then established which uses the Eulerian formulation for analysis as well as sensitivity analysis, to design the shape of a uniformly loaded beam to minimize the area subjected to geometric and stress constraints. The results show that the proposed sensitivity equations and the design scheme work well for this example.
ACKNOWLEDGMENTS

The author wishes to express his deep gratitude to his thesis advisor, Dr. Gene J. W. Hou for his advice, guidance and assistance throughout the research and preparation of this dissertation. The helpful suggestions and criticisms received from the remaining committee members, Drs. Chuh Mei, Nahil A. Sobh, Hideaki Kaneko and Duc T. Nguyen are gratefully acknowledged. Last, but not least, the financial support provided by NSF Grant No. DDM-865-7917 and NASA Task Order No. NAS-18584-74 are gratefully acknowledged.
<table>
<thead>
<tr>
<th>TABLE OF CONTESTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>LIST OF SYMBOLS</td>
<td>xii</td>
</tr>
</tbody>
</table>

Chapter 1. INTRODUCTION ............................................. 1

1.1 Overview and Objectives .................................. 1
1.2 Scope ................................................... 3
1.3 Literature Review ........................................ 5
   1.3.1 Literature Review for Eigensensitivity Analysis and 
         Design Optimization ............................ 5
   1.3.2 Literature Review for Nonlinear Sensitivity Analysis 
         and Design Optimization ..................... 8

Chapter 2. EIGENVALUE SENSITIVITY ANALYSIS OF SKELETAL 
STRUCTURES WITH VARIABLE JOINT AND SUPPORT 

2.1 Governing Equations of Skeletal Structures ........... 12
   2.1.1 Planar Frame .................................... 12
   2.1.2 Planar Truss .................................... 15
   2.1.3 Continuous Beam .............................. 16
2.2 The Concept of the Material Derivative .................. 17
2.3 Continuum Approach ..................................... 21
   2.3.1 Domain Method (DM) ............................ 21
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3.2</td>
<td>Boundary Method (BM)</td>
<td>27</td>
</tr>
<tr>
<td>2.4</td>
<td>Discrete Approach</td>
<td>32</td>
</tr>
<tr>
<td>2.5</td>
<td>Examples</td>
<td>33</td>
</tr>
<tr>
<td>2.5.1</td>
<td>Analytical Example: A Cantilever Beam</td>
<td>33</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Numerical Examples</td>
<td>39</td>
</tr>
<tr>
<td>2.6</td>
<td>Application Example: Design Optimization of a Vibrating Beam</td>
<td>49</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>EIGENVECTOR SENSITIVITY ANALYSIS OF SKELETAL STRUCTURES WITH VARIABLE JOINTS AND SUPPORT LOCATIONS</td>
<td>54</td>
</tr>
<tr>
<td>3.1</td>
<td>Shape Sensitivity Analysis of Eigenvectors of a Continuous Beam</td>
<td>54</td>
</tr>
<tr>
<td>3.1.1</td>
<td>Domain Method (DM)</td>
<td>55</td>
</tr>
<tr>
<td>3.1.2</td>
<td>Boundary Method (BM)</td>
<td>58</td>
</tr>
<tr>
<td>3.2</td>
<td>Analytical Example: A Simply-Supported Beam</td>
<td>60</td>
</tr>
<tr>
<td>3.3</td>
<td>Discrete Approach</td>
<td>65</td>
</tr>
<tr>
<td>3.4</td>
<td>Eigenvector Sensitivity Analysis of Continuous Beam: Numerical Study</td>
<td>66</td>
</tr>
<tr>
<td>3.5</td>
<td>Shape Sensitivity Analysis of Eigenvector of a Planar Truss Using the Domain Method</td>
<td>69</td>
</tr>
<tr>
<td>3.6</td>
<td>Eigenvector Sensitivity Analysis of Planar Truss: Numerical Study</td>
<td>72</td>
</tr>
<tr>
<td>Chapter 4</td>
<td>SHAPE SENSITIVITY ANALYSIS OF GEOMETRICALLY NONLINEAR SOLIDS</td>
<td>75</td>
</tr>
<tr>
<td>4.1</td>
<td>Analysis of Geometrically Nonlinear Solids</td>
<td>76</td>
</tr>
<tr>
<td>4.1.1</td>
<td>Nomenclature</td>
<td>76</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
4.1.2 Principle of Virtual Work ...................................... 77
4.1.3 Incremental Equations for Nonlinear Analysis ....... 81
4.1.4 Finite Element Solution Procedure ..................... 83
4.2 Shape Sensitivity Analysis : Direct Differentiation Method . 90
   4.2.1 Using the Total Lagrangian Formulation for Shape Sensitivity Analysis .......................... 90
   4.2.2 Using the Eulerian Formulation for Shape Sensitivity Analysis .......................... 94
4.3 Shape Sensitivity analysis : Adjoint Variable Method ...... 97
   4.3.1 Shape Sensitivity Analysis of $\Phi(0V)$ ..................... 98
   4.3.2 Shape Sensitivity Analysis of $\Psi(\Delta tV)$ ..................... 99
4.4 Analytical Example ............................................................ 100
   4.4.1 Nonlinear Analysis of a Prismatic Bar .................. 101
   4.4.2 Sensitivity Analysis of $\Psi(\ell)$ .................. 104
   4.4.3 Sensitivity Analysis of $\Phi(L)$ .................. 105
Chapter 5. SHAPE OPTIMIZATION OF GEOMETRICALLY NONLINEAR SOLIDS DEFINED IN THE DEFORMED CONFIGURATION .. 108
5.1 Problem Statements. ..................................................... 109
5.2 Shape Sensitivity Analysis ............................................. 110
   5.2.1 Shape Sensitivity Analysis of Objective and Constraint Functionals ........................................... 110
   5.2.2 Numerical Studies of Shape Sensitivity Analysis .......................... 114
5.3 A Computational Scheme for Shape Optimization of Nonlinear Solids in the Deformed Configuration ........ 118
5.4 Numerical Results ............................................................ 122
<table>
<thead>
<tr>
<th>TABLE</th>
<th>Description</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Eigenvalue Sensitivity Coefficients of Four-Member Frame with with respect to Movement of Support I</td>
<td>40</td>
</tr>
<tr>
<td>2.2</td>
<td>Eigenvalue Sensitivity Coefficients of Four-Member Frame with respect to Simultaneous Movements of Joints J, K, L</td>
<td>41</td>
</tr>
<tr>
<td>2.3</td>
<td>Eigenvalue Sensitivity Coefficients of Nineteen-Member Frame</td>
<td>43</td>
</tr>
<tr>
<td>2.4</td>
<td>Computational Times of Sensitivity Analysis with respect to Support I</td>
<td>43</td>
</tr>
<tr>
<td>2.5</td>
<td>Eigenvalue Sensitivity Coefficients of Fifty-one Bar Truss</td>
<td>45</td>
</tr>
<tr>
<td>2.6</td>
<td>Eigenvalue Sensitivity Coefficients of a Uniform Beam</td>
<td>47</td>
</tr>
<tr>
<td>2.7</td>
<td>Eigenvalue Sensitivity Coefficients of a Stepped Beam</td>
<td>48</td>
</tr>
<tr>
<td>2.8</td>
<td>Eigenvalue Sensitivity Coefficients of a Continuous Beam with a Spring Support</td>
<td>48</td>
</tr>
<tr>
<td>2.9</td>
<td>Eigenvalue Sensitivity Coefficients of a Stepped Beam with a Spring Support</td>
<td>49</td>
</tr>
<tr>
<td>3.1</td>
<td>Eigenvalues of a Simply-Supported Beam</td>
<td>63</td>
</tr>
<tr>
<td>3.2</td>
<td>Eigenvector Sensitivity Coefficients of First Mode</td>
<td>63</td>
</tr>
<tr>
<td>3.3</td>
<td>Eigenvector Sensitivity Coefficients of Third Mode</td>
<td>64</td>
</tr>
<tr>
<td>3.4</td>
<td>Computational Times of Sensitivity Analysis of Simply-Supported Beam</td>
<td>64</td>
</tr>
<tr>
<td>3.5</td>
<td>Eigenvector Sensitivity Coefficients of a Uniform Beam</td>
<td>67</td>
</tr>
<tr>
<td>3.6</td>
<td>Eigenvector Sensitivity Coefficients of a Stepped Beam</td>
<td>67</td>
</tr>
<tr>
<td>3.7</td>
<td>Eigenvector Sensitivity Coefficients of a Continuous Beam with a Spring Support</td>
<td>68</td>
</tr>
</tbody>
</table>
3.8 Eigenvector Sensitivity Coefficients of a Stepped Beam with a Spring Support ......................................................... 68
3.9 Computational Times of Sensitivity Analysis of a Stepped Beam with a Spring Support ........................................ 69
3.10 Eigenvector Sensitivity Coefficients of Design Variable Set 1 at Joint A 73
3.11 Eigenvector Sensitivity Coefficients of Design Variable Set 2 at Joint A 73
3.12 Eigenvector Sensitivity Coefficients of Design Variable Set 3 at Joint A 74
3.13 Eigenvector Sensitivity Coefficients of Design Variable Set 4 at Joint A 74
3.14 Computational Times of Sensitivity Analysis with Case 1 ............ 74
5.1 Sensitivity Coefficients of Cost Functional ................................. 115
5.2 Sensitivity Coefficients of Geometric Constraint Functional ........... 116
5.3 Sensitivity Coefficients of Von-Mises Stress Functional of Element Number 10 .............................................................. 117
5.4 Von-Mises Stress Functions of Various Stage Design : case 1 ........... 123
5.5 Von-Mises Stress Functions of Various Stage Design : case 2 ........... 126
<table>
<thead>
<tr>
<th>FIGURES</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Coordinate Systems of a Beam in Two-Dimensional Space</td>
<td>13</td>
</tr>
<tr>
<td>2.2 A Cantilever Beam</td>
<td>34</td>
</tr>
<tr>
<td>2.3 Convergence Study of Eigenvalue Analysis</td>
<td>36</td>
</tr>
<tr>
<td>2.4 The Second Mode Shape of the Cantilever Beam</td>
<td>36</td>
</tr>
<tr>
<td>2.5 Convergence Study of First Eigenvalue Sensitivity Analysis</td>
<td>37</td>
</tr>
<tr>
<td>2.6 Convergence Study of Second Eigenvalue Sensitivity Analysis</td>
<td>37</td>
</tr>
<tr>
<td>2.7 Convergence Study of Third Eigenvalue Sensitivity Analysis</td>
<td>38</td>
</tr>
<tr>
<td>2.8 A Four-Member Frame</td>
<td>39</td>
</tr>
<tr>
<td>2.9 A Nineteen-Member Frame</td>
<td>42</td>
</tr>
<tr>
<td>2.10 A Fifty-one Bar Truss</td>
<td>44</td>
</tr>
<tr>
<td>2.11 Continuous Beams with Various Support Conditions</td>
<td>46</td>
</tr>
<tr>
<td>2.12 Example Problems for Design Optimization</td>
<td>52</td>
</tr>
<tr>
<td>3.1 A Simply-Supported Beam</td>
<td>60</td>
</tr>
<tr>
<td>4.1 Various Types of Configurations in a Stationary Cartesian Coordinate System</td>
<td>77</td>
</tr>
<tr>
<td>4.2 Finite Element Model of a Uniformly Loaded Beam</td>
<td>86</td>
</tr>
<tr>
<td>4.3 Relationship of Total Lagrangian Formulation and the Eulerian Formulation</td>
<td>87</td>
</tr>
<tr>
<td>4.4 Convergence History for Nonlinear Analysis</td>
<td>88</td>
</tr>
<tr>
<td>4.5 Various Configurations Resulted from Finite Element Analysis</td>
<td>89</td>
</tr>
<tr>
<td>4.6 A Prismatic Bar</td>
<td>101</td>
</tr>
</tbody>
</table>
5.1 Design Variables and Design Boundaries of a Beam .......... 110
5.2 A Conceptual Model of Design Optimization ................. 120
5.3 Flow Chart of Computational Scheme for Shape Optimization ...... 121
5.4 Various Stages in the Shape Optimization : case 1 .......... 124
5.5 Convergence History of Shape Optimization : case 1 .......... 125
5.6 Various Stages in the Shape Optimization : case 2 .......... 127
5.7 Convergence History of Shape Optimization : case 2 .......... 128
LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_i ), ( E_i ), ( I_i )</td>
<td>cross-sectional area, Young's modulus and moment of inertia of member ( i ), respectively</td>
</tr>
<tr>
<td>( \ell_i ), ( \theta_i ), ( \rho_i )</td>
<td>length, orientational angle and mass density of member ( i ), respectively</td>
</tr>
<tr>
<td>( b )</td>
<td>design variable</td>
</tr>
<tr>
<td>( m_i ), ( n_i )</td>
<td>cosine and sine of member ( i )</td>
</tr>
<tr>
<td>( N )</td>
<td>total number of members</td>
</tr>
<tr>
<td>( N_{\text{E}} )</td>
<td>total number of finite elements</td>
</tr>
<tr>
<td>( \bar{N} )</td>
<td>number of members connected at a joint</td>
</tr>
<tr>
<td>( P_i ), ( S_i ), ( M_i )</td>
<td>internal axial force, shear force and bending moment of member ( i ), respectively</td>
</tr>
<tr>
<td>( \tau )</td>
<td>monitoring parameter of the location variation</td>
</tr>
<tr>
<td>( u_i ), ( w_i )</td>
<td>eigenfunctions in the local coordinate system of member ( i )</td>
</tr>
<tr>
<td>( U_i ), ( W_i )</td>
<td>eigenfunctions in the global coordinate system of member ( i )</td>
</tr>
<tr>
<td>( v_i )</td>
<td>design velocity function for member ( i )</td>
</tr>
<tr>
<td>( (s_i, \theta_i) )</td>
<td>length and orientation of member ( i )</td>
</tr>
<tr>
<td>( (X, Y) )</td>
<td>location of a joint in the global coordinate system</td>
</tr>
<tr>
<td>([K],[M])</td>
<td>global stiffness and mass matrices</td>
</tr>
<tr>
<td>([K_i],[M_i])</td>
<td>elementary stiffness and mass matrices</td>
</tr>
<tr>
<td>( {X_i}, {x_i} )</td>
<td>eigenvectors in the global and local coordinate systems</td>
</tr>
<tr>
<td>([T_i])</td>
<td>elementary transformation matrix of element ( i )</td>
</tr>
<tr>
<td>( \hat{u}_i ), ( \hat{w}_i )</td>
<td>total shape derivatives of eigenfunctions in which the orientation, ( \theta_i ), of the member ( i ) remains unchanged</td>
</tr>
<tr>
<td>( u_{\tau}, w_{\tau} )</td>
<td>relative shape derivatives of eigenfunctions in the local coordinate system</td>
</tr>
</tbody>
</table>
\[ \dot{\lambda} \] total shape derivative of eigenvalue \( \lambda \)

\( J, \dot{J} \) a line functional \( J \) and its total shape derivative

\( \pi \) weak variational functional of an eigenvalue problem

\( \phi_i, \varphi_i \) testing functions in the local coordinate system

\( \Phi_i, \Psi_i \) testing functions in the global coordinate system

\( \Omega_i, R_i \) slopes at joint \( i \)

\( ^0x_i, ^1x_i, ^{+\Delta t}x_i \) Cartesian coordinate of a point in configuration time \( 0, t \) and \( t + \Delta t \)

\( ^{+\Delta t}u_{i,j}, ^{+\Delta t}u_{i,j} \) Derivative of \( i^{th} \) components of displacement vector in configuration time \( t + \Delta t \) with respect to the \( j^{th} \) component of coordinate vector \( ^0x, ^{+\Delta t}x \)

\( ^Iu_i, ^{+\Delta t}u_i \) Cartesian component of displacement vector in configuration \( t \) and \( t + \Delta t \)

\( ^0u_i \) Cartesian component of incremental displacement from \( t \) to \( t + \Delta t \)

\( ^{+\Delta t}\sigma_{ij} \) Cartesian component of the Cauchy stress tensor

\( ^{+\Delta t}\epsilon_{ij} \) Cartesian component of an infinitesimal strain tensor

\( \delta \) variation in

\( ^{+\Delta t}f^B_i, ^{+\Delta t}f^T_i \) components of the externally applied body and surface force vectors in configuration time \( t + \Delta t \)

\( u^p_i \) prescribed displacement on the assigned portions of the surface \( a_u \)

\( ^{+\Delta t}R \) external virtual work in configuration \( t + \Delta t \)

\( ^0V, ^{+\Delta t}V \) volume of body in configuration \( 0 \) and \( t + \Delta t \)

\( ^{+\Delta t}C_{ij}, ^{+\Delta t}C_{ij} \) component of stress-strain material property tensor at time \( t + \Delta t \) referred to configuration time \( 0 \) and \( t + \Delta t \)

\( ^{+\Delta t}\epsilon_{rs} \) component of Eulerian strain tensor measured at the deformed configuration \( t + \Delta t \)

\( ^{+\Delta t}\sigma_{0ij} \) component of the 2nd Piola-Kirchhoff stress tensor
\( t^+\Delta t \varepsilon_{ij} \) component of the Green-Lagrange strain tensor

\( \rho^{t^+\Delta t} \rho \) ratio of the mass densities at time 0 and time \( t + \Delta t \)

\( \dot{t}^{t^+\Delta t} \mathbf{R} \) external virtual work in configuration 0

\( u^p_i \) prescribed displacement on the assigned portions of the surface \( A_u \)

\( o S_{ij} \) incremental of the 2nd Piola-Kirchhoff stresses referred to time \( t \)

\( o e_{ij} \) incremental of the Green-Lagrange strains referred to time \( t \)

\( o e_{ij} \) linear incremental strain referred to the configuration at time 0

\( o \eta_{ij} \) nonlinear incremental strain referred to the configuration at time 0

\( o \hat{\epsilon}_{ij} \) linear incremental strain referred to the configuration at time \( t \)

\( o \hat{\eta}_{ij} \) nonlinear incremental strain referred to the configuration at time 0

\( o \sigma_{ij} \) incremental of the Cauchy stresses referred to time \( t \)

\( o E_{ij} \) incremental of the Eulerian strains referred to time \( t \).

\( \varepsilon_d \) pre-set tolerance for nonlinear finite element analysis

\( o \mathbf{v}(\mathbf{x}_i) \) design velocity field defined in the initial configuration

\( t^+\Delta t \mathbf{v}(t^+\Delta t \mathbf{x}_i) \) design velocity field defined in the deformed configuration
Chapter 1

INTRODUCTION

1.1 Overview and Objectives

The advanced state of art of the finite element structural analysis has provided a reliable tool for evaluation of the structural responses in the design stage. In its present form, however, it is used to identify technical problems, but it gives the designer little help in identifying ways to modify the design to avoid unexpected problems or improve design quality. Therefore, the common method of structural design involves decisions made by the designer, based on experience and intuition. This conventional way of structural design can be substantially enhanced if the designer is provided with design sensitivity information that explains the influence of design changes, without requiring trial and error.

Design sensitivity analysis aims to find the effects of the rate of design variables due to small modifications on the responses of structural systems. The structural responses such as displacement, stress, natural frequency, and buckling load are governed by state equations which are determined by the law of mechanics. In general, the structural responses are expressed as the nonlinear, implicit functions of design variables. The design variables are classified into two groups: sizing variables and shape variables. The sizing variables are related to local modifications of structures such as member's cross-sectional area, thickness or laminate angles in the case of composite material structures, etc. On the other hand, shape variables are concerned with the changes of the structural shape or configuration, which may be further classified into two types. One is the contour profile of a continuous solid, which can be described as a continuous function. The other is the joint
or support location of a skeletal structure, such as beam, truss and frame, etc. The shape design variables in this type are simply distinct parameters.

Many methodologies have been developed in the area of sensitivity analysis. Depending upon whether to differentiate or discretize first, there appears to be two major categories, the continuum approach and the discrete approach. In the first approach, the continuum approach, the sensitivity equation is first derived based upon the weak variational form of the governing differential equation which is subsequently discretized in order to obtain sensitivity coefficients numerically. On the other hand, the discrete approach derives sensitivity equation based upon the matrix equation obtained from spatial discretization of the governing differential equation. The continuum approach can be further classified into two methods which are the domain method and the boundary method. The sensitivity equations derived by the domain method are usually expressed by integral forms in terms of field variables. On the other hand, shape sensitivity equations derived by the boundary method are usually algebraic equations in terms of quantities defined at the boundaries. In the discrete approach, the sensitivity equation can be obtained either by differentiating the matrix equations with respect to the design variables (the discrete analytical method) or by using the finite difference method.

Another way to classify the method for sensitivity analysis is based upon the type of linear equation to be solved for sensitivity coefficients. Two methods are generally mentioned in reference: the direct differentiation method and the adjoint variable method. The direct differentiation method sets up a linear equation in which the unknown parameters to be solved are the derivatives of state variables with respect to design variables. The adjoint variable method, on the other hand, derives an adjoint equation to calculate the adjoint variables which can facilitate the computation of sensitivity coefficients of the functionals of concern.

The main objective of this dissertation is to develop unified formulations and computational schemes for shape design sensitivity analysis of skeletal structures and
geometrically nonlinear solids. Throughout the presentation, the continuum approach plays a central role in the development of various sensitivity equations. In this study, the direct differentiation method is employed for eigenvalue and eigenvector sensitivity analysis for skeletal structures and the adjoint variable method is applied to find the sensitivity equation of a general functional defined in the deformed configuration of a geometrically nonlinear solid.

In summary, the first objective of this dissertation is to derive explicit design sensitivity equations of eigenvalues/vectors with respect to configuration parameters such as joint and support locations of skeletal structures by the domain method as well as the boundary method. The second objective of this dissertation is to derive an appropriate shape sensitivity equation and a design optimization scheme suitable for designing the shape of a geometrically nonlinear solid whose performance criteria are defined in the deformed configuration. The resultant sensitivity equations are first validated by simple problems whose analytical solutions can be easily obtained. These sensitivity equations will then be investigated for their numerical accuracy, computational efficiency and their implementation in design optimization. All the analysis works are supported by the finite element method.

1.2 Scope

The dissertation is organized into two parts corresponding to the two objectives outlined in the previous section. In the first part including Chapters 2 and 3, specific attention is given to develop shape design sensitivity equations of eigenvalues and eigenvectors, by both the domain and the boundary methods, with respect to support and joint locations of skeletal structures. In Chapter 2, the governing differential equations of skeletal structures and associated boundary conditions are first outlined. The concept of the material derivative is employed in this study to develop the basic relations for shape variation of structural members. Two sets of equations are derived based upon the
continuum approach for eigenvalue sensitivity analysis of skeletal structures with variable joint and/or support locations. An analytical example of a cantilever beam is provided to study the accuracy of the obtained formulations. Several numerical examples are also presented to investigate the performances of those derived sensitivity equations. At the end of Chapter 2, the developed sensitivity equation is implemented into a design optimization routine, LINRM, to study the effects of support variables on the design of a vibrating continuous beam. Furthermore, in Chapter 3, the methodology presented in Chapter 2 is extended and applied to eigenvector sensitivity equations of skeletal structures with respect to joint and/or support locations.

In the second part of this work, including Chapters 4 and 5, a new shape sensitivity analysis and a new optimization scheme for the design of geometrically nonlinear elastic solids will be presented. This development is centered around a notion that the shape design optimization formulation of a geometrically nonlinear solid should be defined in the deformed configuration. In Chapter 4, both the total Lagrangian and Eulerian formulations for analysis of an elastic solid subjected to large displacements, large rotations and small strains are outlined. It then introduces a new shape sensitivity formulation in which the functional interest is defined in the deformed configuration. At the end of Chapter 4, an analytical example of a one dimensional prismatic bar is presented to validate the derived sensitivity equations. The sensitivity equation derived in Chapter 4 is then implemented in a new design optimization procedure in Chapter 5 in order to produce a optimal shape of a nonlinear solid. In this procedure, the deformed configuration of the solid is considered as an intermediate design variable. To demonstrate and validate the new design optimization procedure, a shape optimization problem of a uniformly loaded beam is used to serve as a numerical example in which the geometry and stress constraints are concerned.

Conclusions from this dissertation and recommendations for future research are presented in Chapter 6.
1.3 Literature Review

Sensitivity analysis has been emerging as a fruitful area of engineering research recently. The reason for this interest is the recognition of the variety of uses for sensitivity derivatives. In the early stages, sensitivity analysis found its predominant use in assessing the effect of varying parameters in the mathematical models of control systems [1,2,3] and Randanovic [4] gave discussions of the early development of sensitivity theory. Interest in optimal control in the early 1960s [5] and automated structural optimization [6] led to the use of gradient-based mathematical programming methods in which derivatives were used to find search directions toward optimum solutions. The voluminous publications [7,8,9] in engineering applications which are related to sensitivity analysis are available. In the literature review to follow, however, the focus of attention is in shape sensitivity analysis and design optimization associated with the eigenfunctions of skeletal structures and geometrically nonlinear solids.

1.3.1 Literature Review for Eigensensitivity Analysis and Design Optimization

Eigensensitivity analysis is concerned specifically with the rates of changes of eigenvalues and eigenvectors with respect to design variables. The rates of changes are expressed as partial or full derivatives of eigenvalues/vectors with respect to design variables. In most treatments of eigenvalues/vectors design sensitivity problems, the discrete approach has been routinely applied in the literature. The first result on eigenvalue derivatives was developed by Jacobi [10]. One of the earliest and still very popular method for calculating mode shape derivatives was developed by Fox and Kapoor [11]. Fox and Kapoor's method, sometimes referred to as the modal expansion method, represents an eigenvector derivative as a linear combination of all the eigenvectors. In most engineering practices this is not possible and must be approximated by a subset of total eigenvectors. Nelson [12] was the first to develop an exact method for calculating the eigenvalue and
eigenvector derivatives that required only the associated eigenvalue/vector pair and other known quantities. Vanhonacker [13] has used the theory of adjoint structures to derive formulas for derivatives of eigenvalues and eigenvectors of structures. Cardani and Mantegazza [14] extended Nelson's method to transcendental flutter eigenvalue problems. Derivatives of nonlinear buckling eigenvalues were obtained by Kamat and Ruangsiliasingha [15]. Sutter et al. [16] presented four methods for the calculation of derivatives of vibration mode shapes (eigenvectors) with respect to design variables. These are finite difference method, modal expansion method, a modified modal expansion method and Nelson's method. Results indicated an advantage in using Nelson's method because this method is exact and requires less CPU times, especially when derivatives with respect to several design variables are computed.

The discrete approach for eigensensitivity analysis, as surveyed above, is very general and applicable to either sizing or shape variables. However, for structural applications they draw two disadvantages [17]. First, not all structural analysis solution methods resort to the discretized equations. For example, shell-of-revolution codes such as FASOR [18] directly integrate the equations of equilibrium without first converting them to the system of algebraic equations. Second, operating on the discretized equations often requires access to the source code of the structural analysis program which implements these equations. Unfortunately, many of the popular structural analysis programs do not provide such access to most users. It is desirable, therefore, to have sensitivity analysis methods that are generally applicable and can be implemented without extensive access to any knowledge of the insides of structural analysis programs. The continuum approach achieves this goal by differentiating the equations governing the structure before they are discretized. The resulting sensitivity equations can then be solved with the aid of a structural analysis program.

A textbook dedicated to design sensitivity analysis of structural systems was published by Haug and co-authors [19]. It provided an excellent section of the continuum
approach for shape sensitivity analysis of eigenvalues. The material derivative idea of continuum mechanics is used in the textbook to predict the effects of shape changes on functionals that define structural responses. In this book, an example of simply-supported beam is provided. The length of a beam is considered as a design variable to develop the eigenvalue sensitivity equation in terms of boundary quantities by the boundary method. Besides this textbook, only a limited number of publications reported the results done by the continuum approach for shape sensitivity analysis with respect to support and joint locations. Garstecki and Mroz [20] derived sensitivity equations with respect to the support variables. However, their efforts are limited to the static responses of structures. Recently, Choi and Twu [21] used the domain method to derive sensitivity equations of static responses of built-up structures in which the joint locations are considered as design variables.

A large amount of literature related to the design optimization of structures with frequency constraints are available. Most of these papers presented design algorithms based on single or multiple frequency constraints. Khan et al. [22] and Grandhi et al. [23], and a few others had successfully applied the optimality criterion method for the minimum weight design of mechanical and structural systems subject to stress and natural frequency constraints. On the other hand, Rubin [24], Brach [25], Haug [26,27], Turner [28], Cassis [29,30] and Vanderplaats [31] used the mathematical programming methods for natural frequency constrained problems. The design variables studied in these papers are usually the sizing variables. Publications are also available to deal with another type of design variables such as the shape, position or layout of structural members. Mitchel [32] in 1904, presented perhaps the best-known classical treatise on shape optimization in frame structures, while the first work using modern numerical techniques was possibly presented by Dorn, Gemory and Greenberg in 1964 [33]. This was followed by other works, notably that of Dobbs and Felton [34], Pederson [35], Vanderplaats and Moses [36], Lipson, et al. [37], Spillers [38], Imai and Schmit [39], Lev [40], and Felix and Vanderplaats [41]. Each
of the foregoing referenced works dealt with shape optimization of trusses. Moreover, Topping [42] presented a state of art review of methods of topological design applied to skeletal elastic structures. The sensitivities or derivatives of stresses, deflections and natural frequencies in the above papers were determined with respect to member areas and joint coordinates by the discrete approach.

1.3.2 Literature Review for Nonlinear Analysis. Nonlinear Sensitivity Analysis and Design Optimization

The increased importance of nonlinear analysis is largely due to the emphasis placed by agencies on realistic modeling and accurate analysis of critical structural components as they arise, for example, in the safety deliberations of strategic structures and nuclear reactor components, and the design of satellites. Basically, two different approaches have been pursued in nonlinear solid/structural analysis. The first one is the Lagrangian formulation in which both static and kinematic variables are referred to the initial configuration. Three incremental forms of the Lagrangian description are noted: the total Lagrangian formulation [43,44,45], the updated Lagrangian formulation [43,45,46] and the general Lagrangian formulation [47]. The second one is generally called the Eulerian formulation. In the Eulerian formulation, the static and kinematic variables are referred to the current configuration. The Lagrangian formulation is commonly used in solids and structures, while the Eulerian formulation is usually employed in the analysis of fluid mechanics problems [48,49], in which attention is focused on the motion of the material through a stationary control volume. Only little effort concerns the development of an Eulerian formulation for analyzing solids and structures, not to mention the shape sensitivity analysis. Gadala et al. [50] presented a consistent Eulerian formulation of large deformation in static and dynamics structural problems. The final incremental form is obtained from the energy balance equation. However, no example can be found in the paper [50] to demonstrate the use of the incremental form. Furthermore, a mixed Eulerian-Lagrangian...
displacement model of large-deformation analysis in solid mechanics has been proposed by Haber [51] to frictional contact and fracture mechanics.

The theory of sizing and shape design sensitivity analysis for a linear elastic structural system has been well developed [9,19,52 and literature cited therein] over the last decade and a half. However, sensitivity analysis for nonlinear systems is just beginning to be investigated and the literature on the subject is beginning to grow. The discrete approach of design sensitivity analysis of nonlinear structures has been developed by Arora and co-authors [53,54]. The continuum approach of the sensitivity analysis with sizing variables has also been investigated by several workers. Mroz and co-workers [55] presented a general variational formulation for design sensitivity analysis of geometrically and material nonlinear beams and plates using the total Lagrangian formulation. They applied their theory to optimize a clamped-clamped beam under a uniformly distributed load. Arora and co-authors [56] performed a design sensitivity analysis of nonlinear structures using ADINA for analysis. A unified structural design sensitivity analysis method for nonlinear structural systems were presented by Choi and Santos [57].

For sensitivity analysis of nonlinear solids with shape variables, however, only a few articles have appeared in the literature. The papers in References 54 and 58 briefly discussed shape variations. General shape variation problems were treated in Mroz [59] using the material derivative approach. Santos in his dissertation [60] using the continuum approach and the material derivative concept derived the shape design sensitivity equations. It should be noted that the derivations of nonlinear sensitivity analysis presented in the cited papers are based on displacement-based finite element models and the Lagrangian formulation. Further, the design variables used are quantities referenced to the undeformed (initial) configuration. However, there are cases for which design criteria may be necessary to be defined in the deformed configuration. One such a case is in the design of a vehicle tire which is always deformed in the working environment. Phelan [61] was the first researcher, to the author's knowledge, to look into the shape sensitivity analysis using the
Eulerian formulation. The incremental form for the Eulerian formulation is presented by Phelan for determining the responses of a nonlinear elastic solid. Based upon the mutual Hu-Washizu energy principle, shape design sensitivity analysis based upon this Eulerian formulation was also developed by Phelan using the adjoint variable method. In his work, instead of using the concept of material derivative for shape sensitivity analysis, Phelan first translated all the functionals to the reference configuration and took variations which included the variation of Jacobian matrix. Phelan's work provided only a simple analytical example involving a prismatic bar to demonstrate the Eulerian formulation in nonlinear analysis. Issues regarding numerical implementation of shape design sensitivity analysis and integration of sensitivity equation to a design optimization problem were not discussed in his work. These issues are to be investigated here.
Chapter 2

EIGENVALUE SENSITIVITY ANALYSIS OF SKELETAL STRUCTURES WITH VARIABLE JOINT AND SUPPORT LOCATIONS

Various equations are developed in this chapter for eigenvalue sensitivity analysis of skeletal structures with variable joint and support locations. In this work, the emphasis is placed upon the sensitivity equations derived by the continuum approach based on the weak variational form of the governing differential equation. The governing differential equations and their boundary conditions of skeletal structures, such as planar frames, planar trusses and continuous beams are outlined in Section 2.1. The variational form of a general eigenvalue equation is first derived for a typical element in which all of the quantities are expressed in the local coordinate system which is attached to each member. Subsequently, the basic concept of the material derivative and the fundamental relations of shape variations with respect to the length and the orientation of a component member are introduced in Section 2.2. Material derivative of the variational form is then sought to account for changes in member's length and orientation resulting from the perturbations of joint and support locations. In Section 2.3, the eigenvalue sensitivity equations for skeletal structures are formulated in domain quantities by the domain method and in boundary quantities by the boundary method. Both the domain and the boundary methods are categorized as the continuum approach. The finite difference method and the discrete analytical method which are summarized in Section 2.4 are generally categorized as the discrete approach. Analytical and numerical examples are provided in Section 2.5 to investigate the performance of the derived sensitivity equations.

At the end of this chapter, the design optimization problem of a vibrating beam is investigated. The objective function of the design optimization problem is to minimize the
combination of the weight of the structure and the stiffness of the support springs, subjected to the constraints that require the first three frequencies of the beam equal to the assigned values. The design variables include sizing variables, support locations and spring stiffness constants. Results suggest that a small amount of adjustment in support locations and support spring stiffness constants can greatly improve the quality of the design.

2.1 Governing Equations of Skeletal Structures

2.1.1 Planar Frame

Let a planar frame consist of N straight members, each of which is confined by a pair of joints. A local coordinate systems, (s,θ), can be introduced for each member, in which the s-axis is the member axis and the θ-axis is the orientation angle of the member. Accordingly, the free vibration of each member of a planar frame can be decomposed into axial and lateral components. More specially, the governing differential equations of free vibration of a planar frame structure with N members can be stated as follows:

\[ E_i A_i u''_i + \lambda \rho_i A_i u_i = 0, \quad i = 1, 2, 3, \ldots, N \]  \hspace{1cm} (2.1)

\[ E_i I_i w'''_i - \lambda \rho_i A_i w_i = 0, \quad i = 1, 2, 3, \ldots, N \]  \hspace{1cm} (2.2)

where \( \lambda, E_i, I_i, \rho_i \) and \( A_i \) are the eigenvalue, Young's modulus, moment of inertia, mass density and cross-sectional of each member, respectively. Furthermore, the superscript "\( ' \)" denoted differentiation along the s-axis. Equation (2.1) describes an axial vibration where \( u_i(s) \) is the corresponding eigenfunction. Similarly, Eq. (2.2) denotes a lateral vibration with \( w_i(s) \) as the corresponding eigenfunction. To uniquely specify \( u_i(s) \) and \( w_i(s) \), the eigenfunctions should satisfy the normalization condition

\[ \sum_{i=1}^{N} \int_{0}^{\ell_i} \rho_i A_i (u_i^2 + w_i^2) ds = 1 \]  \hspace{1cm} (2.3)

where \( \ell_i \) is the length of member i.
The weak variational form, $\pi$, of Eqs. (2.1) and (2.2) can be stated as

$$\pi = 0$$

$$= \sum_{i=1}^{N} \int_{0}^{l_i} \left[ (E_i A_i u''_i + \lambda \rho_i A_i u_i) \phi_i + (\lambda \rho_i A_i w_i - E_i I_i w'''_i) \varphi_i \right] ds$$

$$= \sum_{i=1}^{N} \int_{0}^{l_i} \left[ \lambda \rho_i A_i (u_i \phi_i + w_i \varphi_i) - E_i (A_i u'_i \phi'_i + I_i w''_i \varphi'_i) \right] ds$$

$$+ \sum_{i=1}^{N} (E_i A_i u'_i \phi_i - E_i I_i w''_i \varphi_i + E_i I_i w''_i \varphi_i) \bigg|_{0}^{l_i}$$

(2.4)

where $\phi_i$ and $\varphi_i$ are any testing functions with proper regularity. Note that all quantities in the above equations are defined in terms of the local coordinates system of the corresponding member.

![Figure 2.1 Coordinate Systems of a Beam in Two-Dimensional Space](image)

Next, in order to properly define the boundary conditions, the boundary terms in Eq. (2.4) should be expressed in the global coordinate system. Equations that relate the local eigenfunctions, $(u_i, w_i, u'_i, w'_i)$, to their counterparts in the global coordinate system, $(U_i, W_i, U'_i, W'_i)$, are defined as
where $m_i = \cos \theta_i$ and $n_i = \sin \theta_i$, in which $\theta_i$ is defined as the orientation angle between the global $X$-axis and the local $s$-axis of member $i$ as shown in Fig. 2.1. Note that Eqs. (2.5) and (2.6) are also applicable for testing functions, $(\Phi_i, \Psi_i)$, in the local coordinate system as well as their counterparts in the global coordinate system, $(\Phi_i', \Psi_i')$.

With the aid of the above relations, the boundary terms in Eq. (2.4) can be rewritten as

$$\sum_{i=1}^{N} \left[ (P_i m_i - S_i n_i) \Phi_i + (P_i n_i + S_i m_i) \Psi_i + M_i \Omega_i \right] = 0$$

(2.7)

where $P_i$, $S_i$ and $M_i$ are defined as the internal axial force, shear force and bending moment at the end point of member $i$, respectively, i.e.,

$$P_i = E_i A_i u_i'$$
$$S_i = -E_i I_i w_i''$$
$$M_i = E_i I_i w_i'''$$

(2.8)

and $\Omega_i$ is the slope at the same point,

$$\Omega_i = -n_i \Phi_i' + m_i \Psi_i'$$

(2.9)

It should be noted that $(\Phi_i, \Psi_i, \Omega_i)$ have unique values at each joint. Therefore, either the kinematic boundary conditions can be defined at the support joints or the natural boundary conditions which require the balance of internal forces at the interior joints can be specified. The natural boundary conditions at each interior joint are in fact given as
\[
\sum_{i=1}^{N} (P_i m_i - S_i n_i) = 0 \\
\sum_{i=1}^{N} (P_i n_i + S_i m_i) = 0 \\
\sum_{i=1}^{N} M_i = 0
\] (2.10)

where \( N \) symbolically denotes the number of members connected as a joint.

### 2.1.2 Planar Truss

The state equation of free vibration of a planar truss can be expressed in the same form as those of a planar frame presented in Section 2.1.1. In a planar truss, however, only axial vibration needs to be considered. The free vibration of a planar truss structure with \( N \) member is, in this case,

\[
E_i A_i u''_i + \lambda \rho_i A_i u_i = 0, \quad i = 1, 2, \ldots, N
\]

(2.11)

The normalization condition of the truss structure can be stated as

\[
\sum_{i=1}^{N} \int_0^{\ell_i} \rho_i A_i u_i^2 \, ds = 1
\]

(2.12)

where \( \ell_i \) is the length of member \( i \). Referring to Eq. (2.4), it should be noted that the weak variational form of the governing differential equation, \( \pi \), can be simplified as

\[
\pi = 0 = \sum_{i=1}^{N} \int_0^{\ell_i} (\lambda \rho_i A_i u_i \phi_i - E_i A_i u_i \dot{\phi}_i \dot{u}_i) \, ds + \sum_{i=1}^{N} E_i A_i u_i \phi_i \bigg|_{0}^{\ell_i}
\]

(2.13)

where \( \phi_i \) is any arbitrary function with proper regularity. The local eigenfunction, \( u_i \), and their corresponding eigenfunctions in the global coordinate, \((U_i, W_i)\), has the relation shown in Fig. 2.1 as

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ u_i = m_i U_i + n_i W_i \] (2.14)

where \( m_i = \cos \theta_i \) and \( n_i = \sin \theta_i \), in which \( \theta_i \) is defined as the orientation angle between the global X-axis and the local s-axis of member \( i \). The same relation is applicable for the arbitrary function \( \Phi_i \) in the local coordinate system and their respective quantities, \( (\Phi_i, \Psi_i) \), defined in the global coordinate system.

With the aid of the above relations, the boundary terms in Eq. (2.13) can be rewritten as

\[ \sum_{i=1}^{N} (P_i m_i \Phi_i + P_i n_i \Psi_i) \phi_i \] (2.15)

where \( P_i \) is the internal axial force, i.e., \( P_i = E_i A_i u'_i \). It should be noted that \( (\Phi_i, \Psi_i) \) have unique values at each joint. Thus, either the kinematic boundary conditions defined at the support joints or the natural boundary conditions which require the balance of interior forces at the joints can be specified. The natural boundary conditions at each interior joint are given as

\[ \sum_{i=1}^{N} P_i m_i = 0 \]
\[ \sum_{i=1}^{N} P_i n_i = 0 \] (2.16)

where \( N \) symbolically denotes the number of members connected at a joint.

### 2.1.3 Continuous Beam

The continuous beam is a special case of a planar frame system in which only transverse effect is considered. The free vibration of a continuous beam with \( N \) member is

\[ E_i I_i w'''_i - \lambda \rho_i A_i w_i = 0, \quad i = 1, 2, 3, \ldots, N. \] (2.17)

The normalization condition of the eigenfunctions of a continuous beam can be stated as

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where \( \xi_i \) is the length of member \( i \). The weak variational form of the continuous beam system can be obtained directly from Eq. (2.4) as

\[
\pi = 0 = \sum_{i=1}^{N} \int_0^{\xi_i} \rho_i A_i w_i^2 ds = 1
\]  

(2.18)

where \( \varphi_i \) is any arbitrary function with proper regularity.

It should be mentioned that no coordinate transformation is needed in one-dimensional continuous beam system. The boundary conditions can be specified at each support location as

\[
\varphi_i |_{s^-} = \varphi_i |_{s^+},
\]

and

\[
\varphi_i |_{s^-} = 0
\]  

(2.20)

where \( s^- \) and \( s^+ \) are the locations just on the left-hand side and the right-hand side of the support point. The boundary conditions state that the slope has to be continuous at the support.

### 2.2 The Concept of the Material Derivative

The concept of the material derivative has been proven to be very useful in shape sensitivity analysis [19,57]. In this work, this concept will be extended to derive eigenvalue sensitivity equations of skeletal structures. Before doing so, however, a brief introduction of this concept is in order.

The shape variation of any point in a varied domain may be viewed as a continuous transformation phenomenon between its final location, \( x^* \), and the original one, \( x \). The position vector of the point at any intermediate stage can be expressed as
where \( x \) is the position vector and \( \tau \) is a monitoring parameter, ranging from 0 to 1 with \( x(0)=x \) and \( x(1)=x^* \). Note that only the linear variation is retained in Eq. (2.21). Mathematically, the term \( v(x) \) is defined as \( \dot{x} = v(x) = \frac{dx(t)}{dt} \) evaluated at \( \tau=0 \) and is called the design velocity function [19]. As an example, the variations of a joint location, \((X,Y)\), can be given as

\[
\begin{align*}
X(\tau) &= X + \tau \dot{X} \\ Y(\tau) &= Y + \tau \dot{Y}
\end{align*}
\] (2.22)

where the shape derivatives of \( \dot{X} \) and \( \dot{Y} \) are identical to the perturbations of a joint location, i.e.,

\[
\begin{align*}
\dot{X} &= X^* - X = \Delta X \\ \dot{Y} &= Y^* - Y = \Delta Y
\end{align*}
\] (2.23)

With the preceding definitions, one can further obtain the shape derivative of the orientation, \( \theta_i \), as

\[
\dot{\theta}_i = -\frac{n_i}{L_i}(\dot{X}_2 - \dot{X}_1) + \frac{m_i}{L_i}(\dot{Y}_2 - \dot{Y}_1)
\] (2.24)

The detailed derivation of the above equation is shown in Appendix A.

The definition of Eq. (2.21) can be directly extended to any domain-dependent function. In any stage of domain transformation, a function, \( z(x) \), can be represented as \( z(x(\tau), \tau) \) whose shape derivative at \( \tau=0 \) is given as
\[ \dot{z}(x) = \dot{z}(x(\tau), \tau) |_{\tau=0} \]
\[ = \frac{dz}{d\tau} |_{\tau=0} \]
\[ = \lim_{\tau \to 0} \frac{z(x + \tau v, \tau) - z(x)}{\tau} \]
\[ = \frac{\partial z}{\partial \tau} + \frac{\partial z}{\partial x} \cdot v \]
\[ = z_{\tau} + \nabla z \cdot v \]

(2.25)

where the subscript "," and the notation "\nabla" denote the partial and spatial derivatives, respectively. The last identity states that the total shape variation of a domain-dependent function is a combination of the variation of the function itself and the variation induced by the perturbations of locations of points in the domain. The detailed discussion of the aforementioned equation can be found in Reference 19 in which \( z_{\tau} \) is called the relative shape derivative and \( \dot{z} \) is called the total shape derivative of function \( z(x) \).

Consider a two-dimensional skeletal structure whose configuration is defined by not only the length between a pair of joints but also its orientation. Therefore, Eq. (2.25), which only accounts for the variation in a fixed coordinate system, should be applied herein with modifications. Using a planar frame as an example, the total shape derivatives of eigenfunctions, \( \mathbf{U}_j \) and \( \mathbf{W}_j \) in Eq. (2.5) should become

\[
\begin{bmatrix}
\dot{\mathbf{u}}_i \\
\dot{\mathbf{w}}_i
\end{bmatrix} = \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \mathbf{U}_i + \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \mathbf{W}_i \\
\end{bmatrix} + \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \dot{\mathbf{U}}_i + \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \dot{\mathbf{W}}_i \\
\end{bmatrix}
\]

(2.26)

\[
\begin{bmatrix}
\dot{\mathbf{u}}_i \\
\dot{\mathbf{w}}_i
\end{bmatrix} = \begin{bmatrix}
-\mathbf{m}_i \\
-\mathbf{n}_i
\end{bmatrix} \mathbf{U}_i + \begin{bmatrix}
-\mathbf{m}_i \\
-\mathbf{n}_i
\end{bmatrix} \mathbf{W}_i \\
\end{bmatrix} + \begin{bmatrix}
\mathbf{m}_i \\
\mathbf{n}_i
\end{bmatrix} \dot{\mathbf{U}}_i + \begin{bmatrix}
\mathbf{m}_i \\
\mathbf{n}_i
\end{bmatrix} \dot{\mathbf{W}}_i \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\mathbf{u}}_i \\
\dot{\mathbf{w}}_i
\end{bmatrix} = \begin{bmatrix}
\mathbf{m}_i \\
\mathbf{n}_i
\end{bmatrix} \mathbf{U}_i + \begin{bmatrix}
\mathbf{m}_i \\
\mathbf{n}_i
\end{bmatrix} \mathbf{W}_i \\
\end{bmatrix}
\]

(2.27)

where the symbols, \( \dot{\mathbf{u}}_{s,i} \) and \( \dot{\mathbf{w}}_{s,i} \), are defined as

\[
\begin{bmatrix}
\dot{\mathbf{u}}_{s,i} \\
\dot{\mathbf{w}}_{s,i}
\end{bmatrix} = \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \dot{\mathbf{U}}_i + \begin{bmatrix}
m_i \\
n_i
\end{bmatrix} \dot{\mathbf{W}}_i \\
\end{bmatrix}
\]

(2.28)
It should be noted that $\dot{u}_i$ and $\dot{w}_i$ are completely different from $\dot{u}_{s,i}$ and $\dot{w}_{s,i}$. The former are the total shape derivatives of the eigenfunctions whereas the latter are the shape derivatives of the eigenfunctions in which the orientation of the member, $\theta_0$, remains unchanged. The preceding equations clearly indicate that the domain variation of a function pertaining to a planar frame member is comprised of two parts. One is related to the effect of orientation variation, $\theta_0$, and the other is due to the domain variations, $\dot{u}_{s,i}$ and $\dot{w}_{s,i}$, along the local coordinate system. Furthermore, with the aid of Eq. (2.25), the total shape derivatives of $u_i$ and $w_i$ can also be represented in terms of the relative shape derivatives, $u_{i,\tau}$ and $w_{i,\tau}$ as

$$
\begin{bmatrix}
\dot{u}_i \\
\dot{w}_i 
\end{bmatrix} = \dot{\theta}_i \begin{bmatrix}
-w_i \\
-w_{i,\tau}
\end{bmatrix} + v_i \begin{bmatrix}
u'_i \\
w'_i
\end{bmatrix}
$$

(2.28)

where $v_i$ is the design velocity function defined along the s-axis of the local coordinate system. In the same manner, the total shape derivatives of other quantities such as $u'_s$, $w'_s$ and $w''_s$, can be derived. These are given in the Appendix A for reference.

Next, one may proceed to find the total shape derivative of a typical line functional defined over a straight line in a two-dimensional frame system as

$$J(u, w, u', w'', \tau) = \int_0^\ell f(u, w, u', w'', \tau) ds$$

(2.29)

where functions $u(s)$ and $w(s)$ are defined with respect to the local coordinate system. Its total shape derivative can be shown in terms of $\dot{\theta}$, $\dot{u}_s$ and $\dot{w}_s$ as

$$J = \int_0^\ell \left( \frac{df}{d\tau} + \left( \frac{df}{du} w - \frac{df}{dw} u + \frac{df}{du'} u' + \frac{df}{du''} u'' \right) \dot{\theta} + \frac{df}{du} \dot{u}_s + \frac{df}{dw} \dot{w}_s \\
+ \frac{df}{du'} \dot{u}_s' + \frac{df}{dw'} \dot{w}_s' - \left( \frac{df}{du'} u' + 2 \frac{df}{dw'} w'' - f \right) \nu' - \frac{df}{dw''} w' \nu'' \right) ds$$

(2.30)

or in terms of $\dot{\theta}$, $u_\tau$ and $w_\tau$ as
The detailed derivation of the preceding equations is given in the Appendix A. These equations lay the groundwork for the derivation of eigenvalue sensitivity equations of a skeletal structure with respect to joint or support locations. Equation (2.30) is the basic equation used in the domain method, while Eq. (2.31) is used in the boundary method.

2.3 Continuum Approach

In the following section, eigenvalue shape sensitivity equations are derived by the domain method as well as the boundary method. Although these two methods follow a similar derivation procedure, they result in sensitivity equations of different forms.

2.3.1 Domain Method (DM)

Planar Frame

Let the admissible function \( \phi_i \) and \( \phi_i \) in Eq. (2.4) be the eigenfunctions themselves, i.e., \( u_i \) and \( w_i \). Consequently, the weak variational form of free vibration is then reduced to

\[
\pi = 0 \\
= \sum_{i=1}^{N} \int_0^l \left( \lambda \rho_i A_i (u_i^2 + w_i^2) - E_i (A_i u_i'^2 + I_i w_i''^2) \right) ds
\]

(2.32)

where the boundary terms have been dropped because the kinematic and boundary conditions are satisfied at the supports as well as the interior joints.

Since Eq. (2.32) is a special case of Eq. (2.29), Eq. (2.30) can be applied here to find the total shape derivative of \( \pi \). The resultant equation, \( \pi = 0 \), yields the following identity
where \( \dot{\lambda} \) is the total shape derivative of the eigenvalue.

Incorporating with the normalization condition of Eq. (2.3), the left side of the above equation is simply equal to \( \dot{\lambda} \). Furthermore, the last two terms in Eq. (2.33) can be simplified by means of integration by parts, yielding

\[
\begin{align*}
\sum_{i=1}^{N} \int_0^{t_f} & 2(\lambda i A_i (u_{i}'' + w_{i}''') - E_i I_i w_{i}'''') \, \text{d}s - \sum_{i=1}^{N} \int_0^{t_f} 2(\lambda i A_i w_{i}'' w_{i}'') \, \text{d}s \\
= & -\sum_{i=1}^{N} \int_0^{t_f} 2(\lambda i A_i u_i + E_i A_i u_i'') \, \text{d}s - \sum_{i=1}^{N} \int_0^{t_f} 2(\lambda i A_i w_i - E_i I_i w_{i}'''') \, \text{d}s \\
+ & \sum_{i=1}^{N} 2(\lambda i A_i u_i' u_i + E_i I_i w_{i}'''') \, \text{d}s
\end{align*}
\]

(2.34)

The integrals in Eq. (2.34) are dropped as the terms in the parentheses are exactly identical to the state equations presented by Eqs. (2.1) and (2.2). Further simplification of Eq. (2.34) is in order. The first step in this effort is to replace the terms, \( E_i A_i u_i' + E_i I_i w_{i}''' \) and \( E_i I_i w_{i}'' \) by internal forces, \( P_i, S_i \), and \( M_i \), respectively, as defined by Eq. (2.8). The next step is to convert the terms of \( \dot{u}_{s,i}, \dot{w}_{s,i} \) to their counterparts in the global coordinate system. In the end, one obtains the following equality

\[
\sum_{i=1}^{N} 2(\lambda i A_i u_i' u_i + E_i I_i w_{i}'''') \, \text{d}s
= 2 \sum_{i=1}^{N} \left[ (P_i m_i - S_i n_i) \dot{U}_i + (P_i n_i + S_i m_i) \dot{W}_i + M_i (\ddot{R}_i + u_i' \dot{\theta}_i + v_i' \omega_i) \right]_0^{t_f}
\]  

(2.35)
where $\hat{R}_i$ is the shape derivative of the slope, $R_i$, i.e., $R_i = -n_i U'_i + m_i W'_i$. Now, note that $U_i$, $W_i$, and $R_i$ are equal to zero at each of the built-in supports. Hence, one has $U_i = W_i = R_i = 0$ at the supports. Furthermore, since $U_i$, $W_i$, and $R_i$ have unique values at the interior joints, the natural boundary conditions stated in Eq. (2.10) can be applied here to eliminate all the terms in Eq. (2.35) except the last two, $\sum_{i=1}^{N} 2M_i (u'_i \dot{\theta}_i + v'_i w'_i) \|_{\theta=0}^t$.

Finally, after this manipulation, the sensitivity equation given by Eq. (2.33) can be rewritten in a shorter form as

$$\dot{\lambda} = -\sum_{i=1}^{N} \int_{0}^{t} 2(E_i I_i u'_i w''_i - E_i A_i u'_i w'_i) \dot{\theta}_i ds + \sum_{i=1}^{N} 2E_i I_i w''_i (u'_i \dot{\theta}_i + v'_i w'_i) \|_{\theta=0}^t$$

$$-\sum_{i=1}^{N} \int_{0}^{t} [(\lambda_\rho A_i (u_i^2 + w_i^2) + E_i A_i u_i'^2 + 3E_i I_i w''_i^2) v'_i + 2E_i I_i w''_i w'_i v'_i] ds$$

(2.36)

which states that the shape derivative of $\lambda$ is a linear functional of $v_i$ and $\dot{\theta}_i$. Nevertheless, further simplification is still possible if the following integration by parts is performed

$$\int_{0}^{t} E_i I_i u'_i w''_i \dot{\theta}_i ds = E_i I_i w''_i u'_i \dot{\theta}_i \|_{\theta=0}^t - \int_{0}^{t} E_i I_i u'_i w''_i \dot{\theta}_i ds$$

(2.37)

where $\dot{\theta}_i$ is independent of $s$. Thus, the first term of the first integral in Eq. (2.36) can be replaced by the above equality to produce the following eigenvalue sensitivity equation

$$\dot{\lambda} = \sum_{i=1}^{N} \int_{0}^{t} 2(E_i I_i u'_i w''_i + E_i A_i u'_i w'_i) \dot{\theta}_i ds + \sum_{i=1}^{N} 2E_i I_i w''_i v'_i v'_i \|_{\theta=0}^t$$

$$-\sum_{i=1}^{N} \int_{0}^{t} [(\lambda_\rho A_i (u_i^2 + w_i^2) + E_i A_i u_i'^2 + 3E_i I_i w''_i^2) v'_i + 2E_i I_i w''_i w'_i v'_i] ds.$$

(2.38)

The term $\dot{\theta}_i$ in Eq. (2.38) has been shown to be related to the variation of joint locations by Eq. (2.24). However, the other term $v_i$, which is the velocity function defined along the $s$-axis in each member requires further investigation. In this study, $v_i$ is specified as a cubic polynomial

$$v_i(s) = [1 - 3(\frac{s}{\ell_i})^2 + 2(\frac{s}{\ell_i})^3] v_1 + [3(\frac{s}{\ell_i})^2 - 2(\frac{s}{\ell_i})^3] v_2$$

(2.39)
where \( v_1(0) = v_1 \) and \( v_i(\ell_i) = v_2 \) account for the axial movements of the end joints of member \( i \). Mathematically, \( v_1 \) and \( v_2 \) can be represented by the following relations

\[
\begin{align*}
  v_1 &= m_i \dot{X}_1 + n_i \dot{Y}_1 \\
  v_2 &= m_i \dot{X}_2 + n_i \dot{Y}_2.
\end{align*}
\]

(2.40)

The definition of \( v_1(s) \) given by Eq. (2.39) provides a nice feature, \( v'_1(0) = v'_2(\ell_i) = 0 \), that eliminates the only boundary term from Eq. (2.38). As a result, the eigenvalue sensitivity equation of a planar frame derived by the domain method is given as

\[
\begin{align*}
  \lambda &= \sum_{i=1}^{N} \int_0^\ell_i 2(E_i I_i u''_i w''_i + E_i A_i u'_i w'_i) \dot{\theta}_i ds \\
  &- \sum_{i=1}^{N} \int_0^\ell_i \left[ (\lambda \rho_i A_i (u''_i + w''_i) + E_i A_i u''_i + 3E_i I_i w''_i v'_i + 2E_i I_i w''_i v'_i) ds \right] (2.41)
\end{align*}
\]

which is completely expressed in terms of line integrals.

**Planar Truss**

In a planar truss structure, only the total shape derivative of the axial displacement needs to be considered in the derivation. The total shape derivatives of the axial displacement and its derivative can be summarized as

\[
\begin{align*}
  \dot{u}_i &= \dot{\varphi}_i \dot{\theta}_i + \dot{u}_{s,i} \\
  u'_i &= \varphi \dot{\theta}_i + \dot{u}'_{s,i} - v'_i u_i
\end{align*}
\]

(2.42)

where \( \varphi_i \) is a function of \( s \), i.e., \( \varphi_i = m_i W_i - n_i U_i \), which is pertaining to the effect of orientation variation.

The total shape derivative of the functional \( J \)

\[
J(u, u', \tau) = \int_0^\ell f(u, u', \tau) ds
\]

(2.43)

of a planar truss can be shown to be in terms of \( \dot{\theta} \) and \( \dot{u}_s \) as
Let the admissible function \( \phi_i \) be the axial eigenfunction of \( u_i \) in Eq. (2.13). As a consequence, the weak variational form of the free vibration of a planar truss is simplified as

\[
\pi = 0 = \sum_{i=1}^{N} \int_{0}^{t_i} (\lambda \rho_i A_i u_i^2 - E_i A_i u_i'^2) \, ds \tag{2.45}
\]

where the boundary terms are dropped because the natural boundary conditions are satisfied at the interior joints and the kinematic boundary conditions are satisfied at the supports.

The total shape derivative of \( \pi \) may be found by using Eq. (2.44). The equation, \( \dot{\pi} \), yields the following identity

\[
\lambda \sum_{i=1}^{N} \int_{0}^{t_i} \rho_i A_i u_i^2 \, ds = -\sum_{i=1}^{N} \int_{0}^{t_i} 2(\lambda \rho_i A_i u_i \dot{\theta}_i - E_i A_i u_i' \dot{\theta}_i') \dot{\theta}_i \, ds \\
-\sum_{i=1}^{N} \int_{0}^{t_i} (\lambda \rho_i A_i u_i^2 + E_i A_i u_i'^2) v_i \, ds - \sum_{i=1}^{N} \int_{0}^{t_i} 2(\lambda \rho_i A_i u_i \dot{u}_{s,i} - E_i A_i u_i' \dot{u}_{s,i}') \, ds \tag{2.46}
\]

where \( \dot{\lambda} \) is the total shape derivative of the eigenvalue. The normalization condition of Eq. (2.12) can be directly employed to simplify the left-hand side as \( \dot{\lambda} \). Furthermore, the technique of integration by parts can be applied to the last term in the right-hand side of Eq. (2.46). The resultant equation becomes

\[
-\sum_{i=1}^{N} \int_{0}^{t_i} 2(\lambda \rho_i A_i u_i \dot{u}_{s,i} - E_i A_i u_i' \dot{u}_{s,i}') \, ds \\
= -\sum_{i=1}^{N} \int_{0}^{t_i} 2(\lambda \rho_i A_i u_i + E_i A_i u_i'') \dot{u}_{s,i} \, ds + \sum_{i=1}^{N} 2E_i A_i u_i' \dot{u}_{s,i}^i \bigg|_{0}^{t_i} \tag{2.47} \\
= \sum_{i=1}^{N} 2E_i A_i u_i' \dot{u}_{s,i}^i \bigg|_{0}^{t_i} .
\]
The integral in Eq. (2.47) is dropped as the terms in the parentheses are exactly identical to the state equation presented by Eq. (2.11). Replacing the terms \( E_i A_i u_i' \) and \( \dot{u}_{si} \) by the internal axial force, \( P_i \), and \( m_i \dot{U}_i + n_i \dot{W}_i \), Eq. (2.47) becomes

\[
\sum_{i=1}^{N} 2P_i (m_i \dot{U}_i + n_i \dot{W}_i)_{0}^{T_i}
\]

which are indeed equal to zero. This can be concluded from the boundary conditions. At the built-in supports, one has, \( \dot{U}_i = \dot{W}_i = 0 \). On the other hand, at the interior joint points, \( U_i \) and \( W_i \) have unique values and \( \sum_{i=1}^{N} P_i m_i = 0 \) and \( \sum_{i=1}^{N} P_i n_i = 0 \) as indicated by Eq. (2.16).

Finally, the eigenvalue sensitivity equation can be obtained in a simpler form

\[
\dot{\lambda} = -\sum_{i=1}^{N} \int_{0}^{T_i} 2(\lambda \rho_i A_i u_i \dot{\Theta}_i - E_i A_i u_i' \dot{\Theta}_i') \dot{\Theta}_i \, ds - \sum_{i=1}^{N} \int_{0}^{T_i} (\lambda \rho_i A_i u_i'^2 + E_i A_i u_i'^2) v_i' \, ds \quad (2.48)
\]

which is completely expressed in terms of line integrals.

The velocity function \( v_i(s) \) in the truss structure is specified as a linear function here for convenience

\[
v_i(s) = \left( 1 - \frac{s}{\ell_i} \right) v_1 + \frac{s}{\ell_i} v_2 \quad (2.49)
\]

where \( v_i(0)=v_1 \) and \( v(\ell_i)=v_2 \) account for the axial movements of the end joints of member \( i \).

**Continuous Beam**

A continuous beam is a special case of a planar frame in which the orientation of the member in the whole structure is held constant during the variation. Therefore, \( \dot{\Theta} = 0 \). Thus, the eigenvalue sensitivity of a continuous beam can then be obtained from Eq. (2.38), that is,

\[
\dot{\lambda} = \sum_{i=1}^{N} 2E_i I_i w_i'' v_i' w_i'_{0}^{T_i} - \sum_{i=1}^{N} \int_{0}^{T_i} (\lambda \rho_i A_i w_i'^2 + 3E_i I_i w_i''^2) v_i' + 2E_i I_i w_i'' w_i' v_i' \, ds \quad (2.50)
\]

\[
= -\sum_{i=1}^{N} \int_{0}^{T_i} (\lambda \rho_i A_i w_i'^2 + 3E_i I_i w_i''^2) v_i' + 2E_i I_i w_i'' w_i' v_i' \, ds
\]
where the boundary terms drop in the first equality as \( w'_i \) and \( E_i I_i w''_i \) are continuous at the intermediate supports.

### 2.3.2 Boundary Method (BM)

#### Planar Frame

We now turn our attention to the boundary method. An eigenvalue sensitivity equation in term of relative shape derivatives can be obtained by applying the general formula, Eq. (2.31) to the specific functional, Eq. (2.32), as

\[
\dot{\lambda} = - \sum_{i=1}^{N} \int_0^t 2\left( E_i I_i u'_i w''_i - E_i A_i u'_i w'_i \right) \dot{\theta}_i ds \\
- \sum_{i=1}^{N} \int_0^t \left( (\lambda \rho_i A_i u_i u_{i,\tau} - E_i A_i u'_i ur_{i,\tau}) + (\lambda \rho_i A_i w_i w_{i,\tau} - E_i I_i w''_i w'_{i,\tau}) \right) ds \\
- \sum_{i=1}^{N} \left[ (\lambda \rho_i A_i (u_i^2 + w_i^2) - E_i A_i u''_i - E_i I_i w'_{i,\tau}^2) \right] v_{i,\tau} ds
\]

(2.51)

where the normalization condition of Eq. (2.3) has been incorporated. Integration by parts of the second integral in the last equation leads to the following result

\[
\begin{align*}
- \sum_{i=1}^{N} \int_0^t & 2\left( (\lambda \rho_i A_i u_i u_{i,\tau} - E_i A_i u'_i ur_{i,\tau}) + (\lambda \rho_i A_i w_i w_{i,\tau} - E_i I_i w''_i w'_{i,\tau}) \right) ds \\
= & - \sum_{i=1}^{N} \int_0^t 2\left( (\lambda \rho_i A_i u_i + E_i A_i u'') u_{i,\tau} + (\lambda \rho_i A_i w_i - E_i I_i w''') w_{i,\tau} \right) ds \\
& + \sum_{i=1}^{N} 2\left( E_i A_i u'_i u_{i,\tau} - E_i I_i w''_i w'_{i,\tau} + E_i I_i w''_i w'_{i,\tau} \right) ds
\end{align*}
\]

(2.52)

Two observations can be made here. First, the integral on the right-hand side is dropped because the terms in the parentheses are identical to the state equations of Eqs. (2.1) and (2.2). Second, the unknown boundary terms \( u_{i,\tau}, w_{i,\tau} \) and \( w'_{i,\tau} \), can be replaced by the known quantities based upon the relations of Eqs. (2.26), (2.28), (2.35) and (A.4).
\[
\sum_{i=1}^{N} 2(E_i A_i u[i,u_{i}, - E_i I_i w_i'''' w_i + E_i I_i w_i'' w_i'''] \bigg|_{0}^{i}
\]
\[
= \sum_{i=1}^{N} 2(E_i A_i u[i,u_{i}, - E_i I_i w_i'''' w_i + E_i I_i w_i'' w_i'''] \bigg|_{0}^{i}
\]
\[
- \sum_{i=1}^{N} [2(E_i A_i u_i'^2 - E_i I_i w_i'''' w_i + E_i I_i w_i''^2) v_i + 2E_i I_i w_i'''' w_i' v_i] \bigg|_{0}^{i} (2.53)
\]
\[
= \sum_{i=1}^{N} 2E_i I_i w_i''(u_i^i \dot{\theta}_i + v_i' w_i') \bigg|_{0}^{i}
\]
\[
- \sum_{i=1}^{N} [2(E_i A_i u_i'^2 - E_i I_i w_i'''' w_i + E_i I_i w_i''^2) v_i + 2E_i I_i w_i'''' w_i' v_i] \bigg|_{0}^{i}.
\]

Combination of the last three equations, Eqs. (2.51) to (2.53), yields the basic equation for eigenvalue shape sensitivity
\[
\dot{\lambda} = \sum_{i=1}^{N} 2(E_i I_i u_i' w_i'''' - E_i A_i u_i' w_i') \dot{\theta}_i ds
\]
\[
- \sum_{i=1}^{N} [2(E_i A_i u_i'^2 + w_i'^2) + E_i A_i u_i'^2 + E_i I_i w_i''^2 - 2E_i I_i w_i'''' w_i' v_i] \bigg|_{0}^{i}. \quad (2.54)
\]

An alternative form of the above equation can be obtained by integrating by parts of the remaining integral in the last equation as,
\[
\sum_{i=1}^{N} \int_{0}^{i} 2(E_i I_i u_i'''' w_i'' - E_i A_i u_i' w_i') \dot{\theta}_i ds
\]
\[
= \sum_{i=1}^{N} \int_{0}^{i} 2(E_i I_i w_i''' u_i' + E_i A_i u_i' w_i') \dot{\theta}_i ds \quad (2.55)
\]
\[
+ \sum_{i=1}^{N} 2(E_i I_i w_i'''' u_i' - E_i I_i u_i'' w_i'' - E_i A_i u_i' w_i') \dot{\theta}_i \bigg|_{0}^{i}
\]

where the integral is dropped because of the following identity
\[
E_i I_i w_i''' u_i' + E_i A_i u_i'' w_i'
\]
\[
= (\lambda \rho_i A_i u_i + E_i A_i u_i'') w_i \quad (2.56)
\]
\[
= 0
\]
which is proven by the state equations, Eqs. (2.1) and (2.2). Hence, the integral term in
Eq. (2.54) can be completely replaced by the boundary terms given in Eq. (2.55). That is,
\[
\dot{\lambda} = \sum_{i=1}^{N} 2 \left( E_i A_i u_i \dot{w}_i + E_i I_i u_i \ddot{w}_i \right) \hat{\theta}_i \big|_{0}^{\tilde{t}} \\
- \sum_{i=1}^{N} \left[ \lambda \rho_i A_i \left( u_i^2 + w_i^2 \right) + E_i A_i u_i' + E_i I_i \dddot{w}_i - 2E_i I_i w_i' \dddot{w}_i \right] v_i \big|_{0}^{\tilde{t}}. \tag{2.57}
\]

It is understood that Eq. (2.57) is identical to Eq. (2.54) only when Eqs. (2.1) and (2.2)
can be exactly satisfied. Hence, in conjunction with the finite element method, Eq. (2.57)
can only be expected to provide approximate eigenvalue sensitivity coefficients. Numerical
examples will be presented later to demonstrate the approximate nature of Eq. (2.57).
Furthermore, it should be noted that the derived sensitivity equations, Eqs. (2.41) and
(2.57), are valid not only for joint location variations but also for support location
variations.

**Planar Truss**

To develop the eigenvalue sensitivity equation using the boundary method, we
recall that the total shape derivatives of \( u_i \) and \( u_i' \) can also be represented in terms of the
relative shape derivatives, \( u_i' \) and \( u_i'' \) as
\[
\dot{u}_i = \hat{\theta}_i \dot{\theta}_i + u_i' \dot{v}_i + v_i u_i' \\
\ddot{u}_i = \hat{\theta}_i \ddot{\theta}_i + u_i'' \dot{v}_i + v_i u_i''. \tag{2.58}
\]

Furthermore, the total shape derivative of functional \( J \) pertaining to a planar truss in Eq.
(2.43) can be obtained in terms of \( \hat{\theta} \) and \( u_i' \) as
\[
J = \int_{0}^{\tilde{t}} \left[ \frac{\partial f}{\partial \tau} + \left( \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \hat{\theta}} \hat{\theta} + \frac{\partial f}{\partial \dot{u}} \dot{u} + \frac{\partial f}{\partial u'} u' + \frac{\partial f}{\partial u''} u'' \right) ds + (fv)_{0}^{\tilde{t}}. \tag{2.59}
\]

Equation (2.59) can be directly employed to express the total shape derivative of the weak
variational form of Eq. (2.45). With the aid of the normalization condition, Eq. (2.12), one
has
The last integral may be simplified if integration by parts is employed

\[
\dot{\lambda} = -\sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i \dot{\theta}_i - E_i A_i u'_i \dot{\theta}'_i) \dot{\theta}_i \, dt - \sum_{i=1}^{N} (\lambda \rho_i A_i u_i^2 - E_i A_i u'_i^2) \nu_i |_{t_0}^{t_f} \\
\quad - \sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i u_i' - E_i A_i u_i' u_i') \, dt.
\]  

(2.60)

The last integral may be simplified if integration by parts is employed

\[
\sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i u_i' - E_i A_i u_i' u_i') \, dt = \sum_{i=1}^{N} 2E_i A_i u'_i u_i |_{t_0}^{t_f} - \sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i + E_i A_i u_i'') u_i' \, dt.
\]

(2.61)

where the integral on the right-hand side is dropped because the term in the parentheses is identical to the state equation of Eq. (2.11). To further simplify the resultant equation, the term \(u_i\) is substituted by \(u_i = \hat{u}_{i,1} + \nu_i u_i\), then the remaining terms in the above equation become \(\sum_{i=1}^{N} 2E_i A_i u'_i (\hat{u}_{i,1} - \nu_i u_i') |_{t_0}^{t_f}\). The boundary conditions discussed in Section 2.1.2 can be employed here to help by dropping the term \(\sum_{i=1}^{N} 2E_i A_i u'_i \hat{u}_{i,1} |_{t_0}^{t_f}\). In the end, Eq. (2.61) is simplified to \(-\sum_{i=1}^{N} 2E_i A_i u_i^2 \nu_i |_{t_0}^{t_f}\). Substituting this result to Eq. (2.60), the basic equation for eigenvalue sensitivity is obtained as

\[
\dot{\lambda} = -\sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i \dot{\theta}_i - E_i A_i u'_i \dot{\theta}'_i) \dot{\theta}_i \, dt - \sum_{i=1}^{N} (\lambda \rho_i A_i u_i^2 + E_i A_i u_i'^2) \nu_i |_{t_0}^{t_f}.
\]

(2.62)

Integrating by parts, the integral of Eq. (2.62) becomes

\[
-\sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i \dot{\theta}_i - E_i A_i u'_i \dot{\theta}'_i) \dot{\theta}_i \, dt = \sum_{i=1}^{N} 2E_i A_i u'_i \dot{\theta}_i |_{t_0}^{t_f} - \sum_{i=1}^{N} \int_{t_0}^{t_f} 2(\lambda \rho_i A_i u_i + E_i A_i u_i'') \dot{\theta}_i \, dt.
\]

(2.63)

\[
= \sum_{i=1}^{N} 2E_i A_i u'_i \dot{\theta}_i |_{t_0}^{t_f}.
\]
where again the integral on the right-hand side of the first equality is dropped because of the state equation, Eq. (2.11). Thus, an alternative form of the eigenvalue sensitivity equation can be stated as

\[ \dot{\lambda} = \sum_{i=1}^{N} 2E_i A_i u_i^2 \dot{\theta}_i |_{i \to 0}^{i} - \sum_{i=1}^{N} (\lambda \rho_i A_i u_i^2 + E_i A_i u_i^4) v_i |_{i \to 0}^{i} \]  

(2.64)

which is completely presented in terms of boundary quantities.

**Continuous Beam**

The support locations of a continuous beam can be viewed as the special case of the joint locations of a frame structure in which the orientations of the members remain unchanged. Therefore, the eigenvalue sensitivity equation for a continuous beam can be reduced from Eq. (2.57) as

\[ \dot{\lambda} = -\sum_{i=1}^{N} (\lambda \rho_i A_i w_i^2 + E_i I_i w_i^2 - 2E_i I_i w_i^2 w_i''') v_i |_{i \to 0}^{i} \]  

(2.65)

where the axial deformation is neglected.

The last equation can be directly applied to find the eigenvalue sensitivity of a continuous beam supported by intermediate spring supports. The eigenvalue equation and its boundary conditions of a continuous beam with spring supports are the same as those given in Eqs (2.17) and (2.20). However, the kinematic condition, \( w(s) = 0 \), at a rigid support should be replaced by a natural boundary condition

\[ S(s^-) - S(s^+) = -kw(s) \]  

(2.66)

where \( S(s^-) - S(s^+) \) is the jump of the shear force at the spring support and \( k \) is the spring constant. With the aid of Eq. (2.66), the eigenvalue sensitivity equation of the new problem is identified as
where $N_1$ is the number of rigid supports and $N_2$ is the number of spring supports.

### 2.4 Discrete Approach

The discrete analytical method (DAM) is a discrete approach commonly used for sensitivity analysis in which the sensitivity equations are derived based upon the discretized state equation. For the purpose of comparison, an eigenvalue sensitivity equation derived by the discrete analytical method will be presented hereafter.

The matrix equation of a skeletal structure under free vibration is given as

$$[K - \lambda M] \{X\} = \{0\}$$  \hspace{1cm} (2.68)

It may be expressed in the form of Rayleigh's quotient as

$$\{X\}^T[K]\{X\} - \lambda \{X\}^T[M]\{X\} = \{0\}.$$ \hspace{1cm} (2.69)

In the above equations, $\{X\}$ is the eigenvector defined in the global coordinate system, and $[K]$ and $[M]$ are the global stiffness and mass matrices, respectively. These matrices are the assemblies of elementary stiffness and mass matrices, $[K_i]$ and $[M_i]$ as

$$[K] = \sum_{i=1}^{NE} [T_i]^T[K_i][T_i]$$

$$[M] = \sum_{i=1}^{NE} [T_i]^T[M_i][T_i]$$ \hspace{1cm} (2.70)

where $NE$ is the total number of the elements and $[T_i]$ is the elementary transformation matrix. Note that the matrices $[K_i]$, $[M_i]$ and $[T_i]$, are explicit functions of either the length or the orientation of element $i$. Let $b$ denote the design variable pertaining to the $X$- or $Y$-coordinate of a joint. The eigenvalue sensitivity equation [12] then becomes

$$\dot{\lambda} = -\sum_{i=1}^{N_1} (\lambda \rho_i A_i w_i^2 + E_i I_i w_i'^2) v_i |_0^t + \sum_{i=1}^{N_2} 2k_i w_i'(s) v_i(s)$$  \hspace{1cm} (2.67)
\[
\frac{d\lambda}{db} = 2\sum_{i=1}^{NE} \{x_i\}^T \frac{d[T_i]}{db} (\{K_i\} - \lambda \{M_i\})\{x_i\} \\
+ \sum_{i=1}^{NE} \{x_i\}^T (\frac{d[K_i]}{db} - \lambda \frac{d[M_i]}{db})\{x_i\} 
\] (2.71)

where \(\{x_i\}\) is the eigenvector of element \(i\) in the local coordinate system. The gradients of the elementary stiffness, mass and transformation matrices with respect to the design variable related to joint location are given in Appendix B.

### 2.5 Examples

Examples are collected here to investigate the numerical performance of the sensitivity equations derived in the previous sections. The eigenvalues and eigenvectors are mainly evaluated by the finite element analysis. In Section 2.5.1, a cantilever beam whose eigenvalues and eigenvectors can be analytically derived provides an opportunity to show that both the domain method and the boundary method are capable to compute the exact eigenvalue sensitivity coefficients. In Section 2.5.2 several numerical examples are presented to study the computational accuracy and efficiency of the methods presented for eigenvalue sensitivity analyses.

#### 2.5.1 Analytical Example: A Cantilever Beam

A cantilever beam with its geometric parameters is depicted in Fig. 2.2. Only the lateral vibration is considered in this study. The eigensolutions [62] of this problem are given as

\[
\lambda_r = \frac{(aL)^4}{L^4} \frac{EI}{\rho A}, \quad r = 1, 2, 3, \ldots 
\] (2.72)

and

\[
X_r(s) = C_r \{\cosh(a_s) - \cos(a_s) - K_r \sinh(a_s) - \sin(a_s)\} 
\] (2.73)
where the arc length, s, is varied between 0 and L, the amplitude constant, \( C_r \), is determined by the normalization condition, and parameter, \( K_r \), is determined by the following equation

\[
K_r = \frac{\cosh(a_rL) + \cos(a_rL)}{\sinh(a_rL) + \sin(a_rL)}, \quad r = 1, 2, 3, \ldots
\]  

(2.74)

The terms \((a_rL)\) are the roots of a transcendental equation. The first three roots are given as

\[
\begin{align*}
a_1L &= 1.8751 \\
a_2L &= 4.6941 \\
a_3L &= 7.8548
\end{align*}
\]

Figure 2.2 A Cantilever Beam

In this example the orientation of the beam remains unchanged, this is, \( \theta = 0 \). Therefore, the sensitivity equations, Eq. (2.50) and (2.65), may be simplified to result in the following forms, respectively

\[
\dot{\lambda}_r = -\int_0^L \left\{ \left( \lambda_r \rho A X_r'^2 + 3 E I X_r'' \right) v' + 2 E I X_r' X_r'' v'' \right\} ds, \quad r = 1, 2, 3, \ldots
\]

(2.75)

and

\[
\dot{\lambda}_r = -\left( \lambda_r \rho A X_r'^2 + E I X_r'' \right) v|_0^L, \quad r = 1, 2, 3, \ldots
\]

(2.76)

where \( v(s) \) is defined by Eq. (2.39) with \( v(0) = 0 \) and \( v(L) = 1 \). Substituting the exact values of the first three eigenvalues and eigenvectors, \( \lambda_r \) and \( X_r \), into the above two equations, it
can be shown that Eq. (2.75) and Eq. (2.76) provide identical results. Furthermore, these sensitivity coefficients are the same as those obtained by directly taking the derivative of Eq. (2.72), i.e.,

\[
\frac{d\lambda_r}{dL} = -\frac{4(a_r L)^4 \frac{E I}{L^5}}{\rho A}, \quad r = 1,2,3,\ldots
\]  

This study thus confirms that the sensitivity equations expressed by Eqs. (2.75) and (2.76) are identical as long as the exact eigensolutions are used for evaluation. In many engineering applications, however, the eigensolutions can only be obtained approximately. Therefore, it is necessary to study the effect of analysis inaccuracy on sensitivity analysis. In order to do so, the finite element solutions of various finite element meshes ranging from one to thirty elements are used to evaluate the sensitivity coefficients based upon sensitivity equations derived by different methods, i.e., Eqs. (2.71), (2.75), (2.76), and (2.77).

The results of this study are given in Figs. 2.3 to 2.7. The cantilever beam, 1.0 in length, is assumed to have a solid circular section with radius 0.1. Young's modulus and mass density are selected as 10000.0 and 1.0, respectively. Figure 2.3 shows the data associated with the convergence study of eigenvalue analysis. It is revealed that the one-element model is good enough to obtain an accurate first eigenvalue, whereas at least two- or three-element models are required in order to obtain satisfactory second or third eigenvalues. Figure 2.4 provides an opportunity to study the convergence of eigenvectors. As shown in Figure 2.4, the second eigenvector of the one-element model is quite different from the analytical one. As the number of elements is progressively increased, the eigenvector obtained by finite element method converges to the exact one.

Figures 2.5 to 2.7 contain the data associated the convergence of the first, second and third eigenvalue sensitivities, respectively. The ordinate, y-axis, in these figures is the ratio of the computed eigenvalue sensitivity coefficients to the analytical one, while the abscissa, x-axis, is the number of elements used in the finite-element eigenvalue analysis.
Figure 2.3 Convergence Study of Eigenvalue Analysis

Figure 2.4 The Second Mode Shape of the Cantilever Beam
Figure 2.5 Convergence Study of First Eigenvalue Sensitivity Analysis

Figure 2.6 Convergence Study of Second Eigenvalue Sensitivity Analysis
The following conclusions can be drawn from these figures.

1. The sensitivity coefficients calculated by the domain method are more accurate than those found by the boundary method. In fact, the sensitivity coefficients calculated by the boundary method cannot converge to the exact values, even with a thirty-element model.

   The above conclusion is expected, though. As mentioned in Reference 63, the boundary method does not, but the domain method does consider the variations of the across-element discontinuities in its derivation, which are inevitable in the finite element analysis.

2. Eigenvalue sensitivity coefficients of higher modes converge to the exact values slower than those of lower modes, particularly in the case of the boundary method.

   This may be attributed to the fact that, using the same mesh, the finite element method calculates lower modes more accurately than higher modes.
3. The convergence rate of the sensitivity coefficients calculated by the domain method is very similar to that of the eigenvalue analysis. In order words, the sensitivity coefficients calculated by the domain method will not be accurate if the eigenvalue analysis is not accurate. Therefore, the accuracy of the sensitivity coefficients calculated by the domain method may be used as an error indicator to measure the accuracy of the finite element analysis.

2.5.2 Numerical Examples

Four-Member Frame

The layout of a four-member frame and its geometric data are shown in Fig. 2.8. Each member has a solid circular section with a radius of 0.075. The Young's modulus and the mass density are given as 10000.0 and 0.25, respectively.

![Figure 2.8 A Four-Member Frame](image)

Three finite element meshes with 1, 4 and 8 elements in each member, respectively, are considered in this example. Two cases are studied here. In the first case, the location of
support I is considered as design variable. In the second case, the locations of joints J, K and L are simultaneously considered as design variables. Table 2.1 and 2.2 document the numerical results of the first three eigenvalue sensitivities. Note that the labels, "(X)" and "(Y)", indicate the X and Y coordinates, respectively, of the joint being considered as the design variable. The first columns of these tables list the first three eigenvalues of the example frame. The second and third columns give the eigenvalue derivatives approximated by the central difference method (CDM), i.e.,

\[
\frac{d\lambda}{db} = \frac{\lambda(b + \Delta b) - \lambda(b - \Delta b)}{2\Delta b}
\]

(2.78)

Table 2.1 Eigenvalue Sensitivity Coefficients of Four-Member Frame with respect to Movement of Support I

<table>
<thead>
<tr>
<th>Mesh</th>
<th>eigenvalue</th>
<th>CDM (X)</th>
<th>CDM (Y)</th>
<th>DAM (X)</th>
<th>DAM (Y)</th>
<th>DM (X)</th>
<th>DM (Y)</th>
<th>BM (X)</th>
<th>BM (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 = 1.587$</td>
<td>-0.197</td>
<td>0.628</td>
<td>-0.197</td>
<td>0.628</td>
<td>-0.197</td>
<td>0.632</td>
<td>-0.197</td>
<td>0.611</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 6.168$</td>
<td>-1.049</td>
<td>1.764</td>
<td>-1.049</td>
<td>1.764</td>
<td>-1.049</td>
<td>1.757</td>
<td>-1.049</td>
<td>1.611</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 33.32$</td>
<td>-2.998</td>
<td>5.786</td>
<td>-2.998</td>
<td>5.786</td>
<td>-2.992</td>
<td>5.746</td>
<td>-2.994</td>
<td>4.089</td>
</tr>
<tr>
<td>4</td>
<td>$\lambda_4 = 1.565$</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
</tr>
<tr>
<td></td>
<td>$\lambda_5 = 6.124$</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
</tr>
<tr>
<td>8</td>
<td>$\lambda_7 = 1.565$</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
<td>-0.188</td>
<td>0.608</td>
</tr>
<tr>
<td></td>
<td>$\lambda_8 = 6.123$</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
<td>-1.053</td>
<td>1.753</td>
</tr>
</tbody>
</table>
where \( \Delta b \), given a value of 0.005 units, represents the perturbation of the joint location. The central difference method is selected here over the more commonly used forward difference method for the sake of numerical accuracy. The fourth to the ninth columns in these tables list the eigenvalue sensitivity coefficients calculated by Eqs. (2.71), (2.41) and (2.57), respectively. Note that since the structure is symmetric with respect to the movements of joints J, K and L in the X direction, the eigenvalue sensitivities with respect to these movements become zero. This fact is observable in all the sensitivity analysis methods, as indicated by the column of zeros in Table 2.2.

Table 2.2 Eigenvalue Sensitivity Coefficients of Four-Member Frame with respect to Simultaneous Movements of Joints J, K, L

<table>
<thead>
<tr>
<th>Mesh</th>
<th>eigenvalue</th>
<th>CDM (X)</th>
<th>CDM (Y)</th>
<th>DAM (X)</th>
<th>DAM (Y)</th>
<th>DM (X)</th>
<th>DM (Y)</th>
<th>BM (X)</th>
<th>BM (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \lambda_1 = 1.587 )</td>
<td>0.0</td>
<td>-1.256</td>
<td>0.0</td>
<td>-1.256</td>
<td>0.0</td>
<td>-1.265</td>
<td>0.0</td>
<td>-1.425</td>
</tr>
<tr>
<td></td>
<td>( \lambda_2 = 6.168 )</td>
<td>0.0</td>
<td>-3.528</td>
<td>0.0</td>
<td>-3.528</td>
<td>0.0</td>
<td>-3.515</td>
<td>0.0</td>
<td>-4.867</td>
</tr>
<tr>
<td></td>
<td>( \lambda_3 = 33.322 )</td>
<td>0.0</td>
<td>-11.57</td>
<td>0.0</td>
<td>-11.57</td>
<td>0.0</td>
<td>-10.29</td>
<td>0.0</td>
<td>-10.74</td>
</tr>
<tr>
<td>4</td>
<td>( \lambda_1 = 1.565 )</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.279</td>
</tr>
<tr>
<td></td>
<td>( \lambda_2 = 6.124 )</td>
<td>0.0</td>
<td>-3.507</td>
<td>0.0</td>
<td>-3.507</td>
<td>0.0</td>
<td>-3.506</td>
<td>0.0</td>
<td>-3.677</td>
</tr>
<tr>
<td></td>
<td>( \lambda_3 = 23.608 )</td>
<td>0.0</td>
<td>-7.229</td>
<td>0.0</td>
<td>-7.229</td>
<td>0.0</td>
<td>-7.224</td>
<td>0.0</td>
<td>-6.152</td>
</tr>
<tr>
<td>8</td>
<td>( \lambda_1 = 1.565 )</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.216</td>
<td>0.0</td>
<td>-1.248</td>
</tr>
<tr>
<td></td>
<td>( \lambda_2 = 6.123 )</td>
<td>0.0</td>
<td>-3.506</td>
<td>0.0</td>
<td>-3.506</td>
<td>0.0</td>
<td>-3.506</td>
<td>0.0</td>
<td>-3.579</td>
</tr>
<tr>
<td></td>
<td>( \lambda_3 = 23.590 )</td>
<td>0.0</td>
<td>-7.222</td>
<td>0.0</td>
<td>-7.222</td>
<td>0.0</td>
<td>-7.221</td>
<td>0.0</td>
<td>-6.932</td>
</tr>
</tbody>
</table>

The conclusions drawn from the cantilever beam example are also presented here. For instance, with the coarse mesh, none of the methods yield an acceptable eigenvalue
sensitivity of $\lambda_3$. In addition, the study of this example confirms that the sensitivity equations, Eqs. (2.41) and (2.57), are valid not only for the joint locations but also for the support locations.

**Nineteen-Member Frame**

A relatively complex frame structure is proposed in this example to validate the sensitivity equations. The layout of a nineteen-member frame is given in Fig. 2.9 along with geometric data. The Young's modulus and the mass density are 100000.0 and 4.0, respectively. The frame members are solid bars with two different circular sections. The vertical members have a radius of 0.3 and the horizontal members have a radius of 0.5. The finite element model used here has discretized each member of the structure into four elements. This amounts to 76 elements in total.

![Figure 2.9 A Nineteen-Member Frame](image)
The locations of the support, I, and the joint, J, are considered as design variables. The numerical results listed in Table 2.3 generally agree with the conclusions stated in the cantilever beam example. The additional information provided by Table 2.4 is the computational times (CPU seconds) required by various methods for sensitivity analysis, which are normalized with respect to the computational time of the boundary method. Note that the results presented in Table 2.4 do not include the CPU times for the eigenvalue analysis of the baseline design.

Table 2.3 Eigenvalue Sensitivity Coefficients of Nineteen-Member Frame

<table>
<thead>
<tr>
<th>Joint</th>
<th>Eigenvalue</th>
<th>CDM (X)</th>
<th>CDM (Y)</th>
<th>DAM (X)</th>
<th>DAM (Y)</th>
<th>DM (X)</th>
<th>DM (Y)</th>
<th>BM (X)</th>
<th>BM (Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(\lambda_1 = 4.657)</td>
<td>-0.611</td>
<td>0.579</td>
<td>-0.611</td>
<td>0.579</td>
<td>-0.611</td>
<td>0.579</td>
<td>-0.611</td>
<td>0.579</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2 = 33.20)</td>
<td>0.760</td>
<td>2.554</td>
<td>0.760</td>
<td>2.554</td>
<td>0.760</td>
<td>2.553</td>
<td>0.760</td>
<td>2.553</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3 = 73.685)</td>
<td>1.563</td>
<td>3.264</td>
<td>1.563</td>
<td>3.264</td>
<td>1.562</td>
<td>3.263</td>
<td>1.562</td>
<td>3.263</td>
</tr>
<tr>
<td>J</td>
<td>(\lambda_1 = 4.657)</td>
<td>0.056</td>
<td>-0.271</td>
<td>0.056</td>
<td>-0.271</td>
<td>0.056</td>
<td>-0.271</td>
<td>0.056</td>
<td>-0.274</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2 = 33.20)</td>
<td>-1.634</td>
<td>5.718</td>
<td>-1.634</td>
<td>5.718</td>
<td>-1.634</td>
<td>5.717</td>
<td>-1.631</td>
<td>5.760</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3 = 73.657)</td>
<td>-10.0</td>
<td>-4.7</td>
<td>-10.0</td>
<td>-4.7</td>
<td>-10.0</td>
<td>-4.69</td>
<td>-9.993</td>
<td>-4.64</td>
</tr>
</tbody>
</table>

Table 2.4 Computational Times of Sensitivity Analysis with respect to Support I

(CYBER 930 NOS/VE 1.4.1)

<table>
<thead>
<tr>
<th></th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU(sec)</td>
<td>19.104</td>
<td>3.197E-1</td>
<td>2.736E-2</td>
<td>4.228E-3</td>
</tr>
<tr>
<td>normalized</td>
<td>4518.45</td>
<td>75.61</td>
<td>6.47</td>
<td>1</td>
</tr>
</tbody>
</table>
Fifty-One Bar Truss

To validate the sensitivity equations of a truss system, a fifty-one bar truss is given for this study. The layout of this truss with its geometry data are shown in Fig. 2.10. The Young's modulus and the mass density are 5,000,000 and 0.28, respectively. The truss members are solid bars with radius of 1.0. The first three eigenvalues of the structure are 49.87, 302.50 and 4877.01, respectively.

![Figure 2.10 A Fifty-One Bar Truss](image)

Four cases are studied. In the first case, the locations of support I and joint J in X-direction are simultaneously considered as design variables. In the second case, the locations of joint K and L in X-direction are simultaneously considered as design variables. In the third case, the location of L in Y-direction is considered as the design variable. In the last case, the design variable is the location of support I in Y-direction. The marked arrows in Fig. 2.10 indicate the position and direction of each design variable. Table 2.5 documents the numerical results of sensitivity analysis by the central difference method (CDM), the discrete analytical method (DAM), the domain method (DM) and the boundary method (BM), respectively. The numerical results reported in Table 2.5 generally confirm the validity of the derived sensitivity equations.
Table 2.5 Eigenvalue Sensitivity Coefficients of Fifty-One Bar Truss

<table>
<thead>
<tr>
<th>design variable</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>8.532E-1</td>
<td>8.532E-1</td>
<td>8.532E-1</td>
<td>8.477E-1</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-3.083</td>
<td>-3.083</td>
<td>-3.083</td>
<td>-3.053</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>5.126</td>
<td>5.126</td>
<td>5.126</td>
<td>4.863</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>-1.635E-2</td>
<td>-1.635E-2</td>
<td>-1.635E-2</td>
<td>-1.614E-2</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.372E-1</td>
<td>1.372E-1</td>
<td>1.372E-1</td>
<td>1.327E-1</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>1.851E-1</td>
<td>1.851E-1</td>
<td>1.851E-1</td>
<td>1.689E-1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>7.093E-2</td>
<td>7.093E-2</td>
<td>7.093E-2</td>
<td>7.078E-2</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.378</td>
<td>1.378</td>
<td>1.377</td>
<td>1.336</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>-2.175E-1</td>
<td>-2.175E-1</td>
<td>-2.175E-1</td>
<td>-2.104E-1</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>-1.127</td>
<td>-1.127</td>
<td>-1.127</td>
<td>-1.127</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>-3.769</td>
<td>-3.769</td>
<td>-3.769</td>
<td>-3.784</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>1.158E1</td>
<td>1.158E1</td>
<td>1.158E1</td>
<td>1.157E1</td>
</tr>
</tbody>
</table>

Continuous Beam

Four different beam models are presented in this study, shown in Fig. 2.11. All of the beams are simply-supported and divided by an intermediate support located at a distance of 1.5 units from the left end. The beam is made of a circular section with a length of 3.5 units. In the first case, the beam is a continuous beam with Young's modulus, $E=1.E+4$, mass per unit length, $\rho=1$, and the radius, $r=1$. In the second case, the radius of the right span of the beam is increased to 1.2 to make it a stepped beam. Case 3 and 4 are similar to case 1 and 2 in which the intermediate support is replaced by a spring support. Three finite element meshes are considered in each of the first two cases. The coarse mesh has a 1-2
distribution, i.e., one element on the left span and two on the right span. The finer and the finest meshes correspond to a 3-6 and a 6-12 distribution, respectively. Only the finer mesh with a 3-6 distribution is implemented in cases 3 and 4 where two different values of spring constants have been adopted for numerical studies.

Tables 2.6 to 2.9 are organized to document the numerical results of eigenvalue sensitivity analysis for each of the four cases. In those tables, the first column lists the first three eigenvalues of the beam, The second column gives the eigenvalue derivatives approximated by the central difference method (CDM), Eq. (2.78), which employs a value of 0.005 units as perturbation. The third to the fifth columns provide the eigenvalue sensitivity coefficients calculated by the discrete analytical method (DAM), the boundary method (BM) and the domain method (DM), respectively.

![Continuous Beams with Various Support Conditions](image)

**Figure 2.11** Continuous Beams with Various Support Conditions
The numerical results reported in Tables 2.6 to 2.9 generally confirm the validity of the derived sensitivity formulations except the case with a coarse mesh. This inconsistency is due to the inaccuracy of the finite element model to calculate the eigenvalue. This inaccuracy of the finite element analysis can not be realized by the discrete analytical method. On the other hand, the sensitivity equations derived directly by the variational equations, may be equipped with a better "sense" of numerical error than the discrete analytical method. As demonstrated in Tables 2.6 and 2.7 the sensitivity coefficients calculated by the boundary method and the domain method are dramatically different from those by the central difference method as long as the eigenvalues are not accurately computed.

As those studied before, the examples studied here also reveal a general trend that the accuracy of the sensitivity equation obtained by domain method is slightly better than that by the boundary method.

Table 2.6 Eigenvalue Sensitivity Analysis of a Uniform Beam

<table>
<thead>
<tr>
<th>Mesh</th>
<th>eigenvalues</th>
<th>CDM</th>
<th>DAM</th>
<th>BM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>$\lambda_1 = 2.168E4$</td>
<td>2.9183E4</td>
<td>2.9184E4</td>
<td>2.9054E4</td>
<td>3.1974E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 1.129E5$</td>
<td>-1.6953E5</td>
<td>-1.6954E5</td>
<td>-1.1839E5</td>
<td>-6.7369E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 4.334E5$</td>
<td>5.0357E5</td>
<td>5.036E5</td>
<td>2.4744E5</td>
<td>5.6296E5</td>
</tr>
<tr>
<td>3-6</td>
<td>$\lambda_1 = 2.129E4$</td>
<td>2.7198E4</td>
<td>2.72E4</td>
<td>2.7888E4</td>
<td>2.7217E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 8.225E4$</td>
<td>-1.4187E5</td>
<td>-1.4188E5</td>
<td>-1.3395E5</td>
<td>-1.4206E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 3.140E5$</td>
<td>4.6076E5</td>
<td>4.6078E5</td>
<td>4.5695E5</td>
<td>4.6164E5</td>
</tr>
<tr>
<td>6-12</td>
<td>$\lambda_1 = 2.129E4$</td>
<td>2.7181E4</td>
<td>2.7183E4</td>
<td>2.7398E4</td>
<td>2.7184E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 8.207E4$</td>
<td>-1.4164E5</td>
<td>-1.4164E5</td>
<td>-1.3997E5</td>
<td>-1.4164E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 3.131E5$</td>
<td>4.5874E5</td>
<td>4.5876E5</td>
<td>4.6222E5</td>
<td>4.5882E5</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 2.7 Eigenvalue Sensitivity Analysis of a Stepped Beam

<table>
<thead>
<tr>
<th>Mesh</th>
<th>eigenvalues</th>
<th>CDM</th>
<th>DAM</th>
<th>BM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>$\lambda_1 = 2.672E4$</td>
<td>3.7493E4</td>
<td>3.7495E4</td>
<td>2.8324E4</td>
<td>4.078E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 1.437E5$</td>
<td>-2.378E5</td>
<td>-2.3781E5</td>
<td>-1.8041E3</td>
<td>-1.0158E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 5.394E5$</td>
<td>6.2347E5</td>
<td>6.2351E5</td>
<td>9.0272E4</td>
<td>6.7085E5</td>
</tr>
<tr>
<td>3-6</td>
<td>$\lambda_1 = 2.62E4$</td>
<td>3.4361E4</td>
<td>3.4364E4</td>
<td>3.4513E4</td>
<td>3.4381E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 9.751E4$</td>
<td>-1.8017E5</td>
<td>-1.8017E5</td>
<td>-1.6761E5</td>
<td>-1.8045E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 4.057E5$</td>
<td>6.0547E5</td>
<td>6.0551E5</td>
<td>5.5373E5</td>
<td>6.0652E5</td>
</tr>
<tr>
<td>6-12</td>
<td>$\lambda_1 = 2.621E4$</td>
<td>3.4333E4</td>
<td>3.4336E4</td>
<td>3.4453E4</td>
<td>3.4337E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 9.725E4$</td>
<td>-1.7976E5</td>
<td>-1.7976E5</td>
<td>-1.7810E5</td>
<td>-1.7976E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 4.046E5$</td>
<td>6.0138E5</td>
<td>6.0142E5</td>
<td>5.9811E5</td>
<td>6.0151E5</td>
</tr>
</tbody>
</table>

Table 2.8 Eigenvalue Sensitivity Analysis of a Continuous Beam with a Spring Support

<table>
<thead>
<tr>
<th>Spring constant</th>
<th>eigenvalues</th>
<th>CDM</th>
<th>DAM</th>
<th>BM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=10^4$</td>
<td>$\lambda_1 = 3.304E3$</td>
<td>8.2949E2</td>
<td>8.2951E2</td>
<td>8.2912E2</td>
<td>8.3096E2</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 2.634E4$</td>
<td>-2.6741E3</td>
<td>-2.6743E3</td>
<td>-2.7047E3</td>
<td>-2.6432E3</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 1.328E5$</td>
<td>5.3791E3</td>
<td>5.3796E3</td>
<td>4.8458E3</td>
<td>5.8895E3</td>
</tr>
<tr>
<td>$k=10^{10}$</td>
<td>$\lambda_1 = 2.129E4$</td>
<td>2.7198E4</td>
<td>2.72E4</td>
<td>2.7201E4</td>
<td>2.7217E4</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2 = 8.225E4$</td>
<td>-1.4187E5</td>
<td>-1.4188E5</td>
<td>-1.4235E5</td>
<td>-1.4206E5</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3 = 3.140E5$</td>
<td>4.6076E5</td>
<td>4.6078E5</td>
<td>4.6165E5</td>
<td>4.6165E5</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 2.9 Eigenvalue Sensitivity Analysis of a Stepped Beam with a Spring Support

<table>
<thead>
<tr>
<th>Spring constant</th>
<th>eigenvalues</th>
<th>CDM</th>
<th>DAM</th>
<th>BM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k = 10^6)</td>
<td>(\lambda_1 = 2.52E4)</td>
<td>3.6287E4</td>
<td>3.6289E4</td>
<td>3.6023E4</td>
<td>3.6302E4</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2 = 7.718E5)</td>
<td>-1.5047E5</td>
<td>-1.5047E5</td>
<td>-1.4542E5</td>
<td>-1.5085E5</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3 = 2.863E5)</td>
<td>3.9973E5</td>
<td>3.9976E5</td>
<td>3.6528E5</td>
<td>3.9963E5</td>
</tr>
<tr>
<td>(k = 10^{10})</td>
<td>(\lambda_1 = 2.622E3)</td>
<td>3.4361E4</td>
<td>3.4364E4</td>
<td>3.4056E4</td>
<td>3.4381E4</td>
</tr>
<tr>
<td></td>
<td>(\lambda_2 = 9.751E4)</td>
<td>-1.8017E5</td>
<td>-1.8017E5</td>
<td>-1.7247E5</td>
<td>-1.8045E5</td>
</tr>
<tr>
<td></td>
<td>(\lambda_3 = 4.057E5)</td>
<td>6.055E5</td>
<td>6.0554E5</td>
<td>5.6806E5</td>
<td>6.0654E5</td>
</tr>
</tbody>
</table>

2.6 Application Example: Design Optimization of a Vibrating Beam

Once the sensitivity coefficients are accurately calculated, design optimization is obviously the next step. In this section, a gradient based mathematical programming algorithm, called LINRM, will be used to investigate design optimization problems that include sizing variables, support locations and support stiffness constants as design variables. The LINRM, is a recursive quadratic programming that has been proven to converge to a local minimum as long as the \(L^2\) norm of the perturbations approaches zero [64].

The structure considered here is a continuous beam with five supports, shown in Fig. 2.12 where \(\ell_1, \ell_2, \ell_3\) denote the locations of the intermediate supports. The design task is to adjust its cross-sectional areas, support locations and stiffness constants so that the beam can achieve some specific vibration characteristics. More specifically, the objective function of the design optimization problem is defined as the sum of the structural weight and spring constants, subjected to the constraints that require the first three...
frequencies of the beam equal to preset values. Mathematically, the optimization problem can be expressed as

\[
\min \varphi_0 = p_1 \pi r_1^2 \ell_1 + p_2 \pi r_2^2 (\ell_2 - \ell_1) + p_3 \pi r_3^2 (\ell_3 - \ell_2) + p_4 \pi r_4^2 (L - \ell_3) + k_1 + k_2 + k_3
\]

subject to

\[
\begin{align*}
\varphi_1 &= \frac{\omega_1}{200} - 1 = 0 \\
\varphi_2 &= \frac{\omega_2}{250} - 1 = 0 \\
\varphi_3 &= \frac{\omega_3}{300} - 1 = 0 \\
\varphi_4 &= \frac{\ell_1}{\ell_2} - 1 \leq 0 \\
\varphi_5 &= \frac{\ell_2}{\ell_3} - 1 \leq 0
\end{align*}
\]

The objective function, \( \varphi_0 \), may be viewed as the manufacturing cost of the beam structure. As for the constraints, \( \varphi_1 \) to \( \varphi_3 \) strictly specify the frequency values to be achieved at the final design, and \( \varphi_4 \) to \( \varphi_5 \) prevent supports from switching with each other.

The finite element model of the beam has an 8-8-8-8 mesh distribution.

Each of the three design examples discussed in the following has a different set of design variables. The first example considers only the radius of beams in each span as a design variable. It amounts to having four design variables \( r_1, r_2, r_3 \) and \( r_4 \). The second example adds the locations of the three intermediate supports to its design space. The third example further extends the design space by including the three spring constants \( k_1, k_2 \) and \( k_3 \) as design variables.

All the examples start with the same initial design:

- total length : \( L = 16.0 \)
- section radii : \( r_1 = 2.0, r_2 = 3.0, r_3 = 2.0, r_4 = 3.0 \)
- support locations : \( \ell_1 = 4.0, \ell_2 = 8.0, \ell_3 = 12.0 \)
• spring constants: \( k_1 = 1.02 \times 10^8 \), \( k_2 = 1.35 \times 10^8 \), \( k_3 = 8.78 \times 10^7 \)

• beam weight: 326.73

The first three frequencies, \( \omega_1 \), \( \omega_2 \) and \( \omega_3 \), of the initial design for examples 1 and 2 are 250.9, 290.57 and 355.19 respectively. However, due to the presence of spring constants, the initial frequencies of the third example give slightly different values, \( \omega_1 = 250.35 \), \( \omega_2 = 290.37 \) and \( \omega_3 = 354.22 \). The results of design optimization are hence summarized:

**Example 1**

- convergence of LINRM: 9 iterations
- sectional radii: \( r_1 = 1.55, r_2 = 2.36, r_3 = 1.93, r_4 = 2.64 \)
- beam weight: 234.58
- frequencies: \( \omega_1 = 200.36, \omega_2 = 248.87, \omega_3 = 298.65 \)

**Example 2**

- convergence of LINRM: 6 iterations
- sectional radii: \( r_1 = 1.59, r_2 = 2.62, r_3 = 1.57, r_4 = 2.53 \)
- beam weight: 225.96
- frequencies: \( \omega_1 = 199.44, \omega_2 = 249.0, \omega_3 = 300.7 \)
- support locations: \( \xi_1 = 4.02, \xi_2 = 8.23, \xi_3 = 12.43 \)

**Example 3**

- convergence of LINRM: 93 iterations
- sectional radii: \( r_1 = 1.76, r_2 = 2.40, r_3 = 1.59, r_4 = 2.41 \)
- beam weight: 214.61
- frequencies: \( \omega_1 = 200.0, \omega_2 = 250.0, \omega_3 = 300.0 \)
- support locations: \( \xi_1 = 3.85, \xi_2 = 8.07, \xi_3 = 12.3 \)
- spring constants: \( k_1 = 1.02 \times 10^8 \), \( k_2 = 1.35 \times 10^8 \), \( k_3 = 8.78 \times 10^7 \)
Figure 2.12 Example Problems for Design Optimization
A local minimum has been reached in each of the three examples. The last example, however, yields the best design in terms of the reduction of weight and precision of frequencies. This study confirms the observation that the design of the beam subject to frequency constraints can be greatly improved by slightly varying the support stiffness constants and locations.
Chapter 3

EIGENVECTOR SENSITIVITY ANALYSIS OF SKELETAL STRUCTURES WITH VARIABLE JOINTS AND SUPPORT LOCATIONS

In Chapter 2 eigenvalue sensitivity equations for several skeletal structural systems with configuration parameters have been developed. The configuration parameters concerned are joint and support locations. Those equations are derived by using the continuum approach as well as the discrete approach. Moreover, the accuracy and efficiency of those equations have been investigated by example structures including continuous beam, planar truss and planar frame.

This chapter is a continuous work of the last one to derive eigenvector sensitivity equations for skeletal structure systems. However, the focus is placed upon the planar truss and the continuous beam system. Eigenvector sensitivity equations of a continuous beam are first developed by both the domain method and the boundary method, respectively. The results show that the domain method is superior to the boundary method in terms of computational efficiency. Subsequently, only the domain method is extended to find the eigenvector sensitivity equation of a planar truss. Several examples are then presented to investigate the accuracy and the efficiency of the derived eigenvector sensitivity equations.

3.1 Shape Sensitivity Analysis of Eigenvectors of a Continuous Beam

In the last chapter, a procedure of using the concept of the material derivative has been developed to derive an eigenvalue sensitivity equation based upon the weak variational form of the governing differential equation. Here, the same concept will be extended to
derive the eigenvector sensitivity equations of a continuous beam with variable support locations.

3.1.1 Domain Method (DM)

The derivation of the eigenvector sensitivity equation of a continuous beam with variable support locations starts from the weak variational form of the eigenvalue problem, Eq. (2.19). Using the basic relation, Eq. (A.9), the total material derivative of this weak variational form yields

\[ \lambda \sum_{i=1}^{N} \int_{0}^{l_i} \rho_i A_i w_i \phi_i \, ds = \sum_{i=1}^{N} \int_{0}^{l_i} (EI_i \dddot{w}_i \phi_i - \lambda \rho_i A_i \dddot{w}_i \phi_i) \, ds \]

\[ - \sum_{i=1}^{N} \int_{0}^{l_i} [EI_i \dddot{w}_i \phi_i' + w_i \dddot{w}_i \phi_i' + 3w_i \dddot{w}_i \phi_i'] + \lambda \rho_i A_i w_i \phi_i \, ds \]

\[ + \sum_{i=1}^{N} EI_i (\dddot{w}_i \phi_i - \dddot{w}_i \phi_i) \, ds \]

Let the total shape derivatives of the mode shape, \( \dot{w}_i \), be equal to \( \eta_i + c_1 w_i \) and the arbitrary function, \( \phi_i \), be equal to \( \zeta_i + c_2 w_i \). Note that the functions of \( \eta_i \) and \( \zeta_i \) are required to be orthogonal to the mode shape of \( w_i \) with the following orthogonal conditions

\[ \sum_{i=1}^{N} \int_{0}^{l_i} \rho_i A_i \eta_i w_i \, ds = 0, \]

\[ \sum_{i=1}^{N} \int_{0}^{l_i} \rho_i A_i \zeta_i w_i \, ds = 0. \] (3.2)

In order to simplify the notation, the following definitions are introduced

\[ R_1(w, \varphi) = \sum_{i=1}^{N} \int_{0}^{l_i} \rho_i A_i w_i \varphi_i \, ds \]

\[ = c_2 R_1(w, w) + R_1(w, \zeta) \] (3.3)

and

\[ R_2(w, \varphi) = \sum_{i=1}^{N} \int_{0}^{l_i} (EI_i \dddot{w}_i \phi_i' + w_i \dddot{w}_i \phi_i' + 3w_i \dddot{w}_i \phi_i') + \lambda \rho_i A_i w_i \phi_i \, ds \]

\[ = c_2 R_2(w, w) + R_2(w, \zeta) \] (3.4)
Furthermore, the eigenvalue sensitivity equation in Eq. (2.50) can be shortened as
\[ \dot{\lambda} = -R_2(w, w) \]. Substituting the relations of Eqs. (3.2), (3.3), (3.4) and (2.18) into Eq. (3.1), results in a simpler equation
\[ 0 = N \sum_{i=1}^{N} (E_i I_i \dot{w}_i \varphi_i'' - \lambda \rho \varphi_i \dot{w}_i) ds - R_2(w, \zeta) \]
\[ + \sum_{i=1}^{N} E_i I_i (\ddot{w}_i^\alpha \varphi_i'' - \ddot{w}_i^\alpha \varphi_i' + w_i^\alpha \varphi_i') |_0^{t_f} \]. (3.5)

The integral term in the above equation can be further simplified by integrating by parts and using the relations, \( \dot{w}_i = \eta_i + c_1 w_i \) and \( \varphi_i = \zeta_i + c_2 w_i \), to obtain
\[ \sum_{i=1}^{N} \int_0^{t_f} (E_i I_i \dot{w}_i \varphi_i'' - \lambda \rho \varphi_i \dot{w}_i) ds = \sum_{i=1}^{N} c_2 E_i I_i (w_i^\alpha \dot{w}_i' - w_i^\alpha \dot{w}_i) |_0^{t_f} \]
\[ + \sum_{i=1}^{N} c_1 E_i I_i (w_i^\alpha \zeta_i'' - w_i^\alpha \zeta_i') |_0^{t_f} + \sum_{i=1}^{N} \int_0^{t_f} (E_i I_i \eta_i^\alpha \zeta_i''' - \lambda \rho \varphi_i \eta_i \zeta_i) ds. \] (3.6)

Note that the state equation, Eq. (2.17), has been applied in the previous derivation. Arranging all the boundary terms in Eqs. (3.5) and (3.6), the equation to determine the function \( \eta_i \) can be derived from Eqs. (3.5) and (3.6) as
\[ \sum_{i=1}^{N} \int_0^{t_f} E_i I_i (\eta_i^\alpha \zeta_i''' - \lambda \rho \varphi_i \eta_i \zeta_i) ds = R_2(w, \zeta) - \sum_{i=1}^{N} c_2 E_i I_i (w_i^\alpha \dot{w}_i' - w_i^\alpha \dot{w}_i) |_0^{t_f} \]
\[ - \sum_{i=1}^{N} c_1 E_i I_i (w_i^\alpha \zeta_i'' - w_i^\alpha \zeta_i') |_0^{t_f} - \sum_{i=1}^{N} c_2 E_i I_i (\ddot{w}_i^\alpha w_i' - w_i^\alpha \dot{w}_i' + w_i^\alpha w_i' \dot{w}_i' + w_i^\alpha \ddot{w}_i') |_0^{t_f} \]. (3.7)

Although the boundary terms of the last equation seem very complicated, the material derivatives of the boundary conditions of the original continuous beam will provide the necessary equalities to simplify them. The boundary conditions of the continuous beam
\[ w|_s = 0 \]
\[ w'|_s = w'|_s. \] (3.8)

where \( s \) denotes the support location and whose total material derivatives give
Furthermore, the arbitrary function, \( \varphi_i \) also satisfies kinematic boundary condition as \( w_i \) does at the support location at \( s \), that is,

\[
\varphi_i|_{s-} = \varphi_i|_{s+} = 0
\]

\[
\varphi_i'|_{s-} = \varphi_i'|_{s+}.
\]

(3.10)

which imply

\[
(\zeta + c_2 w)|_{s-} = (\zeta + c_2 w)|_{s+} = 0
\]

\[
(\zeta' + c_2 w')|_{s-} = (\zeta' + c_2 w')|_{s+}
\]

(3.11)

The above equations result in the following boundary conditions \( \zeta'|_{s-} = \zeta'|_{s+} = 0 \) and \( \zeta'|_{s-} = \zeta'|_{s+} \) at the intermediate support point. Based upon the above equations, the boundary terms of Eq. (3.7) can be canceled by each other at the intermediate support points. Finally, the Eq. (3.7) yields a linear equation to find \( \eta_i(s) \)

\[
\sum_{i=1}^{N} \int_{0}^{l_i} E_i I_i (\eta_i'' \zeta'' - \lambda \rho_i A_i \eta_i \zeta_i) ds = R_2(w, \zeta)
\]

(3.12)

To complete the derivation, the constant \( c_i \) in the relation, \( \dot{w}_i = \eta_i + c_i w_i \), can be determined by taking the total material derivative of the normalization condition and employing the orthogonal condition. The resultant relation is given as

\[
c_i = -\frac{1}{2} \sum_{i=1}^{N} \int_{0}^{l_i} \rho_i A_i w_i^2 v_i ds.
\]

(3.13)
3.1.2 Boundary Method (BM)

Derivation of eigenvector sensitivity equation using the boundary method can be proceeded following the same procedure as the one described in Section 2.3.2. Many fundamental equations and definitions needed for such a derivation have been introduced previously and will not be repeated here.

To start the derivation, taking the total material derivative of the weak variational form, Eq. (2.19), yields

\[\lambda \sum_{i=1}^{N} \int_{0}^{t_i} \rho_i A_i w_i \varphi_i \, ds = \sum_{i=1}^{N} \int_{0}^{t_i} (E_i I_i w_i^{\prime \prime} \varphi_i^{\prime} - \lambda \rho_i A_i w_i \varphi_i) \, ds + \sum_{i=1}^{N} \int_{0}^{t_i} E_i I_i (w_i^{\prime \prime} \varphi_i^{\prime} - w_i^{\prime} \varphi_i^{\prime}) \, ds \]  

which is expressed in terms of the relative material derivative, \(w_i\). Let the relative material derivative of the eigenvector, \(w_i\), be equal to \(\eta_i + c_4 w_i\) and the arbitrary function \(\varphi_i\), be equal to \(\xi_i + c_5 w_i\). Note that the functions, \(\eta_i\) and \(\xi_i\), are also required to satisfy the orthogonal conditions

\[\sum_{i=1}^{N} \int_{0}^{t_i} \rho_i A_i \eta_i w_i \, ds = 0,\]

\[\sum_{i=1}^{N} \int_{0}^{t_i} \rho_i A_i \xi_i w_i \, ds = 0.\]  

To further simplify the derivation procedure, let the following identities be

\[R_3(w, \varphi) = \sum_{i=1}^{N} \int_{0}^{t_i} \rho_i A_i w_i \varphi_i \, ds \]

\[= c_5 R_3(w, w) + R_3(w, \xi) \]  

and

\[R_4(w, \varphi) = \sum_{i=1}^{N} \int_{0}^{t_i} E_i I_i w_i^{\prime \prime} \varphi_i^{\prime} \, ds \]

\[= c_5 R_4(w, w) + R_4(w, \xi).\]  

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
The eigenvalue sensitivity equation of the continuous beam by the boundary method can then be given by

\[
\lambda = -\sum_{i=1}^{N} \left( \lambda \rho_i A_i w_i^2 + E_i I_i w_i''^2 - 2E_i I_i w_i''' w_i' \right) v_i|_{t_0}^{t_i} \\
= R_4 (w, w) + \sum_{i=1}^{N} E_i I_i (w_i''' w_i' - w_i''^2) v_i|_{t_0}^{t_i}. 
\]  
(2.65)  
(3.17)

Substituting the above equations into Eq. (3.14) yields

\[
0 = \sum_{i=1}^{N} \int_{t_0}^{t_i} \left( E_i I_i w_i'' \phi_i'' - \lambda \rho_i A_i w_i \phi_i \right) ds + R_4 (w, \zeta) \\
- \sum_{i=1}^{N} c_4 E_i I_i (w_i''' - w_i''^2) v_i|_{t_0}^{t_i} + \sum_{i=1}^{N} E_i I_i (w_i''' - w_i'' \phi_i') v_i|_{t_0}^{t_i}. 
\]  
(3.18)

The first integral of Eq. (3.18) can be reduced to

\[
\sum_{i=1}^{N} \int_{t_0}^{t_i} \left( E_i I_i w_i'' \phi_i'' - \lambda \rho_i A_i w_i \phi_i \right) ds = \sum_{i=1}^{N} \int_{t_0}^{t_i} \left( E_i I_i \eta_i'' \zeta_i'' - \lambda \rho_i A_i \eta_i \zeta_i \right) ds \\
+ \sum_{i=1}^{N} c_4 E_i I_i (w_i''' \zeta_i - w_i'' \zeta_i) v_i|_{t_0}^{t_i} + \sum_{i=1}^{N} c_2 E_i I_i (w_i''' w_i' - w_i'' w_i') v_i|_{t_0}^{t_i} 
\]  
(3.19)

where the integration by parts and the state equation have been employed. Now, after replacing the total shape derivatives in the above equation by the relative shape derivatives, the general form of the eigenvector sensitivity equation can be given as

\[
\sum_{i=1}^{N} \int_{t_0}^{t_i} \left( E_i I_i \eta_i'' \zeta_i'' - \lambda \rho_i A_i \eta_i \zeta_i \right) ds = -R_4 (w, \zeta) + \sum_{i=1}^{N} c_4 E_i I_i (w_i''' \zeta_i - w_i'' \zeta_i) v_i|_{t_0}^{t_i} \\
+ \sum_{i=1}^{N} c_3 E_i I_i \left( w_i''' \dot{w}_i + w_i'' \ddot{w}_i - w_i''' \dot{w}_i \right) v_i|_{t_0}^{t_i} + \sum_{i=1}^{N} E_i I_i \left( \zeta_i \ddot{w}_i - \zeta_i \dot{w}_i \right) v_i|_{t_0}^{t_i}. 
\]  
(3.20)

The summation of all the boundary terms in the above equation can be proved to be zero based upon Eqs. (3.8) to (3.11). Simpler form to find \( \eta_i(s) \) can then be obtained as

\[
\sum_{i=1}^{N} \int_{t_0}^{t_i} \left( E_i I_i \eta_i'' \zeta_i'' - \lambda \rho_i A_i \eta_i \zeta_i \right) ds = -R_4 (w, \zeta). 
\]  
(3.21)
The boundary condition, \( \eta_i = -v_i w'_i \), at the support location of the above equation is obtained by taking the total material derivative of the original boundary condition, i.e., \( w_i = 0 \).

The constant \( c_4 \) in the relation \( w_i = \eta_i + c_4 w_i \) is obtained by taking the relative material derivative of the normalization condition of eigenvector. The result is given as

\[
c_4 = -\frac{1}{2} \sum_{i=1}^{N} \rho_i A_i w_i^2 v_i |t_0|.
\]  

3.2 Analytical Example: A Simply-Supported Beam

In this section, a uniform simply-supported beam, is investigated here. Note that the length of the beam will be changed in this study. This simply-supported beam whose eigenvalues and eigenvectors can be analytically derived will be used to investigate the accuracy of eigenvector sensitivity coefficients obtained by both the domain method and the boundary method. The geometric and the material parameters of this simply-supported beam are depicted in Fig. 3.1. Only the lateral vibration is considered in this study.

![Figure 3.1 A Uniform Simply-Supported Beam](image)

The eigensolutions [62] of this problem are given as

\[
\lambda_r = \frac{EI}{\rho A} b^4, \quad r = 1, 2, 3, \ldots
\]  

and
\[ X_r(s) = c_s \sin bs, \quad r = 1, 2, 3, \ldots. \quad (3.24) \]

where \( s \) ranges from 0 to \( L \), \( b = \frac{\pi}{L} \) and the amplitude constant, \( c_s \), is

\[ c_s = \sqrt{\frac{2}{\rho A L}}. \quad (3.25) \]

According to Eq. (2.21), the arc length \( s \) and the total length \( L \) of the beam in the deformed domain are described as

\[ s(\tau) = s + \tau v(s) \quad (3.26) \]

and

\[ L(\tau) = L + \tau v(L) \quad (3.27) \]

where \( v(L) = v_0 \), the movement of the end point. Therefore, the eigenfunction \( X_r(\tau) \) in the deformed domain is given as

\[ X_r(\tau) = \sqrt{\frac{2}{\rho A L(\tau)}} \sin \frac{\pi}{L(\tau)} s(\tau) \quad (3.28) \]

Subsequently, the total shape derivative of an eigenvector can be expressed in the following form, as given by Eq. (2.25)

\[ \dot{X}_r = \frac{\partial X_r}{\partial \tau} + v X'_r \quad (3.29) \]

where \( X'_r \) is the spatial derivative, \( X'_r = \frac{\partial X_r}{\partial s} \), and the relative material derivative \( \frac{\partial X_r}{\partial \tau} \) is obtained by

\[ \frac{\partial X_r}{\partial \tau} = \frac{\partial X_r}{\partial L} \frac{dL}{d\tau} = -\left(\frac{s}{L} c_s b \cos bs + \frac{c_s}{2L} \sin bs\right) v_0. \quad (3.30) \]

Substituting Eq. (3.30) into Eq. (3.29), the eigenvector sensitivity of this problem become
\[ \dot{X}_r = (v - \frac{s}{L}v_0)c_s \cos \beta s - \frac{c_s}{2L}v_0 \sin \beta s \]  
(3.31)

where \( X_r' \) is equal to \( c_s \cos \beta s \). Furthermore, two kinematic boundary conditions of Eq. (3.31), \( \dot{X}_r(0) = \dot{X}_r(L) = 0 \) are obtained from the total material derivatives of the kinematic boundary conditions of simply-supported beam, \( X_r(0) = X_r(L) = 0 \).

Employing the boundary conditions \( \dot{X}_r(0) = \dot{X}_r(L) = 0 \) into Eq. (3.31), the boundary conditions of velocity function \( v(s) \) is specified by \( v(0)=0 \) and \( v(L)=v_0 \), respectively. Then, the velocity function, \( v(s) \), can be specified as a linear function, that is,
\[ v(s) = \frac{s}{L}v_0. \]  
(3.32)

Finally, the shape derivative of the eigenvector of the simply-supported beam is
\[ \dot{X}_r = -\frac{c_s}{2L} \sin \beta s = -\frac{1}{2L}X_r. \]  
(3.33)

Note that the material derivative of the slope of an eigenvector can be derived from the relation
\[ \ddot{X}_r = \dot{X}_r' - v'X_r' \]
\[ = -\frac{1}{2L}X_r' - \frac{1}{L}X_r' \]
\[ = -\frac{3}{2L}X_r'. \]  
(3.34)

Let the simply-supported beam shown in Fig 3.1 be made of a circular section with a length, \( L=2.0 \), Young's modulus, \( E=1.0 \times 10^4 \), mass per unit length, \( \rho = 1.0 \), and the radius, 0.25. This beam is uniformly divided by 8 elements. Table 3.1 shows that these eigenvalues obtained by this finite element model are very close to the analytical one. In other words, 8 elements is a valid FE model. The results of the first and third eigenvector sensitivity coefficients are tabulated in Table 3.2 and Table 3.3. Obviously, the coefficients of eigenvector sensitivity calculated from the continuum approach, Eqs. (3.12) and (3.21),
have the same accuracy as that calculated from the analytical solution, Eqs. (3.33) and (3.34).

Table 3.1 Eigenvalues of a Simply-Supported Beam

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Analytical</th>
<th>Finite Element Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>951.2606546</td>
<td>951.2972412</td>
</tr>
<tr>
<td>2</td>
<td>15220.17047</td>
<td>15228.07324</td>
</tr>
<tr>
<td>3</td>
<td>77052.11303</td>
<td>77250.52344</td>
</tr>
</tbody>
</table>

Table 3.2 Eigenvector Sensitivity Coefficients of First Mode

<table>
<thead>
<tr>
<th>position(s)</th>
<th>Analytical</th>
<th>Domain method</th>
<th>Boundary Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>-2.658</td>
<td>-2.659</td>
<td>-2.659</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.216</td>
<td>-0.216</td>
<td>-0.216</td>
</tr>
<tr>
<td></td>
<td>-2.457</td>
<td>-2.456</td>
<td>-2.456</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.399</td>
<td>-0.399</td>
<td>-0.399</td>
</tr>
<tr>
<td></td>
<td>-1.881</td>
<td>-1.880</td>
<td>-1.880</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.521</td>
<td>-0.521</td>
<td>-0.521</td>
</tr>
<tr>
<td></td>
<td>-1.017</td>
<td>-1.017</td>
<td>-1.017</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.564</td>
<td>-0.564</td>
<td>-0.564</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table 3.3 Eigenvector Sensitivity Coefficients of Third Mode

<table>
<thead>
<tr>
<th>position(s)</th>
<th>Analytical</th>
<th>Domain method</th>
<th>Boundary Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>X'</td>
<td>-8.0</td>
<td>-7.976</td>
<td>-7.976</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.523</td>
<td>-0.521</td>
<td>-0.52</td>
</tr>
<tr>
<td>X</td>
<td>-3.06</td>
<td>-3.053</td>
<td>-3.051</td>
</tr>
<tr>
<td>X'</td>
<td>5.654</td>
<td>5.64</td>
<td>5.637</td>
</tr>
<tr>
<td>0.75</td>
<td>0.217</td>
<td>0.216</td>
<td>0.216</td>
</tr>
<tr>
<td>X</td>
<td>7.388</td>
<td>7.369</td>
<td>7.368</td>
</tr>
<tr>
<td>X'</td>
<td>0.564</td>
<td>0.564</td>
<td>0.564</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 3.4 Computational Times of Sensitivity Analysis of Simply-Supported Beam (CYBER 930 NOS/VE 1.4.1)

<table>
<thead>
<tr>
<th></th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU(sec)</td>
<td>0.0458</td>
<td>0.07</td>
</tr>
<tr>
<td>normalized</td>
<td>0.654</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.4 lists the CPU times requested for each of the computational methods. It clearly indicates that the boundary method is not compared favorably to the domain method. This is because an additional computation is required to relate the solution, $\eta_j$, of the eigenvector sensitivity equation derived by the boundary method to the total shape derivative, $\bar{w}_{ij}$, as

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ \dot{w}_i = w_{i,i} + v_i w_i' = \eta_i + c_i w_i + v_i w_i'. \]  

(3.35)

### 3.3 Discrete Approach

The discrete analytical method (DAM) of the discrete approach has been discussed for the eigenvalue sensitivity analysis in Section 2.4. The eigenvector derivative can be assumed to have the form

\[ \frac{dX}{db} = cX + \overline{X} \]  

(3.36)

where \( b \) is the design variable and \( c \) is determined by the normalization condition, that is

\[ c = -\frac{1}{2} X^T \frac{dM}{db} X. \]  

(3.37)

The vector \( \overline{X} \) is specified by the following

\[ \begin{bmatrix} K - \lambda M & MX \\ X^T M & 0 \end{bmatrix} \begin{bmatrix} \overline{X} \\ \mu \end{bmatrix} = \begin{bmatrix} -(\frac{dK}{db} - \lambda \frac{dM}{db})X \\ 0 \end{bmatrix}. \]  

(3.38)

The second row of the proceeding equation implies a constraint regarding the orthogonal condition

\[ X^T M \overline{X} = 0. \]  

(3.39)

The central difference method (CDM) uses the following equation to approximate eigenvector derivatives

\[ \frac{dX}{db} = \frac{X(b + \Delta b) - X(b - \Delta b)}{2\Delta b} \]  

(3.40)
where $\Delta b$ represents the perturbation of the joint location. In the following numerical examples, the $\Delta b$ is given 0.005.

The left hand side integrals of the sensitivity equations of Eqs. (3.12) and (3.21) can be numerically implemented in the matrix form as $[K]-\lambda[M]$. The additional orthogonal condition, Eq. (3.39), should be employed to avoid the singularity of the matrix $[K]-\lambda[M]$.

### 3.4 Eigenvector Sensitivity Analysis of Continuous Beam: Numerical Study

As those used in Section 2.5.2, a continuous beam, a stepped beam, a continuous beam with a spring support and a stepped beam with a spring support are employed here again to numerically validate the eigenvector sensitivity equations presented in this chapter. The geometry and material properties have been listed in Section 2.5.2. Here, the finer mesh corresponds to a 3-6 distribution used for all cases. The spring constant, 1.0E4, is specified at the intermediate support for those cases considering the spring effects for eigenvector sensitivity analysis.

Four tables, Tables 3.5 to 3.8, are organized to document the numerical results. The first three eigenvector sensitivity coefficients are calculated for the slope of intermediate support at the point A, $A(y_*)$ and the displacement at the point B, $B(y)$, of the middle point at the right-hand side span. The results of four different methods, the central difference method (CDM), discrete analytical method (DAM), the domain method (DM) and the boundary method (BM), respectively, are listed for comparison.

The additional information provided by Table 3.9 is the computational times (CPU seconds) required by various methods for sensitivity analysis, which are normalized with respect to the computational time of the boundary method.

In Table 3.9, consider only the problem case of a stepped beam with a spring support. It shows that the most efficient way to calculate the eigenvector sensitivity
coefficients is the domain method. The numerical results reported in Tables 3.5 to 3.8 generally confirm the validity of the derived sensitivity formulations. However the performance of sensitivity equation derived by the boundary method is worse than that by the other methods. For this reason, only the domain method is used in the next section to derive the sensitivity equation of eigenvector of planar truss.

**Table 3.5 Eigenvector Sensitivity Coefficients of a Uniform Beam**

<table>
<thead>
<tr>
<th>mode</th>
<th>location</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A(y')$</td>
<td>-0.8057</td>
<td>-0.8058</td>
<td>-0.7957</td>
<td>-0.7971</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>0.8246E-1</td>
<td>0.8241E-1</td>
<td>0.8157E-1</td>
<td>0.8241E-1</td>
</tr>
<tr>
<td>2</td>
<td>$A(y')$</td>
<td>2.784</td>
<td>2.178</td>
<td>2.114</td>
<td>2.2257</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>0.6502</td>
<td>0.6502</td>
<td>0.6325</td>
<td>0.6246</td>
</tr>
<tr>
<td>3</td>
<td>$A(y')$</td>
<td>2.6443</td>
<td>2.6440</td>
<td>2.509</td>
<td>2.4897</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>-0.2160</td>
<td>-0.2160</td>
<td>-0.1998</td>
<td>-0.2211</td>
</tr>
</tbody>
</table>

**Table 3.6 Eigenvector Sensitivity Coefficients of a Stepped Beam**

<table>
<thead>
<tr>
<th>mode</th>
<th>location</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A(y')$</td>
<td>-0.5967</td>
<td>-0.5967</td>
<td>-0.5902</td>
<td>-0.5115</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>0.7712E-1</td>
<td>0.7706E-1</td>
<td>0.7601E-1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>$A(y')$</td>
<td>1.9542</td>
<td>1.954</td>
<td>1.8955</td>
<td>1.1881</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>0.5192</td>
<td>0.5192</td>
<td>0.5057</td>
<td>0.5758</td>
</tr>
<tr>
<td>3</td>
<td>$A(y')$</td>
<td>-1.992</td>
<td>-1.992</td>
<td>-1.8867</td>
<td>-1.1935</td>
</tr>
<tr>
<td></td>
<td>$B(y)$</td>
<td>0.1866</td>
<td>0.1866</td>
<td>0.1716</td>
<td>0.1270</td>
</tr>
</tbody>
</table>
Table 3.7 Eigenvector Sensitivity Coefficients of a Continuous Beam with a Spring Support

<table>
<thead>
<tr>
<th>mode</th>
<th>location</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A(\ddot{y})</td>
<td>-0.3836</td>
<td>-0.3837</td>
<td>-0.3801</td>
<td>-0.3888</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>-0.1532</td>
<td>-0.1532</td>
<td>-0.1517</td>
<td>-0.1532</td>
</tr>
<tr>
<td>2</td>
<td>A(\ddot{y})</td>
<td>-0.6019</td>
<td>-0.6019</td>
<td>-0.5966</td>
<td>-0.6110</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>0.6888</td>
<td>0.6888</td>
<td>0.6705</td>
<td>0.7096</td>
</tr>
<tr>
<td>3</td>
<td>A(\ddot{y})</td>
<td>-2.4014</td>
<td>-2.4015</td>
<td>-2.2950</td>
<td>-2.6086</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>0.6543</td>
<td>0.6543</td>
<td>0.6053</td>
<td>0.7359</td>
</tr>
</tbody>
</table>

Table 3.8 Eigenvector Sensitivity Coefficients of a Stepped Beam with a Spring Support

<table>
<thead>
<tr>
<th>mode</th>
<th>location</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A(\ddot{y})</td>
<td>-0.3490</td>
<td>-0.3490</td>
<td>-0.3462</td>
<td>-0.2774</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>-0.1115</td>
<td>-0.1115</td>
<td>-0.1100</td>
<td>-0.1029</td>
</tr>
<tr>
<td>2</td>
<td>A(\ddot{y})</td>
<td>-0.2261</td>
<td>-0.2262</td>
<td>-0.2286</td>
<td>-0.1784</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>-0.7201E-1</td>
<td>-0.7202E-1</td>
<td>-0.7056E-1</td>
<td>-0.9260E-1</td>
</tr>
<tr>
<td>3</td>
<td>A(\ddot{y})</td>
<td>2.7556</td>
<td>2.7556</td>
<td>2.6313</td>
<td>2.0123</td>
</tr>
<tr>
<td></td>
<td>B(y)</td>
<td>0.3873</td>
<td>0.3873</td>
<td>0.3745</td>
<td>0.3867</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 3.9 Computational Times of Sensitivity Analysis of a Stepped Beam with a Spring Support
(CYBER 930 NOS/VE 1.4.1)

<table>
<thead>
<tr>
<th></th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
<th>BM</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU(sec)</td>
<td>0.24</td>
<td>0.056</td>
<td>0.049</td>
<td>0.094</td>
</tr>
<tr>
<td>normalized</td>
<td>2.553</td>
<td>0.595</td>
<td>0.521</td>
<td>1</td>
</tr>
</tbody>
</table>

3.5 Shape Sensitivity Analysis of Eigenvector of a Planar Truss
Using the Domain Method

The eigenvalue equation of a planar truss and its weak variational form have been mentioned in Section 2.1.2. Now taking the total material derivative of the weak variational form, Eq. (2.13), results in

\[
\dot{\pi} = 0
\]

\[
= \sum_{i=1}^{N} \int_{0}^{l_i} \left( \lambda \rho A_i u_i \dot{\phi}_i + \lambda \rho A_i (\dot{u}_i + u_i \phi_i) - E_i A_i (\ddot{u}'_i \phi'_i + u'_i \ddot{\phi}'_i) \right) ds + \sum_{i=1}^{N} \int_{0}^{l_i} (\lambda \rho A_i u_i \phi_i - E_i A_i u'_i \phi'_i) v'_i ds
\]

(3.41)

where the boundary terms are dropped because of the specified boundary conditions of Eq. (2.15).

It should be noted that the total shape derivatives of \( u_i \), \( \phi_i \), \( u'_i \) and \( \phi'_i \) can be obtained from the definition of the total material derivative, i.e.,

\[
\dot{u}_i = \ddot{u}_i + \vartheta_i \dot{\theta}_i
\]

\[
= \eta_i + cu_i + \vartheta_i \dot{\theta}_i
\]

\[
\dot{\phi}_i = \ddot{\phi}_i + \varphi_i \dot{\theta}_i
\]

(3.42)

and
\[
\dot{u}_i = \dot{u}_{s,i} + \phi_i' \hat{\theta}_i
\]
\[
= u_i' - \nu_i u_i' + \phi_i' \hat{\theta}_i
\]
\[
= \eta_i + cu_i' - \nu_i u_i' + \phi_i' \hat{\theta}_i
\]

(3.43)

\[
\dot{\phi}_i = \dot{\phi}_{s,i} + \phi_i' \hat{\theta}_i
\]
\[
= \phi_i' - \nu_i \phi_i' + \phi_i' \hat{\theta}_i.
\]

where \( \phi_i = m_i W_i - n_i U_i \) as given in Eq. (2.42). Moreover, the material derivative of the
eignmode, \( \dot{u}_{s,i} \), has been replaced by a summation of a function \( \eta_i \) and the eigenfunction \( u_i \)
with constant \( c \), i.e., \( \eta_i = cu_i \). Note that the functions, \( \eta_i(s) \) and \( \phi_i(s) \), are required to be
orthogonal to the eigenmode, \( u_i(s) \), with the following conditions

\[
\sum_{i=1}^{N} \int_{0}^{t} \rho_i A_i \eta_i u_i ds = 0,
\]
\[
\sum_{i=1}^{N} \int_{0}^{t} \rho_i A_i \phi_i u_i ds = 0.
\]

(3.44)

With the aid of Eqs. (3.42) to (3.44), Eq. (3.41) can be rewritten in term of \( \eta_i \) and \( \phi_{s,i} \) as

\[
0 = \sum_{i=1}^{N} \int_{0}^{t} \left( \rho_i A_i \left( u_i \phi_i + \phi_i \dot{\theta}_i \right) - E_i A_i (u_i' \phi_i' + \phi_i' \dot{\theta}_i) \right) ds
\]
\[
+ \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i \phi_i + E_i A_i u_i' \phi_i' \right) v_i ds + \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i \eta_i \dot{\theta}_i - E_i A_i \eta_i' \phi_i' \right) ds
\]
\[
+ \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i \phi_i' \theta_i - E_i A_i u_i' \phi_i' \right) ds + \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i \phi_i - E_i A_i u_i' \phi_i' \right) ds
\]

(3.45)

Next, using integration by parts simplifies the last two terms in Eq. (3.45) as

\[
\sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i \phi_i' \theta_i - E_i A_i u_i' \phi_i' \right) ds + \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i \phi_i - E_i A_i u_i' \phi_i' \right) ds
\]
\[
= \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i + E_i A_i u_i' \right) \phi_i ds + \sum_{i=1}^{N} \int_{0}^{t} \left( \lambda \rho_i A_i u_i + E_i A_i u_i' \right) \phi_i ds
\]
\[
- \sum_{i=1}^{N} E_i A_i u_i' \phi_i ds - \sum_{i=1}^{N} c E_i A_i u_i' \phi_i ds
\]

(3.46)
The integrals are dropped in Eq. (3.46) as the terms in the parentheses are exactly identical to the state equation presented by Eq. (2.11). The remaining boundary terms in Eq. (3.46) can be converted to their counterparts in the global coordinate system as

\[- \sum_{i=1}^{N} (P_i m_i \Phi_i + P_i n_i \Psi_i) \delta_0 - \sum_{i=1}^{N} c(P_i m_i \Phi_i + P_i n_i \Psi_i) \delta_i. \]  

(3.47)

The terms in Eq. (3.47) are dropped off counting on the kinematic boundary conditions at supports or the natural boundary conditions at interior joints. Thus, the eigenvector sensitivity equation using the domain method has the form

\[
\sum_{i=1}^{N} \int_{0}^{t} (E_i A_i \eta_i \dot{\phi}_i - \lambda \rho_i A_i \eta_i \dot{\phi}_i) ds = \sum_{i=1}^{N} \int_{0}^{t} (\lambda \rho_i A_i u_i \dot{\phi}_i + E_i A_i u_i \dot{\phi}_i) v_i ds \\
+ \sum_{i=1}^{N} \int_{0}^{t} \lambda \rho_i A_i (u_i \dot{\phi}_i + \theta_i \ddot{\phi}_i) - E_i A_i (u_i \dot{\phi}_i + \theta_i \ddot{\phi}_i) \dot{\theta}_i ds. \]  

(3.48)

The above equation yields a unique solution, \( \eta_i \), which is subjected to the boundary conditions, \( \eta_i = 0 \), at the support location. The constant \( c \) in the relation of \( \dot{u}_{n,i} = \eta_i + cu_i \) can be determined by the following equality which is obtained by taking the material derivative of the normalization condition, Eq. (2.12). That is,

\[
\sum_{i=1}^{N} \int_{0}^{t} 2 \rho_i A_i u_i \dot{u}_i ds + \sum_{i=1}^{N} \int_{0}^{t} \rho_i A_i u_i^2 v_i ds = 0. \]  

(3.49)

Employing the relation \( \dot{u}_i = \eta_i + cu_i + \theta_i \ddot{\phi}_i \) and the orthogonal conditions in Eq. (3.44), then the constant \( c \) is given as

\[
\sum_{i=1}^{N} \int_{0}^{t} 2 \rho_i A_i u_i \dot{u}_i ds + \sum_{i=1}^{N} \int_{0}^{t} \rho_i A_i u_i^2 v_i ds = 0. \]  

(3.50)
3.6 Eigenvector Sensitivity Analysis of Planar Truss: Numerical Study

In this section, a fifty-one member truss is employed here again to test the numerical performance of the derived eigenvector sensitivity equations. The geometry, material information and design variables of the example have been given in Section 2.5.2.

Four sets of design variables are specified in this example. The design variables of each set have been mentioned in Section 2.5.2 and shown in Fig. 2.10. The sensitivity coefficients of the first three eigenvectors with respect to the movements of joint A are investigated. The location of joint A is shown in Fig. 2.10. Table 3.10 to Table 3.13 document the numerical results of eigenvector sensitivity analysis by the central difference method (CDM), discrete analytical method (DAM) and the domain method (DM), respectively.

The additional information provided by Table 3.14 is the computational times (CPU seconds) required by various methods for sensitivity analysis, which are the normalized with respect to the computational time of the domain method. In Table 3.14, consider only design case 1. The numerical results presented in Table 3.10 through Table 3.13 show that the central finite difference method (CDM), the discrete analytical method (DAM) and the domain method (DM) agree with each other. Finally, in terms of the overall accuracy and efficiency, the domain method has the best performance among all of the methods, though the difference in accuracy among those methods is not significant.
Table 3.10 Eigenvector Sensitivity Coefficients of Design Variable Set 1 at Joint A

<table>
<thead>
<tr>
<th>mode</th>
<th>direction</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>-0.1677E-3</td>
<td>-0.1677E-3</td>
<td>-0.1677E-3</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>0.7808E-4</td>
<td>0.7808E-4</td>
<td>0.7808E-4</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>-0.1388E-3</td>
<td>-0.1388E-3</td>
<td>-0.1388E-3</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>0.2271E-3</td>
<td>0.2271E-3</td>
<td>0.2271E-3</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>0.1072E-3</td>
<td>0.1072E-3</td>
<td>0.1072E-3</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-0.4595E-3</td>
<td>-0.4595E-3</td>
<td>-0.4595E-3</td>
</tr>
</tbody>
</table>

Table 3.11 Eigenvector Sensitivity Coefficients of Design Variable Set 2 at Joint A

<table>
<thead>
<tr>
<th>mode</th>
<th>direction</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>0.2984E-6</td>
<td>0.2984E-6</td>
<td>0.2984E-6</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-0.1964E-5</td>
<td>-0.1964E-5</td>
<td>-0.1964E-5</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>0.1208E-4</td>
<td>0.1208E-4</td>
<td>0.1208E-4</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>0.2634E-4</td>
<td>0.2634E-4</td>
<td>0.2634E-4</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>-0.2545E-4</td>
<td>-0.2545E-4</td>
<td>-0.2545E-4</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-0.1095E-4</td>
<td>-0.1095E-4</td>
<td>-0.1095E-4</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 3.12 Eigenvector Sensitivity Coefficients of Design Variable Set 3 at Joint A

<table>
<thead>
<tr>
<th>mode</th>
<th>direction</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>6.4E-6</td>
<td>6.4E-6</td>
<td>6.4E-6</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>2.47E-4</td>
<td>2.47E-4</td>
<td>2.47E-4</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>2.35E-4</td>
<td>2.35E-4</td>
<td>2.35E-4</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-1.68E-3</td>
<td>-1.68E-3</td>
<td>-1.68E-3</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>2.20E-4</td>
<td>2.20E-4</td>
<td>2.20E-4</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>1.54E-4</td>
<td>1.54E-4</td>
<td>1.54E-4</td>
</tr>
</tbody>
</table>

Table 3.13 Eigenvector Sensitivity Coefficients of Design Variable Set 4 at Joint A

<table>
<thead>
<tr>
<th>mode</th>
<th>direction</th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>3.63E-3</td>
<td>3.63E-3</td>
<td>3.63E-3</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>4.60E-4</td>
<td>4.60E-4</td>
<td>4.60E-4</td>
</tr>
<tr>
<td>2</td>
<td>X</td>
<td>8.78E-5</td>
<td>8.78E-5</td>
<td>8.78E-5</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-3.34E-3</td>
<td>-3.34E-3</td>
<td>-3.34E-3</td>
</tr>
<tr>
<td>3</td>
<td>X</td>
<td>9.59E-3</td>
<td>9.59E-3</td>
<td>9.59E-3</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>-9.95E-3</td>
<td>-9.95E-3</td>
<td>-9.95E-3</td>
</tr>
</tbody>
</table>

Table 3.14 Computational Times of Sensitivity Analysis with Case 1

(CYBER 930 NOS/VE 1.4.1)

<table>
<thead>
<tr>
<th></th>
<th>CDM</th>
<th>DAM</th>
<th>DM</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU(sec)</td>
<td>2.34</td>
<td>0.293</td>
<td>0.184</td>
</tr>
<tr>
<td>normalized</td>
<td>12.684</td>
<td>1.58</td>
<td>1</td>
</tr>
</tbody>
</table>
Chapter 4
SHAPE SENSITIVITY ANALYSIS OF GEOMETRICALLY NONLINEAR SOLIDS

The presentation of the second part of this dissertation which concerns the formulation of shape sensitivity analysis of nonlinear solids starts from this chapter. This chapter addresses the first-order sensitivity analysis of a geometrically nonlinear system by using the concept of material derivative. The sensitivity expressions for the functionals defined in the deformed as well as the undeformed configurations are derived using the continuum approach.

The equilibrium equation of a geometrically solid can generally be posed in either the Lagrangian or Eulerian form. To solve this equation, however, it is necessary to resort to an incremental formulation. The total Lagrangian formulation relates all the static and kinematic variables to the undeformed configuration. On the other hand, the Eulerian formulation refers all the static and kinematic variables to the deformed configuration. These formulations are used later for the derivation of design sensitivity equations. In Section 4.1, the general formulations based on the total Lagrangian and the Eulerian formulations are described. A textbook written by Bathe [45] has given a complete description for nonlinear structural analysis based on the Lagrangian formulation. The definition of nomenclature given in Bathe's book will be employed here. The incremental formulations of the total Lagrangian and the Eulerian formulations and the corresponding nonlinear finite element solution procedure are also presented in Section 4.1. To validate these nonlinear finite element formulations, a uniformly loaded beam with fixed ends is adopted at the end of the section to evaluate the performances of the total Lagrangian
formulation and the Eulerian formulation. In the following section, Section 4.2, the shape sensitivity equations of the total Lagrangian and the Eulerian formulations with respect to shape variables are derived by using the direct differentiation method. Section 4.3 concerns the shape sensitivity analysis of a general functional, which can be defined in either the undeformed or the deformed configuration, with respect to the design variables which again can be referred to either the deformed or undeformed configurations. These sensitivity equations are validated in Section 4.4 by an example of a prismatic bar undergoing large deformation.

4.1 Analysis of Geometrically Nonlinear Solids

4.1.1 Nomenclature

The notations which will be employed in the following work are briefly introduced here for future reference [45]. The motion of a body is considered in a fixed Cartesian coordinate system, Fig. 4.1, in which all kinematic and static variables are defined. The coordinates describing the configuration of the body point, p, at time 0 are \( \dot{x}_1, \dot{x}_2, \dot{x}_3 \), at time t are \( x_1, x_2, x_3 \), and at time \( t + \Delta t \) are \( x_1^{+\Delta t}, x_2^{+\Delta t}, x_3^{+\Delta t} \). A "left superscript" denotes the time of the configuration in which the quantity occurs and a "left subscript" indicates the time of the reference configuration with respect to which the quantity is measured. If the reference configuration is the same as the one indicated by the left superscript, the left subscript will be omitted. Right lower case subscripts denote the components of a tensor or vector. Components are referred to a fixed Cartesian coordinate: \( i, j, \ldots = 1,2,3 \). Differentiation is denoted by a right lower case subscript following a comma notation with the subscript indicating the coordinate with respect to which is differentiated; for example \( \frac{\partial^+\Delta u_i}{\partial \dot{x}_j} \). The notation for the displacements of the body is similar to the notation for the coordinates; namely, at time t the displacements are \( u_i, i=1,2,3 \) and at time \( t + \Delta t \) the displacements are \( u_i^{+\Delta t}, i=1,2,3 \); therefore we have
\[ \begin{align*}
\dot{t}x_i &= 0 \cdot x_i + t^i u_i \\
_{t+\Delta t}x_i &= 0 \cdot x_i + t^{i+\Delta t} u_i \\
_{0}u_i &= t^{i+\Delta t} u_i - t^{i} u_i \\
\end{align*} \]

where \(0 u_i\) are the increments of the displacements from time \(t\) to time \(t + \Delta t\).

---

**Figure 4.1 Various Types of Configurations in a Stationary Cartesian Coordinate System**

**4.1.2 Principle of Virtual Work**

The equation of equilibrium of a body in configuration \(t + \Delta t\) based upon the principle of virtual work [45,70] can be expressed as

\[ \int_{t+\Delta V} \delta_{i+\Delta t} \sigma_{ij} \delta_{i+\Delta t} e_{ij}^{t+\Delta t} dV = t^{i+\Delta t} R \]  

(4.2)
where the $t+\Delta t\sigma_{ij}$ are the Cartesian components of the Cauchy stress tensor, the $t+\Delta t\varepsilon_{ij}$ are the Cartesian components of an infinitesimal strain tensor, and the $\delta$ means "variation in", i.e.,

$$
\delta_{t+\Delta t}e_{ij} = \delta \frac{1}{2} \left( \frac{\partial_0 u_i}{\partial (t+\Delta t)x_j} + \frac{\partial_0 u_j}{\partial (t+\Delta t)x_i} \right)
= \frac{1}{2} \left( \frac{\partial \delta_0 u_i}{\partial (t+\Delta t)x_j} + \frac{\partial \delta_0 u_j}{\partial (t+\Delta t)x_i} \right)
$$

(4.3)

where $\delta_0 u_i$ denotes the components of the incremental displacement from time $t$ to $t + \Delta t$.

The corresponding external virtual work is defined as $t+\Delta t\mathbf{R}$,

$$
t+\Delta t\mathbf{R} = \int_{t+\Delta t\mathbf{V}} t+\Delta t\mathbf{e}^B_i \delta_0 u_i t+\Delta t\mathbf{dV} + \int_{t+\Delta t\mathbf{\Gamma}} t+\Delta t\mathbf{f}^\Gamma_i \delta_0 u_i t+\Delta t\mathbf{d\Gamma}
$$

(4.4)

where the $t+\Delta t\mathbf{e}^B_i$ and $t+\Delta t\mathbf{f}^\Gamma_i$ are the components of the externally applied body and surface force vectors, respectively, and $\delta_0 u_i$ is the $i$-th component of the virtual displacement vector. It should be noted here that the virtual strains used in Eq. (4.3) are those corresponding to the imposed body and surface virtual displacements, and that these displacements can be any compatible set of displacements that satisfy the prescribed displacement boundary conditions $\mathbf{u}_p$, i.e. $\mathbf{u}_i = \mathbf{u}_p^i$, on the assigned portions of the surface $a_u$.

Equation (4.2) is the basis of the Eulerian formulation which presents the equilibrium state equation at current configuration $t + \Delta t$. The difficulty of using Eq. (4.2) for analysis is that not only the displacements but also the configuration at $t + \Delta t$ are unknown. This is the major difference between the nonlinear and the linear analysis. In the linear analysis, displacements are infinitesimally small so that the configuration of the body does not change; i.e., $(t+\Delta t)\mathbf{V} = \mathbf{\delta V}$.

The Cauchy stress tensor in Eq. (4.2) is defined as

$$
t+\Delta t\sigma_{ij} = t+\Delta t\mathbf{C}_{ijrs} t+\Delta t\mathbf{E}_{rs}
$$

(4.5)
where \( t^+ \Delta t C_{ijr} \) and \( t^+ \Delta t E_{rs} \) are the components of the constant elastic tensor and Eulerian strain which are measured at the deformed configuration \( t^+ \Delta t V \). The Eulerian strain is written as

\[
t^+ \Delta t E_{ij} = \frac{1}{2} \left( t^+ \Delta t u_{i,j} + t^+ \Delta t u_{j,i} - t^+ \Delta t u_{k,i} t^+ \Delta t u_{k,j} \right).
\] (4.6)

The common way of solving Eq. (4.2) is using Lagrangian formulations in which all of the state variables are referred to a known equilibrium configuration. The total Lagrangian formulation is one of such well-known formulations in which all of the state variables are referred to the initial configuration of the body at time 0. In this formulation, the internal virtual strain energy of the left side of Eq. (4.2) is transformed to [45]

\[
\int_{t^+ \Delta t V} t^+ \Delta t \sigma_{ij} \delta_{t^+ \Delta t} e_{ij} t^+ \Delta t dV = \int_{0V} t^+ \Delta t S_{ij} \delta_{0} e_{ij} 0 dV
\] (4.7)

where \( t^+ \Delta t S_{ij} \) and \( \delta_{t^+ \Delta t} e_{ij} \) are the Cartesian components of the 2nd Piola-Kirchhoff stress tensor and the variations in the Cartesian components of the Green-Lagrange strain tensor, \( \delta_{t^+ \Delta t} e_{ij} \), corresponding to the configuration at time \( t + \Delta t \) but measured in the initial configuration at time 0. The 2nd Piola-Kirchhoff stress tensor, \( t^+ \Delta t S_{ij} \), can be given as

\[
t^+ \Delta t S_{ij} = \frac{0}{t^+ \Delta t} \frac{\partial}{\partial t} \frac{0}{t^+ \Delta t} x_{ij} t^+ \Delta t \sigma_{ij} t^+ \Delta t x_{rs} = t^+ \Delta t C_{ijr} t^+ \Delta t e_{rs} \] (4.8)

where \( \frac{0}{t^+ \Delta t} \frac{\partial}{\partial t} \frac{0}{t^+ \Delta t} \) represents the ratio of the mass densities at time 0 and time \( t + \Delta t \), and \( t^+ \Delta t C_{ijr} \) and \( t^+ \Delta t e_{rs} \) are the components of the constant elastic tensor and the Green-Lagrange strain tensor corresponding to the configuration at time \( t + \Delta t \) but measured in the initial configuration at time 0. The mass density ratio in Eq. (4.8) can be evaluated since the mass of the particles considered is conserved:

\[
\int_{t^+ \Delta t V} t^+ \Delta t p t^+ \Delta t dx_1 t^+ \Delta t dx_2 t^+ \Delta t dx_3 = \int_{0V} 0 \frac{0}{t^+ \Delta t} dx_1 0 \frac{0}{t^+ \Delta t} dx_2 0 \frac{0}{t^+ \Delta t} dx_3.
\] (4.9)
But \( t^+\Delta t \text{ } dt_1 \text{ } t^+\Delta t \text{ } dx_2 \text{ } t^+\Delta t \text{ } dx_3 = (\det t^+\Delta t x_{i,j})^0 \text{ } dx_1 \text{ } 0 \text{ } dx_2 \text{ } 0 \text{ } dx_3 \) and since the relation in Eq. (4.9) must hold for any arbitrary number of particles, we have

\[
0 \text{ } p = (\det t^+\Delta t x_{i,j})^t+\Delta t \text{ } p
\]

(4.10)

where \( \det t^+\Delta t x_{i,j} \) is the determinant of the deformation gradient \( t^+\Delta t x_{i,j} \). The Green-Lagrange strain tensor, \( t^+\Delta t e_{ij} \), is

\[
t^+\Delta t e_{ij} = \frac{1}{2} (t^+\Delta t u_{i,j} \text{ } t^+\Delta t u_{j,i} \text{ } t^+\Delta t u_{k,i} \text{ } t^+\Delta t u_{k,j})
\]

(4.11)

and the variation of the Green-Lagrange strain is

\[
\delta t^+\Delta t e_{ij} = \frac{1}{2} (\delta t^+\Delta t u_{i,j} \text{ } \delta t^+\Delta t u_{j,i} \text{ } \delta t^+\Delta t u_{k,i} \text{ } \delta t^+\Delta t u_{k,j})
\]

\[
= \frac{1}{2} (\delta_0 u_{i,j} \text{ } \delta_0 u_{j,i} \text{ } \delta_0 u_{k,i} \text{ } \delta_0 u_{k,j})
\]

(4.12)

where \( \delta t^+\Delta t u_{i,j} = \delta_0 u_{i,j} \) is due to fact that \( t^+\Delta t u_{i,j} \) is known in the configuration at time \( t \), i.e., \( \delta t^+\Delta t u_{i,j} = \delta(\delta^0 u_{i,j} + t^0 u_{i,j}) = \delta_0 u_{i,j} \).

It should be noted that the external work of the right-hand side of Eq. (4.2) is defined over the current configuration of the body at time \( t + \Delta t \), which may also be evaluated in the initial configuration as [45,65]

\[
\int_{t^+\Delta t v} t^+\Delta t f_{i} \text{ } \delta_0 u_{i} t^+\Delta t dV + \int_{t^+\Delta t v} t^+\Delta t f_{i} \text{ } \delta_0 u_{i} t^+\Delta t d\Gamma
\]

\[
\int_{v} t^+\Delta t f_{i} \text{ } \delta_0 u_{i} 0 dV + \int_{v} t^+\Delta t f_{i} \text{ } \delta_0 u_{i} 0 d\Gamma
\]

(4.13)

where it is assumed that the direction and the magnitude of the forces are independent of the deformation.

Substituting the relations in Eqs. (4.7) and (4.13) into Eq. (4.2), the equation of equilibrium for the body in the configuration at \( t + \Delta t \) but referred to the configuration at time \( 0 \) is obtained:
where these displacements can be any compatible set of displacements that satisfy the prescribed displacement boundary conditions $u_i^p$, i.e. $u_i = u_i^p$, on the assigned portions of the surface $A_u$.

### 4.1.3 Incremental Equations for Nonlinear Analysis

**Total Lagrangian Formulation**

Since the 2nd Piola-Kirchhoff stresses $\sigma_{ij}^0$ and Green-Lagrange strains $\varepsilon_{ij}^0$ at time $t$ are known, the following incremental decompositions are possible

\[
\begin{align*}
\sigma_{ij}^{t+\Delta t} &= \sigma_{ij} + \sigma_{ij}^{\Delta} \\
\varepsilon_{ij}^{t+\Delta t} &= \varepsilon_{ij} + \varepsilon_{ij}^{\Delta}
\end{align*}
\]  
(4.15)  
(4.16)

where $\sigma_{ij}^0$ and $\varepsilon_{ij}^0$ are the corresponding incremental stresses and strains at time $t$. From the displacement definition of the Green-Lagrange strain tensor, the incremental Green-Lagrange strain can be written as

\[
\varepsilon_{ij}^{\Delta} = \varepsilon_{ij} - \varepsilon_{ij}^0 + \eta_{ij}^0
\]  
(4.17)

where

\[
\begin{align*}
\eta_{ij} &= \frac{1}{2} (u_{i,j}^0 + u_{j,i}^0 + u_{i,k,j}^0 + u_{j,k,i}^0 + u_{k,i}^0 + u_{k,j}^0) \\
\eta_{ij} &= \frac{1}{2} (u_{i,k}^0 + u_{k,i}^0)
\end{align*}
\]  
(4.18)

The incremental 2nd Piola-Kirchhoff stresses, $\sigma_{ij}^0$, are related to the incremental Green-Lagrange strain, $\varepsilon_{ij}^0$, that is

\[
\sigma_{ij}^{t+\Delta t} = \sigma_{ij}^0 + C_{ijrs} \varepsilon_{rs}^\Delta
\]  
(4.19)
Equation (4.14) can now be written in terms of incremental quantities as
\[
\int_{\Omega} (1+\Delta t)
\begin{align*}
\Delta_t C_{ijrs} \Delta_t \varepsilon_{ij} &= \int_{\Omega} \theta S_{ij} \varepsilon_{ij} \Delta_t \gamma_{ij} dV - \int_{\Omega} \theta S_{ij} \varepsilon_{ij} \Delta_t \gamma_{ij} dV \quad (4.20)
\end{align*}
\]
where the relation \( \Delta_t \varepsilon_{ij} = \varepsilon_{ij} \) has been employed. Note that Eq. (4.20) represents a nonlinear equation for the incremental displacement \( \Delta u_i \). Bathe [45] suggested a scheme in which the nonlinear quantity, \( \varepsilon_{ij} \), is replaced by a linear one, \( \varepsilon_{ij} \), in Eq. (4.20). As a result, the equilibrium equation to be solved becomes
\[
\int_{\Omega} (1+\Delta t)
\begin{align*}
\Delta_t C_{ijrs} \Delta_t \varepsilon_{ij} &= \int_{\Omega} \theta S_{ij} \varepsilon_{ij} \Delta_t \gamma_{ij} dV - \int_{\Omega} \theta S_{ij} \varepsilon_{ij} \Delta_t \gamma_{ij} dV \quad (4.21)
\end{align*}
\]

**Eulerian Formulation**

Since the Cauchy stresses \( \sigma_{ij} \) and Eulerian strains \( \varepsilon_{ij} \) at time \( t \) are known, the following incremental decompositions are represented
\[
\begin{align*}
\Delta_t \sigma_{ij} &= \sigma_{ij} + 0 \sigma_{ij} \\
\Delta_t \varepsilon_{ij} &= \varepsilon_{ij} + 0 \varepsilon_{ij} \\
\end{align*}
(4.22)
\]
where \( 0 \sigma_{ij} \) and \( 0 \varepsilon_{ij} \) are the corresponding incremental stresses and strains at time \( t \). From the displacement definition of the Eulerian strain tensor, the incremental Eulerian strain can be written as
\[
0 \varepsilon_{ij} = 0 \varepsilon_{ij} + 0 \gamma_{ij} \\
(4.23)
\]
where
\[
0 \varepsilon_{ij} = \frac{1}{2} (0 u_{i,j} + 0 u_{j,i} - u_{i,k} u_{j,k}) \\
0 \gamma_{ij} = \frac{1}{2} 0 u_{k,i} u_{k,j} \\
(4.24)
\]
The incremental Cauchy stresses \( 0 \sigma_{ij} \) are related to the incremental Eulerian strains \( 0 \varepsilon_{ij} \), that is
\[
0 \sigma_{ij} = (1+\Delta t) C_{ijrs} 0 \varepsilon_{rs}. \\
(4.25)
\]
Equation (4.2) can now be written in terms of incremental quantities as

\[ \int_{i+\Delta t}^{i+\Delta t} C_{ij} \delta \varepsilon_{ij} dV + \int_{i+\Delta t}^{i+\Delta t} C_{ij} \delta \varepsilon_{ij} dV = \Delta t \int_{i+\Delta t}^{i+\Delta t} \varepsilon_{ij} dV \]

which represents a nonlinear equation. The nonlinearity comes from two parts; one is the term, \( \delta \hat{H}_{ij} \), and the other comes from the deformed volume and surface, \( i+\Delta t V \) and \( i+\Delta t \Gamma \), which are functions of the Eulerian displacements, \( i+\Delta t U \). Such a nonlinearity makes the Eulerian formulation less attractive than the Lagrangian formulation in terms of numerical implementation. However, in a regular shape design optimization routine, the new domain of the solid to be designed is usually known before starting the new optimization iteration. Therefore, if the shape design variables are defined to describe the deformed configuration of the structure to be designed, the Eulerian formulation may become more nature to be used for such a shape design optimization scheme. This is because, in this case, the deformed domain and surface, \( i+\Delta t V \) and \( i+\Delta t \Gamma \), are already prescribed as a result of a shape design optimization iteration and the statement of the analysis problem is to find the displacements for the given deformed configuration. Now, assuming the \( i+\Delta t V \) and \( i+\Delta t \Gamma \) are known, then Eq. (4.26) can be linearized by dropping the second integral on the left-hand side to obtain

\[ \int_{i+\Delta t}^{i+\Delta t} C_{ij} \delta \varepsilon_{ij} dV = \Delta t \int_{i+\Delta t}^{i+\Delta t} \varepsilon_{ij} dV \]  

4.1.4 Finite Element Solution Procedure

In the finite element method for nonlinear analysis, the basic equation to be solved at \( t+\Delta t \) is

\[ i+\Delta t R - i+\Delta t F = 0 \]
where \( t+\Delta t \mathbf{R} \) is vector of externally applied nodal loads at time \( t + \Delta t \) and \( t+\Delta t \mathbf{F} \) is vector of nodal point forces equivalent to the element stresses at time \( t + \Delta t \). The vector of nodal point forces \( t+\Delta t \mathbf{F} \) is dependent on the nonlinear quantity of nodal point displacement. It is necessary to iterate in the solution of Eq. (4.28). The well-known solution scheme is the modified Newton-Raphson iteration [45] which has been commonly used in many numerical application.

At the condition time \( t \), the following vector equations obtained by linearizing the response of the governing equation are established

\[
\Delta \mathbf{R}^{(i-1)} = t+\Delta t \mathbf{R} - t+\Delta t \mathbf{F}^{(i-1)} \\
\mathbf{t} \mathbf{K} \Delta \mathbf{u}^{(i-1)} = \Delta \mathbf{R}^{(i-1)} \\
t+\Delta t \mathbf{u}^{(i)} = t+\Delta t \mathbf{u}^{(i-1)} + \Delta \mathbf{u}^{(i)}
\]  

(4.29) (4.30) (4.31)

with \( t+\Delta t \mathbf{u}^{(0)} = \mathbf{u} \) and \( t+\Delta t \mathbf{F}^{(0)} = \mathbf{F} \). In each iteration, the out-of-balance load vector which yields an increment in displacements obtained in Eq. (4.30) is calculated, and the iteration procedure is employed until the out-of-balance load vector \( \Delta \mathbf{R}^{(i-1)} \) or the displacement increments \( \Delta \mathbf{u}^{(i)} \) are sufficiently small. The major part of this solution scheme is the calculation of the tangent stiffness matrix \( \mathbf{t} \mathbf{K} \) and the vector of \( \mathbf{F}^{(i-1)} \). In the total Lagrangian formulation, the left-hand side and the last term of the right hand side in Eq. (4.21) presented the tangent stiffness matrix \( \mathbf{t} \mathbf{K} \) and the vector of \( \mathbf{F}^{(i-1)} \), respectively. Similarly, Eq. (4.27) in the Eulerian formulation also provides the same information for these quantities. Careful investigation of the tangent stiffness matrix given in Eq. (4.21) for the total Lagrangian formulation, it is symmetric and sparse. The common solution algorithm of the skyline reduction method [45] is implemented in finite element analysis.

However, due to the different terms \( \delta t+\Delta e_{ii} \) and \( \delta t+\Delta e_{ij} \) in Eq. (4.27), the tangent stiffness matrix in the Eulerian formulation is unsymmetric and sparse. A modified skyline reduction method [71] is employed in numerical implementation of the Eulerian formulation.
The incremental solution formulations, Eqs. (4.29) to (4.31), yield iterative procedures which are terminated by a convergence criterion. The convergence criterion used here is given as

\[ \frac{\| \Delta u^{(i)} \|_2}{\| t^{(i+\Delta t)} \|_2} \leq \varepsilon_d \]  

(4.32)

where \( \varepsilon_d \) is a pre-set tolerance. The vector \( t^{(i+\Delta t)} \) is unknown and must be approximated. In numerical implementation, the last calculated value \( t^{(i+\Delta t)} \) is specified as an approximation to \( t^{(i+\Delta t)} \).

It should be noted here that the elastic constitutive tensors in the total Lagrangian formulation and the Eulerian formulation are equivalent to a linear elastic one under the assumption of large displacement, large rotation but small strain. This is because the stress tensors and strain tensors are invariant under rigid body rotations [45]. Thus, only the actual straining of the material will yield an increase in the components of the stress tensor, and as long as this material straining is small, the elastic constitutive tensor is completed equivalent to using Hook's law in infinitesimal displacement conditions. The more detailed study about the elastic material behavior and the numerical implementation in geometrically nonlinear problem has been presented in Bathe's works [43,45].

In the following, a uniformly loaded beam with fixed ends will be described, which will serve as a numerical example to evaluate the numerical performances of the total Lagrangian and the Eulerian formulations. The finite element model of the beam shown in Fig. 4.2 consists of 103 nodes and 20 plane stress elements. Each element is an isoparametric element with eight variable-number-nodes. The geometric dimension of the beam is a 200 units by 16 units subjected to a uniform loading of \( P_0=500 \) units and with the following material properties: \( E=2.0E6 \) and \( \nu=0.0 \). The basic assumptions are made as

1. The beam undergoes large deformation and rotation but small strain;
2. The material is linear elastic, isotropic;
(3) The elements are initially straight and have uniform cross-section;
(4) The plane of loading coincides with the plane of bending and the direction of
the loading is independent of the deformation configuration.

The purpose of this example is to demonstrate the relationship of the total
Lagrangian formulation and the Eulerian formulation. As shown in Figure 4.3, the
deformed configuration can be obtained by using the total Lagrangian formulation with a
known initial configuration; e.g.,

\[ t + \Delta t x_i = x_i + (t + \Delta t) u_i(x_i). \]  

(4.33)

On the other hand, the Eulerian formulation can be employed to recover the initial
configuration with a given deformed configuration; e.g.,

\[ x_i = (t + \Delta t) x_i - (t + \Delta t) u_i(x_i). \]  

(4.34)

Comparing Eqs. (4.33) and (4.34), the relationship of the displacements between these two
formulations is given as

\[ u_i(x_i) = u_i(x_i). \]  

(4.35)

Figure 4.2 Finite Element Model of a Uniformly Loaded Beam

\[ P_0 = 500 \]
Based on the above discussion, the total Lagrangian formulation is first used to calculate the displacements and the deformed configuration of the beam. Then, the Eulerian formulation is applied to recover the initial configuration based on the deformed configuration obtained from the total Lagrangian formulation.

In Fig. 4.4, the convergence history of both formulations is presented with the convergence tolerance being set up to be 0.001. Both formulations require 7 iterations to reach at the convergence. Moreover, the computational times required for the total Lagrangian formulation and the Eulerian formulations are 5.33 and 4.6 CPU seconds (Apollo Domain 5500), respectively. The maximum displacements of the linear analysis, nonlinear analysis using the total Lagrangian formulation and the Eulerian formulation are 3.01, 3.27 and 3.23 respectively, which happen at the top of the midspan of the beam. In addition to the deformed configuration, the initial configuration used for the total Lagrangian formulation and the initial configuration obtained from the Eulerian formulation are shown in Fig. 4.5. They are almost identical to each other. Therefore, the overall
performances in nonlinear analysis of both formulations, the total Lagrangian and the Eulerian formulations, work well in this example.

Figure 4.4 Convergence History for Nonlinear Analysis
Figure 4.5 Various Configurations Resulted from Finite Element Analysis

(a) Initial Unloaded Configuration

(b) Deformed Configuration Based Upon Linear Analysis

(c) Deformed Configuration Based Upon Nonlinear Analysis
   (Total Lagrangian Formulation)

(d) Recovered Configuration Based Upon Nonlinear Analysis
   (Eulerian Formulation)
4.2 Shape Sensitivity Analysis: Direct Differentiation Method

The concept of the material derivative has been discussed in Section 2.2 and successfully applied to derive the eigenvalue and eigenvector sensitivity equations of skeletal structures with variable joint and support locations. Here, the concept of the material derivative will be applied again to the functions and functionals defined in a continuous domain. With the definition of Eq. (2.21), the following basic relation can be easily proved [19]

\[
\frac{\partial z_{i,j}}{\partial t} = \frac{\partial z_{i,j}}{\partial t} - v_{i,k}z_{k,j} \quad (4.36)
\]

where \( v_{i} \) is the design velocity field function defined in the domain and \( k \) is a dummy index ranging from 1 to 3. Equation (4.36) needs further elaboration. In the total Lagrangian formulation, one has

\[
\frac{\partial}{\partial t} z_{i,j} = \frac{\partial z_{i,j}}{\partial t} + \frac{\partial z_{i,j}}{\partial x_{i}} v_{i,j} + v_{i,k} \frac{\partial z_{i,j}}{\partial x_{k}} \quad (4.37)
\]

where \( v_{i}(\partial x_{i}) \) is the design velocity field defined in the initial configuration. On the other hand, in the Eulerian formulation, one has

\[
\frac{\partial}{\partial t} z_{i,j} = \frac{\partial z_{i,j}}{\partial t} + \frac{\partial z_{i,j}}{\partial x_{i}} v_{i,k} + v_{i,k} \frac{\partial z_{i,j}}{\partial x_{k}} \quad (4.38)
\]

where \( v_{i}(\partial x_{i}) \) is the design velocity field defined in the deformed configuration.

4.2.1 Using the Total Lagrangian Formulation for Shape Sensitivity Analysis

The state equation resulted from the total Lagrangian formulation has been presented in Section 4.1.2. The state equation, Eq. (4.14), can be rewritten here in the following form
The total material derivative of the above equation leads to

\[
\dot{t}^{\alpha} \Pi = \int_\Omega \dot{t}^{\alpha} \sigma_{ij} \dot{\varepsilon}_{ij}^0 \, dV - \int_\Omega \dot{t}^{\alpha} \sigma_{ij} \delta_0 \dot{u}_i^0 \, dV - \int_\Gamma \dot{t}^{\alpha} \Gamma \delta_0 \dot{u}_i^\Gamma \, d\Gamma \\
= 0. \tag{4.39}
\]

where the externally applied body force, \(\dot{t}^{\alpha} \sigma_{ij}^B\), and the externally surface force, \(\dot{t}^{\alpha} \Gamma \), are assumed design independent. Moreover, the material derivatives of the domain and surface boundaries are defined by \(\alpha\) and \(\beta\)

\[
\dot{\sigma} = (\nabla \cdot \sigma)^0 \, dV = \alpha^0 \, dV
\]

and

\[
\dot{\sigma} = \left[\nabla \cdot \sigma - (\nabla \sigma \cdot n) \cdot n\right]^0 \, d\Gamma = \beta^0 \, d\Gamma
\]

where \(n\) is the unit vector normal to the infinitesimal area \(0 \, d\Gamma\). The total material derivative of the 2nd Piola-Kirchhoff stress tensor is a linear function of the total material derivative of Green-Lagrange strain as

\[
\dot{\sigma} = ^{\dot{\sigma}} \sigma_j^{\dot{i}} = \dot{C}_{ijr}^{\sigma} + \dot{C}_{ijr}^{\Gamma}\sigma
\]

where the material constant is independent of the deformation under the assumption that displacements and rotations are large but strain are small.

Based on the relation in Eq. (2.37), the total material derivative of the Green-Lagrange strain tensor defined by Eq. (4.11) can be obtained as
\[
\frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 \varepsilon_{ij}} = \frac{1}{2} \left( \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 u_{i,j}} + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 u_{j,i}} + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 u_{k,i}} + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 u_{k,j}} \right) \\
= \frac{1}{2} \left( \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 P_{ij} (\hat{u}, v) + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 Q_{ij} (v, u)} \right) \\
(4.43)
\]

where
\[
\frac{\delta^{+\Delta t} P_{ij} (\hat{u}, v)}{0 u_{i,j}} = \frac{1}{2} \left( \frac{\delta^{+\Delta t} u_{i,j}}{0 u_{i,j}} + \frac{\delta^{+\Delta t} u_{j,i}}{0 u_{j,i}} + \frac{\delta^{+\Delta t} u_{k,i}}{0 u_{k,i}} + \frac{\delta^{+\Delta t} u_{k,j}}{0 u_{k,j}} \right) \\
\frac{\delta^{+\Delta t} Q_{ij} (v, u)}{0 u_{i,j}} = -\frac{1}{2} \left( \frac{\delta^{+\Delta t} v_{i,j}}{0 v_{i,j}} + \frac{\delta^{+\Delta t} v_{j,i}}{0 v_{j,i}} + \frac{\delta^{+\Delta t} v_{k,i}}{0 v_{k,i}} + \frac{\delta^{+\Delta t} v_{k,j}}{0 v_{k,j}} \right) \\
(4.44)
\]

and the total material derivative of the variation of the Green-Lagrange strain tensor in Eq. (4.12) can be obtained as
\[
\delta^{+\Delta t} \varepsilon_{ij} = \frac{1}{2} \left( \delta^{+\Delta t} u_{i,j} + \delta^{+\Delta t} u_{j,i} + \delta^{+\Delta t} u_{k,i} + \delta^{+\Delta t} u_{k,j} \right) \\
+ \delta^{+\Delta t} \varepsilon_{ij} (\delta^{+\Delta t} u_{k,i}) \\
= \frac{1}{2} \left( \delta^{+\Delta t} \varepsilon_{ij} (\delta, u) + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 W_{ij} (\hat{u}, \delta u) + \frac{\delta^{+\Delta t} \varepsilon_{ij}}{0 G_{ij} (v, \delta u)} \right) \\
\frac{\delta^{+\Delta t} T_{ij} (\delta, u)}{0 t_{ij}} = \frac{1}{2} \left( \delta^{+\Delta t} T_{ij} (\delta, \delta u) + \frac{\delta^{+\Delta t} T_{ij}}{0 W_{ij} (\hat{u}, \delta u) + \frac{\delta^{+\Delta t} T_{ij}}{0 G_{ij} (v, \delta u)} \right) \\
(4.46)
\]

where
\[
\frac{\delta^{+\Delta t} T_{ij} (\delta, u)}{0 u_{i,j}} = \frac{1}{2} \left( \delta^{+\Delta t} u_{i,j} + \delta^{+\Delta t} u_{j,i} + \delta^{+\Delta t} u_{k,i} + \delta^{+\Delta t} u_{k,j} \right) \\
\frac{\delta^{+\Delta t} W_{ij} (\hat{u}, \delta u)}{0 u_{i,j}} = \frac{1}{2} \left( \delta^{+\Delta t} u_{i,j} + \delta^{+\Delta t} u_{k,i} + \delta^{+\Delta t} u_{k,j} \right) \\
\frac{\delta^{+\Delta t} G_{ij} (v, \delta u, u)}{0 u_{i,j}} = -\frac{1}{2} \left( \delta^{+\Delta t} v_{i,j} + \delta^{+\Delta t} v_{j,i} + \delta^{+\Delta t} v_{k,i} + \delta^{+\Delta t} v_{k,j} \right) \\
(4.47)
\]

An important feature about the material derivative of the variation of the displacement which is used in Eq. (4.46) is that the order of taking variation and taking material derivative is interchangeable. That is,
\[
\frac{\delta^{+\Delta t} u_{i,j}}{0 u_{i,j}} = \delta^{+\Delta t} u_{i,j} \\
\frac{\delta^{+\Delta t} v_{i,j}}{0 v_{i,j}} = \delta^{+\Delta t} v_{i,j} - v_{i,k} \delta^{+\Delta t} u_{k,j} \right) \\
(4.50)
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
With proper substitution and using the constitutive relation, Eq. (4.40) can be rearranged to obtain

\[
\frac{1}{\Delta t} \Pi = \int_0^V \left( C_{\text{ijrs}} \frac{1}{\Delta t} P_{\text{ij}} n \delta_{\text{ij}} e_{\text{ij}} g_{\text{ij}} g_{\text{ij}} W_{\text{ij}} \right) \, dv - \int_0^V \delta_{\text{ij}} \left( \delta_{\text{ij}} \delta_{\text{ij}} \right) \, dv - \int_\Gamma \delta_{\text{ij}} \delta_{\text{ij}} \, d\Gamma
\]

(4.51)

where

\[
\frac{1}{\Delta t} H_{\text{ij}}(\delta u, u, v)
\]

(4.52)

Note that the last three terms in Eq. (4.51) satisfy the state equation, \( \frac{1}{\Delta t} \Pi = 0 \), where the virtual displacement is the total material derivatives of the displacements which satisfy the prescribed displacement boundary conditions. Finally, the total material derivative of the state equation defined by the total Lagrangian formulation is

\[
\frac{1}{\Delta t} \Pi = \int_0^V \left( C_{\text{ijrs}} \frac{1}{\Delta t} P_{\text{ij}} n \delta_{\text{ij}} e_{\text{ij}} g_{\text{ij}} g_{\text{ij}} W_{\text{ij}} \right) \, dv - \int_\Gamma \delta_{\text{ij}} \delta_{\text{ij}} \, d\Gamma
\]

(4.53)

where the relation \( \delta_{\text{ij}} g_{\text{ij}} = \delta_{\text{ij}} e_{\text{ij}} \) has been employed. Note that the above equation is a linear equation of \( \frac{1}{\Delta t} \delta_{\text{ij}} \). The feature of this equation is that the cost of computational times will be the same as the linear elastic systems [60]. Moreover, comparing the incremental equation, Eq. (4.20), the stiffness matrix of Eq. (4.53) can be identified as the tangent stiffness matrix at the final equilibrium configuration while the incremental displacement, \( u_1 \), in Eq. (4.20) is replaced by the total material derivative of displacement, \( \frac{1}{\Delta t} \delta_{\text{ij}} \).
4.2.2 Using the Eulerian Formulation for Shape Sensitivity Analysis

A notion has been taken in the following derivation that the shape of the deformed configuration at $t + \Delta t$ is considered as a design variable. As a result, the domain and the surface, $t + \Delta t V$ and $t + \Delta t \Gamma$, are independent of the state variable, $t + \Delta t u_i$; instead, they are determined by the design process. In this way, the nonlinearity in the Eulerian formulation will be greatly reduced. More discussion in this regard can be found in Section 4.1.3.

Based upon the Eulerian formulation in Eq. (4.2), the state equation is rewritten here as

$$
\Pi^{\Delta t} = \int_{V^{\Delta t}} \sigma_{ij}^{\Delta t} \delta^{\Delta t} e_{ij}^{\Delta t} dV - \int_{V^{\Delta t}} \Pi^{\Delta t B} \delta^{\Delta t} u_i^{\Delta t} dV \\
- \int_{\Gamma^{\Delta t}} \Pi^{\Delta t \Gamma} \delta^{\Delta t} u_i^{\Delta t} d\Gamma = 0. \tag{4.54}
$$

The total material derivative of the above equation gives

$$
\Pi^{\Delta t} = \int_{V^{\Delta t}} \left( \sigma_{ij}^{\Delta t} \delta^{\Delta t} e_{ij}^{\Delta t} + \Pi^{\Delta t B} \delta^{\Delta t} u_i^{\Delta t} \right) dV \\
+ \int_{V^{\Delta t}} \left( \sigma_{ij}^{\Delta t} \delta^{\Delta t} e_{ij}^{\Delta t} - \Pi^{\Delta t B} \delta^{\Delta t} u_i^{\Delta t} \right) \delta^{\Delta t} dV \\
- \int_{\Gamma^{\Delta t}} \Pi^{\Delta t \Gamma} \delta^{\Delta t} u_i^{\Delta t} d\Gamma - \int_{\Gamma^{\Delta t}} \Pi^{\Delta t \Gamma} \delta^{\Delta t} u_i^{\Delta t} \delta^{\Delta t} d\Gamma = 0 \tag{4.55}
$$

where the externally applied body force, $t + \Delta t f_i^B$, and the externally surface force, $t + \Delta t f_i^\Gamma$, are assumed to be design independent. Moreover, using short notations, $\hat{\alpha}$ and $\hat{\beta}$, the material derivative of domain is defined as

$$
\frac{\delta}{t + \Delta t} dV = (\nabla \cdot (t + \Delta t \nu))^{t + \Delta t} dV \\
= \hat{\alpha}^{t + \Delta t} dV \tag{4.56}
$$

and the material derivative of surface boundary is

$$
\frac{\delta}{t + \Delta t} d\Gamma = [\nabla \cdot (t + \Delta t \nu - (\nabla \cdot (t + \Delta t \nu \cdot n) \cdot n)]^{t + \Delta t} d\Gamma \\
= \hat{\beta}^{t + \Delta t} d\Gamma \tag{4.57}
$$
where \( n \) is the unit normal vector to the infinitesimal area \( t^{+\Delta t} d\Gamma \). Again, the total material derivative of the Cauchy stress tensor is a linear function of the total material derivative of Eulerian strain as

\[
\dot{\epsilon}_{ij}^{t^{+\Delta t}} = \dot{\epsilon}_{ij}^{t^{+\Delta t}} C_{ijrs} \dot{\epsilon}_{rs}^{t^{+\Delta t}} E_{r}^{s}
\]

where the material constant is independent of the deformation under the assumption that displacements and rotations are large but strains are small. The total material derivative of the Eulerian strain can be derived from its definition, Eq. (4.6), as

\[
\dot{\epsilon}_{ij} = \frac{1}{2} \left( \left( \dot{u}_{ij} + \dot{u}_{ji} - \dot{u}_{k,i} \dot{u}_{k,j} \right) + \left( \dot{v}_{ij} + \dot{v}_{ji} - \dot{v}_{k,l} \dot{v}_{k,l} \right) \right)
\]

Next, the total material derivative of the variation of the infinitesimal strain tensor, Eq. (4.3), is given

\[
\delta_{t^{+\Delta t}} \epsilon_{ij} = \frac{1}{2} \left( \delta_{t^{+\Delta t}} u_{ij} + \delta_{t^{+\Delta t}} v_{ij} \right)
\]

where

\[
\dot{T}_{ij} (\delta \dot{u}) = \frac{1}{2} \left( \delta_{t^{+\Delta t}} u_{ij} + \delta_{t^{+\Delta t}} v_{ij} \right)
\]

\[
\dot{G}_{ij} (v, \delta u) = -\frac{1}{2} \left( \delta_{t^{+\Delta t}} v_{ij} \delta_{t^{+\Delta t}} u_{ij} + \delta_{t^{+\Delta t}} u_{ij} \delta_{t^{+\Delta t}} v_{ij} \right).
\]

With proper substitution, the total material derivative of the state equation becomes
\[ t^{+\Delta t} \hat{\Pi} = \int_{V^{+\Delta t}} t^{+\Delta t} C_{ijr}^{+\Delta t} \delta_{r}^{+\Delta t} e_{ij}^{+\Delta t} dV - t^{+\Delta t} H_{ij}(\delta u, u, v) + \int_{V^{+\Delta t}} t^{+\Delta t} \sigma_{ij}^{+\Delta t} T_{ij}^{+\Delta t} dV - \int_{V^{+\Delta t}} t^{+\Delta t} e^{+\Delta t} \delta_{0}^{+\Delta t} dV - \int_{\Gamma^{+\Delta t}} t^{+\Delta t} \delta_{0}^{+\Delta t} d\Gamma \] (4.64)

\[ = 0 \]

where

\[ t^{+\Delta t} H_{ij}(\delta u, u, v) = \int_{\Omega} \left( t^{+\Delta t} C_{ij}^{+\Delta t} \delta_{0}^{+\Delta t} - t^{+\Delta t} \sigma_{ij}^{+\Delta t} e_{ij}^{+\Delta t} \right) \delta u dV(\delta u, u, v) + \int_{\Gamma} t^{+\Delta t} \delta_{0}^{+\Delta t} d\Gamma \] (4.65)

\[ - \int_{\Omega} \left( t^{+\Delta t} C_{ij}^{+\Delta t} Q_{ij}^{+\Delta t} \delta_{0}^{+\Delta t} - t^{+\Delta t} \sigma_{ij}^{+\Delta t} e_{ij}^{+\Delta t} G_{ij}^{+\Delta t} \right) dV. \]

The last three terms in the right-hand side of Eq. (4.64) satisfy the state equation, Eq. (4.2), as the total material derivative of \( \delta u \), \( \delta \dot{u} \), can be considered as a virtual displacement which satisfies the prescribed displacement boundary conditions. At last, the total material derivative of the state equation described by the Eulerian formulation is given as

\[ t^{+\Delta t} \hat{\Pi} = \int_{V^{+\Delta t}} t^{+\Delta t} C_{ijr}^{+\Delta t} \delta_{r}^{+\Delta t} e_{ij}^{+\Delta t} dV - t^{+\Delta t} H_{ij}(\delta u, u, v) \]

\[ = 0. \] (4.66)

The last equation is a linear equation in terms of \( t^{+\Delta t} \dot{u} \). This sensitivity equation is linear.

This is because no additional efforts are needed to evaluate the stiffness matrix in Eq. (4.66). The stiffness matrix of Eq. (4.66) is in fact equal to the tangent stiffness matrix at the final equilibrium configuration while the incremental displacement, \( \delta u \), in Eq. (4.27) changes to the total material derivative of the displacement, \( t^{+\Delta t} \dot{u} \). Compare to the sensitivity equation expressed in the Eulerian formulation, Eq. (4.66), term by term to that in the Lagrangian formulation, Eq. (4.53), it should be noticed that Eq. (4.66) is simpler than Eq. (4.53). This is because

1. Eq. (4.66) is one term less than Eq. (4.53);
2. The \( G_{ij} \) appearing in \( H_{ij} \) in Eq. (4.65) is four terms less than the counterpart in Eq. (4.52).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
4.3 Shape Sensitivity Analysis: Adjoint Variable Method

The objective of this section is to use the adjoint variable method to derive the shape sensitivity equation for a generic functional of displacements and stresses that are measured in the final equilibrium configuration, though the functional itself can be defined either in the initial configuration, as

$$\Phi(0V) = \int_{V} F^{i+\Delta i} S_{ip} (i+\Delta i u) dV$$ (4.67)

or defined in the final equilibrium configuration

$$\Psi(t+\Delta t V) = \int_{V'} F^{i+\Delta i} \sigma_{ip} (i+\Delta i u) dV.$$ (4.68)

If the shape of the initial configuration is considered as a design variable, it is natural to use the total Lagrangian formulation to find the shape sensitivities of $\Phi(0V)$ as well as $\Psi(t+\Delta t V)$ in which, however, all the Eulerian quantities should be mapped back to the initial configuration via the Jacobian matrix. Nevertheless, there are cases in which the final equilibrium configuration may be directly considered as a design variable. In such a case, it is then beneficial to use the Eulerian formulation for shape sensitivity analysis. This is because the shape sensitivity equation expressed by the Eulerian formulation is easier to compute than that by the total Lagrangian formulation, as concluded in the previous section.

In this section, shape sensitivity analysis of $\Phi(0V)$ with respect to the variation of $0V$ is discussed in Section 4.3.1. It is followed by shape sensitivity analysis of $\Psi(t+\Delta t V)$ with respect to the variation of $t+\Delta t V$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
4.3.1 Shape Sensitivity Analysis of $\Phi(0V)$

The total material derivative of $\Phi(0V)$ can be expressed as

$$
\dot{\Phi}(0V) = \int_{SV} \left( \frac{\partial F}{\partial 0V} \frac{\partial F}{\partial 0V} + \frac{\partial F}{\partial 0U} \right) dV + \int_{SV} F 0V dV
$$

$$
= \int_{SV} \left[ \frac{\partial F}{\partial 0V} C_{ijm} \left( \frac{\partial F}{\partial 0V} P_m + \frac{\partial F}{\partial 0V} Q_n \right) \frac{\partial F}{\partial 0U} \right] dV
$$

$$
+ \int_{SV} F 0V dV. \tag{4.69}
$$

Note that Eqs. (4.42) to (4.45) have been used in the above derivation. The above equation can be appended with the total material derivative of the system virtual work, $t^{+\Delta 0\dot{u}} \dot{\Pi} = 0$, Eq. (4.53), without changing the value of function $\Phi(0V)$. In other words, the following relation holds true

$$
\dot{\Phi}(0V) = \dot{\Phi}(0V) + t^{+\Delta 0\dot{u}} \dot{\Pi}
$$

Therefore,

$$
\dot{\Phi}(0V) = \dot{\Phi}(0V) + t^{+\Delta 0\dot{u}} \dot{\Pi}
$$

$$
= \int_{SV} \left( \frac{\partial F}{\partial 0V} C_{ijm} \frac{\partial F}{\partial 0V} P_m (\dot{u}, u) \delta^{+\Delta 0\dot{u}} e_{ij} (\delta u, u) \right) dV
$$

$$
+ \int_{SV} \left( \frac{\partial F}{\partial 0V} C_{ijm} \left( \frac{\partial F}{\partial 0V} P_m (\dot{u}, u) + \frac{\partial F}{\partial 0U} \frac{\partial F}{\partial 0U} \right) \delta^{+\Delta 0\dot{u}} \right) dV.
$$

To eliminate the terms associated $t^{+\Delta 0\dot{u}}$, an adjoint equation can be introduced by replacing the terms $t^{+\Delta 0\dot{u}}$ and $\delta_{0u}$ by a virtual adjoint variable $\delta^{+\Delta 0\lambda_i}$ and an adjoint variable $+\Delta 0\lambda_i$, respectively. Collecting terms associated with $t^{+\Delta 0\lambda_i}$ and $\delta^{+\Delta 0\lambda_i}$ in Eq. (4.71), the resultant adjoint equation is then obtained as

$$
\int_{SV} \left( \frac{\partial F}{\partial 0V} C_{ijm} \frac{\partial F}{\partial 0V} P_m (\delta \lambda, u) \right) \delta^{+\Delta 0\lambda} e_{ij} (\lambda, u) + \int_{SV} F 0V dV.
$$

$$
= -\int_{SV} \left( \frac{\partial F}{\partial 0V} C_{ijm} \left( \frac{\partial F}{\partial 0V} P_m (\delta \lambda, u) + \frac{\partial F}{\partial 0U} \delta^{+\Delta 0\lambda} \right) \right) dV \tag{4.72}
$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
where the virtual adjoint displacement can be any compatible set of displacements that satisfy the prescribed geometric boundary conditions. Note that the left-hand side of Eq. (4.72) is the tangent stiffness matrix at the final equilibrium configuration which has been discussed in Section 4.2.1. After solving the above adjoint equation, the shape design sensitivity can be easily obtained as

\[
\Phi(0V) = \int_0^V \frac{\partial F}{\partial \sigma_{ij}} T^\Delta_c \nabla \delta^\Delta Q_n \delta V + \int_0^V T^\Delta H_{ij}(\lambda, u, v) \delta V
\]

where all the terms are associated with \( T^\Delta u_i \) and \( \partial V_i \) but only the last term depends on the adjoint variable, \( T^\Delta \lambda_i \).

### 4.3.2 Shape Sensitivity Analysis of \( \Psi(t^\Delta V) \)

In this subsection, the shape sensitivity equation of a domain functional, \( \Psi(t^\Delta V) \), defined in the final equilibrium configuration, \( t^\Delta V \), with respect to the variation of the final equilibrium configuration is sought. The functional is given in Eq. (4.68) and the total material derivative of \( v P(t + \Delta V) \) can be obtained as

\[
\frac{d}{dt} v P(t + \Delta V) = \int_{\sigma_{ij}} \frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{T^\Delta_c \nabla}{\partial \sigma_{ij}} + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \int_{\nabla} \frac{T^\Delta H}{\partial T^\Delta \lambda_i} \cdot T^\Delta \lambda_i \delta V
\]

Since \( T^\Delta \Pi = 0 \) as given in Eq. (4.66), the following relation holds true

\[
\frac{d}{dt} \psi(\tau) = \psi(\tau) T^\Delta \Pi
\]

\[
= \int_{\sigma_{ij}} \frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{T^\Delta_c \nabla}{\partial \sigma_{ij}} + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \int_{\nabla} T^\Delta H_{ij}(\delta u, u, v)
\]

\[
\int_{\sigma_{ij}} \frac{\partial F}{\partial \sigma_{ij}} \cdot \frac{T^\Delta_c \nabla}{\partial \sigma_{ij}} + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \frac{\partial F}{\partial T^\Delta u_i} \cdot T^\Delta u_i + \int_{\nabla} T^\Delta H_{ij}(\delta u, u, v)
\]
An adjoint equation can be formed by collecting all the terms with $t + \Delta t \delta u_i$ together and among them by replacing $t + \Delta t \delta u_i$ and $\delta_0 u_i$ by a virtual adjoint variable $\delta^+ \Delta \lambda_i$ and an adjoint variable $t + \Delta t \delta \lambda_i$, respectively. In this way, all the terms associated with $t + \Delta t \delta u_i$ will be dropped and the shape sensitivity equation for $\Psi$ can be greatly simplified as

$$
{\Psi}(t + \Delta t \lambda) = \int_{\Omega} \frac{\partial F}{\partial (t + \Delta t \sigma_{ij})} C_{ij} \delta^+ \Delta \lambda_i + \int_{\partial \Omega} m(t + \Delta t \sigma_{ij}) dV - \int_{\partial \Omega} \delta^+ \Delta H_i(\lambda, u, v)
$$

(4.76)

where the adjoint variable $\lambda$ is the solution of the following adjoint equation

$$
\int_{\Omega} (\delta^+ \Delta P_{ij} \delta^+ \Delta \sigma_{ij} + \frac{\partial F}{\partial (t + \Delta t \delta u_i)} \delta^+ \Delta \lambda_i) dV = 0
$$

(4.77)

where the virtual adjoint displacement can be any compatible set of displacements that satisfy the prescribed geometric boundary conditions. Note that the stiffness matrix of the left-hand side in Eq. (4.77) is the tangent stiffness matrix at the final equilibrium configuration. The discussion about this matter has been explained in Section 4.2.2.

### 4.4 Analytical Example

An example of axially loaded prismatic bar is presented in this section to validate the shape sensitivity equations derived above. In this example, the design sensitivity coefficient of a stress functional due to change in the length of the prismatic bar is determined. The initial length and the cross-sectional area of the bar are $L$ and $A$, respectively. The axial force $P$ is applied at the tip of the bar, as shown in Fig. 4.6. The bar is made of a linear elastic material with a constitutive relation being given as $S = k \varepsilon$ defined in undeformed configuration where $S$, $k$ and $\varepsilon$ are the 2nd Piola-Kirchhoff stress, a material constant and the Green-Lagrange strain, respectively. The bar is assumed to undergo a large deformation...
but small strain from its original position; so that \( \ell/L >> 1 \) where \( \ell \) is the length of the deformed bar. Moreover, the cross-sectional area \( A \) is assumed to be constant during the deformation process.

\[ A, k \]

\[ X,L \]

\[ x, \ell \]

Figure 4.6 A Prismatic Bar

4.4.1 Nonlinear Analysis of a Prismatic Bar

Before doing stress sensitivity analysis of this prismatic bar, the governing equations of the total Lagrangian formulation and the Eulerian formulation are first presented. The governing equation of the total Lagrangian formulation of this example can be expressed as

\[ x\Pi = \int_0^L S \delta \epsilon \, dX - P \delta u|_{x=L} = 0 \]  

(4.78)

with the following boundary conditions

\[ u|x=0 = 0 \]

\[ JSA|x=L = P \]  

(4.79)

where \( X \) indicates the coordinate system of the undeformed configuration. The relation of the deformed configuration, \( x \), and the undeformed configuration, \( X \), can be stated as
where the axial displacement is linear in \( X \). The deformation gradient \( J \) (Jacobian) can be obtained as

\[
J = \frac{dx}{dX} = \frac{\ell}{L}
\]  

(4.81)

and the Green-Lagrange strain then yields

\[
\varepsilon = u_x + \frac{1}{2} u_x^2 = \frac{1}{2} \left( \frac{\ell}{L} \right)^2 - 1.
\]  

(4.82)

On the other hand, the governing equation based upon the Eulerian formulation can be stated as

\[
\begin{align*}
\delta \Pi &= \int_0^L \sigma \delta e dx - P \delta u_{x=L} \\
&= 0
\end{align*}
\]  

(4.83)

where \( \sigma \) and \( e \) are the Cauchy stress and the infinitesimal strain, respectively, and the following boundary conditions

\[
\begin{align*}
u_{x=0} &= 0 \\
\sigma A_{x=L} &= P.
\end{align*}
\]  

(4.84)

The relation of the undeformed configuration, \( X \), and the deformed configuration, \( x \), can be stated as

\[
\begin{align*}
X &= x - u(x) \\
&= x - (\ell - L) \frac{x}{\ell} \\
&= \frac{L}{\ell} x.
\end{align*}
\]  

(4.85)
where the axial displacement is linear in \( x \). The Eulerian strain yields

\[
E = u_x - \frac{1}{2} u_x^2 \\
= \frac{1}{2} \left[ 1 - \left( \frac{\ell}{L} \right)^2 \right].
\]  

(4.86)

From the final equilibrium position at the deformed configuration, the Cauchy stress is a constant value along the bar, that is

\[
\sigma = \frac{P}{A} = kE.
\]  

(4.87)

The relation of the 2nd Piola-Kirchhoff stress \( S \) and Cauchy stress \( \sigma \) is given from Eq. (4.8) as

\[
S = \frac{\lambda}{A} \frac{\partial \sigma}{\partial \sigma} \frac{dX}{dx} \\
= \frac{\ell}{L} \frac{L}{\ell} \sigma \frac{L}{\ell} \\
= \frac{PL}{\ell A} 
\]  

(4.88)

where the relation of density ratio, Eq. (4.10), has been employed. Since the constitutive relation gives \( S = k\varepsilon \), Eq. (4.82) along with Eq. (4.88) provides the nonlinear equation to determine the deformed length \( \ell \)

\[
\left( \frac{\ell}{L} \right)^3 - \left( \frac{\ell}{L} \right) = \frac{2P}{kA}.
\]  

(4.89)

Note that the ratio \( \frac{\ell}{L} \) is a constant that is a function of \( P, A \) and \( k \) which are fixed.
4.4.2 Sensitivity Analysis of $\Psi(\ell)$

Let a stress functional be defined as

$$
\Psi(\ell) = \int_0^\ell \sigma^2 dx
$$

$$
= \frac{P^2}{A^2} \ell
$$

(4.90)

where Eq. (4.87) is employed and its total material derivative can be obtained as

$$
\overline{\Psi}(\ell) = \int_0^\ell 2 \sigma \dot{\sigma} dx + \int_0^\ell \sigma^2 v_x dx
$$

$$
= \int_0^\ell \left[ \sigma^2 v_x - 2 \sigma k (1 - u_x) v_x u_x \right] dx + \int_0^\ell 2 \sigma k (1 - u_x) \dot{u}_x dx
$$

(4.91)

where

$$
\dot{\sigma} = k \dot{\varepsilon} = k (1 - u_x) (\dot{u}_x - v_x u_x).
$$

(4.92)

and the velocity field function $v(x)$ is given as

$$
v(x) = \frac{\dot{\ell}}{\ell} x
$$

(4.93)

where $\dot{\ell}$ denotes the total length change of the bar.

The total material derivative of $\dot{x} \dot{\Pi}$, Eq. (4.83), is given

$$
\dot{x} \dot{\Pi} = \int_0^\ell k (1 - u_x) \dot{u}_x \delta u_x dx - \int_0^\ell k (1 - u_x) v_x u_x \delta u_x dx
$$

$$
= 0
$$

(4.94)

Combine Eqs. (4.91) and (4.94) together and replace $\dot{u}$ and $\delta u$ by $\delta \lambda$ and $\lambda$, an adjoint equation which is associated with $\lambda$ and $\delta \lambda$, can be written as

$$
\int_0^\ell k (1 - u_x) (\lambda_x + 2 \sigma) \delta \lambda_x dx = 0
$$

(4.95)

where the adjoint variable $\lambda$ satisfies the geometric boundary condition, $\lambda(0) = 0$. In Eq. (4.95), the term $k(1-u_x)$ is a constant which can be proved by Eq. (4.85)
\[ k(1-u_x) = k(1-1+\frac{L}{\ell}) \]
\[ = k\frac{L}{\ell} \]
\[ \neq 0. \]  
(4.96)

Therefore, the gradient of \( \lambda \) can be obtained from Eq. (4.95) as

\[ \lambda_x = -2\sigma. \]  
(4.97)

Finally, the sensitivity of a stress functional is expressed as

\[ \overline{\Psi}(\ell) = \int_0^L \sigma^2 v_x - 2\sigma k(1-u_x)v_x u_x - k(1-u_x)v_x u_x \lambda_x \lambda_x \lambda_x \]  
\[ = \int_0^L \sigma^2 v_x \]  
\[ = \int_0^L \sigma^2 \frac{\dot{\ell}}{\ell} \]  
\[ = \frac{p^2}{A^2} \dot{\ell} \]  
(4.98)

where Eq. (4.87) has been employed to derive the above equation and, from Eq. (4.93), \( v_x = \dot{\ell}/\ell \). The same result as the above equation can be obtained by directly taking the total material derivative of Eq. (4.90).

### 4.4.3 Sensitivity Analysis of \( \Psi(\ell) \)

The same stress functional in Eq. (4.90) is investigated again but it is mapped to the initial configuration

\[ \Psi(\ell) = \int_0^L \sigma^2 dx \]  
(4.90)
\[ = \int_0^L S^2 J^3 dX \]  
(4.99)
\[ = \Phi(L) \]

where \( J \) is the deformation gradient as defined by Eq. (4.81). The total material derivative of Eq. (4.99) is
\[ \overline{\Phi(L)} = \int_0^L (2S\dot{J}^3 + 3S^2J^2\dot{J} + S^2J^2v_x) \, dX \]
\[ = \int_0^L 2SJ^3k(1 + u_x)\dot{u}_x \, dX + \int_0^L S^2J^3v_x \, dX \]
\[ - \int_0^L 2SJ^3k(1 + u_x)u_x v_x \, dX \]

where
\[ \dot{S} = k\dot{e} \]
\[ = k(1 + u_x)(\dot{u}_x - u_x v_x) \quad (4.101) \]

and \( \dot{J} \) is equal to zero because the deformation gradient \( J = \ell/L \) is a constant which is a function of \( P, k \) and \( A \) as shown in Eq. (4.89). According to Eq. (4.78), the total material derivative of \( x\Pi \) can be obtained as,
\[ x\Pi = \int_0^L [k(1 + u_x)^2 + S]\delta u_x \dot{u}_x \, dX - \int_0^L [k(1 + u_x)^2 + S]\delta u_x u_x v_x \, dX \]
\[ = 0. \quad (4.102) \]

Thus, the adjoint equation is obtained by combining Eq. (4.100) and (4.102) together, and replacing \( \dot{u} \) and \( \delta u \) by \( \delta \lambda \) and \( \lambda \) as
\[ \int_0^L [k(1 + u_x)^2 + S]\lambda_{xx} \, dX = 0 \quad (4.103) \]

where the adjoint variable \( \lambda \) satisfies the geometric boundary condition, \( \lambda(0) = 0 \). The gradient of the adjoint variable \( \lambda \) is then determined as
\[ \lambda_{xx} = \frac{-2SJ^3k(1 + u_x)}{S + k(1 + u_x)^2}. \quad (4.104) \]

Finally, the sensitivity of the stress functional, \( \Phi(L) \), becomes
\[ \overline{\Phi(L)} = \int_0^L \left\{ S^2J^3v_x - [S + k(1 + u_x)^2]u_x v_x \lambda_{xx} - 2SJ^3k(1 + u_x)u_x v_x \right\} \, dX \]
\[ = \int_0^L S^2J^3v_x \, dX \]

where the velocity function is defined as
\[ v(X) = \frac{\dot{L}}{L}X . \]  

(4.106)

Therefore, its derivative gives

\[ v_x = \frac{\dot{L}}{L} . \]  

(4.107)

Since \( J = \frac{\ell}{L} \) has a fixed value, the total material derivative of \( J \) provides a relation

\[ \dot{J} = 0 \]

\[ = \frac{\dot{\ell}L - \dot{L}\ell}{L^2} . \]  

(4.108)

Therefore, the velocity function defined in the initial configuration is the same as that defined in the final equilibrium configuration as

\[ v_x = \frac{\dot{L}}{L} \]

(4.107)

\[ = \frac{\dot{\ell}}{\ell} = v_x . \]  

(4.109)

Finally, one may transfer Eq. (4.105) into the final equilibrium configuration as

\[ \Phi(L) = \int_0^L S^2 J^3 v_x dx \]

\[ = \int_0^L \sigma^2 v_x dx \]

\[ = \int_0^L \sigma^2 \frac{\dot{\ell}}{\ell} dx = \left( \frac{P}{A} \right)^2 \dot{\ell} \]  

(4.110)

which is the same as that obtained at the end of Section 4.4.2.
Chapter 5

SHAPE OPTIMIZATION OF GEOMETRICALLY NONLINEAR SOLIDS DEFINED IN THE DEFORMED CONFIGURATION

The main thrust of this chapter is to develop a computational scheme for shape optimization of geometrically nonlinear solids in which the objective, constraints and design variables are defined in the deformed configuration. In most current design optimization applications, the objective and the constraints are all defined in the initial configuration. However, there are cases in which the overall system performance is sensitive to the local deformation of an elastic component. In such case, for the sake of design precision, it may be preferable to specify the design formulation based on the deformed configuration.

To deal with this new design formulation, a design optimization scheme is developed here which uses the Eulerian formulation for analysis and sensitivity analysis. An example problem will be given to demonstrate this scheme. The design optimization formulation of the example, a uniformly loaded beam with fixed ends, is proposed in Section 5.1. The shape sensitivity analysis of the objective and constraint functionals are discussed in Section 5.2. The numerical results of these shape sensitivity coefficients will be validated by comparing with those obtained by the finite difference method. In Section 5.3, a computational scheme based upon the finite element method for shape design optimization of a nonlinear solid with its performance criteria defined in the deformed configuration is then proposed. The numerical study for designing the profile of this uniformly loaded beam will be given in the section which follows.
5.1 Problem Statements

A uniformly loaded beam with fixed ends has been considered previously for nonlinear finite element analysis. Here, the same problem will be used again for studying design optimization application. The finite element model of the beam consists of 103 nodes and 20 elements with 8-node isoparametric element shown in Fig. 4.2. The geometric dimension is a 200 units by 16 units rectangular beam subjected to a uniform loading of $P_0 = 500$ units and with the following material properties: Young's modulus $2.0 \times 10^6$ and poisson's ratio 0.0. The reference datum, $a$, is set to 16 units, and there are 40 design variables assigned to describe the design boundaries $\Gamma_1$ and $\Gamma_2$ shown in Fig. 5.1.

The objective functional of interest is to find the shape of the beam that minimizes the area of the beam after deformation. The first constraint requires that the top surface of the deformed beam is flat. The second constraints are the element stress constraints.

Let $\Omega$ and $\Gamma$ denote the deformed domain and the boundary. Mathematically, the shape optimization formulation can be expressed as

$$\begin{align*}
\text{minimum } & \varphi_0 = \int_{\Omega} d\Omega \\
\text{subject to } & \\
\varphi_1 = & \int_{\Gamma_1} (\bar{y} - a)^2 d\Gamma - \varepsilon \leq 0 \\
\varphi_{k+1} = & \frac{\int_{\Omega_{e,k}} \sigma_M^2 d\Omega}{\int_{\Omega_{e,k}} \sigma_y^2 d\Omega} - 1 \leq 0, \quad k = 1, \ldots, \text{NE}
\end{align*}$$

where $(\bar{y} - a)$ is the deviation between the height of the deformed beam $\bar{y}$ and a given reference datum $a$, and $\sigma_M$ and $\sigma_y$ are the Von-Mises stress and the yield stress in deformed element domain $\Omega_{e,k}$ at element $k$. In fact, the Von-Mises stress is expressed as

$$\sigma_M^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2.$$
It is obvious that the optimal profile of the deformed beam is the one with a flat top and subjects to a stress field which is lower than the yield stress. Note that all the problem functions are defined in the deformed configuration. Nevertheless, the desired design domain is still the shape of the beam before deformation.

![Figure 5.1 Design Variables and Design Boundaries of a Beam](image)

### 5.2 Shape Sensitivity Analysis

In this section, the shape sensitivity analysis of the objective functional and the constraint functionals defined in Eqs. (5.1) to (5.3) will be studied here and the numerical results of those sensitivity coefficients will be verified by comparing them with the traditional finite difference method.

#### 5.2.1 Shape Sensitivity Analysis of Objective and Constraint Functionals

Taking the total material derivatives of Eqs. (5.1) to (5.3), we have

\[ \phi_0 = \int_\Omega \hat{\alpha} d\Omega \]

\[ \phi_1 = \int_{\Gamma} 2(\bar{y} - a) \hat{\gamma} d\Gamma + \int_{\Gamma} \bar{y} \hat{\beta} d\Gamma \]
Note that the Eqs. (5.5) and (5.6) are functionals of coordinates and their material derivatives which are defined in the deformed configuration, and Eq. (5.7) is a functional of Cauchy stress, coordinates and the total material derivative of coordinates which are defined in the deformed configuration.

From the definition of the total material derivative in Section 2.2, the total material derivative of a position vector (coordinate) is equal to the design velocity field. Numerically, it is convenient to express the velocity field as a linear combination of linearly independent velocity vectors, i.e.,

\[ v(\Omega) = \sum_{i=1}^{N} b_i v_i(\Omega) \]  \hspace{1cm} (5.8)

where \( b_i \)'s are the design variables and \( N \) is the total number of design variable. Here, the fictitious load method \[66\] is used to generate the design velocity field in a systematic manner. The fictitious load method treats the perturbations of nodal coordinates of the varied boundary as design variables. The design velocity, \( v_i(\Omega) \), pertaining to each design variable, is taken as the displacement field of a "fictitious" elastic problem with a unique load applied at the corresponding node. The fictitious elastic body is an elastic medium that occupies the domain of concern. The zero-displacement boundary conditions are imposed along the boundaries that are not subjected to design variations. In this example, a simply way to generate the design velocity vectors is using the linear solution of the linear solid defined in the initial undeformed configuration, i.e., \( v_i(\Omega) \).

Numerically, the total material derivative of the objective functional, Eq. (5.5), can be evaluated as
where \( v_x \) and \( v_y \) denote the velocity vectors in X and Y direction. Next, consider the total material derivative of Eq. (5.6), the last term in Eq. (5.6) can be expressed as

\[
\int_{\Gamma_i} (y - a)^2 \hat{\beta} d\Gamma = \int_{\Gamma_i} (y - a)^2 [\nabla \cdot v - (\nabla v \cdot n) \cdot n] d\Gamma
\]  

(5.10)

which is difficult to evaluate numerically. If the boundary of the beam is discretized as a set of piecewise linear segments, the above equation can be simplified as [67]

\[
\int_{\Gamma_i} (y - a)^2 \hat{\beta} d\Gamma = \sum_{i=1}^{N} \ell_i \int_{\Gamma_i} (y - a)^2 d\Gamma
\]  

(5.11)

where \( \ell_i \) is an arc length of the boundary segment and \( \hat{\ell}_i \) is the total material derivative of the arc length. With given nodal coordinates and nodal velocity vectors, the \( \ell_i \) and \( \hat{\ell}_i \) can be calculated without difficulty. For instance, the length of the segment \( i \) with nodes \( j \) and \( k \) whose coordinates are \( (x_j, y_j) \) and \( (x_k, y_k) \) in the deformed domain is

\[
\ell_i = \left[ (\Delta x)^2 + (\Delta y)^2 \right]^\frac{1}{2}
\]  

(5.12)

and its total material derivative is

\[
\hat{\ell}_i = \frac{1}{\ell_i} \left[ -\Delta x, -\Delta y, \Delta x, \Delta y \right] \begin{bmatrix} v_{xj} \\ v_{yj} \\ v_{xk} \\ v_{yk} \end{bmatrix}
\]  

(5.13)

where \( \Delta x \) and \( \Delta y \) are the difference of coordinates, i.e.,

\[
\Delta x = x_k - x_j \\
\Delta y = y_k - y_j.
\]  

(5.14)
Finally, the design sensitivity of the geometric constraint, Eq. (5.6), can be calculated in the following simplified form:

\[
\phi_i = \int_{F_i} 2(y - a) \frac{\dot{y}}{y} d\Gamma + \int_{F_i} (y - a)^2 \beta d\Gamma \\
= \int_{F_i} 2(y - a) \frac{\dot{y}}{y} d\Gamma + \sum_{i=k}^{N} \frac{\dot{\xi}_i}{\xi_i} \int_{F_i} (y - a)^2 d\Gamma. \tag{5.15}
\]

The total material derivative of the element stress constraint of Eq. (5.7) is obtained by using the adjoint variable method derived in Section 4.3.2. Carefully studying the terms in Eq. (5.7) finds that all the terms except the term \(\int_{B_k} \sigma^2 \Omega d\Omega\) have been defined in the above discussion. The shape design sensitivity analysis of a general functional in a deformed configuration has been studied in Section 4.3.2. The result can be applied here to find the required derivation with the functional \(F(t + \Delta t \sigma_{ij}, t + \Delta t u_i)\) being specified as

\[
F(t + \Delta t \sigma_{ij}, t + \Delta t u_i) = \sigma_M^2 \tag{5.16}
\]

and the adjoint equation from Eq. (4.77) becomes

\[
\int_{\Omega} C_{ijkl} (t + \Delta t \sigma_{ij}) \delta t_{kl} \epsilon_{ij} d\Omega = - \int_{\Omega} \frac{\partial F}{\partial t + \Delta t \sigma_{ij}} C_{ijkl} (t + \Delta t \sigma_{ij}) \sigma_{kl} d\Omega + \int_{\Omega} \frac{\partial F}{\partial \sigma_{kl}} C_{ijkl} (t + \Delta t \sigma_{ij}) \sigma_{kl} d\Omega \tag{5.17}
\]

where

\[
\frac{\partial F}{\partial t + \Delta t \sigma_{xx}} = \delta_{t + \Delta t \sigma_{xx}} \sigma_{xx} \tag{5.18}
\]

\[
\frac{\partial F}{\partial t + \Delta t \sigma_{yy}} = \delta_{t + \Delta t \sigma_{yy}} \sigma_{yy} \tag{5.18}
\]

\[
\frac{\partial F}{\partial t + \Delta t \sigma_{xy}} = \delta_{t + \Delta t \sigma_{xy}} \sigma_{xy} \tag{5.18}
\]

and \(t + \Delta t \sigma_{kl}\) is defined in Eq. (4.60). The adjoint and virtual adjoint displacements of the above adjoint structure satisfy the geometry boundary conditions at fixed ends, i.e.,

---

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ t^{+\Delta t} = 0 \quad \text{on} \quad \Gamma_0 \]

and

\[ \delta t^{+\Delta t} = 0 \quad \text{on} \quad \Gamma_0. \]

Finally, the sensitivity equation is obtained according to Eq. (4.76)

\[
\int_{\Omega} \sigma_{\text{M},d} d\Omega = \int_{\Omega} \frac{\partial F}{\partial \sigma_{ij}} t^{+\Delta t} C_{ij} r^{+\Delta t} Q_{rs} t^{+\Delta t} d\Omega - t^{+\Delta t} H_{ij}
\]

where \( t^{+\Delta t} Q_{rs} \) and \( t^{+\Delta t} H_{ij} \) are defined in Eqs. (4.61) and (4.65), respectively. With the aid of the result in Eq. (5.20), the sensitivity coefficients of Eq. (5.7) can be evaluated without difficulty.

5.2.2 Numerical Studies of Shape Sensitivity Analysis

Numerical verification of the shape sensitivity equations derived in Section 5.2.1 will be presented here. The numerical results are tabulated in Tables 5.1 to 5.3. In Table 5.1, the sensitivity coefficients of the cost functional with respect to the design variables, as shown in Eq. (5.5), are listed. The results are compared with those obtained by the finite difference method with each of the design variables being perturbed by 0.001. Similarly, in Table 5.2, the sensitivity coefficients of the geometry constraint, Eq. (5.6), are also calculated and compared with the results obtained by the finite difference method. At last, the sensitivity coefficients of stress in element 10 are evaluated based upon Eq. (5.20). All the computed sensitivity coefficients match well with those obtained by the finite difference method except the first row in Table 5.1. This may be due to the discretization error from the finite difference method.
Table 5.1 Sensitivity Coefficients of Cost Functional

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>FDM</th>
<th>ADJ</th>
<th>(ADJ/FDM)x100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.146484</td>
<td>0.094914</td>
<td>64.795</td>
</tr>
<tr>
<td>4</td>
<td>1.562500</td>
<td>1.615568</td>
<td>103.39</td>
</tr>
<tr>
<td>6</td>
<td>2.050781</td>
<td>2.023810</td>
<td>98.685</td>
</tr>
<tr>
<td>8</td>
<td>2.294922</td>
<td>2.188500</td>
<td>95.363</td>
</tr>
<tr>
<td>10</td>
<td>2.343750</td>
<td>2.265225</td>
<td>96.650</td>
</tr>
<tr>
<td>12</td>
<td>2.246090</td>
<td>2.295506</td>
<td>102.20</td>
</tr>
<tr>
<td>14</td>
<td>2.224609</td>
<td>2.297421</td>
<td>103.27</td>
</tr>
<tr>
<td>16</td>
<td>2.197265</td>
<td>2.281794</td>
<td>103.85</td>
</tr>
<tr>
<td>18</td>
<td>2.246093</td>
<td>2.259120</td>
<td>100.58</td>
</tr>
<tr>
<td>20</td>
<td>2.246094</td>
<td>2.222696</td>
<td>98.958</td>
</tr>
<tr>
<td>21</td>
<td>-6.835937</td>
<td>-6.789319</td>
<td>99.318</td>
</tr>
<tr>
<td>23</td>
<td>-3.076172</td>
<td>-3.086852</td>
<td>100.35</td>
</tr>
<tr>
<td>25</td>
<td>-2.636719</td>
<td>-2.648367</td>
<td>100.44</td>
</tr>
<tr>
<td>27</td>
<td>-2.636719</td>
<td>-2.558787</td>
<td>97.044</td>
</tr>
<tr>
<td>29</td>
<td>-2.490230</td>
<td>-2.526534</td>
<td>101.46</td>
</tr>
<tr>
<td>31</td>
<td>-2.490234</td>
<td>-2.502037</td>
<td>100.47</td>
</tr>
<tr>
<td>33</td>
<td>-2.441406</td>
<td>-2.473574</td>
<td>101.32</td>
</tr>
<tr>
<td>35</td>
<td>-2.441406</td>
<td>-2.439829</td>
<td>99.935</td>
</tr>
<tr>
<td>37</td>
<td>-2.392578</td>
<td>-2.404336</td>
<td>100.49</td>
</tr>
<tr>
<td>39</td>
<td>-2.294922</td>
<td>-2.370598</td>
<td>103.30</td>
</tr>
</tbody>
</table>
Table 5.2 Sensitivity Coefficients of Geometry Constraint Functional

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>FDM</th>
<th>ADJ</th>
<th>(ADJ/FDM) x 100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>614.2578</td>
<td>615.6626</td>
<td>100.23</td>
</tr>
<tr>
<td>4</td>
<td>598.5107</td>
<td>599.8740</td>
<td>100.23</td>
</tr>
<tr>
<td>6</td>
<td>585.8765</td>
<td>578.3959</td>
<td>98.723</td>
</tr>
<tr>
<td>8</td>
<td>573.1812</td>
<td>575.1138</td>
<td>100.34</td>
</tr>
<tr>
<td>10</td>
<td>560.8521</td>
<td>562.5000</td>
<td>100.29</td>
</tr>
<tr>
<td>12</td>
<td>547.6685</td>
<td>549.6271</td>
<td>100.36</td>
</tr>
<tr>
<td>14</td>
<td>535.0342</td>
<td>536.8936</td>
<td>100.35</td>
</tr>
<tr>
<td>16</td>
<td>522.9492</td>
<td>524.9426</td>
<td>100.38</td>
</tr>
<tr>
<td>18</td>
<td>513.3057</td>
<td>515.1429</td>
<td>100.36</td>
</tr>
<tr>
<td>20</td>
<td>504.3335</td>
<td>506.3549</td>
<td>100.40</td>
</tr>
<tr>
<td>21</td>
<td>-629.1504</td>
<td>-630.2010</td>
<td>100.17</td>
</tr>
<tr>
<td>23</td>
<td>-604.4922</td>
<td>-606.1675</td>
<td>100.28</td>
</tr>
<tr>
<td>25</td>
<td>-591.3086</td>
<td>-592.9177</td>
<td>100.27</td>
</tr>
<tr>
<td>27</td>
<td>-579.0405</td>
<td>-580.7161</td>
<td>100.30</td>
</tr>
<tr>
<td>29</td>
<td>-566.5894</td>
<td>-568.3170</td>
<td>100.30</td>
</tr>
<tr>
<td>31</td>
<td>-553.6499</td>
<td>-555.5689</td>
<td>100.35</td>
</tr>
<tr>
<td>33</td>
<td>-540.6494</td>
<td>-542.7209</td>
<td>100.38</td>
</tr>
<tr>
<td>35</td>
<td>-528.4424</td>
<td>-530.3078</td>
<td>100.35</td>
</tr>
<tr>
<td>37</td>
<td>-519.2784</td>
<td>-517.2119</td>
<td>99.602</td>
</tr>
<tr>
<td>39</td>
<td>-508.3008</td>
<td>-510.3586</td>
<td>100.40</td>
</tr>
</tbody>
</table>
Table 5.3 Sensitivity Coefficients of Von-Mises Stress Functional of Element Number 10

<table>
<thead>
<tr>
<th>Design Variable</th>
<th>FDM</th>
<th>ADJ</th>
<th>(ADJ/FDM) x100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.31003E8</td>
<td>-0.33096E8</td>
<td>93.675</td>
</tr>
<tr>
<td>4</td>
<td>-0.09777E8</td>
<td>-0.10342E8</td>
<td>94.543</td>
</tr>
<tr>
<td>6</td>
<td>-0.94432E8</td>
<td>-0.98678E8</td>
<td>95.697</td>
</tr>
<tr>
<td>8</td>
<td>-0.77885E8</td>
<td>-0.83288E8</td>
<td>93.513</td>
</tr>
<tr>
<td>10</td>
<td>-0.75972E8</td>
<td>-0.71033E8</td>
<td>106.954</td>
</tr>
<tr>
<td>12</td>
<td>-0.63166E8</td>
<td>-0.66233E8</td>
<td>95.369</td>
</tr>
<tr>
<td>14</td>
<td>-0.67362E8</td>
<td>-0.71274E8</td>
<td>94.512</td>
</tr>
<tr>
<td>16</td>
<td>-0.68123E8</td>
<td>-0.75573E8</td>
<td>95.579</td>
</tr>
<tr>
<td>18</td>
<td>-0.93426E8</td>
<td>-0.87102E8</td>
<td>107.26</td>
</tr>
<tr>
<td>20</td>
<td>-0.76943E8</td>
<td>-0.80622E8</td>
<td>95.437</td>
</tr>
<tr>
<td>21</td>
<td>0.38254E9</td>
<td>0.408269E9</td>
<td>93.698</td>
</tr>
<tr>
<td>23</td>
<td>0.13077E9</td>
<td>0.137979E9</td>
<td>94.774</td>
</tr>
<tr>
<td>25</td>
<td>0.11022E9</td>
<td>0.101442E9</td>
<td>108.65</td>
</tr>
<tr>
<td>27</td>
<td>0.85237E8</td>
<td>0.798697E8</td>
<td>106.72</td>
</tr>
<tr>
<td>29</td>
<td>0.60022E8</td>
<td>0.638766E8</td>
<td>93.965</td>
</tr>
<tr>
<td>31</td>
<td>0.53228E8</td>
<td>0.557145E8</td>
<td>95.538</td>
</tr>
<tr>
<td>33</td>
<td>0.61632E8</td>
<td>0.572520E8</td>
<td>107.65</td>
</tr>
<tr>
<td>35</td>
<td>0.61685E8</td>
<td>0.643392E8</td>
<td>95.875</td>
</tr>
<tr>
<td>37</td>
<td>0.74598E8</td>
<td>0.795584E8</td>
<td>93.765</td>
</tr>
<tr>
<td>39</td>
<td>0.86166E8</td>
<td>0.808736E8</td>
<td>106.544</td>
</tr>
</tbody>
</table>
5.3 A Computational Scheme for Shape Optimization of Nonlinear Solids in the Deformed Configuration

In this section, a computational scheme for shape optimization of nonlinear solids is investigated in which the objective, constraints and design variables are defined in the deformed configuration. Figure 5.2 shows that the concept of this computational scheme is that the major design process is carried out in the deformed configuration at which shape sensitivity analysis and optimization will be performed. After reaching an optimal design, the developed finite element scheme based upon the Eulerian formulation is then applied to recover the optimal configuration of the unloaded configuration. It should be noted here that the difference of the proposed computational scheme and the traditional computational scheme is hinged upon the domain where the design optimization iterations are carried out. In research works [57, 60, 65], design problems are usually defined in the undeformed configuration and the Lagrangian formulation is used to support analysis. In these works, any design criterion defined in the deformed configuration has to be transferred to the undeformed configuration before starting the design process. This transformation involves Jacobian matrix and complicates the sensitivity analysis.

In the following, the step by step procedures are given to explain the proposed design process. The corresponding flow chart is shown in Figure 5.3.

(1) Set up the design problem and the finite element model.

(2) Find the contour of the deformed configuration by the relation

\[ \mathbf{x} = \mathbf{x} + \mathbf{u}(\mathbf{x}) \quad \mathbf{x} \in \Omega \]  

where \( \mathbf{u}(\mathbf{x}) \) is the displacement of the nonlinear elastic problem. This is done by the total Lagrangian formulation.

(3) Construct the design velocity vectors, \( \mathbf{v}_i(\mathbf{x}) \). Here, the way to generate the design velocity field is to analyze a two-dimensional elastic problem defined in \( \Omega \cup \Gamma \) with a unit load applied to each node along the varied boundary. Each
of such loads will generate a displacement field which will be defined as a design velocity vector. To save the computational cost, the solution of the displacement field is considered as a linear elastic problem.

(4) Calculate the cost functional and the constraint functionals which are defined in the deformed configuration.

(5) Calculate the sensitivity coefficients of the cost and the constraint functionals with respect to design variables which describe the deformed configuration. If the functionals contain only the geometric parameters, for instance, Eq. (5.1) or Eq. (5.2), the sensitivity coefficients of these functionals can be easily calculated by the combination of design velocity vectors, Eqs. (5.9) and (5.15). On the other hand, if the functional involves stresses, the technique of the adjoint variable method is applied. Firstly, the adjoint load vector of each constraint is generated based upon Eq. (5.17). The adjoint structure is then established to solve the adjoint variable vector by using the existed finite element stiffness matrix which is assembled by the last iteration of nonlinear finite element analysis as discussion in Section 4.3.2. Only back and forward substitutions are needed in this adjoint variable method. Secondly, the sensitivity coefficients are evaluated by using the values of the adjoint variable vector, the design velocity vectors and the original displacement vector.

(6) Perform shape optimization by using an existed optimizer, CONMIN [68], to find the optimal solution. The CONMIN is developed by using the combination of the approximation concept [69] and the feasible direction method [17]. The deformed optimal shape is defined by the optimal solution \( b^o_i \), i.e.,

\[
\bar{x}^o = \bar{x} + \sum_{i=1}^{N} b^o_i v(x)
\]

(5.22)
where $\bar{x}$ gives the initial deformed configuration to start the optimization iterations and $\bar{x}^0$ is the final optimal deformed configuration.

(7) Check the convergence criteria. If the convergence criterion has not been satisfied, reanalysis of the structure with the new design profile by using the Eulerian formulation is resumed to start step (2) for next cycle of shape optimization.

(8) If the convergence criterion has been satisfied, the Eulerian formulation is used here again to find the initial optimal configuration. That is

$$x^0 = \bar{x}^0 - u(\bar{x}).$$

(9) Finally, one may verify the optimal shape by imposing loads to the structure and calculating its deformed configuration by using the total Lagrange formulation. The resultant shape should be the same as the one obtained in step 6.

---

Figure 5.2 A Conceptual Model of Design Optimization
Figure 5.3 Flow Chart of Computational Scheme for Shape Optimization
5.4 Numerical Results

A numerical example is presented herein to validate the proposed computational procedures. The example will determine the geometry of an unloaded, fixed-fixed beam such that the top edge will stay straight after deformation. The problem statements and the objective and constraint functionals have been presented in Section 5.2. The height of the deformed beam is set to be 16 units, and there are 40 design variables assigned to describe the boundary. The tolerance value, $e$, of geometric constraint, Eq. (5.2), is specified by 0.001. Based on the fact of the symmetry of geometry and the applied loading, ten elements of Von-Mises stress constraint functionals are employed in the design process. The value of allowable yielding stress is set as $8.8 \times 10^9$ for all of the stress constraint functionals.

Since the constraint such as Eq. (5.2) can be explicitly expressed as a functional of shape variables in the deformed configuration. One may then select a feasible shape that satisfies the geometric constraint to start the design optimization iterations. This may help to ease the computational burden for design optimization of the entire problem. For the example of concern, the area of the deformed beam of the initial design is 3201.56 shown in Fig. 5.4(b). An intermediate design can be selected so that the top edge of the beam is leveled and is matched with the requirement of the height. The bottom edge of the beam is represented by a quadratic polynomial function

$$y = c_1 x^2 + c_2 x + c_3$$

(5.24)

Two cases are offered herein:

**Case 1**

The coefficients $c_1$, $c_2$ and $c_3$ in the first case are selected by passing through three points, (0,0), (50,4) and (100,6), respectively. The area of this new design is reduced to 2466.7 shown in Fig. 5.4(c) and the stresses in the beam are below the allowable stress level as listed in column 3 in Table 5.4. Therefore, this is a feasible
design which will be used for next run of design optimization. After 22 additional design cycles, a final design is reached with the cost functional of the area as 2219.97. The convergence history of this shape optimization process is given in Fig. 5.5. Besides the geometric constraint, the stress constraint of the first element becomes active at the final design. The stress values of elements at various design stages are listed in Table 5.4. The shape of the beam at the final design is shown in Fig. 5.4(d). To recover the unloaded shape of the designed beam, the Eulerian formulation is used to find the displacements, \( u(\mathbf{x}) \) in Eq. (5.23). The optimal initial shape of the unloaded beam is given in Fig. 5.4(e).

### Table 5.4 Von-Mises Stress Functionals at Various Design Stages: case 1

<table>
<thead>
<tr>
<th>Element</th>
<th>Initial Design (Deformed)</th>
<th>Intermediate Design (Deformed)</th>
<th>Optimal Design (Deformed)</th>
<th>Optimal Design (Reproduced)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.603E10</td>
<td>7.916E10</td>
<td>8.787E10</td>
<td>8.749E10</td>
</tr>
<tr>
<td>2</td>
<td>3.327E10</td>
<td>5.183E10</td>
<td>6.894E10</td>
<td>6.874E10</td>
</tr>
<tr>
<td>3</td>
<td>1.397E10</td>
<td>3.131E10</td>
<td>6.504E10</td>
<td>6.493E10</td>
</tr>
<tr>
<td>4</td>
<td>4.928E9</td>
<td>1.819E10</td>
<td>7.008E10</td>
<td>7.016E10</td>
</tr>
<tr>
<td>5</td>
<td>2.867E9</td>
<td>1.115E10</td>
<td>8.287E10</td>
<td>8.253E10</td>
</tr>
<tr>
<td>6</td>
<td>5.124E9</td>
<td>9.814E9</td>
<td>8.349E10</td>
<td>8.326E10</td>
</tr>
<tr>
<td>7</td>
<td>9.573E9</td>
<td>1.171E10</td>
<td>6.655E10</td>
<td>6.614E10</td>
</tr>
<tr>
<td>8</td>
<td>1.452E10</td>
<td>1.599E10</td>
<td>4.749E10</td>
<td>4.729E10</td>
</tr>
<tr>
<td>9</td>
<td>1.864E10</td>
<td>2.124E10</td>
<td>4.107E10</td>
<td>4.087E10</td>
</tr>
<tr>
<td>10</td>
<td>2.096E10</td>
<td>2.543E10</td>
<td>4.257E10</td>
<td>4.239E10</td>
</tr>
</tbody>
</table>

* Based upon Step (9) described in Section 5.3.
Figure 5.4 Various Stages in the Shape Optimization: case 1
Finally, to verify the design, the total Lagrangian formulation is applied to find the displacements for the deformed shape of the beam. The cost functional is subjected to a slight change from 2219.97 to 2224.38. Comparing Figs. 5.4(f) with 5.4(d), the shape as well as the stress values indicated in column 5 of Table 5.4 are very similar to that obtained from the final optimal deformed design.

**Case 2**

The second intermediate design selects the coefficients $c_1$, $c_2$ and $c_3$ by passing through three points, $(0.,0.)$, $(50.,4.5)$ and $(100.,7.)$, respectively. The area of this new design is 2366.7, as shown in Fig. 5.6(c). Again, the second intermediate design yields a
feasible design as the stresses in the beam are below the allowable stress level as listed in column 3 of Table 5.5. After 27 additional design cycles, a much improved design is reached with an area of 2274.85. The convergence history of this shape optimization process is given in Fig. 5.7. Only the geometric constraint becomes active at the final design. The stress values of elements at the final design are also listed in Table 5.5. The shape of the beam at the final design is shown in Fig. 5.6(d). To recover the unloaded shape of the designed beam, the Eulerian formulation is used to find the displacements, \( u(\bar{x}) \) in Eq. (5.23). The optimal shape of the unloaded beam is given in Fig. 5.6(e). The proposed design scheme is again working well for the second intermediate design.

Table 5.5 Von-Mises Stress Functionals at Various Design Stages : case 2

<table>
<thead>
<tr>
<th>Element</th>
<th>Initial Design (Deformed)</th>
<th>Intermediate Design (Deformed)</th>
<th>Optimal Design (Deformed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.603E10</td>
<td>7.654E10</td>
<td>8.465E10</td>
</tr>
<tr>
<td>2</td>
<td>3.327E10</td>
<td>4.984E10</td>
<td>6.514E10</td>
</tr>
<tr>
<td>3</td>
<td>1.397E10</td>
<td>3.012E10</td>
<td>6.232E10</td>
</tr>
<tr>
<td>4</td>
<td>4.928E9</td>
<td>1.653E10</td>
<td>6.823E10</td>
</tr>
<tr>
<td>5</td>
<td>2.867E9</td>
<td>1.043E10</td>
<td>7.954E10</td>
</tr>
<tr>
<td>6</td>
<td>5.124E9</td>
<td>9.613E9</td>
<td>8.144E10</td>
</tr>
<tr>
<td>7</td>
<td>9.573E9</td>
<td>1.074E10</td>
<td>6.655E10</td>
</tr>
<tr>
<td>8</td>
<td>1.452E10</td>
<td>1.395E10</td>
<td>4.622E10</td>
</tr>
<tr>
<td>9</td>
<td>1.864E10</td>
<td>2.013E10</td>
<td>3.932E10</td>
</tr>
<tr>
<td>10</td>
<td>2.096E10</td>
<td>2.272E10</td>
<td>4.032E10</td>
</tr>
</tbody>
</table>
Figure 5.6 Various Stages in the Shape Optimization: case 2
Figure 5.7 Convergence History of Shape Optimization: case 2

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Chapter 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

Formulations and computational schemes for shape design sensitivity analysis and optimization have been developed in this thesis for both skeletal structures and nonlinear elastic solids. The continuum approach based on the concept of material derivative from continuum mechanics plays a central role in this development. The shape variations of skeletal structures as well as nonlinear elastic solids with design parameters are systematically derived here to build fundamental relations for shape sensitivity analysis. Two major objectives are considered in this study, (1) to derive explicit design sensitivity expressions for eigenvalues/vectors with configuration parameters of a skeletal structure, such as joint and support locations by using the domain method as well as the boundary method, and (2) to perform the shape sensitivity analysis and design optimization of a nonlinear elastic solid using the Eulerian formulation. Conclusions are outlined as follows.

Eigensensitivity Analysis and Configuration Optimization

Eigensensitivity analysis is concerned specially with the rates of changes of eigenvalues and eigenvectors with respect to design variables. In this study, the length and orientation of a beam member are considered as the design variables which can completely describe the shape variation of a skeletal structure. The final design sensitivity expressions can be written in either a domain integral form by the domain method or a boundary integral form by the boundary method.
Several examples including frame, truss and continuous beam are used to demonstrate the derived equations. The results generally give correct sensitivity information except in the cases with coarse finite element mesh. This is due to the fact that the finite element model used to calculate the eigenfunctions is not accurate enough.

Conclusions drawn from this study are listed as follows:

1. The continuum-based design sensitivity methods are capable of giving exact eigenvalue/eigenvector sensitivity coefficients, provided that the exact eigensolutions are available. Even using approximate eigensolutions with reasonable accuracy, these sensitivity equations are still able to yield accurate sensitivity coefficients.

2. The continuum-based design sensitivity methods are computationally more efficient than the ones derived by the discrete approach. Particularly, the eigenvalue sensitivity equation derived by the boundary method is extremely fast to compute; however, it is sensitive to the inaccuracy of the eigensolutions.

In summary, it is concluded that the domain method is accurate and efficient to the configuration design sensitivity analysis of skeletal structures.

To study the effects of support locations and the stiffness coefficients of supports on the design, a configuration design optimization of a vibrating beam is presented. The sensitivity coefficients obtained from the domain method are used to obtain an optimal design. It is concluded that support locations and the stiffness coefficients of supports are important to improve the quality of design.

**Shape Sensitivity Analysis and Optimization of Nonlinear Elastic Solids**

A general formulation using the Eulerian formulation for shape design sensitivity analysis and design optimization of a solid undergoing large displacement, large rotation but small strain is presented. This formulation is developed particularly for the class of problems whose design objective and constraints are defined in the deformed configuration. In this new design procedure, the design optimization is first performed in
the deformed domain while the configuration of the deformed domain is treated as the
design variables. The optimal shape of the unloaded configuration can be retrieved from the
obtained optimal deformed shape using the Eulerian formulation.

The Eulerian formulation cited above differs from the traditional one which has
taken into account the fact that the deformed domain is known before in the analysis. This
makes the Eulerian formulation more tractable in nonlinear shape sensitivity analysis and
optimization. In fact, numerical study has shown that the modified Eulerian formulation in
nonlinear analysis performs well and takes about the same amounts of CPU times to
converge as that of the total Lagrange formulation.

6.2 Recommendations

Future studies to extend this research could include the following:

(1) Development of eigensensitivity equations for three-dimensional skeletal
structures with respect to configuration parameters.

(2) Development of a robust design procedure for design optimization problem with
various kinds of design variables, such as thickness, joint and support location.

(3) A systematic study of nonlinear design sensitivity analysis and design
optimization defined in the deformed configuration including the effects of material
nonlinearity.
REFERENCES


Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.


APPENDIX A

Total Shape Derivatives of Length and Orientation Parameters

The length and the orientation of beam member i confined by a pair of end point 
\((X_1, Y_1)\) and \((X_2, Y_2)\), as shown in Fig. 2.1, are defined as

\[
\ell_i = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}
\]

\[
\theta_i = \cos^{-1} \left( \frac{X_2 - X_1}{\ell_i} \right) \quad \text{or} \quad \theta_i = \sin^{-1} \left( \frac{Y_2 - Y_1}{\ell_i} \right).
\]

(A.1)

The total material derivative of the member length can then be obtained as,

\[
\dot{\ell}_i = m_i(\dot{X}_2 - \dot{X}_1) + n_i(\dot{Y}_2 - \dot{Y}_1)
\]

(A.2)

where \(\dot{X}_i = \Delta X_i\) and \(\dot{Y}_i = \Delta Y_i\) for \(i=1\) and \(2\) as defined in Eq. (2.23).

To obtain the total material derivative of the orientation parameter, \(\theta_i\), one can take 
the total material derivative of \(\cos \theta_i\) as

\[
\dot{\theta}_i = -n_i \dot{\theta}_i = \dot{(\frac{X_2 - X_1}{\ell_i})}
\]

\[
= \frac{\dot{X}_2 - \dot{X}_1}{\ell_i} - \frac{\dot{\ell}_i}{\ell_i} m_i
\]

\[
= \frac{n_i^2}{\ell_i} (\dot{X}_2 - \dot{X}_1) - \frac{m_i n_i}{\ell_i} (\dot{Y}_2 - \dot{Y}_1)
\]

(A.3)

where the notation, \(\dot{()}\), is defined as the total material derivative of the quantity, ( ). Note 
that Eq. (A.2) and the condition, \(n_i^2 + m_i^2 = 1\), are employed in the above derivation.

Finally, the preceding equation yields the following relation,

\[
\dot{\theta}_i = -\frac{n_i}{\ell_i} (\dot{X}_2 - \dot{X}_1) + \frac{m_i}{\ell_i} (\dot{Y}_2 - \dot{Y}_1).
\]

(A.4)
A similar equation can be derived based upon the total material derivative of \( \sin \theta_i \).

**Total Shape Derivatives of Derivatives of Joint Displacements**

The total material derivatives of derivatives of joint displacements, \((u_i', w_i')\), can be obtained by taking the total material derivatives of Eq. (2.23) as

\[
\begin{bmatrix}
\dot{u}_i' \\
\dot{w}_i'
\end{bmatrix} = \theta_i + \begin{bmatrix}
w_i' \\
-u_i'
\end{bmatrix} \begin{bmatrix}
\dot{u}_i \\
\dot{w}_i'
\end{bmatrix} + \begin{bmatrix}
\dot{u}_i'' \\
\dot{w}_i''
\end{bmatrix}
\]  

(A.5)

where \((u_{i,i}', w_{i,i}')\) are the total material derivatives of \((u_i', w_i')\) with the orientation of the member \(i\) held unchanged, that is,

\[
\begin{bmatrix}
\dot{u}_{i,i}' \\
\dot{w}_{i,i}'
\end{bmatrix} = \begin{bmatrix}
m_i & n_i \\
-n_i & m_i
\end{bmatrix} \begin{bmatrix}
\dot{U}_i \\
\dot{W}_i
\end{bmatrix}
\]

(A.6)

where \((U_i', W_i')\) are the quantities defined in the global coordinate system, whose material derivatives can be rewritten in terms of either the total or relative shape derivatives [19]. As a result, Eq. (A.5) can be expanded as

\[
\begin{bmatrix}
\dot{u}_i' \\
\dot{w}_i'
\end{bmatrix} = \theta_i + \begin{bmatrix}
w_i' \\
-u_i'
\end{bmatrix} \begin{bmatrix}
\dot{u}_i \\
\dot{w}_i'
\end{bmatrix} - \dot{v}_i \begin{bmatrix}
u_i' \\
w_i'
\end{bmatrix}
\]

(A.7)

and

\[
\begin{bmatrix}
\dot{u}_i' \\
\dot{w}_i'
\end{bmatrix} = \theta_i + \begin{bmatrix}
w_i' \\
-u_i'
\end{bmatrix} \begin{bmatrix}
\dot{u}_i \\
\dot{w}_i'
\end{bmatrix} + \dot{v}_i \begin{bmatrix}
u_i'' \\
w_i''
\end{bmatrix}
\]

(A.8)

The above procedure can also be applied to find the total material derivatives of the second order derivative, \(w_i''\). Only the final relations are given here.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ w_i'' = -u''_i \hat{\theta}_i + w_{s,i}'' \]
\[ = -u''_i \hat{\theta}_i + \hat{u}''_s - v'_i w'_i - 2v'_i w''_i \]  
(A.9)

or

\[ w_i'' = -u''_i \hat{\theta}_i + w_{i,r}'' + v_i w''_i \]  
(A.10)

where the term \( w_{s,i}'' \) is defined by

\[ w_{s,i}'' = -n_i \tilde{U}'_{s,i}^t + m_i \tilde{W}''_i. \]  
(A.11)

**Total Shape Derivative of a Functional Defined by a Line Integral**

Based upon the definition of the total material derivatives given in Eq. (2.22) [19], the total shape derivative of the functional defined in Eq. (2.29) can be found as

\[ \dot{j} = \int_0^t \left( \frac{\partial f}{\partial u} \dot{u} + \frac{\partial f}{\partial w} \dot{w} + \frac{\partial f}{\partial u'} \dot{u'} + \frac{\partial f}{\partial w'} \dot{w'} + \frac{\partial f}{\partial \tau} + f' \right) ds. \]  
(A.12)

Next, Eqs. (2.26), (A.7) and (A.9) can be used to replace the terms \( \dot{u}, \dot{w}, \dot{u}' \) and \( \dot{w}' \), in terms of \( \dot{\theta}, \dot{u}_s, \dot{w}_s, \dot{u}'_s \) and \( \dot{w}'_s \), respectively. The result is stated in Eq. (2.30). On the other hand, Eqs. (2.28), (A.8) and (A.10) can be used to obtain an equation in terms of \( \dot{\theta}, \dot{u}_r, \dot{w}_r, \dot{u}'_r \) and \( \dot{w}'_r \) as

\[ \dot{j} = \int_0^t \left[ \frac{\partial f}{\partial \tau} + \left( \frac{\partial f}{\partial u} w - \frac{\partial f}{\partial w} u + \frac{\partial f}{\partial u'} u' - \frac{\partial f}{\partial w'} w' \right) \dot{\theta} \right. \\
+ \frac{\partial f}{\partial u} \dot{u}_r + \frac{\partial f}{\partial w} \dot{w}_r + \frac{\partial f}{\partial u'} \dot{u}'_r + \frac{\partial f}{\partial w'} \dot{w}'_r \]  
\[ + \int_0^t \left[ \left( \frac{\partial f}{\partial u'} u' + \frac{\partial f}{\partial w'} w' + \frac{\partial f}{\partial u''} u'' + \frac{\partial f}{\partial w''} w'' \right) + f' \right] ds. \]  
(A.13)

It is straightforward to show that the integrand in the second integral of the preceding equation is exactly identical to \( d(fv)/ds \). Therefore, integration by parts of the second integral can simplify the above equation. The result is given in Eq. (2.31).
APPENDIX B

The derivatives of the stiffness, mass and transformation matrices of element \( i \) with respect to the end-point locations may be written in the following forms, where \( b \) denotes as a design variable,

\[
\frac{d[K_i]}{db} = \frac{d\ell_i}{db} \frac{d\Gamma_i}{db}
\]

\[
\frac{d[M_i]}{db} = \frac{d\ell_i}{db} \frac{d\rho A_i}{db}
\]

\[
\frac{d[T_i]}{db} = \frac{d\theta_i}{db}
\]

where the derivatives, \( \frac{d\ell_i}{db} \) and \( \frac{d\theta_i}{db} \), can be found by Eqs. (A.2) and (A.4). More specifically, one has

\[
\begin{bmatrix}
-A_i & 0 & 0 & A_i & 0 & 0 \\
-\frac{36I_i}{\ell_i^2} & -\frac{12\ell_i}{\ell_i} & 0 & \frac{36I_i}{\ell_i^2} & 0 \\
-4I_i & 0 & -\frac{12I_i}{\ell_i} & -2I_i & 0 \\
-A_i & 0 & \frac{36I_i}{\ell_i^2} & \frac{12I_i}{\ell_i} & -4I_i \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
140 & 0 & 0 & 70 & 0 & 0 \\
156 & 44\ell_i & 0 & 54 & -26\ell_i & 0 \\
12\ell_i^2 & 0 & 26\ell_i & -9\ell_i^2 & 0 \\
140 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-n_i & m_i & 0 & 0 & 0 \\
-m_i & -n_i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -n_i & m_i \\
0 & 0 & 0 & -m_i & -n_i \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[ \frac{d\ell_i}{dX_1} = -m_i, \quad \frac{d\ell_i}{dX_2} = m_i \]

\[ \frac{d\ell_i}{dY_1} = -n_i, \quad \frac{d\ell_i}{dY_2} = n_i \]

\[ \frac{d\theta_i}{dX_1} = \frac{n_i}{\ell_i}, \quad \frac{d\theta_i}{dX_2} = -\frac{n_i}{\ell_i} \]

\[ \frac{d\theta_i}{dY_1} = -\frac{m_i}{\ell_i}, \quad \frac{d\theta_i}{dY_2} = \frac{m_i}{\ell_i} \]

(B.4)