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Chen-Fliess Series for Linear Distributed Systems

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CHEN-FLIESS SERIES FOR LINEAR DISTRIBUTED SYSTEMS

by

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ABSTRACT

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Distributed systems like fluid flow and heat transfer are modeled by partial differential equations (PDEs). In control theory, distributed systems are generally reformulated in terms of a linear state space realization, where the state space is an infinite dimensional Banach space or Hilbert space. In the finite dimension case, the input-output map can always be written in terms of a Chen-Fliess functional series, that is, a weighted sum of iterated integrals of the components of the input function. The Chen-Fliess functional series has been used to describe interconnected nonlinear systems, to solve system inversion and tracking problems, and to design predictive and adaptive controllers. The main goal of this thesis is to show that there is a generalized notion of a Chen-Fliess series for linear distributed systems where the weights are now linear operators acting on the iterated integrals. Sufficient conditions for convergence are developed. The method is compared against classical PDE theory using a number of first-order and second-order examples.

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INTRODUCTION

1.1 OVERVIEW ON SOLVING PDES

A partial differential equation (PDE) is a differential equation in terms of an unknown function involving two or more independent variables, for example, a temporal variable and a spatial variable. Such equations are used widely in physics and engineering to describe physical phenomena such as fluid dynamics [35, 38], heat transfer [5, 35], transmission lines [33, 34], electromagnetic fields [33, 35], and flexible structures [5, 38]. In contrast, ordinary differential equations (ODEs) constitute the class of differential equations in terms of a single independent variable. They often provide a suitable lumped approximation for systems that are truly distributed systems in nature. For example, the resistance, capacitance and inductance in a transmission line are not concentrated at specific points in space but rather are distributed continuously over the length of the line. In general, PDEs provide more accurate representations of such systems than models involving only ODEs. However, solving PDEs analytically is usually a much more difficult problem even for the linear case.

The first source of difficulty in solving a PDE is defining what exactly constitutes a solution, that is, what set of functions is admissible as a solution? Many PDEs only have solutions in certain *weak* senses, for example, the solution function may not be differentiable everywhere in its domain. It can also happen that the solution is a generalized function, for example, an impulse function. Finding conditions for the existence and uniqueness of different classes of solutions is a nontrivial subject [27, 28]. The second complicating factor is that the boundary conditions for a PDE are in the form of boundary *functions*. Such functions also belong to different classes. Not surprisingly, the nature of a solution is strongly dependent on the function class of the boundary conditions. Finally, in most applications there is a temporal variable. Thus, it is convenient to view a solution as a trajectory in an infinite dimensional vector space such as a Banach space or a Hilbert space [1,5,37]. This means that many of the mathematical tools used to solve PDEs rely on functional analysis and operator theory [1,5,27,28,37]. Often in this framework, only a series solution in terms of an orthonormal spanning set can be computed analytically.

There are a variety of numerical approaches to solving PDEs, for example, the finite difference method (FDM), the finite element method (FEM), and the finite volume method (FVM) [29,40]. The FDM approximates derivatives by numerically computing finite differences on a grid of nodes [40]. The FEM divides the solution domain into simply shaped regions, approximates the solution for each element individually, and then combines the solutions to form a single solution [4]. The FVM is a discretization method where the PDE is formulated as a volume integral in order to develop finite volume schemes for arbitrary meshes to approximate complex geometries [24]. Other methods include spatial discretization to convert a PDE into an ODE [29] and the semi-analytical solution (SAS) approach [40]. Spatial discretization, more specifically the spectral method, approximates the solution of the PDE so that the spatial and temporal variables can be treated separately. This is done by expanding the solution as a finite summation of products of spatially dependent basis functions and time dependent coefficients [29]. The SAS approach poses the solution in terms of piecewise multivariate polynomials and then determines the coefficients to approximate the solution of the PDE over a given domain [40]. Finally, there are methods for solving PDEs that use symbolic manipulation in software packages such as Mathematica [31] and Maple [30]. Many of the existing methods use Gröbner bases to synthesize formal solutions, i.e., series solutions that may or may not converge on a given domain [3]. For alternative formal approaches, see [26].

1.2 PDES IN CONTROL THEORY

Control theory for distributed systems is generally formulated in a linear state space setting where the state space is modeled as an infinite dimensional Banach space or a Hilbert space [5, 27]. In this context, it is possible to consider standard problems like controllability, observability, and stability [5, 37]. Areas like optimal control for hyperbolic and parabolic linear systems have also been developed [27, 28]. Considerable work has also been done in the frequency domain using suitable notions of the Laplace transform, a transfer function, and the Nyquist criterion [2]. A number of standard controller design techniques such as PI control, LQ optimal control, and H_∞ optimal control have also been extended for distributed systems [27, 28, 32].

Control theory for nonlinear distributed systems is considerably less developed. There are results on the existence and uniqueness of solutions to semilinear state space systems, that is, systems where the state equation has a linear part defining a strongly continuous semigroup and a nonlinear part that satisfies a Lipschitz continuity condition [1, 5]. In this setting, one can develop Lyapunov stability theory and small gain type theorems for closed-loop systems. Volterra series have been used to describe boundary control systems for nonlinear parabolic PDEs [38]. It does not appear that methods for representing the input-output map of a distributed system via a Chen-Fliess functional expansion have been developed even for the linear case.

1.3 CHEN-FLIESS SERIES

A Chen-Fliess functional series is a weighted sum of iterated integrals using a set of input functions indexed by letters of a finite alphabet [10, 12, 22]. The weights are normally real numbers. A Chen-Fliess series can be viewed as a time domain representation of a causal input-output system. The underlying mapping need not be linear or time-invariant. If the series can be shown to be convergent over some interval of time and for a set of admissible input, then the corresponding input-output map is called a *Fliess operator*. Sufficient conditions for convergence are in terms of the growth rate of the weights or coefficients of the series [10, 39].

Every smooth nonlinear finite dimensional state space system which is affine in the controls has a Chen-Fliess series representation [10, 11, 22]. Convergence of the series, however, is not automatic unless, for example, the realization is analytic. The converse claim is false. That is, there exists input-output systems having a Chen-Fliess series representation and no finite dimensional control-affine realization. For the linear time-invariant finite dimensional case, the coefficients of the Chen-Fliess series are the Markov parameters computed from the state space realization [23]. This is most easily seen by inductively integrating the state equation with respect to time. A similar process can be imitated in the infinite dimensional case [5], which suggests that a generalization of the Chen-Fliess series is at least possible in this setting. But to date no explicit development of this idea exists in the literature. It is this fact that motivates the work in this thesis.

Finally, Chen-Fliess series have several applications in control theory. They can be used to describe interconnected nonlinear systems [13, 16–19]. They have been employed for system inversion problems [15], tracking problems [8, 9], and predictive and adaptive control applications [14, 20]. The conjecture is that if a notion of a Chen-Fliess series can be developed for distributed systems, then similar applications can be pursued in this new setting.

1.4 PROBLEM STATEMENT

The main objectives of this thesis are to:

i. Provide a new class of Chen-Fliess series capable of describing the input-output map of a linear distributed system.

ii. Demonstrate the method for the class of first-order two-dimensional linear distributed systems and compare the results against classical methods for solving PDEs.

iii. Demonstrate the method for the class of second-order two-dimensional linear distributed systems and compare the results against classical methods for solving PDEs.

1.5 THESIS OUTLINE

The remainder of the thesis is organized as follows. In Chapter 2, the necessary mathematical tools used throughout the work are presented. First, the classical approach to solving a first-order linear PDE with two independent variables is described, more specifically, the transport equation. Next, its Green's function is derived. In the following section, different types of constant coefficient second-order linear PDEs are presented. In the subsequent section, elements of semigroup theory are introduced, including strongly continuous semigroups and their infinitesimal generators. Using this framework, the classical and mild solutions of the abstract Cauchy problem are treated, both the homogenous and nonhomogeneous cases. The next section describes the classical Chen-Fliess series. This requires the introduction of formal power series, iterated integrals, and the convergence properties of such series. The final section of the chapter describes finite dimensional linear state space realizations. In particular, the solution of the state equation in terms of the Peano-Baker series is presented. This series is closely related to Chen-Fliess series.

Chapter 3 addresses the first objective of the thesis. It begins with a description of the linear state equation in the infinite dimensional case. Using this as motivation, the generalized Chen-Fliess series is then developed for general distributed systems. Finally, sufficient conditions for convergence of such series are provided. More tractable conditions for the special case of linear time-invariant systems are then developed.

Chapter 4 addresses the second and third objectives of the thesis. First, the generalized

Chen-Fliess series representation of the solution of the transport equation is computed and compared against the classical solution. The next section presents the same type of analysis for the solutions of second-order, constant coefficient linear PDEs. The initial focus is on the general case and then numerical examples are provided for specific hyperbolic PDEs.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

2.1 ELEMENTS OF LINEAR PARTIAL DIFFERENTIAL EQUATION THEORY

2.1.1 First-Order Partial Differential Equations

The initial value problem or Cauchy problem for a first-order linear partial differential equation (PDE) with two independent real variables, x and t, consists of finding a solution to

$$ay_x(x,t) + by_t(x,t) + cy(x,t) = u(x,t), \quad y(x,0) = z_0(x), \tag{1}$$

where $a, b, c \in \mathbb{R}$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, y_x and y_t denote the partial derivatives with respect to x and t, respectively, and $u(x,t)$ and $z_0(x)$ are given functions. One specific example is the homogeneous *transport equation*,

$$y_t(x,t) + Vy_x(x,t) = 0, \quad y(x,0) = z_0(x), \tag{2}$$

which models the density of a fluid flowing with velocity $V > 0$ through a straight thin tube with a constant cross sectional area [7]. To find the solution, an auxiliary function, $h(s)$, is first introduced for fixed x and t of the form

$$h(s) = y(x + Vs, t + s) = z(q,r), \tag{3}$$

where $q = x + Vs$ and $r = t + s$. In which case,

$$\begin{aligned} h'(s) &= \frac{\partial y}{\partial q} \cdot \frac{\partial q}{\partial s} + \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial s} \\ &= \frac{\partial y}{\partial q} \cdot V + \frac{\partial y}{\partial r} \cdot 1 \end{aligned}$$

$$
\begin{aligned}
&= \partial_q y(q,r) \cdot V + \partial_r y(q,r) \\
&= 0.
\end{aligned}
$$

Applying the fundamental theorem of integral calculus gives the solution:

$$
\begin{aligned}
\int_{-t}^{0} h'(s)\, ds &= \int_{-t}^{0} 0\, ds \\
h(0) - h(-t) &= 0 \\
y(x,t) - y(x - Vt, 0) &= 0 \\
y(x,t) - z_0(x - Vt) &= 0 \\
y(x,t) &= z_0(x - Vt). \qquad (4)
\end{aligned}
$$

Example 2.1.1 Consider the transport equation with initial condition $y(x,0) = z_0(x) = ae^{bx}$ for some fixed $a, b \in \mathbb{R}$. From (4) the solution is

$$
y(x,t) = z_0(x - Vt) = ae^{b(x-Vt)}.
$$

□

The transport equation with an applied input $u(x,t)$ is

$$
\partial_t y(x,t) + V\partial_x y(x,t) = u(x,t), \qquad y(x,0) = z_0(x). \qquad (5)
$$

To determine the general solution, again utilize the auxiliary function, $h(s)$, in (3). In this case,

$$
\begin{aligned}
h'(s) &= \frac{\partial y}{\partial q} \cdot \frac{\partial q}{\partial s} + \frac{\partial y}{\partial r} \cdot \frac{\partial r}{\partial s} \\
&= \frac{\partial y}{\partial q} \cdot V + \frac{\partial y}{\partial r} \cdot 1 \\
&= \partial_q y(q,r) \cdot V + \partial_r y(q,r) \\
&= u(q,r).
\end{aligned}
$$

Applying the fundamental theorem of integral calculus in this case gives:

$$\begin{aligned}
\int_{-t}^{0} h'(s)\ ds &= \int_{-t}^{0} u(q,r)\ ds \\
h(0) - h(-t) &= \int_{-t}^{0} u(x + Vs, t + s)\ ds \\
y(x,t) - y(x - Vt, 0) &= \int_{-t}^{0} u(x + Vs, t + s)\ ds \\
y(x,t) - z_0(x - Vt) &= \int_{-t}^{0} u(x + Vs, t + s)\ ds \\
y(x,t) &= z_0(x - Vt) + \int_{-t}^{0} u(x + Vs, t + s)\ ds.
\end{aligned}$$

Let $r = s + t$ so that the solution to (5) becomes

$$y(x,t) = z_0(x - Vt) + \int_{0}^{t} u(x + V(r - t), r)\ dr. \tag{6}$$

The first term is the zero-input response, and the second term is the zero-state response.

Example 2.1.2 Consider the transport equation where

$$z_0(x) = 0, \quad u(x,t) = ax + bt, \quad a, b \in \mathbb{R}.$$

From (6) it follows that

$$\begin{aligned}
y(x,t) &= \int_{0}^{t} u(x - Vt + Vr, r)\ dr \qquad (7) \\
&= \int_{0}^{t} [a(x - Vt + Vr) + br]\ dr \\
&= axt - aVt^2 + aV\frac{t^2}{2} + b\frac{t^2}{2} \\
&= axt - aV\frac{t^2}{2} + b\frac{t^2}{2}.
\end{aligned}$$

$\square$

Example 2.1.3 Consider the transport equation where

$$V = 2, \quad z_0(x) = 0, \quad u(x,t) = t \sin x.$$

In this case, equation (6) gives the solution

$$\begin{aligned} y(x,t) &= z_0(x-2t) + \int_0^t u(x-2t+2r, r)\; dr \\ &= \int_0^t r \sin(x-2t+2r)\; dr \\ &= \frac{1}{2}\left(-t\cos(x) + \cos(t-x)\sin(t)\right). \end{aligned}$$

$\square$

2.1.2 Green's Functions

Green's functions are often utilized to find the solution of a linear nonhomogeneous ordinary differential equation or partial differential equation in boundary value problems [34]. In the language of system theory, a Green's function is simply the *impulse response* if the given equation is viewed as an input-output equation. The following definition describes the concept in the current context.

Definition 2.1.1 *When the zero-state solution of a two-dimensional linear partial differential equation is written as the convolution integral*

$$y(x,t) = \int_D G(x,t;\alpha,\beta)u(\alpha,\beta)\; d\alpha\, d\beta, \tag{8}$$

where D is the domain of the equation, then the function $G(x,t;\alpha,\beta)$ is called the ***Green's function*** *of the equation. In particular, every such Green's function satisfies*

$$\mathcal{L}G(x,t) = \delta(x-t), \qquad y(x,0) = 0, \quad \forall (x,t) \in D,$$

where $\mathcal{L}$ is the corresponding linear partial differential operator, and δ is the Dirac delta function.

Example 2.1.4 Consider the nonhomogeneous transport equation rewritten in terms of the linear partial differential operator $\mathcal{L} = \partial_t + V\partial_x$ acting on $y(x,t)$ so that

$$\mathcal{L}y(x,t) = u(x,t), \qquad y(x,0) = 0.$$

Combining (6) and (8) with $D = [0,\infty) \times (-\infty,\infty)$ gives

$$\int_0^\infty \int_{-\infty}^\infty G(x,t;\alpha,\beta)u(\alpha,\beta)\; d\alpha\, d\beta = \int_0^t u(x+V(r-t),r)\; dr. \tag{9}$$

If $u(x,t) = w(x)\delta(t-t_0)$, where $w(x)$ is an arbitrary continuous function and $t_0 > 0$, then

$$\int_0^\infty \int_{-\infty}^\infty G(x,t;\alpha,\beta)w(\alpha)\delta(\beta-t_0)\; d\alpha\, d\beta = \int_0^t w(x-V(t-r))\delta(r-t_0)\; dr$$

$$\int_{-\infty}^\infty G(x,t;\alpha,t_0)w(\alpha)\; d\alpha = w(x-V(t-t_0)). \tag{10}$$

Equation (10) holds only when

$$\begin{aligned} G(x,t;\alpha,t_0) &= \delta(\alpha-(x-V(t-t_0))) \\ &= \delta(-(x-\alpha)+V(t-t_0)). \end{aligned} \tag{11}$$

□

Given that the zero-state response of the transport equation has a representation in terms of the Green's function suggests that there is also a representation for the general classical solution.

Theorem 2.1.1 *The general classical solution of the transport equation can be represented in terms of the Green's function (11) as follows*

$$y(x,t) = \int_{-\infty}^\infty G(x,t;\alpha,t_0)z_0(\theta)\; d\theta + \int_0^t \int_{-\infty}^\infty G(x,t;\alpha,\beta)u(\alpha,\beta)\; d\alpha\, d\beta$$

for $t \geq 0$*, where* $\theta := \alpha - Vt_0$*.*

Proof: It was shown above that this claim is true for the zero-state response, so in light of linearity, it is sufficient to verify only the zero-input response. Observe

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,t;\alpha,t_0)z_0(\theta)\ d\theta &= \int_{-\infty}^{\infty} \delta(\theta-(x-Vt))z_0(\theta)\ d\theta\\ &= z_0(x-Vt)\end{aligned}$$

as expected. ■

Example 2.1.5 Reconsider the transport equation in Example 2.1.2, where $u(x,t) = ax+bt$ and $z_0(x) = 0$. The solution can be computed in terms of its Green's function using (8) and (11). Specifically,

$$\begin{aligned}y(x,t) &= \int_0^t \int_{-\infty}^{\infty} \delta(\alpha - [x - V(t-\beta)])(a\alpha + b\beta)\ d\alpha\, d\beta\\ &= \int_0^t a[x - V(t-\beta] + b\beta\ d\beta\\ &= (ax - aVt)t + (b + aV)\int_0^t \beta\ d\beta\\ &= axt - aVt^2 + b\frac{t^2}{2} + aV\frac{t^2}{2}\\ &= axt + b\frac{t^2}{2} - aV\frac{t^2}{2},\end{aligned}$$

which is consistent with the previous calculation. □

2.1.3 Second-Order Partial Differential Equations

A general second-order linear PDE with two independent variables, x and t, is written as

$$ay_{xx}(x,t) + by_{xt}(x,t) + cy_{tt}(x,t) + dy_x(x,t) + ey_t(x,t) + fy(x,t) = u(x,t), \qquad (12)$$

where $a, b, c, d, e, f \in \mathbb{R}$ and subject to some boundary conditions. Such PDEs can be classified into three categories: hyperbolic, elliptic, and parabolic. This classification scheme is based on the value of the discriminant $b^2 - 4ac$, namely,

$$b^2 - 4ac \begin{cases} < 0 & : \quad \text{elliptic} \\ = 0 & : \quad \text{parabolic} \\ > 0 & : \quad \text{hyperbolic.} \end{cases}$$

The main interest here will be in initial value problems where the coefficients d, e, and f are equal to zero, i.e.,

$$ay_{xx}(x,t) + by_{xt}(x,t) + cy_{tt}(x,t) = u(x,t), \qquad y(x,0) = z_0(x). \tag{13}$$

Examples of second-order PDEs from physics and engineering include the wave equation, the heat equation, and Poisson's equation.

The wave equation

$$y_{tt}(x,t) = \alpha^2 y_{xx}(x,t),$$

where $\alpha > 0$ is a physical constant, is a hyperbolic PDE. It is used to describe the displacement from equilibrium of vibrating objects like strings and membranes [34]. It can be found in applications such as electromagnetics, fluid dynamics, and electric circuits. Derived using Hooke's law in the one-dimensional case, it was first proposed by the French scientist Jean-Baptiste le Rond d'Alembert [33]. Leonhard Euler generalized the concept to three-dimensional [36].

The heat equation

$$y_t(x,t) = \kappa y_{xx}(x,t),$$

where $\kappa > 0$ is the diffusivity coefficient, is a parabolic PDE. It is also known as the diffusion equation. It describes the movement of thermal energy throughout a material like a metal bar [34]. It also appears in quantum mechanics and the material sciences [33, 34]. The heat equation was first derived by Joseph Fourier [33].

The Poisson equation

$$y_{xx}(x,t) + y_{tt}(x,t) = u(x,t)$$

is an elliptic PDE. It is used to describe steady-state temperature distributions and potential fields in material containing matter, charge, or sources of heat or fluid [34]. It has many applications in electrostatics and fluid dynamics. Developed by Siméon Denis Poisson, the equation is derived using Gauss's law.

Analogous to solving linear ODEs, the homogeneous form of (13) is solved by first making an initial guess of the solution in order to establish its characteristic equation. Suppose a solution of the form $y(x,t) = f(t+mx)$ exists, where f is an unknown twice differentiable function. Therefore,

$$\begin{aligned} am^2 f''(t+mx) + bmf''(t+mx) + cf'' &= 0 \\ (am^2 + bm + c)f''(t+mx) &= 0. \end{aligned}$$

Assuming $f''(t+mx) \neq 0$ yields the characteristic equation

$$am^2 + bm + c = 0. \tag{14}$$

The classification scheme described above follows directly from the nature of the roots of this equation via the quadratic equation.

<u>Hyperbolic equations: $b^2 - 4ac > 0$, two distinct real roots</u>: If (14) has two distinct real roots, m_1 and m_2, then the solution to (13) with $u(x,t) = 0$ has the form

$$y(x,t) = f(t + m_1 x) + g(t + m_2 x), \tag{15}$$

where $g(x,t)$ is another arbitrary function that is twice differentiable. To verify this assertion, first compute the necessary partial derivatives:

$$y_{xx} = {m_1}^2 f''(t+m_1 x) + {m_2}^2 g''(t+m_2 x)$$

$$y_{xt} = m_1 f''(t + m_1 x) + m_2 g''(t + m_2 x)$$

$$y_{tt} = f''(t + m_1 x) + g''(t + m_2 x).$$

When these functions are substituted into (13), it follows directly that

$$0 = a[{m_1}^2 f''(t + m_1 x) + {m_2}^2 g''(t + m_2 x)] + b[m_1 f''(t + m_1 x)$$

$$+ m_2 g''(t + m_2 x)] + c[f''(t + m_1 x) + g''(t + m_2 x)]$$

$$0 = (a{m_1}^2 + bm_1 + c) f''(t + m_1 x) + (a{m_2}^2 + bm_2 + c) g''(t + m_2 x).$$

By assumption, $a{m_i}^2 + bm_i + c = 0$, $i = 1, 2$, and thus, the claim is verified.

Parabolic equations: $b^2 - 4ac = 0$, one repeated real root: When (14) has one repeated real root, it can be verified as in the previous case that (13) with $u(x, t) = 0$ has one of two possible forms, either

$$y(x, t) = f(t + m_1 x) + tg(t + m_2 x)$$

or

$$y(x, t) = f(t + m_1 x) + xg(t + m_2 x).$$

Elliptic equations: $b^2 - 4ac < 0$, two distinct complex roots: When (14) has two distinct complex roots, the general solution $y(x, t)$ to (13) with $u(x, t) = 0$ takes the same form as that in the hyperbolic solution except that the solution is now a complex-valued function. Of course, if the coefficients of (14) are real, the roots will be a complex conjugate pair. Therefore, $y(x, t)$ can also be written in terms of real-valued functions.

Solving the nonhomogeneous version of (14) is much less trivial. The following example illustrates the process in one instance.

Example 2.1.6 Consider a vibrating string of length $L > 0$ with an applied sinusoid forcing function described by the wave equation

$$y_{tt}(x, t) = y_{xx}(x, t) + \gamma \sin\left(\frac{\pi x}{L}\right), \quad 0 \le x \le L, \quad t \ge 0, \tag{16}$$

where $\gamma > 0$ is a fixed constant, and subject to the boundary conditions

$$y(0,t) = y(L,t) = 0, \quad t \geq 0$$

$$y(x,0) = y_t(x,0) = 0, \quad 0 \leq x \leq L.$$

Suppose there exists a solution of the form

$$y(x,t) = v(x,t) + \psi(x), \tag{17}$$

where v and ψ are unknown functions. Substituting (17) into the given PDE yields

$$v_{tt}(x,t) + \psi_{tt}(x) = v_{xx}(x,t) + \psi_{xx}(x) + \gamma \sin\left(\frac{\pi x}{L}\right),$$

where the first two boundary conditions are rewritten as

$$v(0,t) + \psi(0) = v(L,t) + \psi(L) = 0.$$

In light of superposition, there is an ODE boundary value problem in $\psi(x)$, namely,

$$\psi_{xx}(x) + \gamma \sin\left(\frac{\pi x}{L}\right) = 0 \tag{18}$$

with the boundary conditions

$$\psi(0) = \psi(L) = 0. \tag{19}$$

Equation (18) can be solved directly by repeated integration:

$$\begin{aligned} \psi_{xx}(x) &= -\gamma \sin\left(\frac{\pi x}{L}\right) \\ \psi_x(x) &= -\gamma\left(-\frac{L}{\pi}\right)\cos\left(\frac{\pi x}{L}\right) + C_1 \\ \psi(x) &= \gamma\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) + C_1 x + C_2. \end{aligned}$$

The constants C_1 and C_2 are determined from (19) so that

$$\psi(x) = \gamma\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right).$$

The solution to the PDE is then updated to give

$$y(x,t) = v(x,t) + \gamma \left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right)$$

with boundary conditions

$$y(0,t) = v(0,t) = y(L,t) = 0, \quad t \geq 0 \tag{20}$$

$$y(x,0) = v(x,0) + \gamma \left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) = 0, \quad 0 \leq x \leq L \tag{21}$$

$$y_t(x,0) = v_t(x,0) = 0, \quad 0 \leq x \leq L. \tag{22}$$

Note that (21) is a nonhomogeneous ODE. The next problem is to solve the homogeneous version of the wave equation to determine $v(x,t)$, namely,

$$v_{tt}(x,t) = v_{xx}(x,t), \quad 0 \leq x \leq L, \quad t \geq 0. \tag{23}$$

Assuming the solution is separable, i.e.,

$$v(x,t) = X(x)T(t),$$

equation (23) becomes

$$X(x)T''(t) = X''(x)T(t),$$

or equivalently,

$$\frac{T''}{T} = \frac{X''}{X} = Q$$

for some constant $Q \in \mathbb{R}$. In which case,

$$\frac{X''}{X} = Q \tag{24}$$

$$\frac{T''}{T} = Q \tag{25}$$

with boundary conditions

$$X(0) = X(L) = 0 \tag{26}$$

$$T'(0) = 0, \tag{27}$$

respectively. Note that $T(0)$ is a free variable. Rewriting (24) in the form

$$X'' - QX = 0,$$

there are three possible cases: $Q = \alpha^2$, $Q = 0$, and $Q = -\alpha^2$, where $\alpha > 0$ is a real constant. The solution $X(x)$ can only satisfy its boundary conditions if $Q = -\alpha^2$, and thus,

$$X(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x).$$

The constants C_1 and C_2 are found using the boundary conditions (26) and yield a family of solutions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

The initial value problem (25) and (27) is similarly solved except now it is known that $Q = Q_n = n\pi/L$. Specifically,

$$T_n(t) = \cos\left(\frac{n\pi t}{L}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

Therefore, the corresponding family of solutions to (23) is

$$v_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right), \quad n = 0, \pm 1, \pm 2, \ldots \tag{28}$$

From superposition it follows that any linear combination of these solutions is also a solution. So in the most general case,

$$v(x,t) = \sum_{n=-\infty}^{\infty} Q_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right),$$

where $Q_n \in \mathbb{R}$ for all integers n. Finally, applying the last remaining boundary condition (21) implies

$$v(x,0) = \sum_{n=-\infty}^{\infty} Q_n \sin\left(\frac{n\pi x}{L}\right) = -\gamma\left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right),$$

so that

$$Q_1 = -\gamma \left(\frac{L}{\pi}\right)^2,$$

and $Q_n = 0$ otherwise. Therefore,

$$v(x,t) = -\gamma \left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi t}{L}\right).$$

Combining $v(x,t)$ and $\psi(x)$ gives the complete solution to (16)

$$y(x,t) = -\gamma \left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi t}{L}\right) + \gamma \left(\frac{L}{\pi}\right)^2 \sin\left(\frac{\pi x}{L}\right).$$

□

2.2 STRONGLY CONTINUOUS SEMIGROUPS

A *semigroup* is a set S with an associative product $S \times S \to S : (a,b) \mapsto ab$. That is, $(ab)c = a(bc)$ for all $a, b, c \in S$. The set of $n \times n$ matrices, for example, forms a semigroup under matrix multiplication. The most important semigroup appearing in finite dimensional linear systems theory is that generated by the matrix exponential.

Example 2.2.1 Let $A \in \mathbb{R}^{n \times n}$ and define $\exp(At) = \sum_{n \geq 0} (At)^n/n!$ for all $t \in \mathbb{R}$. The claim is that $S = \{\exp(At) : t \in \mathbb{R}\}$ is a semigroup. First observe that from the binomial theorem

$$\begin{aligned}
\exp(A(t+s)) &= \sum_{n=0}^{\infty} \frac{A^n (t+s)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^n}{k!(n-k)!} t^k s^{n-k}.
\end{aligned}$$

Next apply the change of variables $n = k + i$ so that

$$\begin{aligned}\exp(A(t+s)) &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{A^{k+i}}{k!i!}t^k s^i \\ &= \sum_{k=0}^{\infty}\frac{A^k}{k!}t^k\sum_{i=0}^{\infty}\frac{A^i}{i!}s^i \\ &= \exp(At)\exp(As).\end{aligned}$$

This verifies that under matrix multiplication S maps back to S. The associativity property follows directly from the known fact that matrix multiplication is associative. It should also be noted that since $\exp(At)|_{t=0} = I$, and I is the unit of matrix multiplication, that S also constitutes a *monoid*. In addition, as $(\exp At)^{-1} = \exp(-At)$, S is also a *group*. $\square$

In infinite dimensional linear system theory, a generalization of the matrix exponential is needed as described next. It utilizes the notion of a *Banach space*, that is, a vector space Z with a norm $\|\cdot\|$ having the property that all Cauchy sequences in Z converge to an element in Z [25]. Such spaces are said to be *complete*. In the event that Z also has an inner product $\langle\cdot,\cdot\rangle$, then it constitutes a *Hilbert space* with $\|z\|^2 = \langle z, z\rangle$ for all $z \in Z$. Let $L(Z)$ denote the set of all bounded linear operators taking Z back to itself.

Definition 2.2.1 *[1, 5] Let Z be a Banach space. A **strongly continuous semigroup** is an operator-valued function $\mathbb{T}(t)$ from $\mathbb{R}^+$ to $L(Z)$ that satisfies the following properties:*

a. $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s)$, $\forall t, s \geq 0$;

b. $\mathbb{T}(0) = \mathrm{I}$;

c. $\lim_{t\to 0^+} ||\mathbb{T}(t)z_0 - z_0|| = 0$, $\forall z_0 \in Z$.

*Henceforth, such a semigroup will be called a **C_0-semigroup**.*

Example 2.2.2 Continuing the previous example, the assertion is that the semigroup of matrix exponentials defines a strongly continuous semigroup from $\mathbb{R}^+$ to $L(Z)$, where $Z = \mathbb{R}^n$ is the n-dimensional $\mathbb{R}$-vector space endowed with the usual norm. The semigroup property has already been verified. As noted above, $\exp(At)|_{t=0} = I$. So only the third property needs to be checked. Observe for any $z_0 \in \mathbb{R}^n$:

$$\begin{aligned}
\|\exp(At)z_0 - z_0\| &\leq \|(\exp(At) - I)\| \|z_0\| \\
&= \left\| \left(\sum_{n=0}^{\infty} \frac{(At)^n}{n!} - I \right) \right\| \|z_0\| \\
&\leq \sum_{n=1}^{\infty} \frac{\|A\|^n t^n}{n!} \|z_0\| \\
&= [\exp(\|A\|t) - 1] \, \|z_0\|.
\end{aligned}$$

The continuity of the exponential function implies that the desired property holds.

□

Example 2.2.3 Let $Z = L_2(0, \infty)$ be the Hilbert space of square integrable functions on $[0, \infty)$ with the usual inner product. Consider the left-shift semigroup

$$(\mathbb{T}(t)h)(x) = h(t + x), \qquad h \in Z, \;\; x \geq 0.$$

The first two properties in Definition 2.2.1 clearly hold, so only the third property is checked. For any continuous function h with compact support, observe

$$||\mathbb{T}(t)h - h||_2 = \left(\int_0^{\infty} |h(t + x) - h(x)|^2 \, dx \right)^{1/2}.$$

Taking the limit as $t \to 0^+$ results in zero. The set of continuous functions with compact support is a dense subset of $L_2(0, \infty)$. Thus, for any function $f \in L_2(0, \infty)$ and any positive ε, there exists a continuous function h with compact support such that $||f - h||_2 \leq \varepsilon$. Now observe that T is bounded, specifically, $||\mathbb{T}(t)|| \leq 1$ for all $t \geq 0$. Therefore,

$$||\mathbb{T}(t)f - f||_2 = ||\mathbb{T}(t)(f - h) + \mathbb{T}(t)h - h + h - f||_2$$

$$\leq ||f - h||_2 + ||\mathbb{T}(t)h - h||_2 + ||h - f||_2$$
$$\leq 3\varepsilon$$

for sufficiently small $t > 0$. So,

$$\lim_{t \to 0^+} ||\mathbb{T}(t)f - f||_2 = 0,$$

and thus, $\mathbb{T}(t)$ defines a C_0-semigroup on $L_2(0, \infty)$. □

The next definition will be central to describing a state space realization in the infinite dimensional sense.

Definition 2.2.2 *[1, 5] The **infinitesimal generator** A of a C_0-semigroup on a Hilbert space Z is defined by*

$$Az = \lim_{t \to 0^+} \frac{1}{t}(\mathbb{T}(t) - I)z, \tag{29}$$

where the domain of A, $D(A)$, is the set of elements in Z for which the limit exists.

Example 2.2.4 The infinitesimal generator is computed for the left-shift semigroup defined in Example 2.2.3. Consider any $z \in L_2(0, \infty)$ which is absolutely continuous on any finite subinterval and whose derivative is also in $L_2(0, \infty)$. Then

$$\begin{aligned} Az(x) &= \lim_{t \to 0^+} \frac{1}{t}(\mathbb{T}(t) - I)z(x) \\ &= \lim_{t \to 0^+} \frac{\mathbb{T}(t)z(x) - z(x)}{t} \\ &= \lim_{t \to 0^+} \frac{z(t + x) - z(x)}{t}. \end{aligned}$$

The limit exists as z is differentiable, and thus

$$Az = \frac{d}{dx}z,$$

where $D(A) \subset L_2(0, \infty)$ is the subset of functions satisfying all the specified requirements.

□

The following definition will be useful in the next section in understanding in what sense a Cauchy problem has a well defined solution.

Definition 2.2.3 *[1, 5] An operator A on a Banach space Z is* ***closed*** *if its graph $G(A)$ is closed in $Z \times Z$, where*

$$G(A) := \{(z, Az) : z \in D(A)\}.$$

Thus, A is closed if and only if whenever $z_n \in D(A)$ for $n \in \mathbb{N}$,

$$\|z_n - z\| \to 0, \qquad \|Az_n - y\| \to 0,$$

it follows that $z \in D(A)$ and $Az = y$.

Example 2.2.5 Let Z be the Hilbert space $L_2(0, 1)$ and consider the operator $A = d/dx$ with domain

$$D(A) = \left\{ z \in Z \,\middle|\, z \text{ is absolutely continuous with } z(0) = 0 \text{ and } \frac{dz}{dx} \in L_2(0, 1) \right\}.$$

Let $\{z_n\} \subset D(A)$ be a sequence such that z_n converges to z and dz_n/dx converges to y, i.e.,

$$\|z_n - z\| \to 0, \qquad \text{and} \qquad \left\| \frac{dz_n}{dx} - y \right\| \to 0.$$

Define a function

$$f(\zeta) = \int_0^\zeta y(x)\, dx.$$

It is clear that $f \in D(A)$ and $df/dx = y$. The claim is that $z = f$ almost everywhere, and thus, A is closed.

First observe that given functions h and g, the Cauchy-Schwartz inequality gives

$$\left| \int_0^1 h(x)g(x)\, dx \right|^2 \le \left(\int_0^1 h^2(x)\, dx \right) \left(\int_0^1 g^2(x)\, dx \right).$$

If $h(x) = 1$ on the interval $[0, \zeta]$ and zero otherwise, then

$$\begin{aligned}\left|\int_0^1 h(x)g(x)\ dx\right|^2 &\leq \left|\int_0^1 1_{[0,\zeta]}g(x)\ dx\right|^2 \\ &\leq \left(\int_0^1 1_{[0,\zeta]}\ dx\right)\left(\int_0^1 g^2(x)\ dx\right),\end{aligned}$$

where $1_{[a,b]}$ denotes the indicator function. Thus,

$$\begin{aligned}\left|\int_0^\zeta g(x)\ dx\right|^2 &\leq \left(\int_0^\zeta 1\ dx\right)\left(\int_0^1 g^2(x)\ dx\right) \\ &\leq \int_0^1 g^2(x)\ dx.\end{aligned}$$

Next, use this inequality to derive an upper bound on the distance between f and z:

$$\begin{aligned}\|f - z\| &= \|f - z_n + z_n - z\| \\ &\leq \|f - z_n\| + \|z_n - z\| \\ &\leq \left[\int_0^1 |f(\zeta) - z_n(\zeta)|^2\ d\zeta\right]^{1/2} + \|z_n - z\| \\ &= \left[\int_0^1 \left|\int_0^\zeta y(x)\ dx - z_n(\zeta)\right|^2\ d\zeta\right]^{1/2} + \|z_n - z\| \\ &= \left[\int_0^1 \left|\int_0^\zeta y(x) - \frac{dz_n}{dx}(x)\ dx\right|^2\ d\zeta\right]^{1/2} + \|z_n - z\| \\ &\leq \left[\int_0^1 \int_0^1 \left(y(x) - \frac{dz_n}{dx}(x)\right)^2\ dx\ d\zeta\right]^{1/2} + \|z_n - z\| \\ &\leq \left[\int_0^1 \left\|y - \frac{dz_n}{dx}\right\|^2\ d\zeta\right]^{1/2} + \|z_n - z\| \\ &\leq \left\|y - \frac{dz_n}{dx}\right\| + \|z_n - z\|.\end{aligned}$$

As both terms on the right-hand side approach zero as $n \to \infty$, it follows that $z = f$ almost everywhere.

$\square$

2.3 ABSTRACT DIFFERENTIAL EQUATIONS

This section describes in what sense a linear differential equation on a Banach space has a well defined solution. The focus is on the initial value or Cauchy problem. First the homogenous case is considered. Then the nonhomogeneous case is addressed.

Let A be an operator on a Banach space Z with domain $D(A)$. The *abstract homogeneous Cauchy problem* consists of finding a solution z to

$$\dot{z}(t) = Az(t), \qquad z(0) = z_0 \tag{30}$$

for $t \geq 0$ when $z_0 \in Z$. The solution z is viewed as a trajectory in the function space Z. In infinite dimensional problems, there are a variety of different senses in which there can be a solution. The following definition gives one such instance.

Definition 2.3.1 *[1] The abstract homogeneous Cauchy problem has a* ***classical solution*** *if there exists a function $z \in C^1(R_+, Z)$ which satisfies (30) such that $z(t) \in D(A)$ for all $t \geq 0$.*

Alternatively, there can be a solution in a weaker sense as given below.

Definition 2.3.2 *[1] The abstract homogeneous Cauchy problem has a* ***mild solution*** *if there exists a function $z \in C(\mathbb{R}_+, Z)$ such that*

$$\int_0^t z(s)\ ds \in D(A)$$

and

$$z(t) = z_0 + A \int_0^t z(s)\ ds$$

holds for $t \geq 0$.

The following theorem relates the two forms of the solution.

Theorem 2.3.1 *[1] A mild solution z of the abstract homogeneous Cauchy problem (30) is a classical solution if and only if $z \in C^1(R_+, Z)$.*

The next theorem shows the importance of C_0-semigroups in ensuring the existence of a solution.

Theorem 2.3.2 *[1] Let $\mathbb{T}$ be a C_0-semigroup on Z and let A be its generator. Then the following properties hold:*

a. $z(t) = \mathbb{T}(t)z_0$ is a classical solution of (30) if and only if $z_0 \in D(A)$.

b. $z(t) = \mathbb{T}(t)z_0$ is a mild solution of (30) for every $z_0 \in Z$.

Uniqueness of solutions is addressed in the next result. It uses the additional property of closedness and the following definition.

Definition 2.3.3 *Let Z be a Banach space over $\mathbb{C}$ and $A : D(A) \to Z$ a linear map. A number $\lambda \in \mathbb{C}$ is in the* ***resolvent set****, $\rho(A)$, if $\lambda I - A$ is invertible.*

Theorem 2.3.3 *[1] Let A be a closed operator. The following statements are equivalent:*

a. The operator A generates a C_0-semigroup.

b. For all $z_0 \in D(A)$ there exists a unique classical solution of (30).

c. $\rho(A) \neq \emptyset$ and for all $z_0 \in Z$ there exists a unique mild solution of (30).

When these assertions hold, the mild solution of (30) is given by $z(t) = \mathbb{T}(t)z_0$.

Example 2.3.1 Consider the closed operator $A = d/dx$ on $Z = L_2(0, 1)$ in Example 2.2.5. As it defines C_0-semigroup, by Theorem 2.3.3, the corresponding Cauchy problem has a unique classical solution for all $z_0 \in D(A)$ and a unique mild solution for all $z_0 \in Z$. $\square$

Next the *abstract nonhomogeneous Cauchy problem* is considered. Let Z be a Banach space and $A : D(A) \to Z$ a closed linear operator. Fix $T > 0$ and select a $v \in L_1([0,T], Z)$. The abstract nonhomogeneous Cauchy problem involves finding a solution to

$$\dot{z}(t) = Az(t) + v(t), \quad z(0) = z_0 \tag{31}$$

for $t \in [0,T]$ when $z_0 \in Z$. As in the homogeneous case, the solution can exist in different senses.

Definition 2.3.4 *[1] The abstract nonhomogeneous Cauchy problem has a* ***classical solution*** *for a given $v \in C([0,T], Z)$ if there exists a function $z \in C^1([0,T], Z)$ which satisfies (31) such that $z(t) \in D(A)$ for all $t \in [0,T]$.*

Definition 2.3.5 *[1] The abstract nonhomogeneous Cauchy problem has a* ***mild solution*** *for a given $v \in L_1([0,T], Z)$ if there exists a function $z \in C([0.T], Z)$ such that*

$$\int_0^t z(s)\ ds \in D(A)$$

and

$$z(t) = z_0 + A\int_0^t z(s)\ ds + \int_0^t v(s)\ ds$$

holds for $t \in [0,T]$.

The relationship between the two solutions is given by the following theorem.

Theorem 2.3.4 *[1] Fix $v \in C([0,T], Z)$, and let z be a mild solution of (31). Then z is a classical solution if and only if $z \in C^1([0,T], Z)$.*

The final theorem of this section will be the most useful in this thesis.

Theorem 2.3.5 *[1] Let A be the infinitesimal generator of a C_0-semigroup $\mathbb{T}$ on a Banach space Z. If $v \in L_1([0,T],Z)$, then the abstract nonhomogeneous Cauchy problem has a unique mild solution of the form*

$$z(t) = \mathbb{T}(t)z_0 + \int_0^t \mathbb{T}(t-s)v(s)\ ds \tag{32}$$

for all $t \in [0,T]$.

Proof: The proof of uniqueness is given in [1, Proposition 3.1.16]. The existence part of the proof, which is also presented in [1], will be explicitly addressed here.

From Theorem 2.3.2b, $\mathbb{T}(\cdot)z_0$ is a mild solution of the homogeneous Cauchy problem. By setting the initial value $z_0 = 0$ in (32), it only needs to be shown that $w(t) := \int_0^t \mathbb{T}(t-s)v(s)\ ds$ is the mild solution when $z_0 = 0$. Observe

$$\begin{aligned}
A\int_0^t w(s)\ ds &= A\int_0^t \int_0^s \mathbb{T}(s-r)v(r)\ dr\, ds \\
&= A\int_0^t \int_r^t \mathbb{T}(s-r)v(r)\ ds\, dr \\
&= \int_0^t A\int_0^{t-r} \mathbb{T}(s)v(r)\ ds\, dr \\
&= \int_0^t [\mathbb{T}(t-r)v(r) - v(r)]\, dr \\
&= \int_0^t \mathbb{T}(t-r)v(r)\ dr - \int_0^t v(r)\ dr \\
&= w(t) - \int_0^t v(r)\ dr,
\end{aligned}$$

which proves the claim. ■

Example 2.3.2 The goal of this example is to write the transport equation (5) as an abstract Cauchy problem on $L_2(-\infty,\infty)$. First observe that a C_0-semigroup on $L_2(-\infty,\infty)$ can be identified for the transport equation in Example 2.1.4 by comparing Theorems 2.1.1 and 2.3.5. The claim is that

$$z(t) = \mathbb{T}(t)z_0 = z_0(x - Vt).$$

This assertion is verified by checking the three conditions in Definition 2.2.1. Regarding the first condition, observe

$$\mathbb{T}(t+s)z_0(x) = z_0(x - V(t+s)) = \mathbb{T}(t)[\mathbb{T}(s)z_0(x)],$$

so that $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s)$ as required. Concerning the second condition, it is immediate that

$$\mathbb{T}(0)z_0(x) = z_0(x),$$

and thus, $\mathbb{T}(0) = I$. Finally, consider the third condition. Observe

$$\begin{aligned}
\lim_{t\to 0^+} ||\mathbb{T}(t)z_0(x) - z_0(x)|| &= \lim_{t\to 0^+} ||z_0(x - Vt) - z_0(x)|| \\
&= \lim_{t\to 0^+} \left[\int_{-\infty}^{\infty} (z_0(x - Vt) - z_0(x))^2 \, dx\right]^{1/2} \\
&= 0
\end{aligned}$$

provided that $z_0(0) \in L_2(-\infty, \infty)$, and z_0 is uniformly continuous.

Using Theorem 2.2.2, the infinitesimal generator A for the transport equation can be computed directly from $\mathbb{T}(t)$, i.e.,

$$\begin{aligned}
Az_0(x) &= \lim_{t\to 0^+} \frac{1}{t}\left(\mathbb{T}(t) - I\right) z_0(x) \\
&= \lim_{t\to 0^+} \frac{1}{t}\left(\mathbb{T}(t)z_0(x) - z_0(x)\right) \\
&= \lim_{t\to 0^+} \frac{z_0(x - Vt) - z_0(x)}{t} \\
&= \lim_{t\to 0^+} \frac{-V(z_0(x) - z(x - Vt))}{Vt} \\
&= -V\frac{d}{dx}z_0(x).
\end{aligned}$$

Therefore, the transport equation (5) can be rewritten as the abstract Cauchy problem

$$\dot{z}(t) = Az(t) + v(t), \quad z(0) = z_0, \tag{33}$$

where $y = z$. $\square$

2.4 CHEN-FLIESS SERIES

In this section, the basic elements of classical Chen-Fliess series are presented. First, the notion of a formal power series is presented. The concept of an iterated integral is then introduced. Finally, the definition of a Chen-Fliess series is given along with various notions of convergence that yield a Fliess operator. The treatment is largely based on [10, 12, 21].

2.4.1 Formal Power Series

An *alphabet* is a nonempty set of noncommuting symbols denoted by $X = \{x_0, x_1, \ldots, x_m\}$. Each element in X is called a *letter*. A finite sequence of these letters is called a *word* over X and is denoted as $\eta = x_{i_k} \ldots x_{i_1}$. The *length* of the word, $|\eta|$, is the number of letters in η, in this case, $|\eta| = k$. The length of the empty word, $\emptyset$, is zero. The set of all words over X is written as X^*.

A formal power series, c, is any function of the form

$$c : X^* \to \mathbb{R}^\ell,$$

where $\ell \geq 1$ is an integer. The superscript will be omitted when $\ell = 1$. Formal power series are often written in terms of a formal sum

$$c = \sum_{\eta \in X^*} (c, \eta)\eta,$$

where (c, η) is the *coefficient* of c at $\eta \in X^*$. A series c is said to be *proper* when the coefficient $(c, \emptyset)$, also known as the *constant term*, is zero. The *support* of c is the set of all words in X^* whose coefficients are nonzero. It will be written as $\mathrm{supp}(c)$. The set of all such formal power series is denoted as $\mathbb{R}^\ell\langle\langle X \rangle\rangle$. The subset of series having finite support, that is, the set of all polynomials, is denoted as $\mathbb{R}^\ell\langle X \rangle$.

2.4.2 Iterated Integrals

Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $||u||_{\mathfrak{p}} = \max\{||u_i||_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $||u_i||_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions u defined on $[t_0, t_1]$ having a finite $||\cdot||_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R_u)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : ||u||_{\mathfrak{p}} \leq R_u]$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the iterated integral $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] := 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The argument t_0 will often be suppressed when $t_0 = 0$.

Example 2.4.1 If $\eta = x_i$, then

$$E_{x_i}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)\, d\tau.$$

In particular,

$$E_{x_0}[u](t, t_0) = \int_{t_0}^{t} 1\, d\tau = t - t_0,$$

so that

$$\begin{aligned} E_{x_0^2}[u](t, t_0) &= \int_{t_0}^{t} u_0(\tau) E_{x_0}[u](\tau, t_0)\, d\tau \\ &= \int_{t_0}^{t} (\tau - t_0)\, d\tau \\ &= \frac{(t - t_0)^2}{2!}. \end{aligned}$$

$\square$

The following lemmas give some general results.

Lemma 2.4.1 *For every $k \geq 0$,*

$$E_{x_0^k}[u](t,t_0) = \frac{(t-t_0)^k}{k!}, \quad t \geq t_0.$$

Proof: The proof is by induction on word length. The claim is trivial when $k = 0$. As seen previously, $E_{x_0}[u](t,t_0) = t - t_0$ and $E_{x_0{}^2}[u](t,t_0) = \frac{(t-t_0)^2}{2!}$ when $k = 1$ and $k = 2$, respectively. Now assume the identity holds up to a fixed $k \geq 0$. Applying the induction hypothesis gives

$$\begin{aligned} E_{x_0^{k+1}}[u](t,t_0) &= \int_{t_0}^{t} E_{x_0^k}[u](\tau,t_0)\, d\tau \\ &= \int_{t_0}^{t} \frac{(\tau-t_0)^k}{k!}\, d\tau \\ &= \frac{(t-t_0)^{k+1}}{(k+1)!}. \end{aligned}$$

Therefore, the identity holds for all $k \geq 0$. ■

Lemma 2.4.2 *For all $k \geq 0$*

$$E_{x_0^k x_1}[u](t) = \int_{t_0}^{t} \frac{(t-\tau)^k}{k!} u_1(\tau_1)\ d\tau.$$

Proof: For brevity, the subscript on u_1 is dropped and without lost of generality $t_0 = 0$. The proof is by induction on k. When $k = 0$ observe by definition

$$E_{x_1}[u](t) = \int_0^t u(\tau)\ d\tau.$$

When $k = 1$,

$$\begin{aligned} E_{x_0 x_1}[u](t) &= \int_0^t E_{x_1}[u](\tau)\ d\tau \\ &= \int_0^t \int_0^{\tau_1} u(\tau_2)\ d\tau_2\, d\tau_1. \end{aligned}$$

Applying integration by parts gives

$$\int_0^t \int_0^{\tau_1} u(\tau_2)\ d\tau_2\, d\tau_1 = \left(\int_0^{\tau_1} u(\tau_2)\ d\tau_2\right) \tau_1 \Big|_0^t - \int_0^t \tau_1 u(\tau_1)\ d\tau_1$$

$$
\begin{aligned}
&= t \int_0^t u(\tau_1) d\tau_1 - \int_0^t \tau_1 u(\tau_1)\ d\tau_1 \\
&= \int_0^t (t-\tau_1) u(\tau_1)\ d\tau_1 \\
&= E_{x_0 x_1}[u](t).
\end{aligned}
$$

Now assume the identity holds up to some fixed $k \geq 0$. In which case, from the induction hypothesis,

$$
\begin{aligned}
E_{x_0{}^{k+1}x_1}[u](t) &= \int_0^t E_{x_0{}^k x_1}[u](\tau)\ d\tau \\
&= \int_0^t \int_0^{\tau_1} \frac{(\tau_1-\tau_2)^k}{k!} u(\tau_2)\ d\tau_2\, d\tau_1.
\end{aligned}
$$

The unit step function $\mathbb{U}(t)$ is used to change the order of integration, i.e.,

$$
\begin{aligned}
E_{x_0{}^{k+1}x_1}[u](t) &= \int_0^t \int_0^t \frac{(\tau_1-\tau_2)^k}{k!} u(\tau_2)\mathbb{U}(\tau_1-\tau_2)\ d\tau_2\, d\tau_1 \\
&= \int_0^t \int_0^t \frac{(\tau_1-\tau_2)^k}{k!} \mathbb{U}(\tau_1-\tau_2)\ d\tau_1\, u(\tau_2)\ d\tau_2 \\
&= \int_0^t \left[\int_{\tau_2}^t \frac{(\tau_1-\tau_2)^k}{k!}\ d\tau_1 \right] u(\tau_2)\ d\tau_2 \\
&= \int_0^t \left. \frac{(\tau_1-\tau_2)^{k+1}}{(k+1)!} \right|_{\tau_2}^{t} u(\tau_2)\ d\tau_2 \\
&= \int_0^t \frac{(t-\tau_2)^{k+1}}{(k+1)!} u(\tau_2)\ d\tau_2.
\end{aligned}
$$

Therefore the identity holds for all $k \geq 0$. ■

2.4.3 Chen-Fliess Series

Given any $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ one can often associate a causal m-input, ℓ-output operator, F_c. First consider the following definition.

Definition 2.4.1 *[10] Given a series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, its corresponding **Chen-Fliess series** is*

$$
y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0). \tag{34}
$$

The series c is called the *generating series* of F_c. In general, the summation above is only formal unless conditions are imposed on c to ensure convergence. For example, if the coefficients of c satisfy the growth bound

$$|(c,\eta)| \leq KM^{|\eta|}(|\eta|!)^s, \quad \forall \eta \in X^* \tag{35}$$

for some real numbers $K, M > 0$ with $s = 1$, then there exists real numbers $R, T > 0$ such that F_c is a mapping from $B_{\mathfrak{p}}^m(R)[t_0, t_0+T]$ into $B_{\mathfrak{p}}^\ell(S)[t_0, t_0+T]$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, +\infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ [21]. The set of all such *locally convergent* series is denoted by $\mathbb{R}_{LC}^\ell\langle\langle X\rangle\rangle$, and F_c is referred to as a *Fliess operator*. If c satisfies the more stringent growth condition where $0 \leq s < 1$, then the series (34) defines an operator from the extended space $L_{\mathfrak{p},e}^m(t_0)$ into $C[t_0, \infty)$, where

$$L_{\mathfrak{p},e}^m(t_0) := \{u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L_{\mathfrak{p}}^m[t_0, t_1], \ \forall t_1 \in (t_0, \infty)\},$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ [39]. In this case, the operator is said to be *globally convergent*, and the set of all such generating series is designated by $\mathbb{R}_{GC}^\ell\langle\langle X\rangle\rangle$. In this case convergence is assured for *any* fixed $T > 0$ and $u \in L_1[t_0, t_0+T]$.

2.5 FINITE DIMENSIONAL LINEAR STATE SPACE REALIZATIONS

Consider a linear time-invariant state space system

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \quad z(t_0) = z_0, \\ y &= Cz(t), \end{aligned} \tag{36}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$. The dimension of the system, n, is assumed to be finite. It is first shown that the input-output map of any such system has a Fliess operator representation whose generating series is globally convergent.

Observe that the state equation can be rewritten in integral form as

$$z(t) = z(t_0) + \int_{t_0}^{t} Az(\tau)\ d\tau + \int_{t_0}^{t} Bu(\tau)\ d\tau.$$

Substituting for $z(t)$ on the right-hand side gives

$$\begin{aligned}
z(t) &= z(t_0) + \int_{t_0}^{t} A\left[z(t_0) + \int_{t_0}^{\tau_2} Az(\tau_1)\ d\tau_1 + \int_{t_0}^{\tau_2} Bu(\tau_1)\ d\tau_1\right]\ d\tau_2 \\
&\qquad + \int_{t_0}^{t} Bu(\tau)\ d\tau \\
&= z(t_0) + Az(t_0)\int_{t_0}^{t} 1\ d\tau + A^2\int_{t_0}^{t}\int_{t_0}^{\tau_2} z(\tau_1)\ d\tau_1\,d\tau_2 \\
&\qquad + AB\int_{t_0}^{t}\int_{t_0}^{\tau_2} u(\tau_1)\ d\tau_1\,d\tau_2 + B\int_{t_0}^{t} u(\tau)\ d\tau.
\end{aligned}$$

Continuing in this way gives the solution to the state equation in the form of a Peano-Baker series

$$\begin{aligned}
z(t) &= z_0 + \sum_{k=1}^{\infty} A^k z_0 \int_{t_0}^{t}\int_{t_0}^{\tau_k}\cdots\int_{t_0}^{\tau_2} 1\ d\tau_1\,d\tau_2\cdots d\tau_k \\
&\qquad + \sum_{k=0}^{\infty} A^k B\int_{t_0}^{t}\int_{t_0}^{\tau_{k+1}}\cdots\int_{t_0}^{\tau_2} u(\tau_1)\ d\tau_1\,d\tau_2\cdots d\tau_{k+1},
\end{aligned}$$

or equivalently,

$$z(t) = \sum_{k=0}^{\infty} A^k z_0 E_{x_0{}^k}[u](t,t_0) + \sum_{k=0}^{\infty} A^k B E_{x_0{}^k x_1}[u](t,t_0). \tag{37}$$

From the output equation it then follows that

$$y(t) = \sum_{k=0}^{\infty} CA^k z_0 E_{x_0{}^k}[u](t,t_0) + \sum_{k=0}^{\infty} CA^k B E_{x_0{}^k x_1}[u](t,t_0). \tag{38}$$

This proves in part the following theorem.

Theorem 2.5.1 *A finite dimensional linear time-invariant system (36) has an input-output given by the Chen-Fliess series*

$$y(t) = \sum_{\eta\in X^*} (c,\eta) E_\eta[u](t,t_0),$$

where

$$(c,\eta) = \begin{cases} CA^k z_0 & : \quad \eta = x_0^k,\ k \geq 0 \\ CA^k B & : \quad \eta = x_0^k x_1,\ k \geq 0 \\ 0 & : \quad \textit{otherwise.} \end{cases}$$

Furthermore, this generating series is globally convergent.

Proof: It only remains to be shown that the generating series satisfies a global convergence growth rate (35) with $0 \leq s < 1$. In particular, the claim is that $s = 0$ is sufficient. Observe that for $\eta = x_0^k$

$$|(c, x_0^k)| = |CA^k z_0| \leq \|C\|\|A^k\|\|z_0\| = \|C\|\|z_0\|\|A\|^{|\eta|}.$$

Thus, global growth bound is satisfied with constants $K_1 = \|C\|\|z_0\|$, $M = \|A\|$, and $s = 0$. Similarly, if $\eta = x_0^k x_1$ and assuming $A \neq 0$, then

$$|(c, x_0^k x_1)| = |CA^k B| \leq \|C\|\|A^k\|\|B\| = \|C\|\|B\|\|A\|^{-1}\|A\|^{|\eta|}.$$

So the corresponding global growth constants in this case are $K_2 = \|C\|\|B\|\|A\|^{-1}$, $M = \|A\|$, and $s = 0$. Therefore, setting $K = \max(K_1, K_2)$ implies that c is globally convergent.

■

The triple (A, B, C) is called a *differential representation* of the generating series c [12]. The underlying semigroup is defined by the mapping $\mathbb{T} : \mathbb{R}^+ \mapsto \exp(At)$, where $\exp(At)$ is the matrix exponential as presented in Examples 2.2.1-2.2.2.

CHAPTER 3

GENERALIZED CHEN-FLIESS SERIES FOR LINEAR DISTRIBUTED SYSTEMS

The goal of this chapter is to introduce a generalized notion of a Chen-Fliess series suitable for representing the input-output map of a linear distributed system. In the first section, the notion of an infinite dimensional linear state space realization is described. The generalization of the Chen-Fliess series is introduced in the next section. The final section describes sufficient conditions for the convergence of such series.

3.1 LINEAR DISTRIBUTED STATE SPACE SYSTEMS

Let A be an infinitesimal generator of a C_0-semigroup defined on a subset $D(A)$ of a Banach space Z. Here the state $z(t) \in Z$ at time $t \in \mathbb{R}^+$ will not be an element in $\mathbb{R}^n$ as in Section 2.5, but rather an element in some function space, for example, $Z = L_2[t_0, t_1]$. In this sense, the state space is not finite dimensional. The notation $z(x,t)$ will be used to indicate the element $z(t)$ in Z evaluated at $x \in [t_0, t_1]$.

Let $u \in L_1([t_0, t_1], U^m)$ for some input space U^m corresponding to an m input system. Consider a bounded linear operator $B : U^m \to Z$. In light of Theorem 2.3.5, the infinite dimensional state equation

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(t_0) = z_0 \tag{39}$$

has at least a unique mild solution on $[t_0, t_1]$. In addition, assume that C is a bounded linear operator from Z to Y, where Y is some suitable output space. The corresponding output equation is

$$y(t) = Cz(t). \tag{40}$$

As is customary in the finite dimensional case, this state space model will be written concisely as the triple (A, B, C).

Example 3.1.1 Reconsider the transport equation (5) when written as the abstract differential equation (33). Here $U = Z = Y = L_2(-\infty, \infty)$ so that u and v coincide as well as z and y. Thus, the state space realization is $(A, B, C) = (-V\, d/dx, I, I)$, where I is the identity map on Z. $\square$

3.2 GENERALIZED CHEN-FLIESS SERIES

In this section, a generalization of the classical Chen-Fliess series described in Section 2.4 is introduced that is capable of describing under certain conditions the input-output map of a linear distributed system.

Let $[t_0, t_1]$ be a subinterval of $\mathbb{R}$ and U a normed linear function space of real-valued functions defined on some interval $[a, b] \subseteq \mathbb{R}$. U is also assumed to be a unital algebra under componentwise multiplication. The unit is $\mathbf{1} : [a, b] \to \mathbb{R} : x \mapsto 1$. Fix an alphabet $X = \{x_0, x_1, \ldots, x_m\}$. Associate with each letter x_i a mapping $u_i : [t_0, t_1] \to U$ which has a well defined integral of the same form. In particular, let $u_0(t) = \mathbf{1}$. Define $u(t) = [u_1(t)\, u_2(t) \cdots u_m(t)]^T \in U^m$. For any $\eta \in X^*$, one can define inductively an iterated integral for a given $u \in L_1([t_0, t_1], U^m)$ by

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

assuming $E_\emptyset = \mathbf{1}$ and provided that each integral is well defined for every $t \in [t_0, t_1]$ and an element in U.

Example 3.2.1 If $X = \{x_0, x_1\}$ and $u_1(x, t) = t \sin x$, then

$$E_{x_0^k}[u](x, t, 0) = \frac{t^k}{k!}$$

$$E_{x_0^k x_1}[u](x,t,0) = \frac{t^{k+1}}{(k+1)!}\sin x$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$. $\square$

Now for any $\eta \in X^*$, let $(c,\eta) : U \to Y(\eta)$ be a linear operator, where $Y(\eta)$ is simply the range of (c,η) on U. Let $L(U,Y)$ denote all the set of all such linear operators with $Y := \bigcup_{\eta \in X^*} Y(\eta)$. Let $c \in L(U,Y)\langle\langle X \rangle\rangle$ denote a formal power series whose coefficients are operator-valued. In this context, consider the following definition.

Definition 3.2.1 *Given a generating series $c \in L(U,Y)\langle\langle X \rangle\rangle$, its corresponding (generalized) Chen-Fliess series is*

$$y(t) = F_c[u](t) = \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t,t_0), \tag{41}$$

where each (c,η) is viewed an operator acting on the element $E_\eta[u](t,t_0) \in U$.

Of course, the summation above can only be viewed as a formal object without providing conditions for convergence in some sense. This issue is addressed in the next subsection.

Example 3.2.2 Classical Chen-Fliess series can be viewed as a special case of the above definition where the operator action is given by scalar multiplication. For example, (c,η) : $C[t_0,t_1] \to C[t_0,t_1] : E_\eta[u](t,t_0) \mapsto c_\eta E_\eta[u](t,t_0)$, where $c_\eta \in \mathbb{R}$ and $C[t_0,t_1]$ denotes the set of real-valued continuous functions on $[t_0,t_1]$. $\square$

Example 3.2.3 Reconsider Example 3.2.1. Suppose that the coefficients of c are the purely multiplicative operator

$$(c, x_0^k) = \left(\frac{\partial}{\partial x}\right)^k z_0(x),$$

where z_0 is a smooth function in U, and the partial differential operator

$$(c, x_0^k x_1) = \left(\frac{\partial}{\partial x}\right)^{k+1}.$$

Assume $z_0(x) = \cos(x)$ and as before $u_1(x,t) = t\sin x$. From Definition 3.2.1, the corresponding Chen-Fliess series is

$$
\begin{aligned}
y(x,t) &= F_c[u](x,t) \\
&= \sum_{k=0}^{\infty}(c, x_0^k)E_{x_0^k}[u](x,t,0) + \sum_{k=0}^{\infty}(c, x_0^k x_1)E_{x_0^k x_1}[u](x,t,0) \\
&= \sum_{k=0}^{\infty}(c, x_0^k)\frac{t^k}{k!} + \sum_{k=0}^{\infty}(c, x_0^k x_1)\frac{t^{k+1}}{(k+1)!}\sin x \\
&= \sum_{k=0}^{\infty}\left(\frac{\partial}{\partial x}\right)^k \cos(x)\frac{t^k}{k!} + \sum_{k=0}^{\infty}\left(\frac{\partial}{\partial x}\right)^{k+1} \frac{t^{k+1}}{(k+1)!}\sin x \\
&= \cos(x) - \sin(x)t - \cos(x)\frac{t^2}{2} + \sin(x)\frac{t^3}{3!} + \cos(x)\frac{t^4}{4!} + \cdots \\
&\quad + t\cos(x) - \sin(x)\frac{t^2}{2} - \cos(x)\frac{t^3}{3!} + \sin(x)\frac{t^4}{4!} + \cos(x)\frac{t^5}{5!} + \cdots \\
&= \cos(x)(1+t) - \sin(x)\left(t + \frac{t^2}{2}\right) - \cos(x)\left(\frac{t^2}{2} + \frac{t^3}{3!}\right) + \sin(x)\left(\frac{t^3}{3!} + \frac{t^4}{4!}\right) \\
&\quad + \cos(x)\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right) + \cdots
\end{aligned}
$$

$\square$

The next theorem is the main result of this thesis.

Theorem 3.2.1 *The state space realization (39)-(40) defines an input-output map with a Chen-Fliess series representation in the sense given in Definition 3.2.1.*

Proof: Without loss of generality, assume $m = 1$ and let $u = u_1$. Using the mild solution described in Definition 2.3.5, one can repeat the inductive process described in Section 2.5 so that the solution to (39) can be written in terms of a Peano-Baker type series

$$
\begin{aligned}
z(t) = z_0 &+ \sum_{k=1}^{\infty} A^k z_0 \int_{t_0}^{t}\int_{t_0}^{\tau_k}\cdots\int_{t_0}^{\tau_2} \mathbf{1}\; d\tau_1\, d\tau_2 \cdots d\tau_k \\
&+ \sum_{k=0}^{\infty} A^k B \int_{t_0}^{t}\int_{t_0}^{\tau_{k+1}}\cdots\int_{t_0}^{\tau_2} u(\tau_1)\; d\tau_1\, d\tau_2 \cdots d\tau_{k+1},
\end{aligned}
$$

or equivalently,

$$z(t) = \sum_{k=0}^{\infty} A^k z_0 E_{x_0^k}[u](t) + \sum_{k=0}^{\infty} A^k B E_{x_0^k x_1}[u](t).$$

Applying (40) gives

$$y(t) = \sum_{k=0}^{\infty} CA^k z_0 E_{x_0^k}[u](t) + \sum_{k=0}^{\infty} CA^k B E_{x_0^k x_1}[u](t).$$

Therefore, the input-output map has the Chen-Fliess series representation

$$y(t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0),$$

where

$$(c, \eta) = \begin{cases} CA^k z_0 & : \ \eta = x_0^k, \ k \geq 0 \\ CA^k B & : \ \eta = x_0^k x_1, \ k \geq 0 \\ \quad 0 & : \ \text{otherwise} \end{cases}$$

and

$$CA^k z_0 : U \longrightarrow Y$$

$$CA^k B : U \longrightarrow Y.$$

Note that the action of *operator* $CA^k z_0$ on U is interpreted as simply function multiplication.

■

Specific examples applying this theorem are given in the next chapter.

3.3 CONVERGENCE CONDITIONS

First a general convergence condition for the Chen-Fliess series (41) is given. This is followed by a convergence condition specific to the linear time-invariant case.

Theorem 3.3.1 *Fix an input space U^m defined on $[a, b]$ and let Y be the corresponding output space. Let $c \in L(U, Y)\langle\langle X \rangle\rangle$ be a formal power series with the property that there* *exist real numbers* $K, M > 0$ *such that for any* $u \in L_1([t_0, t_1], U^m)$

$$|(c,\eta)E_\eta[u](x,t,t_0)| \leq KM^{|\eta|}\frac{1}{|\eta|!}, \quad \forall \eta \in X^*$$

for all $x \in [a, b]$ *and* $t \in [t_0, t_1]$*. Then the Chen-Fliess series (41) converges absolutely and uniformly to an element in* Y*.*

Proof: For a fixed $x \in [a, b]$ and $t \in [t_0, t_1]$, observe

$$\begin{aligned}
|y(x,t)| &= \left|\sum_{\eta \in X^*} (c,\eta)E_\eta[u](x,t,t_0)\right| \\
&\leq \sum_{\eta \in X^*} |(c,\eta)E_\eta[u](x,t,t_0)| \\
&= \sum_{k=0}^{\infty}\sum_{\eta \in X^k} |(c,\eta)E_\eta[u](x,t,t_0)| \\
&\leq \sum_{k=0}^{\infty}\sum_{\eta \in X^k} KM^k\frac{1}{k!} \\
&= K\sum_{k=0}^{\infty}((m+1)M)^k\frac{1}{k!} \\
&= K\exp((m+1)M) \\
&< \infty.
\end{aligned}$$

Therefore, $y(x, t)$ is well defined, and the series defining $F_c[u]$ converges absolutely and uniformly. ■

The condition above in general is quite strong and many practical systems may fail to meet it. But for linear systems this condition can be satisfied as described in the next theorem.

Theorem 3.3.2 *Fix* $u \in L_1([0, T], U^m)$*, where* T *is finite. Let* $c \in L(U, Y)\langle\langle X\rangle\rangle$ *be a generating series with corresponding linear realization* (A, B, C, z_0) *such that* $|CA^k z_0(x)| \leq KM^k$*,* $k \geq 0$ *for every* $x \in [a, b]$ *and* $|CA^k Bu(x, t)| \leq KM^{k+1}$*,* $k \geq 0$ *for all* $x \in [a, b]$*,* $t \in [0, T]$*. Then the corresponding Chen-Fliess series (41) converges absolutely and uniformly.*

Proof: The claim is that Theorem 3.3.1 can be applied under the stated conditions. There are two cases to consider. First suppose that $\eta = x_0^k$, $k \geq 0$. Clearly,

$$\begin{aligned}
|(c, x_0^k)E_{x_0^k}[u](x,t)| &= \left| CA^k z_0(x) \frac{t^k}{k!} \right| \\
&\leq KM^k \frac{T^k}{k!} \\
&= K(MT)^k \frac{1}{k!}.
\end{aligned}$$

Next, suppose that $\eta = x_0^k x_1$, $k \geq 0$. Then

$$\begin{aligned}
|(c, x_0^k x_1)E_{x_0^k x_1}[u](x,t)| &= \left| CA^k B E_{x_0^k x_1}[u](x,t) \right| \\
&= \left| E_{x_0^k x_1}[CA^k Bu](x,t) \right| \\
&\leq E_{x_0^k x_1}[|CA^k Bu|](x,t) \\
&\leq KM^{k+1} E_{x_0^{k+1}}[u](x,t) \\
&\leq KM^{k+1} \frac{T^{k+1}}{(k+1)!} \\
&= K(MT)^{k+1} \frac{1}{(k+1)!}.
\end{aligned}$$

■

Example 3.3.1 Continuing Example 3.2.3, the identified Chen-Fliess series can be shown to converge by applying Theorem 3.3.2. Observe for all $x \in [a,b]$, $t \in [0,T]$, and $k \geq 0$ that

$$|CA^k z_0(x)| = \left| \left(\frac{\partial}{\partial x} \right)^k \cos(x) \right| \leq 1,$$

and

$$|CA^k Bu(x,t)| = \left| \left(\frac{\partial}{\partial x} \right)^{k+1} \sin(x) \right| \leq 1.$$

Thus, the growth constants $K = M = 1$ are applicable and ensure convergence. □

CHAPTER 4

EXAMPLES: FIRST AND SECOND-ORDER LINEAR DISTRIBUTED SYSTEMS

The goal of this chapter is to explicitly compute the generalized Chen-Fliess series representation of the solution of an input-output PDE for a sample of linear distributed systems. The results are then compared against what the classical methods for solving PDEs give. In the first section, the analysis for the transport equation is presented. The next section addresses systems whose input-output equations are second-order, constant coefficient linear PDEs.

4.1 FIRST-ORDER PDES: TRANSPORT EQUATION

Reconsider the transport equation (5) when written as the abstract differential equation (33). The following theorem is really just a special case of Theorem 3.2.1, but the claim will be verified from first principles.

Theorem 4.1.1 *The solution to the transport equation (5) when z_0 and u are smooth functions of x has the Chen-Fliess series representation*

$$y(t) = \sum_{k=0}^{\infty} CA^k z_0 E_{x_0^k}[u](t) + \sum_{k=0}^{\infty} CA^k B E_{x_0{}^k x_1}[u](t),$$

where $(A, B, C) = (-V\,\partial/\partial x, I, I)$ and provided the series converges.

Proof: From linearity, the zero-state and zero-input responses can be checked separately. For the zero-input response, observe

$$y(x,t) = \sum_{k=0}^{\infty} CA^k z_0(x) E_{x_0^k}[u](x,t)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \left(-V\frac{\partial}{\partial x}\right)^k z_0(x)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{\partial^k}{\partial x^k} z_0(x)(-V)^k \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{d^k}{dt^k}(z_0(x-Vt))\bigg|_{t=0} \frac{t^k}{k!} \\
&= z_0(x-Vt),
\end{aligned}
$$

which is the classical solution (4).

For the zero-state response observe

$$
\begin{aligned}
y(x,t) &= \sum_{k=0}^{\infty} CA^k BE_{x_0^k x_1}[u](x,t) \\
&= \sum_{k=0}^{\infty} \left(-V\frac{\partial}{\partial x}\right)^k \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\ d\tau \\
&= \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!}(-V)^k \frac{\partial^k}{\partial x^k} u(x,\tau)\ d\tau \\
&= \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!}\ \frac{d^k}{d(t-\tau)^k} u(x-V(t-\tau),\tau)\bigg|_{t-\tau=0}\ d\tau \\
&= \int_0^t u(x-V(t-\tau),\tau)\ d\tau,
\end{aligned}
$$

which is the classical solution (6) when $z_0(x) = 0$. ■

Example 4.1.1 Revisit Example 2.1.1, where $y(x,0) = z_0(x) = ae^{bx}$ for some fixed $a, b \in \mathbb{R}$.

From Theorem 4.1.1, observe

$$
\begin{aligned}
y(x,t) &= \sum_{k=0}^{\infty} CA^k z_0(x) E_{x_0{}^k}[u](x,t) \\
&= \sum_{k=0}^{\infty} \left(-V\frac{\partial}{\partial x}\right)^k (ae^{-bx})\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} a(-V)^k \left(\frac{\partial^k}{\partial x^k} e^{-bx}\right) \frac{t^k}{k!}.
\end{aligned}
$$

For the first few values of k:

$$
k = 0 : a(-V)^0 \left(\frac{\partial^0}{\partial x^0} e^{-bx}\right) \frac{t^0}{0!} = ae^{-bx}
$$

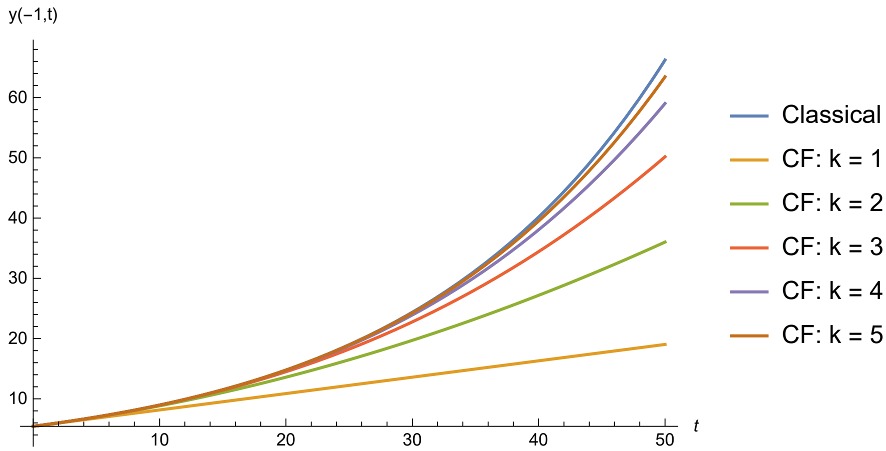

Fig. 1: Example 2.1.1 and 4.1.1 Output Comparison at $x = -1$

$$
\begin{aligned}
k = 1 &: a(-V)^1 \left(\frac{\partial^1}{\partial x^1} e^{-bx} \right) \frac{t^1}{1!} = aVbe^{-bx}t \\
k = 2 &: a(-V)^2 \left(\frac{\partial^2}{\partial x^2} e^{-bx} \right) \frac{t^2}{2!} = a(Vb)^2 ae^{-x}\frac{t^2}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(x,t) &= ae^{-bx} + aVbe^{-bx}t + a(Vb)^2 e^{-bx}\frac{t^2}{2} + \cdots + a(Vb)^k e^{-bx}\frac{t^k}{k!} + \cdots \\
&= ae^{-b(x-Vt)}\big|_{t=0} + \frac{d}{dt}\left(ae^{-b(x-Vt)}\right)\bigg|_{t=0} \frac{t}{1!} + \frac{d^2}{dt^2}\left(ae^{-b(x-Vt)}\right)\bigg|_{t=0} \frac{t^2}{2!} + \cdots \\
&\quad + \frac{d^k}{dt^k}\left(ae^{-b(x-Vt)}\right)\bigg|_{t=0} \frac{t^k}{k!} + \cdots \\
&= ae^{-b(x-Vt)},
\end{aligned}
$$

which agrees with the classical solution. In the context of the convergence criterion given in Theorem 3.3.2, the growth constants are clearly $K = |a|$ and $M = |Vb|$. A plot of the classical solution using Mathematica's `DSolve` command along with Chen-Fliess series approximations involving k terms is shown in Figs. 1 and 2 when the spatial parameter x is fixed at $x = -1$ and when the temporal parameter t is fixed at $t = 25$, respectively. The constants are set as $V = 0.05$, $a = 2$ and $b = -1$ for this computation. Note that the Chen-Fliess series approximations uniformly approach the classical solution as k increases.

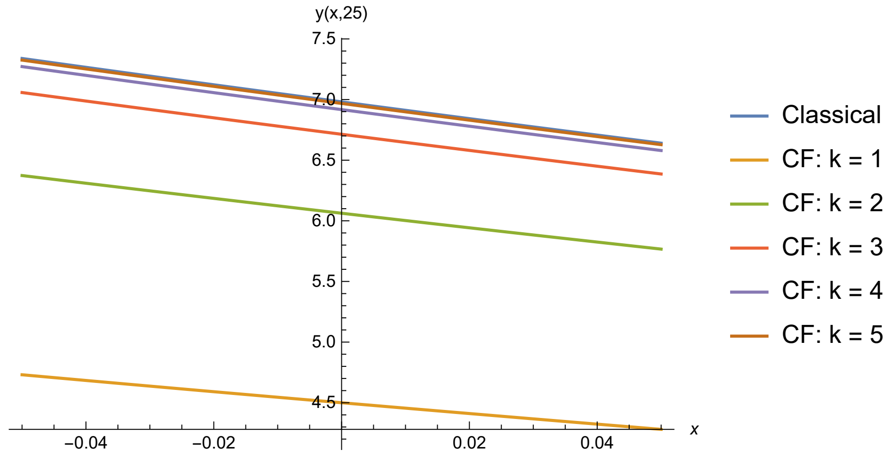

Fig. 2: Example 2.1.1 and 4.1.1 Output Comparison at $t = 25$

$\square$

Example 4.1.2 Reconsider Example 2.1.2, where

$$z_0(x) = 0, \quad u(x,t) = ax + bt, \quad a, b \in \mathbb{R}.$$

Applying Theorem 4.1.1 gives

$$\begin{aligned} y(x,t) &= \sum_{k=0}^{\infty} CA^k B E_{x_0{}^k x_1}[u](t) \\ &= \sum_{k=0}^{\infty} \left(-V\frac{\partial}{\partial x}\right)^k E_{x_0{}^k x_1}[u](x,t) \\ &= \sum_{k=0}^{\infty} E_{x_0{}^k x_1}\left[\left(-V\frac{\partial}{\partial x}\right)^k u\right](x,t). \end{aligned}$$

Observe

$$\begin{aligned} k = 0 &: \left(-V\frac{\partial}{\partial x}\right)^0 u(x,t) = I(ax+bt) = ax+bt \\ k = 1 &: \left(-V\frac{\partial}{\partial x}\right)^1 u(x,t) = -V\frac{\partial}{\partial x}(ax+bt) = -Va \\ k = 2 &: \left(-V\frac{\partial}{\partial x}\right)^2 u(x,t) = 0, \end{aligned}$$

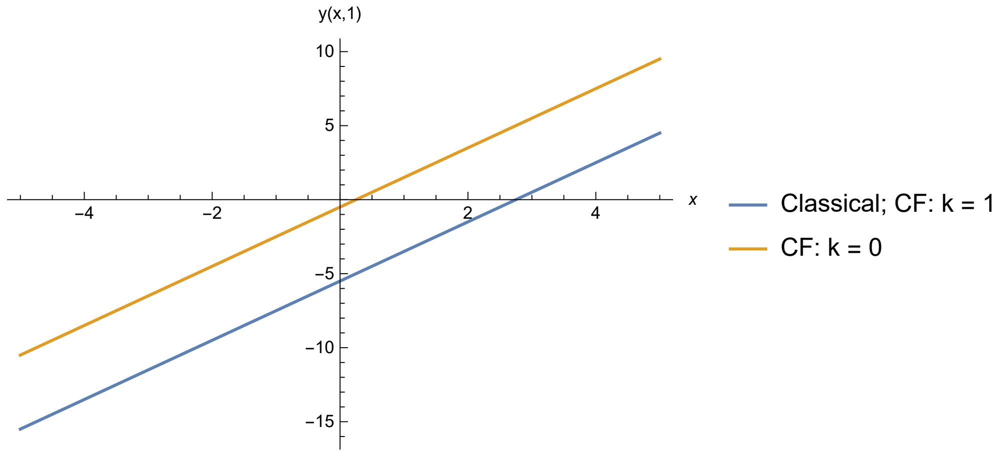

Fig. 3: Example 2.1.2 and 4.1.2 Output Comparison at $t = 1$

so that

$$\begin{aligned} y(x,t) &= E_{x_1}[u](x,t) + E_{z_0 x_1}\left[\left(-V\frac{\partial}{\partial x}\right)u\right](x,t) \\ &= \int_0^t ax + b\tau \; d\tau + \int_0^t \int_0^{\tau_1} -Va \; d\tau_2 \, d\tau_1 \\ &= axt + b\frac{t^2}{2} - Va\frac{t^2}{2}, \end{aligned}$$

which is the classical solution. As the series is finite, there is no convergence issue in this example. As shown in Figs. 3 and 4, the Chen-Fliess solution agrees with the output computed by Mathematica when $k = 1$. □

Example 4.1.3 Reconsider Example 2.1.3, where

$$V = 2, \quad z_0(x) = 0, \quad u(x,t) = t \sin x.$$

From Theorem 4.1.1, the Chen-Fliess series yields

$$y(t) = \frac{1}{2}t^2 \sin(x) - \frac{1}{3}t^3 \cos(x) - \frac{1}{6}t^4 \sin(x) + \frac{1}{15}t^5 \cos(x) + \frac{1}{45}t^6 \sin(x) + \cdots$$

Note that when the Chen-Fliess series is written in its closed-form, it is equivalent to the classical solution. For any fixed $T > 0$, the convergence parameters are $K = T$ and $M = 1$.

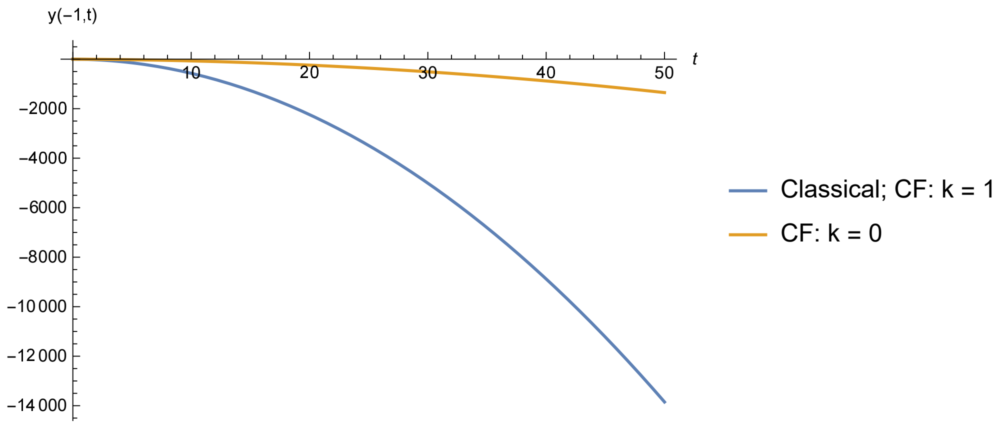

Fig. 4: Example 2.1.2 and 4.1.2 Output Comparison at $x = -1$

Therefore, the series converges uniformly for any $t \in [0, T]$ and $x \in \mathbb{R}$. Fig. 5 compares the Chen-Fliess series approximation of the output against the output computed by Mathematica when $t = 1$. Note the uniform convergence of the series for all $x \in \mathbb{R}$. In contrast, Fig. 6 compares the two outputs when $x = 1$. In this case, there is only uniform convergence over finite intervals of time as expected.

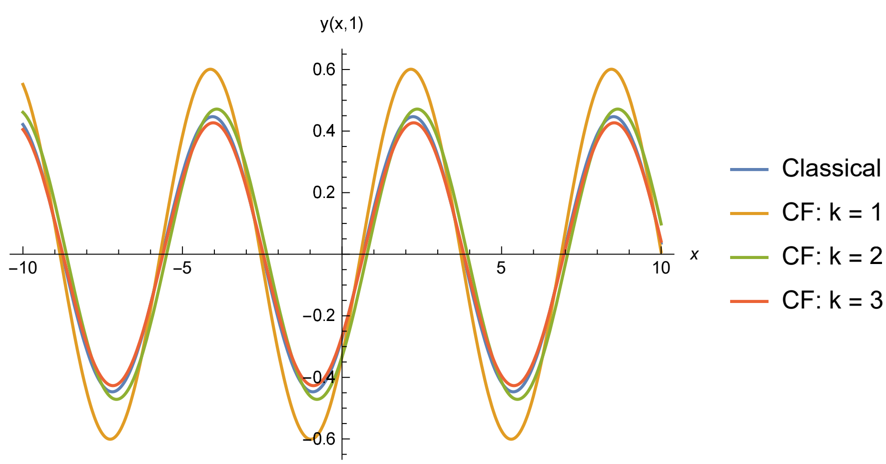

Fig. 5: Example 2.1.3 and 4.1.3 Output Comparison at $t = 1$

$\square$

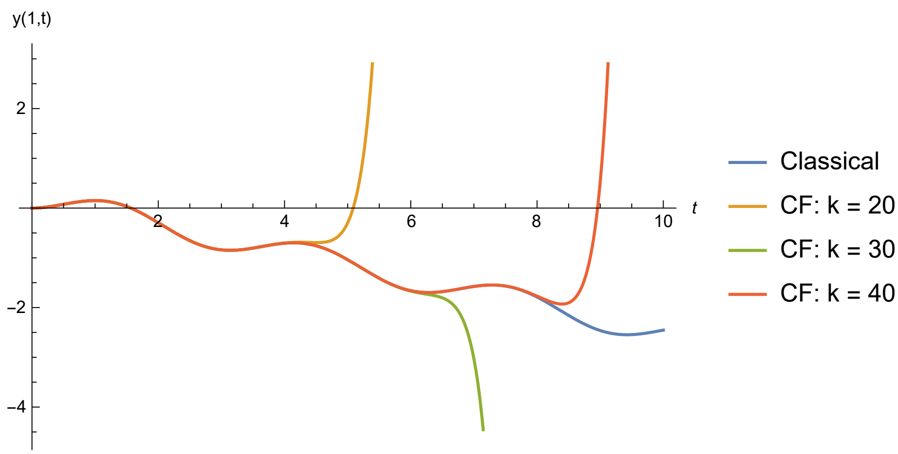

Fig. 6: Example 2.1.3 and 4.1.3 Output Comparison at $x = 1$

4.2 SECOND-ORDER PDES

In this section, Chen-Fliess series representations of the solutions of the second-order partial differential equation

$$a y_{xx}(x,t) + b y_{xt}(x,t) + \hat{c} y_{tt}(x,t) = u(x,t), \qquad y(x,0) = z_0(x). \tag{42}$$

are considered. (Note the slight notation change. Here $\hat{c}$ is used instead of c as in Chapter 2 to avoid the conflict with the symbol for generating series.) The first step is to find a state space realization (A, B, C) for this input-output equation. The problem is split into two cases: $\hat{c} \neq 0$ and $\hat{c} = 0$.

Suppose $\hat{c} \neq 0$ and write (42) in the form

$$y_{tt}(x,t) = -\frac{a}{\hat{c}} y_{xx}(x,t) - \frac{b}{\hat{c}} y_{xt}(x,t) + \frac{1}{\hat{c}} u(x,t), \qquad y(x,0) = z_0(x). \tag{43}$$

Define the state variables $z_1(t) = y(t)$ and $z_2(t) = y_t(t)$ so that

$$\dot{z}_2(t) = -\frac{a}{\hat{c}} \frac{\partial^2}{\partial x^2} z_1(t) - \frac{b}{\hat{c}} \frac{\partial}{\partial x} z_2(t) + \frac{1}{\hat{c}} u(t), \qquad y(x,0) = z_0(x),$$

or equivalently,

$$\dot{z}(t) = \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \frac{1}{\hat{c}} \end{bmatrix} u(t),$$

so as to yield the linear state space realization

$$A := \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix}, \qquad C := \begin{bmatrix} I & 0 \end{bmatrix}. \tag{44}$$

The following theorem then applies.

Theorem 4.2.1 *The solution to the input-output equation (43) when z_0 and u smooth functions of x has the Chen-Fliess series representation*

$$y(t) = \sum_{k=0}^{\infty} CA^k z_0 E_{x_0^k}[u](t) + \sum_{k=0}^{\infty} CA^k B E_{x_0{}^k x_1}[u](t),$$

where

$$(A, B, C) = \left(\begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix}, \begin{bmatrix} I & 0 \end{bmatrix} \right)$$

and provided the series converges.

Proof: Again, this result is a special case of Theorem 3.2.1, but the claim will be verified from first principles. From linearity the proof can be divided into two parts, the zero-input case and the zero-state case.

For the zero-input response, the partial derivatives $y_{tt}(x,t)$ and $y_{xt}(x,t)$ are first computed:

$$y(x,t) = \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^k z_0(x)\frac{t^k}{k!}$$

$$y_t(x,t) = \sum_{k=1}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^k z_0(x)\frac{kt^{k-1}}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} z_0(x)\frac{t^k}{k!}$$

$$y_{tt}(x,t) = \sum_{k=1}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} z_0(x)\frac{kt^{k-1}}{k!}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+2} z_0(x)\frac{t^k}{k!}.$$

Therefore, $y_{tt}(x,t)$ can be expressed in terms of $z(x,t)$ as

$$y_{tt}(x,t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{2} z(x,t).$$

Similarly, $y_{xt}(x,t)$ can also be expressed in terms of $z(x,t)$. Observe

$$y_{xt}(x,t) = \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} z_0(x)\frac{t^k}{k!} \right)$$

$$= \begin{bmatrix} I & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix} z(x,t)$$

$$= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ -\frac{a}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{b}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t).$$

Substituting the above expressions into (42) gives

$$ay_{xx}(x,t) + by_{xt}(x,t) + \hat{c}y_{tt}(x,t) = ay_{xx}(x,t) + b \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ -\frac{a}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{b}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t)$$

$$+ \hat{c} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{2} z(x,t)$$

$$= ay_{xx}(x,t) + \begin{bmatrix} I & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & b\frac{\partial}{\partial x} \\ -\frac{ab}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{b^2}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} \right.$$

$$+\begin{bmatrix} -a\frac{\partial^2}{\partial x^2} & -b\frac{\partial}{\partial x} \\ \frac{ab}{\hat{c}}\frac{\partial^3}{\partial x^3} & (-a+\frac{b^2}{\hat{c}})\frac{\partial^2}{\partial x^2} \end{bmatrix}\Bigg) z(x,t)$$

$$= ay_{xx}(x,t) + \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} -a\frac{\partial^2}{\partial x^2} & 0 \\ 0 & -a\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t)$$

$$= ay_{xx}(x,t) - a\frac{\partial^2}{\partial x^2}\begin{bmatrix} I & 0 \end{bmatrix} z(x,t).$$

Noting that $y(x,t) = Cz(x,t)$ gives

$$\begin{aligned} ay_{xx}(x,t) + by_{xt}(x,t) + \hat{c}y_{tt}(x,t) &= ay_{xx}(x,t) - a\frac{\partial^2}{\partial x^2}Cz(x,t) \\ &= ay_{xx}(x,t) - a\frac{\partial^2}{\partial x^2}y(x,t) \\ &= 0, \end{aligned}$$

as expected.

For the zero-state response, the output $y(x,t)$ is represented by

$$\begin{aligned} y(x,t) &= \sum_{k=0}^{\infty} CA^k BE_{x_0{}^k x_1}[u](t) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^k \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!}u(x,\tau)\,d\tau, \end{aligned}$$

which is now used to find the partial derivatives $y_{tt}(x,t)$ and $y_{xt}(x,t)$ as follows. First observe that

$$\begin{aligned} y_t(x,t) = \frac{\partial}{\partial t}\Bigg(&\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t u(x,\tau)\,d\tau \\ &+ \sum_{k=1}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^k \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!}u(x,\tau)\,d\tau \Bigg) \\ = \sum_{k=1}^{\infty} &\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^k \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{k(t-\tau)^{k-1}}{k!}u(x,\tau)\,d\tau \end{aligned}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\, d\tau$$

and

$$\begin{aligned} y_{tt}(x,t) &= \frac{\partial}{\partial t}\Bigg(\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t u(x,\tau)\, d\tau \\ &\qquad + \sum_{k=1}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\, d\tau \Bigg) \\ &= \frac{1}{\hat{c}}u(x,t) + \sum_{k=1}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{k(t-\tau)^{k-1}}{k!} u(x,\tau)\, d\tau \\ &= \frac{1}{\hat{c}}u(x,t) + \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+2} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\, d\tau. \end{aligned}$$

This last expression can be rewritten in terms of $z(x,t)$ as

$$y_{tt}(x,t) = \frac{1}{\hat{c}}u(x,t) + C \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^2 z(x,t).$$

Computing $y_{xt}(x,t)$ by a similar process yields

$$\begin{aligned} y_{xt}(x,t) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k+1} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\, d\tau \right) \\ &= \frac{\partial}{\partial x} C \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix} \sum_{k=0}^{\infty} \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^{k} \begin{bmatrix} 0 \\ \frac{1}{\hat{c}}I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} u(x,\tau)\, d\tau \\ &= C \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ -\frac{a}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{b}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t). \end{aligned}$$

Again, substituting these expressions into (42) gives

$$
\begin{aligned}
ay_{xx}(x,t) + by_{xt}(x,t) + \hat{c}y_{tt}(x,t) &= ay_{xx}(x,t) + b\left(C \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ -\frac{a}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{b}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t) \right) \\
&\quad + \hat{c}\left(\frac{1}{\hat{c}}u(x,t) + C \begin{bmatrix} 0 & I \\ -\frac{a}{\hat{c}}\frac{\partial^2}{\partial x^2} & -\frac{b}{\hat{c}}\frac{\partial}{\partial x} \end{bmatrix}^2 z(x,t) \right) \\
&= ay_{xx}(x,t) + C \begin{bmatrix} 0 & b\frac{\partial}{\partial x} \\ -\frac{ab}{\hat{c}}\frac{\partial^3}{\partial x^3} & -\frac{bb}{\hat{c}}\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t) + u(x,t) \\
&\quad + C \begin{bmatrix} -a\frac{\partial^2}{\partial x^2} & -b\frac{\partial}{\partial x} \\ \frac{ab}{\hat{c}}\frac{\partial^3}{\partial x^3} & (-a + \frac{b^2}{\hat{c}})\frac{\partial^2}{\partial x^2} \end{bmatrix} z(x,t) \\
&= a\frac{\partial^2}{\partial x^2}y(x,t) - a\frac{\partial^2}{\partial x^2}C \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} z(x,t) + u(x,t) \\
&= a\frac{\partial^2}{\partial x^2}y(x,t) - a\frac{\partial^2}{\partial x^2}Cz(x,t) + u(x,t) \\
&= u(x,t).
\end{aligned}
$$

Thus, the Chen-Fliess series for the zero-state response is established. ■

For the second case where $\hat{c} = 0$, equation (42) simplifies to

$$\dot{z} = -\frac{a}{b}\frac{\partial}{\partial x}z + \frac{1}{b}u,$$

where $z = y_x$. Thus, the corresponding linear state space realization with y_x as the output is

$$A := -\frac{a}{b}\frac{\partial}{\partial x}, \qquad B := \frac{1}{b}I, \qquad C := I.$$

Clearly, y can be computed via direct integration of the solution of the state equation. Hence, this case is very similar to the first-order case treated in the previous section, where now $V = a/b$ in the definition operator A, and B is just scaled by $1/b$. Thus, there is nothing significantly different to consider in this situation.

The following examples apply the above theorem to hyperbolic PDE initial value problems and compare the solutions to the classical solutions.

Example 4.2.1 Reconsider the vibrating string in Example 2.1.6 described by the wave equation

$$y_{tt}(x,t) = y_{xx}(x,t) + \gamma \sin\left(\frac{\pi x}{L}\right), \quad 0 \le x \le L, \quad t \ge 0, \tag{45}$$

where now $\gamma = 1$, $L = 1$, and

$$z_0(z) = \begin{bmatrix} y(x,0) \\ y_t(x,0) \end{bmatrix} = 0, \quad \forall x \in [0,1].$$

This input-output equation corresponds to having the constants $a = -1$, $b = 0$, and $\hat{c} = 1$ in Theorem 4.2.1. Thus, the corresponding Chen-Fliess series for the solution is

$$\begin{aligned} y(x,t) &= \sum_{k=0}^{\infty} CA^k BE_{x_0{}^k x_1}[u](x,t) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ I \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} \sin(\pi x)\, d\tau \\ &= \sin(\pi x) \left[\frac{t^2}{2!} - \pi^2 \frac{t^4}{4!} + \pi^4 \frac{t^6}{6!} - \pi^6 \frac{t^8}{8!} + \cdots \right]. \end{aligned} \tag{46}$$

Note that the closed-form of the Chen-Fliess series is equivalent to its classical solution. Figs. 7 and 8 compare the output computed from Mathematica and various Chen-Fliess approximations with t and then x fixed, respectively. There is clearly uniform convergence spatially but not temporally. This is evident from the separability of the solution (46) in terms of x and t.

Two forms of convergence analysis are presented for this example. First a brute force analysis of the series in (46) using the ratio test. Define

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (-1)^{k+1} \pi^{2(k-1)} \frac{t^{2k}}{(2k)!}.$$

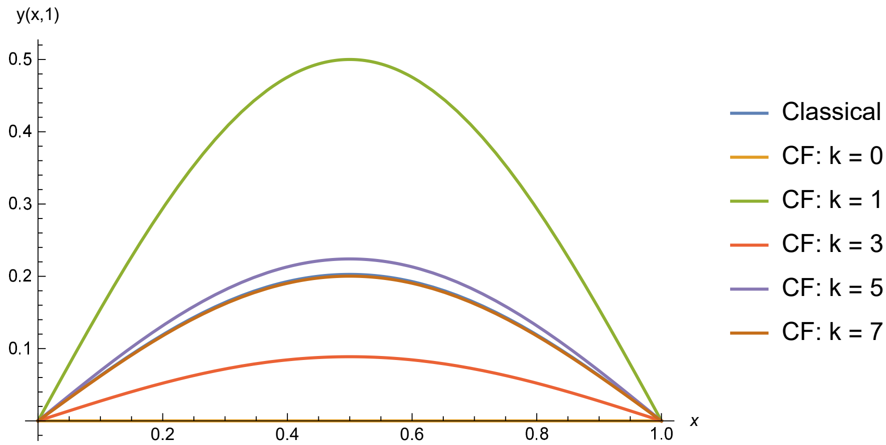

Fig. 7: Example 2.1.6 and 4.2.1 Output Comparison at $t = 1$

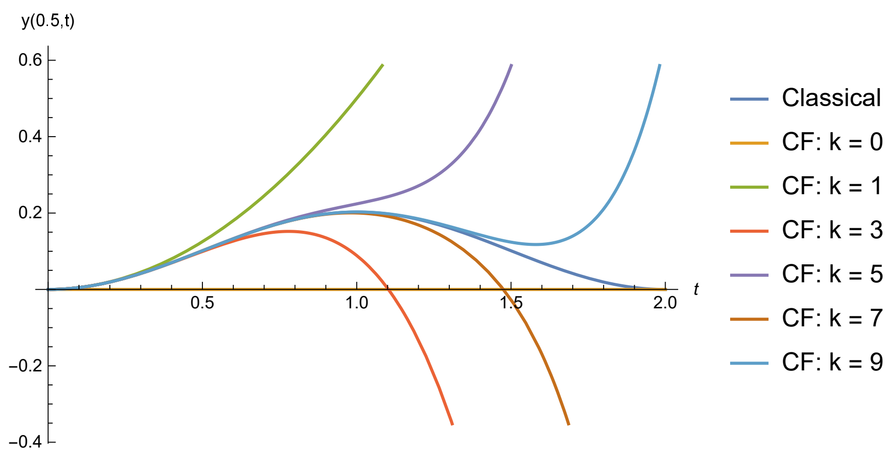

Fig. 8: Example 2.1.6 and 4.2.1 Output Comparison at $x = 0.5$

The ratio test says that the series will converge absolutely if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1.$$

Observe for any finite $t \geq 0$

$$\lim_{k\to\infty}\left|\frac{\pi^{2k}(-1)^{k+2}t^{2k+2}}{(2k+2)!}\frac{(2k)!}{\pi^{2k-2)}(-1)^{k+1}t^{2k}}\right| = \lim_{k\to\infty}\left|\frac{\pi^2 t^2}{(2k+2)(2k+1)}\right|$$
$$= 0 < 1.$$

Thus, the convergence claim is established where (46) converges absolutely for all $x \in [0,1]$ and $t \geq 0$.

The second convergence test is based on Theorem 3.3.2. A straightforward calculation gives

$$CA^kB = \begin{cases} \frac{\partial^j}{\partial x^j} & : \quad k = 2j+1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Therefore,

$$|CA^kBu(x,t)| = \left| \frac{\partial^j}{\partial x^j} \sin(\pi x) \right| \leq \pi^j = \pi^{(k-1)/2} < (\sqrt{\pi})^k.$$

So the growth conditions $K = 1/\sqrt{\pi}$ and $M = \sqrt{\pi}$ apply so that convergence is assured for all $x \in [0,1]$ and any $t \in [0,T]$ with T finite. □

Example 4.2.2 Consider the previous example with an input $u(x,t) = \exp(-t)\sin(\pi x)$ that depends now on both x and t. The classical solution to

$$y_{tt}(x,t) = y_{xx}(x,t) + e^{-t}\sin(\pi x),$$

where $0 \leq x \leq 1$ and $t \geq 0$ and with zero boundary conditions is computed by Mathematica to be

$$y(x,t) = \frac{e^{-t}\pi - \pi\cos(\pi t) + \sin(\pi t)}{\pi + \pi^3}\sin(\pi x).$$

Applying Theorem 4.2.1 with the same state space realization as in the previous example gives the Chen-Fliess series representation

$$\begin{aligned} y(x,t) &= \sum_{k=0}^{\infty} CA^kBE_{x_0{}^k x_1}[u](t) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_0^t \frac{(t-\tau)^k}{k!} e^{-t}\sin(\pi x)\, d\tau \\ &= \sin(\pi x)\left[\frac{t^2}{2!} - \frac{t^3}{3!} + \frac{(1-\pi^2)t^4}{4!} + \frac{(-\pi+\pi^5)t^5}{(\pi+\pi^3)5!} + \frac{(\pi+\pi^7)t^6}{(\pi+\pi^3)6!} + \cdots\right]. \end{aligned}$$

Again, when the Chen-Fliess series is written in its closed-form, it is equivalent to its classical solution. Figs. 9 and 10 compare the output computed from Mathematica and various Chen-Fliess approximations. Note that the convergence for the fixed x case is better behaved compared to the previous example because $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$.

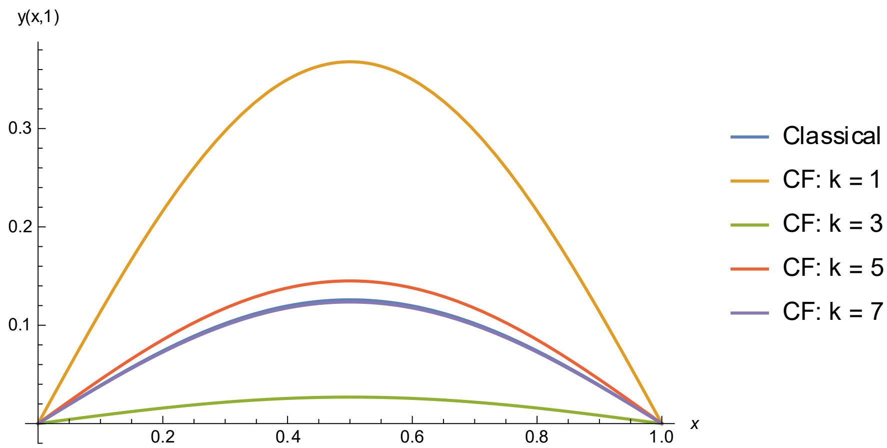

Fig. 9: Example 4.2.2 Output Comparison at $t = 1$

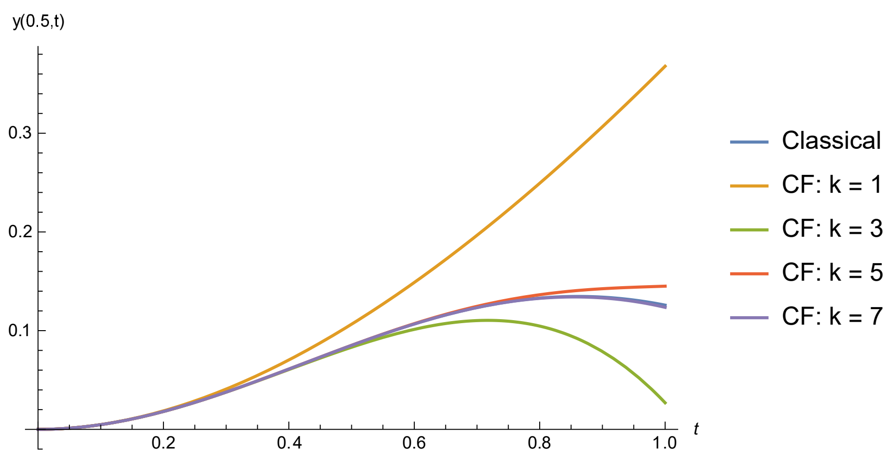

Fig. 10: Example 4.2.2 Output Comparison at $x = 0.5$

Convergence is again verified by two different methods. The first method begins by applying the comparison test for convergence. It states that if for a given series $\sum_{k=0}^{\infty} a_k$, there exists a second series $\sum_{k=0}^{\infty} b_k$ such that $0 \leq |a_k| \leq |b_k|$ for all k and $\sum_{k=0}^{\infty} b_k$ converges, then the series $\sum_{k=0}^{\infty} a_k$ must also converge. The series $\sum_{k=0}^{\infty} a_k$ appearing in the solution has the form

$$a_k = \frac{\pi(-1)^k + \pi^k(-\pi\cos(\frac{k\pi}{2}) + \sin(\frac{k\pi}{2}))}{\pi(1+\pi^2)k!} t^k, \quad \forall k \geq 2.$$

The dominating series b_k is selected to be

$$b_k = \frac{1+\pi^k}{(1+\pi^2)k!} t^k, \quad k \geq 2.$$

Clearly,

$$|a_k| \leq |b_k|, \quad \forall k \geq 2.$$

By the ratio test, $\sum_{k=2}^{\infty} b_k$ converges absolutely for any fixed $t \geq 0$. Therefore, the series appearing in the solution converges absolutely for all $x \in [0,1]$ and any fixed $t \geq 0$.

Alternatively, one can also apply the convergence criterion in Theorem 3.3.2. In fact, since $u(x,t) = e^{-t}\sin(\pi x) \leq \sin(\pi x)$ for all $x \in [0,1]$ and $t \geq 0$, the same growth constants apply as in the previous example. Thus, convergence is assured for all $x \in [0,1]$ and any fixed $t \in [0,T]$ with $T > 0$ fixed. $\square$

CHAPTER 5

CONCLUSIONS

This thesis had three main objectives. The first objective was to provide a new class of Chen-Fliess series capable of describing the input-output map of a linear distributed system. This theory was developed in Chapter 3, where the main innovation was to replace the real coefficients with operator-valued coefficients. Two sufficient conditions for convergence were given, one for the general case, and one for linear systems. The second objective was to demonstrate the method for the class of first-order two-dimensional linear distributed systems and compare the results against classical methods for solving PDEs. This was accomplished in the first section of Chapter 4, where the transport equation was characterized in this setting. There was good agreement between the classical theory and the Chen-Fliess theory. Thus, the method was validated in this context. The final objective was to demonstrate the method for the class of second-order two-dimensional linear distributed systems and compare the results against classical methods for solving PDEs. This analysis was presented in the second section of Chapter 4. Again, there was good agreement between the classical methods and the Chen-Fliess series approach. Thus, in the linear setting, the new Chen-Fliess theory provides a way to explicitly describe the input-output behavior of a distributed system.

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