Summer 1994

Development of Vibration and Sensitivity Analysis Capability Using the Theory of Structural Variations

Ting-Yu Rong
Old Dominion University

DEVELOPMENT OF VIBRATION AND SENSITIVITY ANALYSIS CAPABILITY USING THE THEORY OF STRUCTURAL VARIATIONS

by

Ting-Yu Rong

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirement for the Degree of

DOCTOR OF PHILOSOPHY

ENGINEERING MECHANICS

OLD DOMINION UNIVERSITY

August, 1994

Approved by

Gene J. W. Hou (Director)

Chuh Mei

Leon R. L. Wang

Stephen G. Cupschalk

John E. Kroll

ABSTRACT

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Director: Dr. Gene J. W. Hou

In the author's previous work entitled "General Theorems of Topological Variations of Elastic Structures and the Method of Topological Variation," 1985, some interesting properties of skeletal structures have been discovered. These properties have been described as five theorems and synthesized as a theory, called the theory of structural variations (TSV). Based upon this theory, an innovative analysis tool, called the structural variation method (SVM), has been derived for static analysis of skeletal structures (one-dimensional finite element systems).

The objective of this dissertation research is to extend TSV and SVM from one-dimensional finite element systems to multi-dimensional ones and from statics to vibration and sensitivity analysis. Meanwhile, four new interesting and useful properties of finite element systems are also revealed. One of them is stated as the Gradient Orthogonality

Theorem of Basic Displacements, based upon which a set of explicit formulations are derived for design sensitivities of displacements, internal forces, stresses and even the inverse of the global stiffness matrix of a statically loaded structure. The other three new properties are described as the Evaluation Theorem of Principal Z-Deformations, the Monotonousness Theorem of Principal Z-Deformations and the Equivalence Theorem of Basic Displacement Vectors and Eigenvectors, based upon which a new approach, called the Z-deformation method, is developed for vibration analysis of finite element systems. This method is superior to the commonly used inverse power iteration method when adjacent eigenvalues are close. Explicit formulations for eigenpair sensitivities are also derived in accordance with the Z-deformation method.

The distinct feature of TSV and SVM is that the analysis results for a loaded structure can be obtained without any matrix assembling and inverse operations. This feature gives TSV and SVM an edge over the traditional finite element analysis in many engineering applications, where the repeated analysis is required, such as structural optimization, reliability analysis, elastic-plastic analysis, vibration, contact problems, crack propagation in solids.

ACKNOWLEDGEMENTS

First of all, the author would like to express his gratitude to his adviser, Dr. Gene J-W. Hou for his support throughout this research. Bedsides, the author is also grateful to Dr. Hou for his cordial invitation for the author to conduct a graduate independent study course of the Theory of Structural Topological Variations at ODU in the Summer, 1989 and for his careful review and detailed comments on this theory. It was with his encouragement that the author participated in the Ph.D. program at ODU and chose "Development of Vibration and Sensitivity Analysis Capability Using the Theory of Structural Variations" as the dissertation research project, which is a great extension to the author's original work, " General Theorems of Topological Variations of Elastic Structures and the Method of Topological Variations ", published in 1985.

The assistance of the remaining committee members and the financial supports provided partly by the National Science Foundation of the United States (NSF DDM-8657917) and the Natural Science Foundation of the People's Republic of China (No. 53978376) are also gratefully acknowledged.

Finally, The author would like to express his warmest appreciation to his parents and wife. Without their encouragement, understanding and moral support, this work would never have been possible.

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NOMENCLATURE AND ABBREVIATIONS

NOMENCLATURE

X_r global coordinates, r=1,2,3

x_r local coordinates, r=1,2,3

$(^{\alpha}_{s})$ the s-th subelement of element α

$(^{R}_{t})$ constraint-subelement / support-subelement at node R, acting in direction t

$(^{\ell}_{r})$ the r-th degree of freedom of node ℓ

n the total number of DOFs of a finite element system

m the total number of elements of a finite element system

p the total number of subelements with positive stiffness moduli

N the total number of mass-subelements of a eigensystem

L length of a beam element

A the cross-section area of a beam element or the area of a 2-D element

E Young's modulus

ν Poisson's ratio

θ angle between the local x_1-axis and the global X_1-axis

$\mathbf{d}^{\alpha}$ displacement vector of the end-nodes of element α in local coordinates

$\mathbf{d}^{\alpha}_{i}$ displacement vector at the end-node i of element α

$\mathbf{f}^{\alpha}$ end-force vector of element α in local coordinates

$\mathbf{f}_i^{\alpha}$ end-force vector at the end i of element α

$\mathbf{k}^{\alpha}$ element stiffness matrix of element α in local coordinates

$\mathbf{K}^{\alpha}$ element stiffness matrix of element α in global coordinates

$\mathbf{T}^{\alpha}$ transformation matrix associated with element α

$\mathbf{T}_0^{\alpha}$ transformation matrix associated with a vector

$\mathbf{K}$ global stiffness matrix of a system

$\mathbf{K}_{\lambda}$ global stiffness matrix of an eigensystem

$\mathbf{P}$ applied load vector of a system

$\mathbf{D}$ displacement vector of a system in global coordinates

$\mathbf{e}_s^{\alpha}$ subelement vector of subelement $({}^{\alpha}_{s})$ in local coordinates

$\mathbf{E}_s^{\alpha}$ subelement vector of subelement $({}^{\alpha}_{s})$ in global coordinates

$\mathbf{W}^{\alpha}$ subelement stiffness modulus matrix of element α, diagonal

$\mathbf{W}_s^{\alpha}$ subelement stiffness modulus of subelement $({}^{\alpha}_{s})$

$\mathbf{k}_s^{\alpha}$ stiffness matrix of subelement $({}^{\alpha}_{s})$ in local coordinates

$\mathbf{K}_s^{\alpha}$ stiffness matrix of subelement $({}^{\alpha}_{s})$ in global coordinates

$\mathbf{h}^{\alpha}$ transfer matrix of element α in local coordinates

$\mathbf{h}_i^{\alpha}$ partition of $\mathbf{h}^{\alpha}$ corresponding to the node i of element α

$\mathbf{H}^{\alpha}$ transfer matrix of element α in global coordinates

$\mathbf{H}_i^{\alpha}$ partition of $\mathbf{H}^{\alpha}$ corresponding to the node i of element α

$\mathbf{H}_A^{\alpha}$ partition of $\mathbf{H}^{\alpha}$ corresponding to the group A of DOFs of a simply supported element α

$\mathbf{H}^{\alpha}_{B}$ partition of $\mathbf{H}^{\alpha}$ corresponding to the group B of DOFs of a simply supported element α

$\mathbf{F}^{\alpha}$ generalized internal force vector (GIF vector) of element α due to an applied load

F^{α}_{s} generalized internal force (GIF) of subelement $\binom{\alpha}{s}$

$\mathbf{Z}^{\alpha}$ Z-deformation vector (ZD vector) of element α due to an applied load

Z^{α}_{s} Z-deformation (ZD) of subelement $\binom{\alpha}{s}$ due to an applied load

$\mathbf{F}$ global GIF vector of a system

$\mathbf{Z}$ global ZD vector of a system

$\mathbf{H}$ global transfer matrix of a system

$\ddot{\mathbf{W}}$ global stiffness modulus matrix of a system, diagonal

$\mathbf{w}$ global stiffness modulus vector of a system

$\mathbf{P}^{\alpha}_{s}$ intrinsic load vector of subelement $\binom{\alpha}{s}$

$\bar{\mathbf{P}}^{\ell}_{r}$ unit-load vector applied at DOF $\binom{\ell}{r}$

$\mathbf{V}^{\alpha}_{s}$ basic displacement vector (BD vector) of subelement $\binom{\alpha}{s}$

$V^{\alpha\ell}_{sr}$ the component of $\mathbf{V}^{\alpha}_{s}$ at DOF $\binom{\ell}{r}$

$\mathbf{V}$ global basic displacement matrix of a system

$\mathbf{V}^{\alpha j}_{\cdot\cdot}$ partition of $\mathbf{V}$ corresponding to element α and the DOFs of node j

$\mathbf{V}^{\alpha B}_{\cdot\cdot}$ partition of $\mathbf{V}$ corresponding to element α and the DOFs of group B of a simply supported element α

$\mathbf{V}^{R}_{t}$ BD vector of the constraint-subelement / support-subelement $\binom{R}{t}$

$\mathring{\mathbf{V}}^{R}_{t}$ auxiliary BD vector of the constraint-subelement / support-subelement $\binom{R}{t}$

$\mathring{\mathbf{V}}^{\alpha}_{s}$ auxiliary BD vector of subelement $\binom{\alpha}{s}$

$\bar{F}^{\alpha\ell}_{\cdot r}$ BIF vector of element α due to the unit-load vector $\bar{\mathbf{P}}^{\ell}_{r}$

$\bar{F}^{\alpha\ell}_{sr}$ BIF of subelement $(^{\alpha}_{s})$ due to the unit-load vector $\bar{\mathbf{P}}^{\ell}_{r}$

m^{α}_{s} variation factor of subelement $(^{\alpha}_{s})$

$Z^{\alpha\beta}_{sr}$ Z-deformation of subelement $(^{\alpha}_{s})$ from the BD vector of $(^{\beta}_{r})$

$\dot{Z}^{\alpha\alpha}_{ss}$ principal Z-deformation from the auxiliary BD vector $\dot{\mathbf{V}}^{\alpha}_{s}$

$\dot{Z}^{RR}_{tt}$ principal Z-deformation of constraint-subelement $(^{R}_{t})$ from the auxiliary BD vector $\dot{\mathbf{V}}^{R}_{t}$

$\mathbf{Z}^{R\alpha}_{t\cdot}$ row vector of the three Z-deformations of $Z^{R\alpha}_{ts}$, s=1,2,3

$\mathbf{Z}^{\beta R}_{\cdot t}$ column vector of the three Z-deformations of $Z^{\beta R}_{rt}$, r=1,2,3

$\eta^{R\alpha}_{ts}$ factor modifying BD vectors due to removing a constraint-subelement $(^{R}_{t})$

$\mathbf{R}^{R}_{t}$ projecting vector on the direction $(^{R}_{t})$

$\mathbf{T}^{\beta}_{t}$ vector transferring GIF vectors to the element nodes and projecting them on the direction t

Ω^{α} matrix for calculating displacements due to adding a 1-D branching or a 2-D simply supported element α

ϵ strain vector of a 2-D element

σ stress vector of a 2-D element

B matrix defining the deformation pattern of a finite element

t thickness of a 2-D element

Q matrix making the elastic matrix **M** diagonal

b design variable vector

b a single design variable

λ_i the i-th eigenvalue

$\mathbf{D}_i$ the i-th eigenvector

$\mathbf{Y}_i$ the i-th normalized eigenvector

M mass matrix of an eigensystem or elastic matrix of a static system

C Kronecker δ in the matrix form

ABBREVIATIONS

FEM —— finite element method

TSV —— theory of structural variations

SVM —— structural variation method

DOF —— degree of freedom

BD —— basic displacement

BIF —— basic internal force

GIF —— generalized internal force

ZD —— Z-deformation

Chapter 1

INTRODUCTION

1.1 Historical Backgrounds

Structural analysis, as a branch of engineering science, has had a history of development for more than 100 years. Many methods have been developed for handling stress analysis, vibration analysis, dynamic analysis, buckling analysis and so on. Generally speaking, these methods may be categorized into three groups: displacement method, force method and their combinations. These approaches were widely investigated in a traditional manner in early years. Later, the advances of computing devices have changed the focus of research to search for numerical solutions with the aid of computers, leading to the booming development of the finite element method [1, 2], which is known as the modern structural analysis or the computer aided structural analysis.

However, neither traditional nor modern approaches can avoid assembling and solving a set of simultaneous equations to obtain the responses of a loaded structure. These approaches are inconvenient for structural modifications. When a large-scale structure undergoes some structural modifications, the system equations need to be reassembled and re-solved, demanding a vast amount of computing time. But structural modifications (variations) are indispensable in many engineering applications, such as structural optimization, structural reliability analysis, elastic-plastic analysis, contact problems, crack propagation in solids and so on. Therefore, there has arisen a challenging problem: is it possible to develop an analysis tool which is free from assembling and solving any simultaneous equations? As a part of this effort, engineering scientists had placed their efforts in the past to facilitating, instead of eliminating, this time-consuming and repeated analysis procedure. Many researchers, e.g., Householder [3], Haley [4], Holnicki [5] and others developed various approaches to alleviate the burden of the reanalysis during the past 40 years. Probably, the most interesting advances in this aspect were made by Majid and his coauthors [6]-[8], which partially avoid reanalysis when a structure undergoes certain sort of structural variations. However, none of the methods mentioned above can completely eliminate the need of assembling and solving the simultaneous equations for structural analysis.

Nevertheless, Rong [9] made a breakthrough in this regard in 1985 by establishing a set of General Theorems of Topological Variations of Elastic Structures, which led to the development of an innovative method, the structural variation method, to directly obtain the displacements and stresses of a loaded structure without the need of assembling and solving any simultaneous equations.

About fifteen years ago, the beauty of Green's function [10] of a differential equation lured the author to think about a new technique to answer the seemingly unanswerable question mentioned above. If the Green's function is available, the solution of the differential equation can be obtained extremely easily for any source term of the equation. In structural mechanics, Green's function is also called the influence function. The influence functions of internal forces in a structure are most useful for structural engineers. In the conventional methods, the calculation of an influence function is actually equivalent to the matrix inverse operation. Nevertheless, the author has found that there is a distinctive relationship between the influence function of an internal force in the structure and the stiffness of the structural element with which the internal force is measured. According to this relationship, if the corresponding stiffness is treated as an external load applied to the structure, it will induce a deflection which is exactly the influence function of the internal force of concern. This stiffness-load was named the **two-point load** [9, 11, 12] for skeletal structures , while in this dissertation, it is called the **intrinsic load** for general finite element systems. Based upon this relationship, the author put forth a new and very efficient method for influence function calculations [11, 12] and won the Prize of Advance in Science and Technology awarded by the Ministry of Railroads of China in 1986. A further investigation has shown that the intrinsic load used for constructing influence functions has many useful features related to the properties of structural systems. These features led to the establishment of **the theory of structural variations** (TSV) [9]. A new concept, called the **subelements** of a structural element was introduced in this theory, which paved the way to the development of a new analysis tool, called **the structural variation method** (SVM). The so-called subelements can be viewed as the downward extension of the conventional finite element concept, playing a key role in the new theory.

1.2 Scope of Study

The essence of TSV and SVM is the construction of the Green's functions (influence functions) of the internal forces in a finite element system without matrix inverse operations. This has been achieved in [9] for static analysis of skeletal structures (1-D finite element systems). The focus of the dissertation is the extension of TSV and SVM from 1-D finite element systems to multi-dimensional ones and from static analysis to vibration analysis. Further, static and vibration design sensitivity analyses based upon TSV and SVM are also developed in this dissertation.

1.3 Dissertation Outline

This dissertation has five major parts, Chapters 2 through 6. Chapter 2 presents a concise review of the early work on the theory of structural variations [9], serving as reference for the further developments. Although this work treats only the skeletal structures, it provides the basic concepts and the fundamental theorems applicable to general finite element systems.

Chapter 3 extends the theory of structural variations for static analysis from skeletal structures to 2-D finite element systems. A general approach to establish subelements for any finite element models is also presented in this chapter.

Chapter 4 discusses explicit formulations for design sensitivities of finite element systems in static analysis. Based on the fundamental theorems given in Chapters 2 and 3, this chapter reveals an additional property of finite element systems, which is summarized as the **Orthogonality Theorem of Basic Displacements.** With this new theorem, a set of explicit formulations for design sensitivities are developed for displacements, internal forces, stresses and even the inverse of the global stiffness matrix of a statically loaded finite element system. Another property of finite element systems, stated as the **Evaluation Theorem of Principal Z-Deformations**, is also proven in this chapter. This theorem is important to the practical applications of SVM.

Chapter 5 extends the theory of structural variations for solving vibration problems of finite element systems. Two more interesting and useful properties of finite element systems, described as the **Monotonousness Theorem of Principal Z-Deformations** and the **Equivalence Theorem of BD Vectors and Eigenvectors**, are proven in this chapter, based upon which a new method, called the **Z-deformation method**, is developed for calculating eigenpairs. This new method is superior to the commonly used power iteration method when the adjacent eigenvalues are close.

Chapter 6 discusses design sensitivities of eigenpairs of finite element systems. It provides a set of explicit formulations for the calculation of eigenpair sensitivities, based upon the developments in Chapters 2-5.

The last chapter, Chapter 7, gives a summary of the dissertation and indicates the future direction for research.

An appendix is attached to this dissertation, summarizing the proofs of the fundamental theorems outlined in Chapter 2.

Chapter 2

FUNDAMENTAL THEOREMS OF THE THEORY OF STRUCTURAL VARIATIONS

Any structure can be described by its configuration, rigidity and support condition. Any change of these, called structural variation, will alter the load-carrying capability of the structure. It is the objective of this study to examine the effect of structural variations on the load-carrying capability of the structure. Among all the possible structural variations, the following three types of elementary structural variations are the most important ones:

Type I. Change the rigidity of an element, and if necessary, reduce it to zero, leading to the removal of the element from the structural system;

Type II. Add a new element to the structural system;

Type III. Add a new constraint (or support) to the structural system, or remove an old one from it.

In fact, through the above three types of structural variations, a simple structure can be extended into a complicated one or vise versa. Therefore, study of these three types of structural variations can be a building block for better understanding of structural analysis and modification. The theory of structural variations, abbreviated as

TSV, has been established in [9] to describe how a structure changes its responses, i.e., displacements, internal forces and stresses, when it is undergoing the cited three types of structural variations.

The theory is established based upon a fresh concept, called the **subelements.** Any structural element or typical finite element can be decomposed into subelements. The concept of the subelements can be also viewed as the downward extension of the usual finite element concept. Through the subelement, one can reveal some interesting intrinsic properties of finite element systems, as stated by five fundamental theorems in this dissertation. These theorems constitute a complete set of explicit formulations sufficient to predict the corresponding responses of any structure undergoing structural variations. In fact, a new analysis tool, called **the structural variation method** (SVM) can be developed based upon the theory of structural variations. This method eliminates the need of assembling and solving simultaneous equations which are indispensable in the commonly used finite element solution procedures. The theory is very promising in many engineering applications, such as structural reanalysis, design sensitivity analysis, structural optimization, reliability, elastic-plastic analysis, contact problems, propagation of cracks in solids, etc. This theory has been initiated for skeletal structures in [9] for static analysis, whereas this dissertation will extend it to vibration analysis and vibration sensitivity analysis.

This chapter gives a short description of the fundamental theorems established in TSV, using the planar beam element system as an illustrative example. Chapter 3 will generalize these theorems to general finite element systems.

2.1 Basic Concepts

Basic concepts and terminology used for the development of the theory of structural variations are introduced here.

2.1.1 Subelements

Consider a beam element system with n nodes and m elements. Use Greek letters α, β, ··· to denote the element number and i, j its end-nodes as shown in Fig. 2.1, where the local coordinates of element α as well as the global coordinates are also indicated. The node i is always treated as the origin of the local coordinates of the beam element throughout the dissertation. The formulations for finite element analysis can be found in any finite element analysis textbook (e.g., Ref. 2):

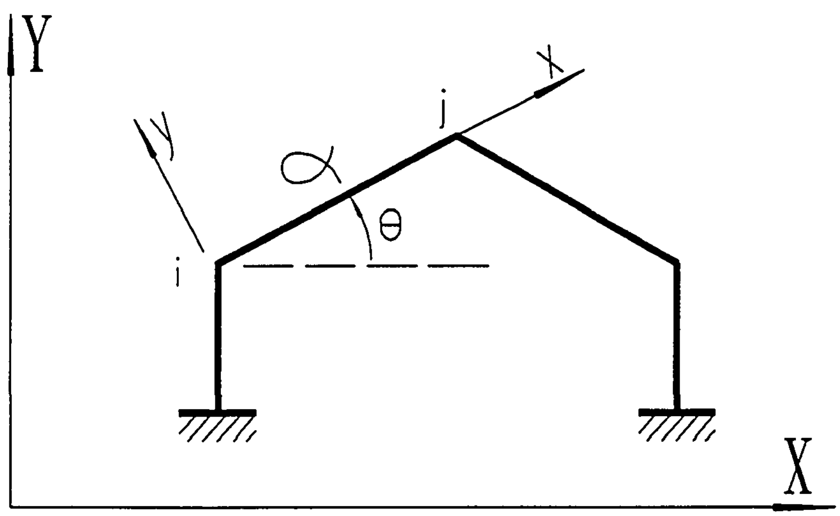

Figure 2.1 A Beam Element System

$$f^{\alpha}=k^{\alpha}d^{\alpha} \tag{2.1}$$

$$K^{\alpha}=(T^{\alpha})^{T}k^{\alpha}T^{\alpha} \tag{2.2}$$

$$K=\sum_{\alpha=1}^{\bar{m}} K^{\alpha}=\sum_{\alpha=1}^{\bar{m}} (T^{\alpha})^{T}k^{\alpha}T^{\alpha} \tag{2.3}$$

$$KD=P \tag{2.4}$$

where $\mathbf{f}^{\alpha}$, $\mathbf{d}^{\alpha}$ (Fig. 2.2) and $\mathbf{k}^{\alpha}$ are the end-force vector, the nodal displacement vector and the element stiffness matrix of element α in local coordinates, respectively, while **P**, **D** and **K** the applied nodal force vector, the nodal displacement vector and the global stiffness matrix of the system in global coordinates, respectively. The superscript T stands for transpose. The symbol $\mathbf{T}^{\alpha}$ denotes the transformation matrix associated with the element α (Fig. 2.1):

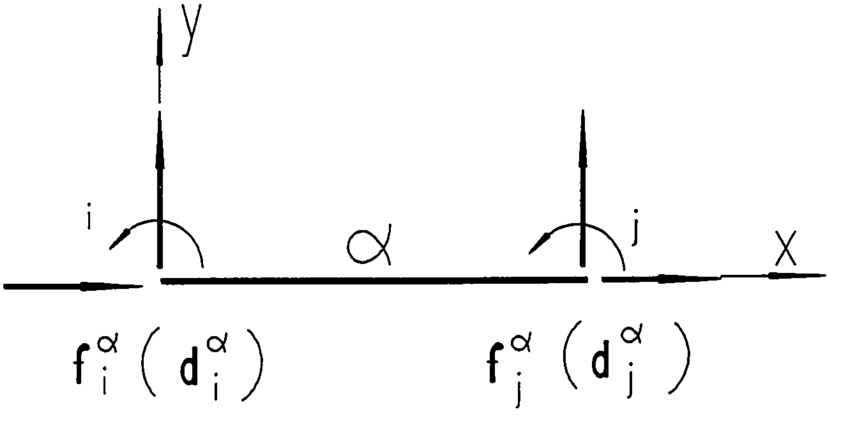

Figure 2.2 A Beam Element

$$T^{\alpha}\equiv\begin{bmatrix} T_0^{\alpha} & 0 \\ 0 & T_0^{\alpha} \end{bmatrix} \tag{2.5a}$$

where $\mathbf{T}_0^{\alpha}$ is the transformation matrix of coordinates:

$$T_0^\alpha \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.5b}$$

and k^α is defined as

$$k^\alpha = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{2.6}$$

where E is Young's modulus, A the cross-section area, I the moment of inertia, L the length of the element and θ the angle between the local x-axis and the global X-axis.

Three special vectors, denoted by e_1^α, e_2^α and e_3^α in local coordinates associated with element α, are introduced here:

$$e_1^\alpha \equiv [\,-1,\ 0,\ 0\ ,1\ ,\ 0\ ,\ 0\ ]^T \tag{2.7a}$$

$$e_2^\alpha \equiv [\ 0,\ 1,\ L/2,\ 0\ ,\ -1,\ L/2\ ]^T \tag{2.7b}$$

$$e_3^\alpha \equiv [\ 0,\ 0,\ -1\ ,\ 0\ ,0\ ,\ 1\ ]^T \tag{2.7c}$$

along with three scalers, denoted by W_1^α, W_2^α and W_3^α, which are defined as

$$W_1^\alpha \equiv \frac{EA}{L}, \qquad W_2^\alpha \equiv \frac{12EI}{L^3}, \qquad W_3^\alpha \equiv \frac{EI}{L}. \tag{2.8}$$

Then, it is easy to prove that

$$\mathbf{k}^{\alpha}=\sum_{s=1}^{3} W_s^{\alpha}\mathbf{e}_s^{\alpha}(\mathbf{e}_s^{\alpha})^T \tag{2.9}$$

or

$$\mathbf{k}^{\alpha}=\sum_{s=1}^{3} \mathbf{k}_s^{\alpha}=\mathbf{h}^{\alpha}\mathbf{W}^{\alpha}(\mathbf{h}^{\alpha})^T \tag{2.10}$$

where

$$\mathbf{k}_s^{\alpha} \equiv W_s^{\alpha}\mathbf{e}_s^{\alpha}(\mathbf{e}_s^{\alpha})^T \tag{2.11}$$

$$\mathbf{h}^{\alpha} \equiv [\ \mathbf{e}_1^{\alpha},\ \mathbf{e}_2^{\alpha},\ \mathbf{e}_3^{\alpha}\ ]^T \tag{2.12}$$

$$\mathbf{W}^{\alpha} \equiv \mathrm{diag}(\ W_1^{\alpha},\ W_2^{\alpha},\ W_3^{\alpha}\ ). \tag{2.13}$$

Therefore, the matrix $\mathbf{k}_s^{\alpha}$ in Eq. (2.11) can be considered as the element stiffness matrix of a subdivided element (having the same length as the parent element α). This subdivided element is called the **subelement** and denoted by the symbol $(^{\alpha}_{s})$, s=1,2,3. The corresponding $\mathbf{e}_s^{\alpha}$ is called the **subelement vector** and W_s^{α} the **subelement stiffness modulus** (or simply modulus) of subelement $(^{\alpha}_{s})$.

In the global coordinate system, the counterparts of $\mathbf{e}_s^{\alpha}$ and $\mathbf{h}^{\alpha}$ are denoted by $\mathbf{E}_s^{\alpha}$ and $\mathbf{H}^{\alpha}$, respectively, and they are related by

$$\mathbf{E}_s^{\alpha}=(\mathbf{T}^{\alpha})^T\mathbf{e}_s^{\alpha} \tag{2.14}$$

$$\mathbf{H}^{\alpha}=(\mathbf{T}^{\alpha})^T\mathbf{h}^{\alpha} \tag{2.15}$$

and therefore, from Eqs. (2.2), (2.10) and (2.15), one has

$$\mathbf{K}^{\alpha}=\mathbf{H}^{\alpha}\mathbf{W}^{\alpha}(\mathbf{H}^{\alpha})^T. \tag{2.16}$$

2.1.2 Generalized Deformations, Internal Forces and Intrinsic loads

Three quantities related to deformation, internal force and a load proportional to the subelement vector are introduced here:

$$\mathbf{Z}^\alpha \equiv [\, Z_1^\alpha,\, Z_2^\alpha,\, Z_3^\alpha \,]^T \equiv (\mathbf{h}^\alpha)^T \mathbf{d}^\alpha = (\mathbf{H}^\alpha)^T \mathbf{D} \tag{2.17}$$

$$\mathbf{F}^\alpha \equiv [\, F_1^\alpha,\, F_2^\alpha,\, F_3^\alpha \,]^T \equiv \mathbf{W}^\alpha \mathbf{Z}^\alpha \tag{2.18}$$

and

$$\mathbf{P}_s^\alpha \equiv W_s^\alpha \mathbf{E}_s^\alpha, \quad s=1,2,3 \tag{2.19}$$

where $\mathbf{Z}^\alpha$ is called the **Z-deformation vector** (ZD vector), $\mathbf{F}^\alpha$ the **generalized internal force vector** (GIF vector) of element α, which has been proven to be the internal forces at the middle section of the beam element [9], and $\mathbf{P}_s^\alpha$ is the **intrinsic load vector** of subelement $(^\alpha_s)$, which was called the two-point load vector in [9]. This load vector is determined as the product of the subelement features $\mathbf{E}_s^\alpha$ and W_s^α only, which does not correspond to any external loading condition of the structural system, but has been proven to be helpful in the development of the theory of structural variations.

Please note that throughout the dissertation, when matrices (or vectors) of different dimensions appear together in an operation, the matrix (or vector) of lower dimension is supposed to be extended to a matrix of the same dimension as the higher one by inserting zero-entries in appropriate locations. For instance, the matrix $(\mathbf{K}^\alpha)_{6x6}$ in $(\mathbf{K})_{3nx3n} = \sum_{\alpha=1}^{m} \mathbf{K}^\alpha$ should be considered to be extended to a matrix $(\mathbf{K}^\alpha)_{3nx3n}$ with some zero-entries inserted in the positions where $(\mathbf{K}^\alpha)_{6x6}$ has no contributions to $\mathbf{K}$; so is the matrix $(\mathbf{H}^\alpha)_{6x3}$ in $\mathbf{Z}^\alpha \equiv (\mathbf{H}^\alpha)^T \mathbf{D}$, where $\mathbf{D}$ is of 3nx1.

From Eqs. (2.1), (2.10), (2.17) and (2.18), one can calculate the nodal forces of element α by using the following formula

$$\mathbf{f}^\alpha = \mathbf{h}^\alpha \mathbf{F}^\alpha. \tag{2.20}$$

Therefore, the matrix $\mathbf{h}^\alpha$ or $\mathbf{H}^\alpha$ can be called the **transfer matrix** of element α. Equation (2.20) may be rewritten in a partition form in accordance with the two end-nodes i and j of element α as:

$$\mathbf{f}^\alpha = \begin{bmatrix} \mathbf{f}_i^\alpha \\ \mathbf{f}_j^\alpha \end{bmatrix} = \mathbf{h}^\alpha \mathbf{F}^\alpha = \begin{bmatrix} \mathbf{h}_i^\alpha \\ \mathbf{h}_j^\alpha \end{bmatrix} \mathbf{F}^\alpha \tag{2.21}$$

where $\mathbf{f}_i^\alpha$ and $\mathbf{f}_j^\alpha$ stand for the end-force vectors at the end-nodes i and j of element α, respectively, and

$$\mathbf{h}_i^\alpha \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L/2 & -1 \end{bmatrix} ; \qquad \mathbf{h}_j^\alpha \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & L/2 & 1 \end{bmatrix} . \tag{2.22}$$

2.1.3 Constraint-Subelements and Support-Subelements

A **constraint-subelement** is a special case of a regular subelement, having the following distinct features:

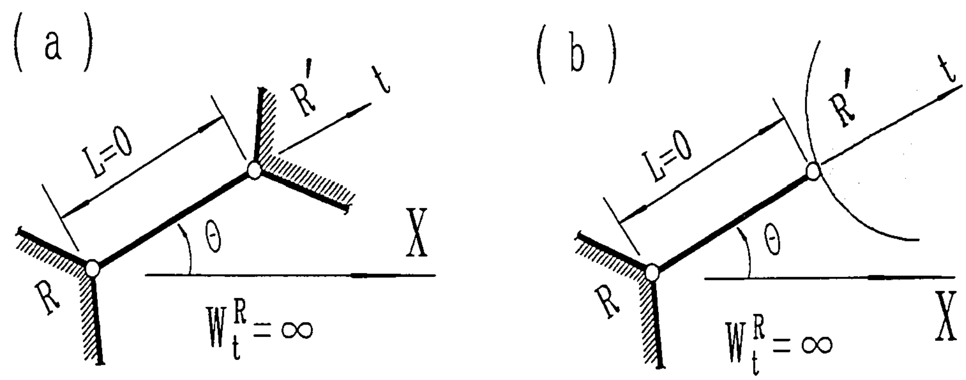

Figure 2.3 Constraint-Subelement / Support-Subelement

(1) A constraint-subelement, denoted by the symbol $\binom{R}{t}$, can connect two nodes R and R$'$ into one in its axial direction t as shown in Fig. 2.3(a);

(2) Its length, L, equals zero, while its stiffness modulus, W_t^R, equals infinite, i.e.,

$$L=0 \quad \text{and} \quad W_t^R=\infty; \tag{2.23}$$

(3) Its subelement vector, e_t^R, in its local coordinates is

$$e_t^R=[\,-1,\ 1\,]^T \tag{2.24}$$

where the values -1 and 1 correspond to the two degrees of freedom of the nodes R and R$'$ in its axial direction t, respectively; in the global coordinates, this subelement vector is symbolized by $\mathbf{E}_t^R$. If node R$'$ in Fig. 2.3(a) is connected to the rigid ground at which the structure is supported, as shown in Fig. 2.3(b), then, the constraint-subelement $\binom{R}{t}$ will function as a support; therefore, in this case it should be called the **support-subelement**, useful to specify a boundary condition.

Note that the constraint-subelement $\binom{R}{t}$ in Fig. 2.3 represents a translational constraint-subelement; if it is a rotational constraint-subelement, then the values -1 and 1 in e_t^R correspond to the rotational degrees of freedom at node R and R$'$, respectively, and the direction t is the z-axis.

Assembling all the subelements, one has from Eq. (2.3)

$$\mathbf{K}=\Sigma\mathbf{K}^{\alpha}=\mathbf{HWH}^T \tag{2.25}$$

where **W** and **H** are the **global stiffness modulus matrix** and the **global transfer matrix**, respectively:

$$\mathbf{W}\equiv\text{diag}(\mathbf{W}^1,\ \mathbf{W}^2,\ \cdots) \tag{2.26}$$

$$\mathbf{H}\equiv[\,\mathbf{H}^1,\ \mathbf{H}^2,\ \cdots\,]. \tag{2.27}$$

It is interesting to see that Eq. (2.25) is quite similar to Eq. (2.16), but standing in the global level.

2.1.4 Basic Displacements and Basic Internal Forces

If a six-component intrinsic load vector $\mathbf{P}^{\alpha}_{s}$ of subelement $\binom{\alpha}{s}$ is placed on the corresponding degrees of freedom of the two nodes of element α, the structural system will deform. The resultant global displacement vector is denoted by $\mathbf{V}^{\alpha}_{s}$ as

$$\mathbf{V}^{\alpha}_{s} \equiv \mathbf{K}^{-1}\mathbf{P}^{\alpha}_{s} \tag{2.28}$$

which is called the **basic displacement vector** of subelement $\binom{\alpha}{s}$. A special quantity pertaining to the Z-deformation of subelement $\binom{\beta}{r}$ is denoted by the symbol $Z^{\beta\alpha}_{rs}$:

$$Z^{\beta\alpha}_{rs} \equiv (\mathbf{E}^{\beta}_{r})^{T}\mathbf{V}^{\alpha}_{s}, \qquad \alpha, \beta = 1,2,\ldots,m; \;\; r, s = 1,2,3. \tag{2.29}$$

where the displacement, **D**, in Eq. (2.17) is substituted by the basic displacement vector, $\mathbf{V}^{\alpha}_{s}$, of subelement $\binom{\alpha}{s}$. In case of $\binom{\alpha}{s} = \binom{\beta}{r}$, $Z^{\alpha\alpha}_{ss}$ is called the **principal Z-deformation.**

The symbol $\binom{\ell}{r}$, r=1,2,3, is used to denote a degree of freedom of a node ℓ, (e.g., r=1 for X-direction, r=2 for Y-direction and r=3 for the rotation about Z-axis, respectively). Note that the symbol $\binom{\ell}{r}$ for a degree of freedom is distinct from the symbol $\binom{\alpha}{s}$ for a subelement in the superscript in Greek.

A unit-load vector is symbolized by $\bar{\mathbf{P}}^{\ell}_{r}$, in which the only non-zero component is placed at $\binom{\ell}{r}$ with a value of 1. The 3x1 generalized internal force vector $\mathbf{F}^{\alpha}$ induced by a unit-load $\bar{\mathbf{P}}^{\ell}_{r}$ is called the **basic internal force vector** of element α and particularly denoted by $\bar{\mathbf{F}}^{\alpha\ell}_{\cdot r} \equiv [\,\bar{F}^{\alpha\ell}_{1r}, \bar{F}^{\alpha\ell}_{2r}, \bar{F}^{\alpha\ell}_{3r}\,]^{T}$ where the components $\bar{F}^{\alpha\ell}_{1r}$, $\bar{F}^{\alpha\ell}_{2r}$ and $\bar{F}^{\alpha\ell}_{3r}$ are related to the subelements of element α. If $\bar{\mathbf{F}}^{\alpha\ell}_{\cdot r}$ is known for all the DOFs, then the generalized internal force vector $\mathbf{F}^{\alpha}$ induced by any external load vector $\mathbf{P}=[\ P_1^1,\ P_2^1,\ P_3^1,\ \cdots,\ P_3^n\ ]^T$ can be calculated by

$$\mathbf{F}^{\alpha}=\sum_{\ell=1}^{n}\sum_{r=1}^{3}\bar{F}^{\alpha\ell}_{\cdot r}P^{\ell}_{r}. \tag{2.30}$$

2.1.5 Additional Explanations on the Notations Used in the Dissertation

At this moment, some explanations should be made to clarify the notations with two columns of subscript and superscript, e.g., $\bar{F}^{\alpha\ell}_{sr}$, $Z^{\alpha\beta}_{sr}$, etc.

(1) A notation with two columns of subscripts and superscripts are needed to indicate a quantity pertaining to both a subelement and a degree of freedom, or involving two subelements. For instance, $\bar{F}^{\alpha\ell}_{sr}$ stands for the basic internal force of subelement $({}^{\alpha}_{s})$ induced by a unit-load vector $\bar{\mathbf{P}}^{\ell}_{r}$ applied at the degree of freedom $({}^{\ell}_{r})$; this case involves one subelement $({}^{\alpha}_{s})$ indicated by the first column of subscript and superscript and one degree of freedom $({}^{\ell}_{r})$ indicated by the second column of subscript and superscript.

Another example for this case is $V^{\alpha\ell}_{sr}$, which stands for the component of the basic displacement vector $\mathbf{V}^{\alpha}_{s}$ of subelement $({}^{\alpha}_{s})$ at the degree of freedom $({}^{\ell}_{r})$. However, one should notice the difference between $\bar{F}^{\alpha\ell}_{sr}$ and $V^{\alpha\ell}_{sr}$; the former is a component of basic internal force vector acting in the subelement $({}^{\alpha}_{s})$, while the latter a component of the basic displacement vector at the degree of freedom $({}^{\ell}_{r})$.

The symbol $Z^{\alpha\beta}_{sr}$ stands for the Z-deformation of the subelement $({}^{\alpha}_{s})$ indicated by the first column of subscript and superscript due to the basis displacement vector of the subelement $({}^{\beta}_{r})$ indicated by the second column of subscript and superscript. Note, as indicated in Subsection 2.1.4, that the second columns of subscript and superscript, ${}^{\ell}_{r}$ and

$^{\beta}_{r}$, in the notations $\bar{F}^{\alpha\ell}_{sr}$ and $Z^{\alpha\beta}_{sr}$, respectively, have different meanings because $(^{\beta}_{r})$ with the Greek letter β stands for a subelement, while $(^{\ell}_{r})$ for a degree of freedom.

(2) Since each beam element α has three subelements $(^{\alpha}_{s})$, s=1,2,3, the three basic internal forces of these subelements, $\bar{F}^{\alpha\ell}_{sr}$, s=1,2,3, constitute a basic internal force vector of the element α induced by the unit-load vector $\bar{P}^{\ell}_{r}$. This vector is denoted by the symbol $\bar{F}^{\alpha\ell}_{\cdot r}$, 3x1, where the dot " . " under the Greek letter α stands for nothing but a space filler to hold the first and the second columns of subscript and superscript in their proper places. Likewise, the three components $V^{\alpha\ell}_{sr}$ of element α, s=1,2,3, at $(^{\ell}_{r})$, constitute a 3x1 vector denoted by $V^{\alpha\ell}_{\cdot r}$.

Another example for the notation with a dot is $Z^{\alpha\beta}_{\cdot r}$, representing the 3x1 Z-deformation vector of element α, which consists of the Z-deformations of the three subelements $(^{\alpha}_{s})$, s=1,2,3, due to the basic displacement vector of subelement $(^{\beta}_{r})$.

(3) When the dot " . " is placed as a subscript in the second column of subscript and superscript, e.g., $Z^{\beta\alpha}_{r\cdot}$, it represents a row vector of the three Z-deformations of element $(^{\beta}_{r})$, $Z^{\beta\alpha}_{rs}$, s=1,2,3, due to the three basic displacement vectors V^{α}_{s} of the subelements $(^{\alpha}_{s})$, s=1,2,3, respectively. Therefore, according to this regulation, the notation $Z^{\beta\alpha}_{\cdot\cdot}$ stands for the 3x3 matrix of the vectors $Z^{\beta\alpha}_{\cdot s}$, s=1,2,3, or the three row vectors $Z^{\beta\alpha}_{r\cdot}$, r=1,2,3.

2.2 Fundamental Theorems

The basic concepts introduced in the previous section bring to light some interesting features of finite element systems. These features are collectively stated in five fundamental theorems which constitute a complete set of explicit formulations sufficient to carry out the elementary structural variations of Types I, II and III. These theorems have been proven in [9] for skeletal structures and are outlined here for reference, and their short proofs are also given in the Appendix of this dissertation. The five fundamental theorems will be extended to general finite element systems in Chapter 3.

2.2.1 General Identity Relationships in Finite Element Systems

There are three general identity relationships among the quantities described in the previous sections for finite element systems. These relationships have been established in [9] and stated as three theorems, which are useful for the conventional structural analysis as well as for the theory of structural variations.

Theorem 1. *The Reciprocal Theorem of Basic Displacements and Basic Internal Forces:*

In a finite element system, the value of the component $V^{\alpha\ell}_{sr}$ of the basic displacement vector V^{α}_{s} at the degree of freedom $\binom{\ell}{r}$ is identical to $\bar{F}^{\alpha\ell}_{sr}$, the s^th^ component of the basic internal force vector $\bar{F}^{\alpha\ell}_{\bullet r}$ of element α, i.e.,

$$V^{\alpha\ell}_{sr}=\bar{F}^{\alpha\ell}_{sr} \quad \text{or} \quad V^{\alpha\ell}_{\bullet r}=\bar{F}^{\alpha\ell}_{\bullet r}\,. \tag{2.31}$$

Theorem 1 indicates that V^{α}_{s} is actually the influence coefficient vector of the generalized internal force F^{α}_{s}. The physical meaning of Theorem 1 is as follows. If the intrinsic load vector P^{α}_{s} is applied to the element α, the displacement component at the degree of freedom $\binom{\ell}{r}$ induced by this load will be always numerically equal to the s^{th} force component of the element α induced by a unit-load applied at the degree of freedom $\binom{\ell}{r}$. A pictorial explanation of Theorem 1 is given in Fig. 2.4 where the intrinsic

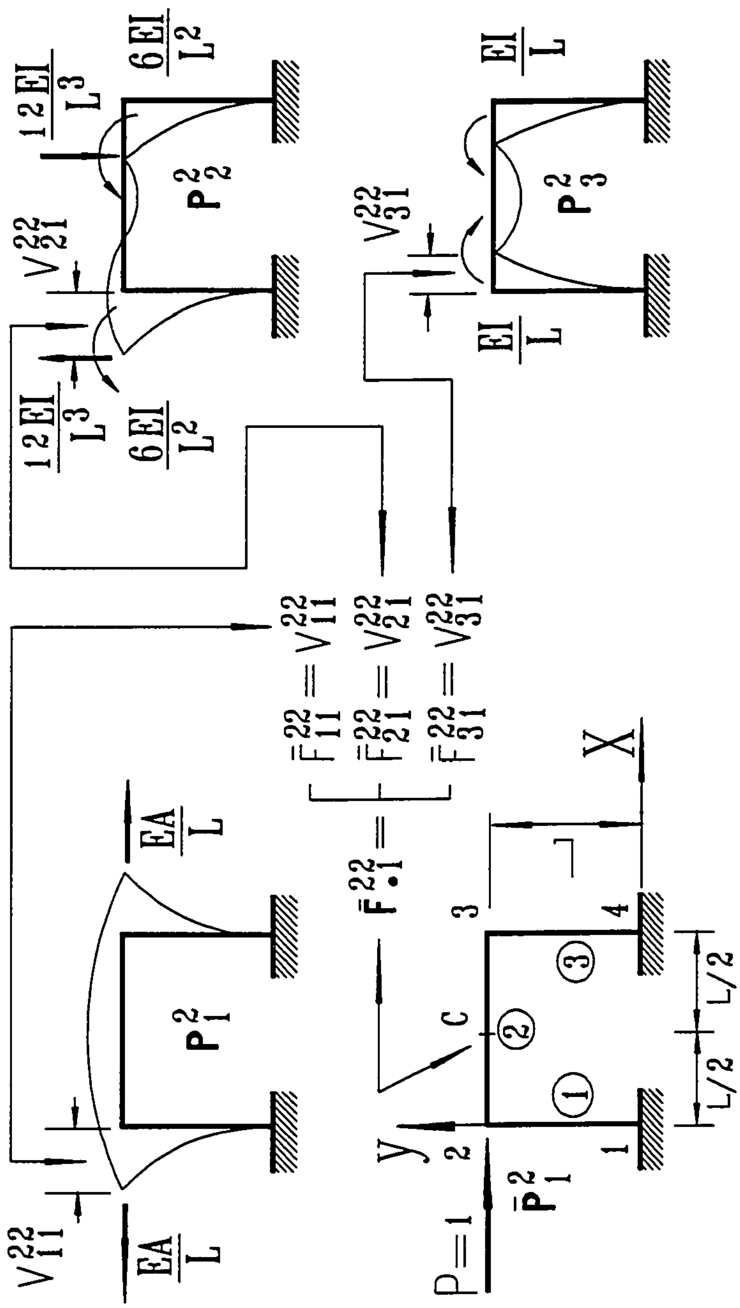

Figure 2.4 Pictorial Statement of Theorem 1

load vectors are applied at element 2, while the unit-load is placed at degree of freedom $\binom{2}{1}$, and the corresponding identities are shown in the middle of the figure.

Therefore, the generalized internal force produced by any external load **P** may be calculated by

$$F_s^\alpha = (\mathbf{V}_s^\alpha)^T \mathbf{P} \text{ or } \mathbf{F}^\alpha = \mathbf{V}^\alpha \mathbf{P} \tag{2.32}$$

where $\mathbf{V}^\alpha$, 3x3*n*, is the matrix of the three row vectors $(\mathbf{V}_s^\alpha)^T$, s=1,2,3. Equation (2.32) can be extended to the entire structure:

$$\mathbf{F} = \mathbf{V}\mathbf{P} = \mathbf{W}\mathbf{Z} \tag{2.33}$$

where Eq. (2.18) has been used; **F**, **V**, **Z** and **W** are the collections of $\mathbf{F}^\alpha$, $\mathbf{V}^\alpha$, $\mathbf{Z}^\alpha$ and $\mathbf{W}^\alpha$, respectively, $\alpha = 1,2,....m$, while **P** is the applied load vector defined in the global coordinate system.

Theorem 2. *The Explicit Decomposition Theorem on the Inverse of the Global Stiffness Matrix*:

The inverse of the global stiffness matrix K of a finite element system can be expressed explicitly in terms of the global basic displacement matrix V and the global diagonal stiffness modulus matrix W, i.e.,

$$\mathbf{K}^{-1} = \mathbf{V}^T \mathbf{W}^{-1} \mathbf{V}. \tag{2.34}$$

Therefore, the displacement vector **D** induced by any external load **P** may be calculated by using any of the following four formulas, as a result of Eqs. (2.33) and (2.34),

$$\mathbf{D} = \mathbf{K}^{-1}\mathbf{P} = \mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\mathbf{P} = \mathbf{V}^T\mathbf{W}^{-1}\mathbf{F} = \mathbf{V}^T\mathbf{Z}. \tag{2.35}$$

Theorem 3. *The Reciprocal Substitution Theorem of Z-deformations*:

In a finite element system, any pair of Z-deformations formed from the basic displacements of any two subelements can be substituted one for another via their stiffness moduli, i.e.,

$$W^{\alpha}_{s}Z^{\alpha\beta}_{sr}=W^{\beta}_{r}Z^{\beta\alpha}_{rs} \quad \text{or} \quad Z^{\beta\alpha}_{rs}=Z^{\alpha\beta}_{sr}W^{\alpha}_{s}/W^{\beta}_{r}\,. \tag{2.36}$$

Theorem 3 is found to be helpful for the discussions of the theorems for the structural variations.

2.2.2 Theorem and Formulation for Structural Variations of Type I

Theorems 1 and 2 indicate that **F** and **D** induced by any external load **P** may be simply calculated via **V**, where **V** is an intrinsic property of the structure and independent of external loads. Therefore, it is possible to obtain the modified responses of a loaded structural system undergoing the structural variations of Types I, II and III by modifying **V** alone. This subsection presents the explicit formulation used to modify **V** when the structure undergoes the structural variations of Type I.

Theorem 4. *The Theorem on the Structural Variations of Type I*:

The new basic displacements of a finite element system subjected to the variation in the stiffness modulus of a subelement $\binom{\alpha}{s}$, $\hat{W}^{\alpha}_{s}=W^{\alpha}_{s}+\Delta W^{\alpha}_{s}$, **are given by**

$$\hat{V}^{\alpha}_{s}=V^{\alpha}_{s}(1+m^{\alpha}_{s})/(1+m^{\alpha}_{s}Z^{\alpha\alpha}_{ss}) \tag{2.37}$$

and

$$\hat{V}^{\beta}_{r}=V^{\beta}_{r}-V^{\alpha}_{s}Z^{\alpha\beta}_{sr}m^{\alpha}_{s}/(1+m^{\alpha}_{s}Z^{\alpha\alpha}_{ss}), \qquad \binom{\beta}{r}\neq\binom{\alpha}{s} \tag{2.38}$$

where $m^{\alpha}_{s}\equiv\Delta W^{\alpha}_{s}/W^{\alpha}_{s}$ is the variation factor of $\binom{\alpha}{s}$; $\hat{V}^{\alpha}_{s}$ and $\hat{V}^{\beta}_{r}$ stand for the new basic displacement vectors of subelements $\binom{\alpha}{s}$ and $\binom{\beta}{r}$, respectively.

Note that hereafter all the new quantities after undergoing structural variations will be denoted by the original symbol with an additional overhead mark " $\wedge$ ".

As indicated by Eq. (2.14), the stiffness modulus W_s^α is a function of the physical properties, E, A and I, of the element α. Consequently, all the three subelements of element α will be altered if the properties E, A and I of element α are changed. In this case, Theorem 4 can be repeatedly applied three times to complete the variations of Type I. Furthermore, setting m_s^α=-1 in Eqs. (2.37) and (2.38) implies the removal of the subelement $(^\alpha_s)$. Therefore, application of Eqs. (2.37) and (2.38) in conjunction with m_s^α=-1 for s=1,2,3 will result in removing the entire element α.

2.2.3 Theorem and Formulation for Structural Variations of Type II

Two types of new elements are considered in the Type II structural variations. The first one is called the **branching element.** This new element is only partially connected to the original structure, as shown in fig. 2.5(a). Thus, this element increases the number of nodal points of the original structure. The second one is called the **connecting element**, as shown in Fig. 2.5(b), which is completely surrounded by the existing structure. Therefore, no new nodes are added to the original structure after adding the element α.

In the case of adding a branching element α with the end-nodes i and j, the basic displacements associated with the added element α itself are:

$$\hat{\mathbf{V}}_{\cdot\cdot}^{\alpha j}=\mathbf{h}_j^\alpha \mathbf{T}_0^\alpha \tag{2.39}$$

$$\hat{\mathbf{V}}_{\cdot\cdot}^{\alpha k}= \mathbf{0}, \qquad k\neq j \tag{2.40}$$

and the basic displacements of element β in the original structure will become

$$\hat{\mathbf{V}}^{\beta j}_{\cdot\cdot} = \mathbf{V}^{\beta i}_{\cdot\cdot}(\Omega^{\alpha})^{T} \qquad (2.41)$$

$$\hat{\mathbf{V}}^{\beta k}_{\cdot\cdot} = \mathbf{V}^{\beta k}_{\cdot\cdot}, \quad k \neq j \qquad (2.42)$$

where

$$\hat{\mathbf{V}}^{\alpha j}_{\cdot\cdot} \equiv [\ \hat{\mathbf{V}}^{\alpha j}_{\cdot 1},\ \hat{\mathbf{V}}^{\alpha j}_{\cdot 2},\ \hat{\mathbf{V}}^{\alpha j}_{\cdot 3}\ ] \qquad (2.43)$$

and likewise for $\hat{\mathbf{V}}^{\alpha k}_{\cdot\cdot}$, $\hat{\mathbf{V}}^{\beta j}_{\cdot\cdot}$, $\mathbf{V}^{\beta i}_{\cdot\cdot}$ and others (see Subsection 2.1.5); and

$$\Omega^{\alpha} \equiv \begin{bmatrix} 1 & 0 & -L\sin\theta \\ 0 & 1 & L\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.44)$$

where L is the length of the element α and θ the angle between the local coordinate, x_1, and the global one, X_1.

Figure 2.5 (a) Branching Beam Element; (b) Connecting Beam Element

In the case when a connecting element α (with the end-nodes i and j) is added to the system, this structural variation can be carried out through the addition of its subelements $\binom{\alpha}{s}$, s=1,2,3. The new basic displacement vector is formulated for the new subelement $\binom{\alpha}{s}$ as

$$\hat{\mathbf{V}}_s^{\alpha}=\mathring{\mathbf{V}}_s^{\alpha}/(1+\mathring{Z}_{ss}^{\alpha\alpha}) \tag{2.45}$$

and for the subelements in the original structure as

$$\hat{\mathbf{V}}_r^{\beta}=\mathbf{V}_r^{\beta}-\mathring{\mathbf{V}}_s^{\alpha}Z_{sr}^{\alpha\beta}/(1+\mathring{Z}_{ss}^{\alpha\alpha}),\quad \binom{\beta}{r}\neq\binom{\alpha}{s} \tag{2.46}$$

where $\mathring{\mathbf{V}}_s^{\alpha}$ is called the **auxiliary basic displacement vector** of subelement $\binom{\alpha}{s}$, defined as

$$\mathring{\mathbf{V}}_s^{\alpha}\equiv\mathbf{K}^{-1}\mathbf{P}_s^{\alpha}=\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\mathbf{P}_s^{\alpha} \tag{2.47}$$

and

$$Z_{sr}^{\alpha\beta}=(\mathbf{E}_s^{\alpha})^T\mathbf{V}_r^{\beta} \text{ and } \mathring{Z}_{ss}^{\alpha\alpha}=(\mathbf{E}_s^{\alpha})^T\mathring{\mathbf{V}}_s^{\alpha}. \tag{2.48}$$

Note that the difference between Eqs. (2.47) and (2.28) is that the effect of the new subelement $\binom{\alpha}{s}$ has not been included in Eq. (2.47), but included in Eq. (2.28).

In short, the theorem pertaining to the structural variations of type II is summarized as follows.

Theorem 5. *The Theorem on the Structural Variations of Type II:*

When a branching element is added to a structural system, the basic displacements remain unchanged except for those associated with the new degrees of freedom of the new element, Eqs. (2.39)-(2.42); if a connecting subelement is added to the structural system, the basic displacements are modified by its auxiliary basic displacement vectors, Eqs. (2.45) and (2.46).

2.2.4 Formulations for Structural Variations of Type III

The Type III structural variations consider the addition as well as the removal of a constraint-subelement / support-subelement $\binom{R}{t}$ between two nodes R and R$'$ along its axial direction t, as shown in Fig. 2.3. The following will discuss them separately.

2.2.4.1 Inserting a Constraint-Subelement / Support-Subelement:

When a constraint-subelement $\binom{R}{t}$ is inserted into a structural system, the basic displacement vector of a subelement $\binom{\beta}{r}$ of the original system is given as:

$$\hat{\mathbf{V}}^{\beta}_{r}=\mathbf{V}^{\beta}_{r}-\mathring{\mathbf{V}}^{R}_{t}Z^{R\beta}_{tr}/\dot{Z}^{RR}_{tt}\ ,\quad \beta=1,2,\ldots,m;\ \ r=1,2,3. \tag{2.49a}$$

or in a 3x3n matrix form

$$\hat{\mathbf{V}}^{\beta}=\mathbf{V}^{\beta}-(Z^{R\beta}_{t\bullet})^{T}(\mathring{\mathbf{V}}^{R}_{t})^{T}/\dot{Z}^{RR}_{tt} \tag{2.49b}$$

where $\quad \hat{\mathbf{V}}^{\beta}=\mathbf{V}^{\beta}-(Z^{R\beta}_{t\bullet})^{T}(\mathring{\mathbf{V}}^{R}_{t})^{T}/\dot{Z}^{RR}_{tt}$

where $\mathring{\mathbf{V}}^{R}_{t}$ is the **auxiliary basic displacement vector** of $\binom{R}{t}$: (2.50)

which can be readily obtained from Eq. (2.35), while $Z^{R\beta}_{tr}=(\mathbf{E}^{R}_{t})^{T}\mathbf{V}^{\beta}_{r}$, $\dot{Z}^{RR}_{tt}=(\mathbf{E}^{R}_{t})^{T}\mathring{\mathbf{V}}^{R}_{t}$ and $Z^{R\beta}_{t\bullet}=(\mathbf{E}^{R}_{t})^{T}(\mathbf{V}^{\beta})^{T}$.

2.2.4.2 Removing a Constraint-Subelement / Support-Subelement:

Upon removing a constraint-subelement $\binom{R}{t}$ from a structural system, the new basic displacements of a subelement $\binom{\alpha}{s}$ in the original structure becomes:

$$\hat{\mathbf{V}}^{\alpha}_{s}=\mathbf{V}^{\alpha}_{s}+\mathbf{V}^{R}_{t}\eta^{R\alpha}_{ts} \tag{2.51}$$

where

$$\eta^{R\alpha}_{ts}\equiv W^{\alpha}_{s}Z^{\alpha R}_{st}\,/\,\Big(\sum_{\beta=1}^{q}\,(\mathbf{T}^{\beta}_{t})^{T}\mathbf{W}^{\beta}\mathbf{Z}^{\beta R}_{\bullet t}\Big) \tag{2.52}$$

$$\mathbf{T}^{\beta}_{t}\equiv-(\mathbf{h}^{\beta}_{R})^{T}\mathbf{R}^{R}_{t} \tag{2.53}$$

$$\mathbf{R}^{R}_{t}\equiv[\ \cos\theta,\ \sin\theta,\ \ 0\ ]^{T}\quad \text{for a translational } \binom{R}{t} \tag{2.54a}$$

or $\quad \mathbf{R}^R_t \equiv [\ 0\ ,\ 0\ ,\ 1\ ]^T \quad$ for a rotational $\binom{R}{t}$ (2.54b)

and, $\mathbf{V}^{\alpha}_{s}$ and $\mathbf{V}^{R}_{t}$ are the original basic displacement vectors of $\binom{\alpha}{s}$ and $\binom{R}{t}$, respectively; q is the total number of the elements connected to the support node R; $\mathbf{R}^R_t$ is the projecting vector and θ the angle between $\binom{R}{t}$ and X-axis (see Fig. 2.3); $(\mathbf{H}^{\beta}_{R})_{3X3}$ is the partition of $\mathbf{H}^{\beta}$ of element β, associated with the node R (see Eq. (2.22)) and $\mathbf{Z}^{\beta R}_{\cdot t}=(\mathbf{H}^{\beta})^T\mathbf{V}^R_t$ is the Z-deformation vector of element β from $\mathbf{V}^R_t$.

2.3 Structural Variation Method (SVM)

Based upon the concepts and theorems introduced in previous sections, an innovative method for structural analysis is developed. The analysis procedure of the new method may be described as follows. Select an arbitrary element from the structural system to be analyzed and fix one of its ends on the ground. This branching element is treated as the initial structure whose basic displacements can be found from Eq. (2.39). Since elements can be added to the initial structure to build the entire structure of interest, Theorem 5 can be repeatedly used allowing to establish the basic displacements of the entire structure. Subsequently, the support conditions can be modified as needed by using Eqs. (2.49) and (2.51). With **V** being available for the entire structure, one can calculate the **F**, $\mathbf{f}^{\alpha}$ and **D** induced by any applied force **P** readily from Eqs. (2.33), (2.20) and (2.35), without incurring any matrix assembly and inversion. If any structural modifications are needed, the corresponding explicit expressions (Theorems 1−5) can be repeatedly used to generate the new **V** and, consequently, the new responses are obtained without any matrix assembly and inversion, either. Since the new analysis procedure is developed based upon the theory of structural variations, it is called the **structural variation method** (SVM) hereafter. An illustrative example is given here to explain the structural variation method.

Illustrative Example

A simple beam system is shown in Fig. 2.6(d). The beam is descretized into two elements and supported at both ends with different boundary conditions. The lengths of both elements are L. The rigidities of the two elements are EI_1 and EI_2, respectively.

Since the axial degrees of freedom are not involved in the problem calculations, they are ignored in the following derivation for simplicity.

With the given data, the subelement vectors and transfer matrices for elements 1 and 2 are obtained as

$$e_1^1 = e_1^2 = [\,1 \;\; L/2 \;\; -1 \;\; L/2\,]^T; \quad e_2^1 = e_2^2 = [\,0 \;\; -1 \;\; 0 \;\; 1\,]^T$$

$$\mathbf{h}^1 = \begin{bmatrix} \mathbf{h}_1^1 \\ \mathbf{h}_2^1 \end{bmatrix} = \mathbf{h}^2 = \begin{bmatrix} \mathbf{h}_2^2 \\ \mathbf{h}_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ L/2 & -1 \\ \cdots & \cdots \\ -1 & 0 \\ L/2 & 1 \end{bmatrix}$$

where a line of dots is used to partition the matrix according to $\mathbf{h}_2^2$ and $\mathbf{h}_3^2$. Furthermore, the subelement stiffness moduli and the coordinate transformation matrix are obtained as

$$\mathbf{W}^1 = \begin{bmatrix} \dfrac{12EI_1}{L^3} & 0 \\ 0 & \dfrac{EI_1}{L} \end{bmatrix}; \qquad \mathbf{W}^2 = \begin{bmatrix} \dfrac{12EI_2}{L^3} & 0 \\ 0 & \dfrac{EI_2}{L} \end{bmatrix}$$

$$\mathbf{T}_0^1 = \mathbf{T}_0^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The structural analysis of the beam structure subjected to p=1 is processed as follows.

Step 1. Select element 1 as the initial structure, as shown in Fig. 2.6(a), whose basic displacement matrix is given by Eq. (2.39) as

$$\hat{\mathbf{V}}^{12}_{\cdot\cdot} = \mathbf{h}_2^1 \mathbf{T}_0^1 = \begin{bmatrix} -1 & 0 \\ L/2 & 1 \end{bmatrix}.$$

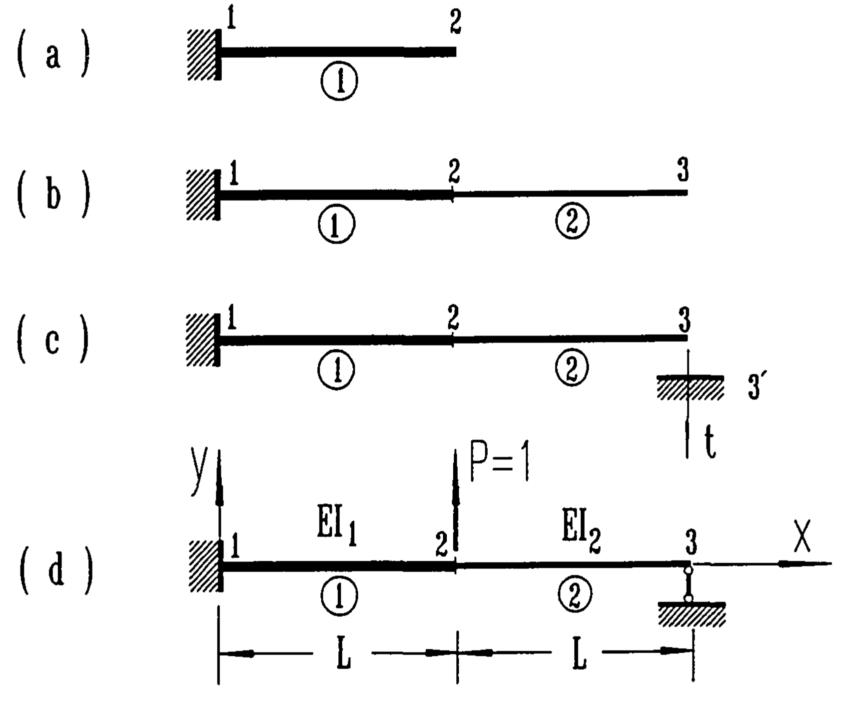

Figure 2.6 Structural Variation Process of a Beam System

Step 2. Add element 2 to the initial structure as a branching element, as shown in Fig. 2.6(b). The basic displacements of the new element are obtained by using Eqs. (2.40), (2.39) and (2.44) as

$$\mathring{V}^{22}_{\cdot\cdot}=[\ 0\ ]; \qquad \mathring{V}^{23}_{\cdot\cdot}=h^2_3T^1_0=\begin{bmatrix} -1 & 0 \\ L/2 & 1 \end{bmatrix}$$

$$\Omega^2=\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

whereas the basic displacements of the old element, element 1, are modified as

$$\mathring{V}^{13}_{\cdot\cdot}=V^{12}_{\cdot\cdot}(\Omega^2)^T=\begin{bmatrix} -1 & 0 \\ 3L/2 & 1 \end{bmatrix}.$$

Therefore, the basic displacement matrix of the new structure shown in Fig. 2.6(b) is

$$V=\begin{bmatrix} V^1 \\ V^2 \end{bmatrix}=\begin{bmatrix} -1 & 0 & \vdots & -1 & 0 \\ L/2 & 1 & \vdots & 3L/2 & 1 \\ 0 & 0 & \vdots & -1 & 0 \\ 0 & 0 & \vdots & L/2 & 1 \end{bmatrix}$$

node 2 node 3

where the line of dots partitions the **V** matrix into two submatrices corresponding to node 2 and node 3 as indicated underneath the matrix. The line of dots is used here for clarification.

Step 3. Add a support-subelement (^R_t) at node 3 as shown in Fig. 2.6(d) to build the final structure. The auxiliary basic displacement vector $\mathring{V}^R_t$ accounting for the insertion of the support-subelement can be found by using Theorem 2 as

$\mathring{V}^R_t = V^T W^{-1} V E^R_t$

$$= \frac{L^3}{EI_2}\begin{bmatrix} -1 & L/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 3L/2 & -1 & L/2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/12\xi & 0 & 0 & 0 \\ 0 & 1/L^2\xi & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & 1/L^2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ L/2 & 1 & 3L/2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & L/2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{L^3}{EI_2}\left[\frac{5}{6\xi},\ \frac{3}{2L\xi},\ \frac{7}{3\xi}+\frac{1}{3},\ \frac{3}{2L\xi}+\frac{1}{2L}\right]^T$$

where $\xi \equiv I_1/I_2$. Therefore, the corresponding Z-deformation, $\mathring{Z}^{RR}_{tt}$, is obtained as

$$\mathring{Z}^{RR}_{tt} = (E^R_t)^T \mathring{V}^R_t = [\,0\ \ 0\ \ 1\ \ 0\,] \begin{bmatrix} \frac{5}{6\xi} \\ \frac{3}{2L\xi} \\ \frac{7}{3\xi}+\frac{1}{3} \\ \frac{3}{2L\xi}+\frac{1}{2L} \end{bmatrix} \frac{L^3}{EI_2} = \frac{(7+\xi)L^3}{3\xi EI_2}$$

whereas $Z^{R1}_{t\bullet}$ and $Z^{R2}_{t\bullet}$ are obtained as

$$Z^{R1}_{t\bullet} = (E^R_t)^T (V^1)^T = [\,-1,\ 3L/2\,]$$

$$Z^{R2}_{t\bullet} = (E^R_t)^T (V^2)^T = [\,-1,\ L/2\,].$$

As a result, the new basic displacements for element 1 can be obtained from Eq. (2.49)

as

$$\mathring{V}^1 = V^1 - (Z^{R1}_{t\bullet})^T (\mathring{V}^R_t)^T / \mathring{Z}^{RR}_{tt} = \frac{1}{7+\xi}\begin{bmatrix} -\frac{(9+2\xi)}{2} & \frac{9}{2L} & \vdots & 0 & \frac{3(3+\xi)}{2L} \\ \frac{(2\xi-1)L}{4} & \frac{(1+4\xi)}{4} & \vdots & 0 & \frac{(1-5\xi)}{4} \\ \multicolumn{2}{c}{\text{node 2}} & & \multicolumn{2}{c}{\text{node 3}} \end{bmatrix}.$$

Similarly, the basic displacement matrix for element 2 can be obtained as

$$\hat{V}^2=V^2-(Z^{R2}_{t\bullet})^T(\mathring{V}^R_t)^T/\mathring{Z}^{RR}_{tt}=\frac{1}{7+\xi}\begin{bmatrix} -\frac{5}{2} & \frac{9}{2L} & \vdots & 0 & \frac{3(3+\xi)}{2L} \\ -\frac{5L}{4} & -\frac{9}{4} & \vdots & 0 & \frac{(19+\xi)}{4} \end{bmatrix}.$$

node 2 node 3

Thus, the final basic displacement matrix of the structure is

$$V_{final}=\begin{bmatrix} V^1 \\ V^2 \end{bmatrix}_{final}=\frac{1}{7+\xi}\begin{bmatrix} -\frac{(9+2\xi)}{2} & \frac{9}{2L} & \vdots & 0 & \frac{3(3+\xi)}{2L} \\ \frac{(2\xi-1)L}{4} & \frac{(1+4\xi)}{4} & \vdots & 0 & \frac{(1-5\xi)}{4} \\ \frac{5}{2} & \frac{9}{2L} & \vdots & 0 & \frac{3(3+\xi)}{2L} \\ -\frac{5L}{4} & -\frac{9}{4} & \vdots & 0 & \frac{19+\xi}{4} \end{bmatrix}.$$

node 2 node 3

Step 4. Subjected to the load, **P**= [1, 0, 0, 0]^T, the generalized internal force vector of the structure is given by Eq. (2.33) as

$$F=VP=\frac{1}{7+\xi}\begin{bmatrix} -(9+2\xi)/2 & 9/2L & 0 & 3(3+\xi)/2L \\ (2\xi-1)L/4 & (1+4\xi)/4 & 0 & (1-5\xi)/4 \\ 5/2 & 9/2L & 0 & 3(3+\xi)/2L \\ -5L/4 & -9/4 & 0 & 19+\xi/4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} -9+2\xi/2 \\ (2\xi-1)L/4 \\ 5/2 \\ -5L/4 \end{bmatrix}\frac{1}{7+\xi}.$$

Step 5. The Z-deformation vector **Z** and the displacement vector **D** are found by using Eqs. (2.33) and (2.35), respectively, as

$$Z=W^{-1}F=\frac{L^3}{EI_2}\begin{bmatrix} 1/12\xi & 0 & 0 & 0 \\ 0 & 1/L^2\xi & 0 & 0 \\ 0 & 0 & 1/12 & 0 \\ 0 & 0 & 0 & 1/L^2 \end{bmatrix}\begin{bmatrix} -(9+2\xi)/2 \\ (2\xi-1)L/4 \\ 5/2 \\ -(5L)/4 \end{bmatrix}\frac{1}{7+\xi}$$

$$= \frac{L^3}{EI_1(7+\xi)} \begin{bmatrix} -\frac{9+2\xi}{24} & \frac{2\xi-1}{4L} & \frac{5\xi}{24} & -\frac{5\xi}{4L} \end{bmatrix}^T$$

$$\mathbf{D=V^TZ=} \frac{L^3}{12EI_1(7+\xi)^2} \begin{bmatrix} 21+31\xi+4\xi^2 \\ -21+39\xi+6\xi^2 \\ 0 \\ -(21+66\xi+9\xi^2) \end{bmatrix}.$$

Chapter 3

GENERALIZATION OF THE THEORY OF STRUCTURAL VARIATIONS TO MULTIDIMENSIONAL FINITE ELEMENT SYSTEMS*

In the previous chapter, the concepts and fundamental theorems of TSV have been described via the skeletal structures (1-D finite element systems). Nevertheless, these concepts and theorems are also applicable to multidimensional finite element systems. However, in this case, the characteristics of subelements (subelement vector and subelement stiffness modulus) must be reestablished for each specific element model of interest. This chapter will discuss how to establish general subelements and show how to generalize the formulations of the fundamental theorems to the multidimensional finite element models. Only a 2-D constant strain triangular element model in linear isotropic elasticity is used as an illustrative example in this chapter. However, the procedure and the formulations developed here are extendable to plate, shell elements and other multidimensional finite element models, provided their subelements can be clearly characterized.

* The contents of this chapter has been presented in [13] and accepted for publication in AIAA Journal.

Note that the formulations of Theorems 1-4 in any finite element system will remain the same as those in 1-D systems, because they do not explicitly involve the specific features of subelements. However, the formulation of Theorem 5 needs some modifications due to the distinct features of each element model under consideration. Therefore, the discussion in this chapter will focus on the basic concepts of 2-D subelements (Sections 3.2 and 3.3) and the formulation of Theorem 5 for 2-D finite element systems (Section 3.4). This study generates new subelements from an existing finite element model, the constant strain triangular element. The basic formulations for constant strain triangular element systems are listed in Section 3.1 for a recollection, while the detail of description can be found in any text books on the finite element method, e.g., [1]. Section 3.5 will give a description of structural variations of Type III in 2-D finite element systems and Section 3.6 will present an illustrative example to show how the structural variation method works for multidimensional finite element systems.

The last section of this chapter will provide a general procedure for generating subelements and their characteristics from finite element models in general.

3.1 Basic Formulations for Constant Strain Triangular Finite Element Systems

Consider a finite element system of n nodes and m triangular elements. Use α, β ··· to denote the element number and i, j, m its vertices as shown in Fig. 3.1. The formulations of the finite element method for a typical constant triangular element system are well-known (see, e.g., [1]):

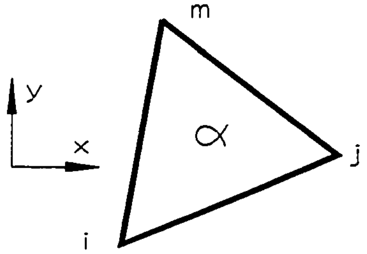

Figure 3.1 A Triangular Element

$$\epsilon = [\ \epsilon_x\ \epsilon_y\ \gamma_{xy}\ ]^T = \mathbf{BD} \tag{3.1}$$

$$\sigma = [\ \sigma_x\ \sigma_y\ \tau_{xy}\ ]^T = \mathbf{M}\epsilon \tag{3.2}$$

$$\mathbf{K}^\alpha = A\,t\,\mathbf{B}^T\mathbf{MB} \tag{3.3}$$

$$\mathbf{f}^\alpha = \mathbf{K}^\alpha\mathbf{D} \tag{3.4}$$

$$\mathbf{K} = \sum_{\alpha=1}^{m} \mathbf{K}^\alpha \tag{3.5}$$

$$\mathbf{KD} = \mathbf{P} \tag{3.6}$$

$$\mathbf{M} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \frac{E}{(1-\nu^2)} \tag{3.7}$$

$$\mathbf{B} = \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \frac{1}{2A} \tag{3.8}$$

$$b_i = y_j - y_m, \quad c_i = -x_j + x_m \tag{3.9}$$

where ϵ is the strain vector, σ the stress vector, **M** the elastic matrix, E the Young's modulus, ν the Poisson's ratio, A the area, $\mathbf{f}^\alpha$ the nodal force vector, $\mathbf{K}^\alpha$ the element stiffness matrix, x_i and y_i the coordinates of the vertices of element α, where *i*, *j* and *m* are in cyclic permutation; **K**, **D** and **P** are the global stiffness matrix, the nodal displacement vector and the applied load vector, respectively. The superscript T stands for transpose. The following will introduce the subelements pertaining to constant triangular elements.

3.2 2-D Subelements

Introduce an orthogonal matrix **Q** such that $\mathbf{Q}^T\mathbf{M}\mathbf{Q}$ becomes a diagonal matrix. For the particular **M** defined in Eq. (3.7), one has

$$\mathbf{Q} \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2} \tag{3.10}$$

with which

$$\mathbf{Q}^T\mathbf{M}\mathbf{Q} = \mathrm{diag}(\ \tfrac{1}{2}E/(1-\nu),\ \tfrac{1}{2}E/(1+\nu),\ \tfrac{1}{2}E/(1+\nu)\ ).$$

Then, Eq. (3.3) can be rewritten as

$$\mathbf{K}^\alpha = \mathbf{H}^\alpha \mathbf{W}^\alpha (\mathbf{H}^\alpha)^T \tag{3.11}$$

where $\mathbf{H}^\alpha$ is the transfer matrix of element α and defined as

$$\mathbf{H}^\alpha \equiv \mathbf{A}\mathbf{B}^T\mathbf{Q}^{-T} = \frac{1}{2} \begin{bmatrix} b_i & c_i & b_j & c_j & b_m & c_m \\ b_i & -c_i & b_j & -c_j & b_m & -c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}^T \tag{3.12}$$

with Q^{-1} being

$$Q^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W^{\alpha} \equiv \text{diag}(\, W_1^{\alpha}, W_2^{\alpha}, W_3^{\alpha}\,) \equiv \frac{t}{A} Q^T M Q \tag{3.13}$$

$$W_1^{\alpha} \equiv \frac{Et}{2A(1-\nu)}\,; \qquad W_2^{\alpha} \equiv \frac{Et}{2A(1+\nu)}\,; \qquad W_3^{\alpha} \equiv \frac{Et}{2A(1+\nu)}\,. \tag{3.14}$$

Denote each column in H^{α} by a vector E_s^{α}, s=1,2,3:

$$E_1^{\alpha} \equiv ½[\, b_i \;\; c_i \;\; b_j \;\; c_j \;\; b_m \;\; c_m \,]^T \tag{3.15a}$$

$$E_2^{\alpha} \equiv ½[\, b_i \; -c_i \;\; b_j \; -c_j \;\; b_m \; -c_m \,]^T \tag{3.15b}$$

$$E_3^{\alpha} \equiv ½[\, c_i \;\; b_i \;\; c_j \;\; b_j \;\; c_m \;\; b_m \,]^T. \tag{3.15c}$$

Thus, one has

$$H^{\alpha} = [\, E_1^{\alpha} \; E_2^{\alpha} \; E_3^{\alpha} \,]. \tag{3.16}$$

Consequently, Eq. (3.11) yields

$$K^{\alpha} = \sum_{s=1}^{3} K_s^{\alpha} \tag{3.17}$$

where the subelement stiffness matrix K_s^{α} is given as

$$K_s^{\alpha} \equiv W_s^{\alpha} E_s^{\alpha} (E_s^{\alpha})^T, \quad s=1,2,3. \tag{3.18}$$

The matrix K_s^{α} in Eq. (3.17) may be regarded as the element stiffness matrix of a subdivided element (having the same vertices as the parent element α), as has been done for beam subelements. Furthermore, the vector, E_s^{α}, and the parameter, W_s^{α}, may be identified as the subelement vector and stiffness modulus, respectively.

Note that the relations between the 2-D triangular element and its subelement stiffness matrices defined by Eqs. (3.17) and (3.18), respectively, are identical to those for the one-dimensional beam elements. Therefore, many concepts and theorems given in the previous chapter for structural variations can be extended here for 2-dimensional triangular elements.

3.3 Generalized Internal Forces, Z-deformations and Intrinsic Loads for 2-D Finite Element systems

The generalized internal force vector, $\mathbf{F}^\alpha$, the Z-deformation vector, $\mathbf{Z}^\alpha$, and the intrinsic load vector, $\mathbf{P}^\alpha_s$, for triangular element systems are defined, respectively, as

$$\mathbf{F}^\alpha \equiv [\, F^\alpha_1 \; F^\alpha_2 \; F^\alpha_3 \,]^T \equiv t\,\mathbf{Q}\sigma \tag{3.19}$$

$$\mathbf{Z}^\alpha \equiv [\, Z^\alpha_1 \; Z^\alpha_2 \; Z^\alpha_3 \,]^T \equiv (\mathbf{H}^\alpha)^T\mathbf{D} \tag{3.20}$$

$$\mathbf{P}^\alpha_s \equiv \mathbf{W}^\alpha_s \mathbf{E}^\alpha_s. \tag{3.21}$$

From Eqs. (3.19), (3.2), (3.1), (3.12), (3.13) and (3.20), one has

$$\mathbf{F}^\alpha = t\,\mathbf{Q}^T\mathbf{MBD} = \mathbf{W}^\alpha(\mathbf{H}^\alpha)^T\mathbf{D} = \mathbf{W}^\alpha\mathbf{Z}^\alpha. \tag{3.22}$$

Therefore, it turns out that $\mathbf{W}^\alpha$ is still the coefficient matrix between the generalized internal force vector $\mathbf{F}^\alpha$ and the Z-deformation vector $\mathbf{Z}^\alpha$ of element α. Furthermore, collecting $\mathbf{F}^\alpha$, $\mathbf{Z}^\alpha$ and $\mathbf{W}^\alpha$, for all elements, $\alpha=1,2,\cdots, m$, to make their global counterparts, denoted by **F**, **Z** and **W** (diagonal), respectively, one has the same global relationship as that for 1-D systems:

$$\mathbf{F} = \mathbf{WZ}. \tag{3.23}$$

With $\mathbf{F}^\alpha$ and $\mathbf{Z}^\alpha$ known, from Eqs. (3.19), (3.1) and (3.12), the stress vector σ and strain vector ϵ are calculated by

$$\sigma=\mathbf{Q}^{-1}\mathbf{F}^\alpha/t; \qquad \epsilon=\mathbf{Q}\mathbf{Z}^\alpha/A \tag{3.24}$$

The other terms introduced in the theory of structural variations, such as the basic displacement vector and the basic internal force vector for 2-D systems, are defined by the same way as those in the previous chapter for 1-D systems. To avoid repetition, they are not reiterated here.

3.4 Theorem 5 for 2-D Finite Element Systems

The fundamental theorems described in Chapter 2 for skeletal structures are also valid for the general finite element systems. Nevertheless, Theorem 5 needs to be modified to account for the specific features of the subelements in use. This section will discuss this aspect in detail. The discussion here is applicable to not only the 2-D element under consideration but also other types of element models.

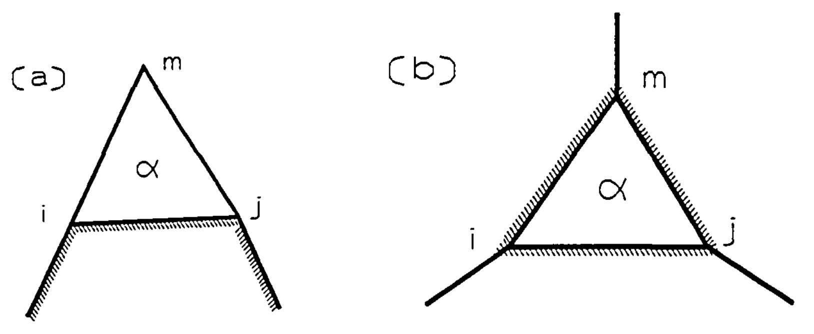

Figure 3.2 (a) Branching Element; (b) Connecting Element

Theorem 5 deals with the responses of a structural system subjected to the structural variations of Type II. This type of structural variations involve two cases as have been described in Chapter 2 for 1-D systems. In the first case, a new triangular element, say α, branches out from two original nodes i and j, and a new node m is added to the structure at the same time; the element added in this way is called the branching element as shown in Fig. 3.2(a). In the second case, a new element α is added to the structure by connecting three existing nodes i, j and m without introducing any new node as shown in Fig. 3.2(b); this type of element is called the connecting element. In the following, formulations will be derived to find the new basic displacement matrix $\hat{V}$ after the structure being added with a branching or a connecting element. To this end, however, the concept of constraint-subelement introduced in Subsection 2.2.3 for 1-D finite element systems will be extended here for 2-D finite element systems.

3.4.1 Addition of a Branching Element

With the concept of the constraint-subelement described in Subsection 2.1.3, the local structure of a hinge joint, say j, can be regarded as a pair of constraint-subelements $\binom{R}{t}$, t=1 (x-direction) and t=2 (y-direction), as shown in Fig. 3.3 (a); the nodes R and j are actually located at the same point. Thus, a branching element can be treated as the combination of a **simply supported element** α as shown in Fig. 3.3(b) or (c) and a constraint-subelement $\binom{R}{t}$ between R and j with t=1 for Fig. 3.3(b) or t=2 for Fig. 3.3-(c). Therefore, adding a branching element to the system can be carried out through two steps. First add a simply supported element α and then a constraint-subelement $\binom{R}{t}$.

Step 1: adding a simply supported element

To add a simply supported element to a 2-D system, one should notice two facts that every intrinsic load vector $\mathbf{P}^{\alpha}_{s}$ is a self-equilibrized load set, which may be verified directly from the definition (3.21), and that any self-equilibrized load set applied to the degrees of freedom of a simply supported element (the new element added to the original system) produces no displacements at the degrees of freedom of the original system. According to these facts, the six degrees of freedom of the simply supported element α may be divided into two groups. For example, based upon the simply supported element shown in Fig. 3.3(b), the first group, group A, includes the constrained degrees of freedom $\binom{i}{1}$, $\binom{i}{2}$ and $\binom{j}{2}$ which are connected to the original system

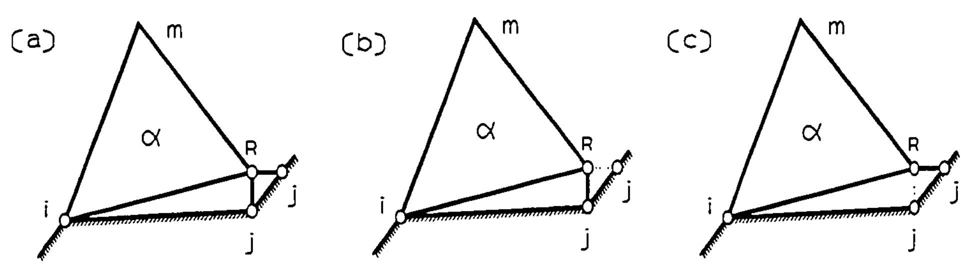

Figure 3.3 (a) A Pair of Constraint-Subelements Acting as a Hinge;

(b) and (c) Simply Supported Elements

and the second group, group B, $\binom{R}{1}$, $\binom{m}{1}$ and $\binom{m}{2}$ which are the free and new degrees of freedom added to the original system. To derive the components of the varied $\hat{\mathbf{V}}^{\alpha}$ pertaining to the degrees of freedom of group B, one may first express the nodal force vector $\mathbf{f}^\alpha$ in terms of the generalized internal force vector $\mathbf{F}^\alpha$ by using Eqs. (3.4), (3.11) and (3.22):

$$\mathbf{f}^\alpha=\mathbf{H}^\alpha\mathbf{W}^\alpha(\mathbf{H}^\alpha)^T\mathbf{D}=\mathbf{H}^\alpha\mathbf{F}^\alpha \tag{3.25}$$

or in the partition form,

$$\mathbf{f}^\alpha=\begin{bmatrix}\mathbf{f}^\alpha_A\\ \mathbf{f}^\alpha_B\end{bmatrix}=\mathbf{H}^\alpha\mathbf{F}^\alpha=\begin{bmatrix}\mathbf{H}^\alpha_A\\ \mathbf{H}^\alpha_B\end{bmatrix}\mathbf{F}^\alpha \tag{3.26}$$

where subscripts A and B indicate that the associated quantities are separated according to the degrees of freedom in groups A and B, respectively. In fact, for the simply supported element shown in Fig. 3.3(b), $\mathbf{H}^\alpha_A$ and $\mathbf{H}^\alpha_B$ are defined by Eq. (3.12) as

$$\mathbf{H}^\alpha_A=\frac{1}{2}\begin{bmatrix}b_i & b_i & c_i\\ c_i & -c_i & b_i\\ c_j & -c_j & b_j\end{bmatrix}\ ;\quad \mathbf{H}^\alpha_B=\frac{1}{2}\begin{bmatrix}b_j & b_j & c_j\\ b_m & b_m & c_m\\ c_m & -c_m & b_m\end{bmatrix} \tag{3.27a}$$

while $\mathbf{H}^\alpha_A$ and $\mathbf{H}^\alpha_B$ for the simply supported element shown in Fig. 3.3(c) are written as

$$\mathbf{H}^\alpha_A=\frac{1}{2}\begin{bmatrix}b_i & b_i & c_i\\ c_i & -c_i & b_i\\ b_j & b_j & c_j\end{bmatrix}\ ;\quad \mathbf{H}^\alpha_B=\frac{1}{2}\begin{bmatrix}c_j & -c_j & b_j\\ b_m & b_m & c_m\\ c_m & -c_m & b_m\end{bmatrix}. \tag{3.27b}$$

Thus, from Eq. (3.26), one has

$$\mathbf{f}^\alpha_B=\mathbf{H}^\alpha_B\mathbf{F}^\alpha. \tag{3.28}$$

One may apply the unit-load $\bar{\mathbf{P}}^\ell_r$ to those degrees of freedom in group B, which may be regarded as $\mathbf{f}^\alpha_B$ in Eq. (3.28). Consequently, $\mathbf{F}^\alpha$ in Eq. (3.28) is equal to $\bar{\mathbf{F}}^{\alpha\ell}_{\cdot r}$ as defined in Subsection 2.2.4, i.e.,

$$\bar{\mathbf{P}}^\ell_r=\mathbf{H}^\alpha_B\bar{\mathbf{F}}^{\alpha\ell}_{\cdot r}\ ,\quad \text{for } (^\ell_r)\in B \tag{3.29}$$

where B represents the group B of degrees of freedom. Collectively, Eq. (3.29) can be expressed in a matrix form as

$$\mathbf{I}=\mathbf{H}^{\alpha}_{B}\bar{\mathbf{F}}^{\alpha B}_{\cdot\cdot} \tag{3.30}$$

where $\mathbf{I}$ is a 3x3 unit matrix and $\bar{\mathbf{F}}^{\alpha B}_{\cdot\cdot}$ stands for the 3x3 matrix of the three basic internal force vectors $\bar{\mathbf{F}}^{\alpha\ell}_{\cdot r}$ induced by the three $\bar{\mathbf{P}}^{\ell}_{r}$ applied at $\binom{\ell}{r}=\binom{R}{1}$, $\binom{m}{1}$ and $\binom{m}{2}$ individually. According to Theorem 1, the desired basic displacement components pertaining to the degrees of freedom in group B of the element α, denoted by $\hat{\mathbf{V}}^{\alpha B}_{\cdot\cdot}$, are identical to those in $\bar{\mathbf{F}}^{\alpha B}_{\cdot\cdot}$. Therefore, $\hat{\mathbf{V}}^{\alpha B}_{\cdot\cdot}$ can be obtained from Eq. (3.29) as $(\mathbf{H}^{\alpha}_{B})^{-1}$,

$$\hat{\mathbf{V}}^{\alpha B}_{\cdot\cdot}=\frac{1}{2c_m A}\begin{bmatrix} (b_m^2+c_m^2) \ , & -(b_j b_m+c_j c_m) \ , & 2A \\ (c_m^2-b_m^2) \ , & (b_j b_m-c_j c_m) \ , & -2A \\ -2b_m c_m \ , & 2b_j c_m \ , & 0 \end{bmatrix}. \tag{3.31}$$

Similarly, for the case of the simply supported element shown in Fig. 3.3(c), group B consists of degrees of freedom $\binom{R}{2}$, $\binom{m}{1}$ and $\binom{m}{2}$, and

$$\hat{\mathbf{V}}^{\alpha B}_{\cdot\cdot}=\frac{1}{2b_m A}\begin{bmatrix} -(b_m^2+c_m^2) \ , & 2A \ , & (b_j b_m+c_j c_m) \\ (b_m^2-c_m^2) \ , & 2A \ , & (c_j c_m-b_j b_m) \\ 2b_m c_m \ , & 0 \ , & -2b_m c_j \end{bmatrix}. \tag{3.32}$$

Next, one can proceed to derive the new basic displacement matrix $\hat{\mathbf{V}}^{\beta}$ associated with the original structure after the simply supported element α being added to it. Before doing so, however, it is worthwhile mentioning that the displacements of the modified structure induced by any load applied to the original structure remain the same as those of the original structure. Thus, the modified basic displacement vector, $\hat{\mathbf{V}}_r^\beta$ of any original subelement $\binom{\beta}{r}$, is identical to $\mathbf{V}_r^\beta$ of the original structure. Furthermore, the simply supported element α is subjected to no deformation if the external force applies at the original degrees of freedom of the system, because the nodal forces at the degrees of freedom of group B are zero. Consequently, $\mathbf{Z}^\alpha=(\mathbf{H}^\alpha)^T\mathbf{D}=\mathbf{0}$ as indicated by Eq. (3.24). Therefore, $(\mathbf{H}_A^\alpha)^T\mathbf{D}_A+(\mathbf{H}_B^\alpha)^T\mathbf{D}_B=\mathbf{0}$, where $\mathbf{D}_A$ and $\mathbf{D}_B$ are the displacement components at degrees of freedom of groups A and B, respectively. Thus, one has

$$\mathbf{D}_B=-(\mathbf{H}_B^\alpha)^{-T}(\mathbf{H}_A^\alpha)^T\mathbf{D}_A=\boldsymbol{\Omega}^\alpha\mathbf{D}_A \tag{3.33}$$

where for the case of Fig. 3.3(b), $\boldsymbol{\Omega}^\alpha$ is obtained as

$$\boldsymbol{\Omega}^\alpha\equiv-(\mathbf{H}_B^\alpha)^{-T}(\mathbf{H}_A^\alpha)^T=\frac{1}{c_m}\begin{bmatrix} c_m & -b_m & b_m \\ c_m & b_j & -b_j \\ 0 & -c_i & -c_j \end{bmatrix} \tag{3.34}$$

whereas for the case of Fig. 3.3(c), $\boldsymbol{\Omega}^\alpha$ is given as

$$\boldsymbol{\Omega}^\alpha=\frac{1}{b_m}\begin{bmatrix} -c_m & b_m & c_m \\ -b_i & 0 & -b_j \\ c_j & b_m & -c_j \end{bmatrix}. \tag{3.35}$$

Equation (3.33) can also be represented as

$$(\mathbf{D}_B)^T=(\mathbf{D}_A)^T(\boldsymbol{\Omega}^\alpha)^T \tag{3.36}$$

which relates any displacements pertaining to the degrees of freedom in group B to those in group A. Therefore, the new components of $\hat{\mathbf{V}}_r^\beta$ at the degrees of freedom of group B, denoted by $\hat{\mathbf{V}}_{r\bullet}^{\beta B}$, a 1x3 matrix, can be obtained by using Eq. (3.36) as

$$\hat{\mathbf{V}}_{r\bullet}^{\beta B}=\mathbf{V}_{r\bullet}^{\beta A}(\boldsymbol{\Omega}^\alpha)^T, \qquad \beta\neq\alpha,\ r=1,2,3 \tag{3.37}$$

where $\mathbf{V}_{r\bullet}^{\beta A}$, a 1x3 matrix, is a row vector of the original components of $\mathbf{V}_r^{\beta}$ at the degrees of freedom of group A. Collecting the three row vector equations (3.37) for element β, one has

$$\hat{\mathbf{V}}_{\bullet\bullet}^{\beta B}=\mathbf{V}_{\bullet\bullet}^{\beta A}(\Omega^{\alpha})^T\ , \quad \beta\neq\alpha \tag{3.38}$$

where $\hat{\mathbf{V}}_{\bullet\bullet}^{\beta B}$ and $\mathbf{V}_{\bullet\bullet}^{\beta A}$ are the matrices of the three row vectors $\hat{\mathbf{V}}_{r\bullet}^{\beta B}$ and $\hat{\mathbf{V}}_{r\bullet}^{\beta A}$, r=1,2,3, respectively.

Step 2: inserting a constraint-subelement

To continue the derivation of adding a 2-D branching element to a structure, a constraint-subelement $\binom{R}{i}$ should be inserted between the node R of the simply supported element α and the node j of the original structure in x-direction (Fig. 3.3(b)), or in y-direction (Fig. 3.3(c)). The procedure of adding a constraint-subelement has been discussed in Subsection 2.2.4 and Eq. (2.49) therein is also applicable for 2-D systems of concern.

3.4.2 Addition of a Connecting Element

When a connecting element α is added to the system, no new nodes are created. Therefore, the formulas derived in Eqs. (2.45) and (2.46) can be directly applied here to derive the new basic displacements by adding one subelement at a time to the structure.

As a conclusion, the derivation given in this section can be summarized in the following theorem.

Theorem 5. *The Theorem on the Structural Variations of Type II* (for 2-D finite element systems):

When a simply supported element is added to a system, the basic displacements of the original structure remain unchanged. However, the additional components, Eqs. (3.36) **and** (3.31) **or** (3.32), **corresponding to the new degrees of freedom should be added to the original ones. If a constraint-subelement or a connecting subelement is inserted among the original nodes, the basic displacement vectors can be determined or modified by the Eqs.** (2.45), (2.46) **and** (2.49).

3.5 Structural Variations of Type III in 2-D Systems

The Type III structural variations in a 2-D system have also two cases to be considered. The first case is to insert a constraint-subelement / support-subelement, symbolized by $\binom{R}{t}$, between two nodes R and R′ of the system along its axial direction t, as shown in Fig. 2.3. Equation (2.49) derived in Subsection 2.2.3 can be directly applied to 2-D systems without modification.

The second case is to remove an existing support-subelement $\binom{R}{t}$ from the system; the corresponding formulation, Eq. (2.51), is still applicable, but modifications are needed to obtain the new coefficient $\eta^{R\alpha}_{ts}$.

Upon removing a support-subelement $\binom{R}{t}$ from the system, the basic displacements of a subelement $\binom{\alpha}{s}$ will become

$$\hat{V}^{\alpha}_{s}=V^{\alpha}_{s}+V^{R}_{t}\eta^{R\alpha}_{ts} \tag{3.39}$$

where

$$\eta^{R\alpha}_{ts}\equiv W^{\alpha}_{s}Z^{\alpha R}_{st} / \left(\sum_{\beta=1}^{q} (T^{\beta}_{t})^{T}W^{\beta}Z^{\beta R}_{\bullet t}\right) \tag{3.40}$$

$$T^{\beta}_{t}\equiv -(H^{\beta}_{R})^{T}R^{R}_{t} \tag{3.41}$$

$$R^R_t \equiv [\ \cos\theta,\ \sin\theta\ ]^T \tag{3.42}$$

where V^α_s and V^R_t are the original basic displacement vectors of $\binom{\alpha}{s}$ and $\binom{R}{t}$, respectively; q is the total number of the elements around the support node R; θ is the angle between $\binom{R}{t}$ and x-axis (Fig. 2.3); $(H^\beta_R)_{2x3}$ is the partition of H^β corresponding to the node R, and $Z^{\beta R}_{t}=(H^\beta)^T V^R_t$ the Z-deformation vector of element β from V^R_t. Equation (3.39) is quite general and can be applied to any finite element system. The proof of Eq. (3.39) is detailed in Appendix.

3.6 Illustrative Example of a 2-D Problem

A plane stress problem with two constant strain triangles, shown in Fig. 3.4(c), has the following properties, E=1.0, ν=0.3 and t=1.0. The problem is to find the stresses σ and the displacements **D** induced by the load **P** given in Fig. 3.4(c). Based upon the given information, one has the initial data: $b_1=-1$, $b_2=1$, $b_3=0$, $c_1=-1$, $c_2=0$, $c_3=1$ and A=0.5 for element 1; and $b_3=-1$, $b_2=0$, $b_4=1$, $c_3=0$, $c_2=-1$, $c_4=1$ and A=0.5 for element 2. Furthermore, $W^1=W^2=\text{diag}(\ 10/7,\ 10/13,\ 10/13\ )$. The solution procedure is listed as follows.

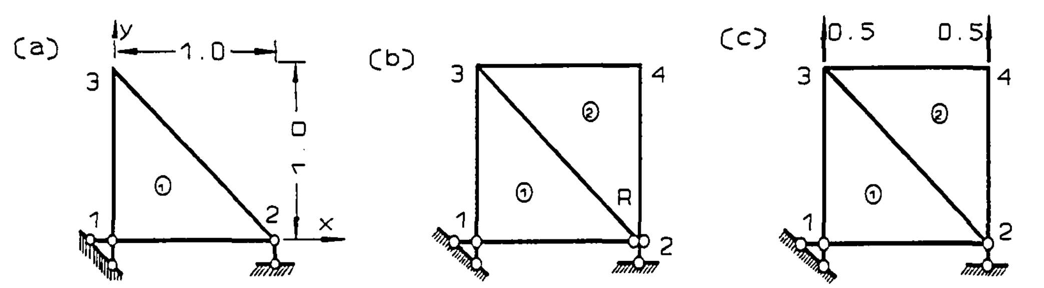

Figure 3.4 Structural Variation Process of a Finite Element System

Step 1. Assume that the initial structure is made of element 1 which is simply supported at points 1 and 2 (Fig. 3.4(a)). The degrees of freedom of group B are $\binom{2}{1}$, $\binom{3}{1}$ and $\binom{3}{2}$ for element 1. Substituting the related data of element 1 into Eq. (3.31) yields

$$\mathbf{V}^1 = \begin{bmatrix} (\mathbf{V}_1^1)^T \\ (\mathbf{V}_2^1)^T \\ (\mathbf{V}_3^1)^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & 0 & 1 \\ 1 & 0 & \vdots & 0 & -1 \\ 0 & 0 & \vdots & 2 & 0 \end{bmatrix} .$$
$$\text{node 2} \qquad \text{node 3}$$

Note that the components of **V** associated with node 1 are all zero. Hence, they are ignored in the **V** matrix.

Step 2. Add the simply supported element 2 to the initial structure, as shown in Fig. 3.4(b), where the double circles surrounding the node R indicate that the node R is free in x-direction. The degrees of freedom in group A are $\binom{3}{1}$, $\binom{3}{2}$ and $\binom{2}{2}$ in this case, while those in group B are $\binom{R}{1}$, $\binom{4}{1}$ and $\binom{4}{2}$. Equation (3.36) provides a means to establish the basic displacements accounting for the new element, element 2.

$$\mathbf{V}^{1B}_{\cdot\cdot} = \mathbf{V}^{1A}_{\cdot\cdot}(\boldsymbol{\Omega}^1)^T = \begin{bmatrix} 0 & 1 & \vdots & 0 \\ 0 & -1 & \vdots & 0 \\ 2 & 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \vdots & 0 & 0 \\ 1 & \vdots & 0 & 0 \\ 2 & \vdots & 2 & 0 \end{bmatrix}$$
$$\text{node 3} \quad \binom{2}{2} \qquad\qquad \binom{R}{1} \quad \text{node 4}$$

$$\mathbf{V}^{2B}_{\cdot\cdot} = \begin{bmatrix} 2 & \vdots & 1 & 1 \\ 0 & \vdots & 1 & -1 \\ -2 & \vdots & 0 & 0 \end{bmatrix}$$
$$\binom{R}{1} \quad \text{node 4}$$

where the matrix $\boldsymbol{\Omega}^1$ is given by Eq. (3.34) as

$$\Omega^1 = \frac{1}{c_4}\begin{bmatrix} c_4 & -b_4 & b_4 \\ c_4 & b_2 & -b_2 \\ 0 & -c_3 & -c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the basic displacement matrix **V** of the structural system in Fig. 3.4(b) is

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}^1 \\ \mathbf{V}^2 \end{bmatrix} = \begin{bmatrix} -1 & \vdots & 1, & 0 & \vdots & 0, & 1 & \vdots & 0, & 0 \\ 1 & \vdots & 1, & 0 & \vdots & 0, & -1 & \vdots & 0, & 0 \\ 2 & \vdots & 0, & 0 & \vdots & 2, & 0 & \vdots & 2, & 0 \\ 2 & \vdots & 0, & 0 & \vdots & 0, & 0 & \vdots & 1, & 1 \\ 0 & \vdots & 0, & 0 & \vdots & 0, & 0 & \vdots & 1, & -1 \\ -2 & \vdots & 0, & 0 & \vdots & 0, & 0 & \vdots & 0, & 0 \end{bmatrix}.$$

$\binom{R}{1}$ node 2 node 3 node 4

Step 3. Construct **V** of a new structure with the constraint-subelement $\binom{R}{1}$ being inserted between nodes R and 2 in x-direction. First, a pair of unit forces, $\mathbf{E}^R_1=[-1, 1]^T$ are applied at $\binom{R}{1}$ and $\binom{2}{1}$ (Fig. 3.4(b)) to calculate the corresponding auxiliary basic displacement vector $\mathring{\mathbf{V}}^R_1$ by using Eq. (2.35): $\mathring{\mathbf{V}}^R_1=\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\mathbf{E}^R_1=[-14.16 \vdots 1.4, 0.0 \vdots -5.2, 1.4 \vdots -6.6, -1.4]^T$ which implies $\mathring{Z}^{RR}_{11}=(\mathbf{E}^R_1)^T\mathring{\mathbf{V}}^R_1=(-1)x(-14.6)+1x(1.4)=16$. Next, use Eq. (2.26) to calculate $Z^{R\beta}_{1r}$ from $\mathbf{V}^\beta_r$ for every $\binom{\beta}{r}$ needed in Eq. (2.49); they are [2, 0, -2, -2, 0, 2]. Finally, Eq. (2.49) is evaluated for the final basic displacement matrix of the desired system (Fig. 3.4(c)):

$$\mathbf{V}_{final} = \begin{bmatrix} .825, & 0 & \vdots & .65, & .825 & \vdots & .825, & .175 \\ 1.0, & 0 & \vdots & 0, & -1.0 & \vdots & 0, & 0 \\ .175, & 0 & \vdots & 1.35, & .175 & \vdots & 1.175, & -.175 \\ .175, & 0 & \vdots & -.65, & .175 & \vdots & .175, & .825 \\ 0, & 0 & \vdots & 0, & 0 & \vdots & 1.0, & -1.0 \\ -.175, & 0 & \vdots & .65, & -.175 & \vdots & .825, & .175 \end{bmatrix}.$$

node 2 node 3 node 4

Step 4. Obtain σ and **D** for the structure subjected to the load, $\mathbf{P}=[\ 0.5,\ 0.5\ ]^T$, applied at $\left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)$. The generalized internal forces are first obtained by using Eq. (2.35) as

$$\mathbf{F}=\mathbf{VP}=[\ \underbrace{0.5,\ -0.5,\ 0.0}_{\text{element 1}}\ \vdots\ \underbrace{0.5,\ -0.5,\ 0.0}_{\text{element 2}}\ ]^T.$$

The stresses in elements 1 and 2 are then calculated separately based on Eq. (3.24) as

$$\sigma^1=\mathbf{Q}^{-1}\mathbf{F}^1/t=\begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0.5\\ -0.5\\ 0.0\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$\sigma^2=\mathbf{Q}^{-1}\mathbf{F}^2/t=[\ 0,\ 1,\ 0\ ]^T.$$

Finally, the displacements of the structure are obtained as

$$\mathbf{D}=\mathbf{V}^T\mathbf{W}^{-1}\mathbf{F}=[\ \underbrace{-0.3,\ 0.0}_{\text{node 2}}\ \vdots\ \underbrace{0.0,\ 1.0}_{\text{node 3}}\ \vdots\ \underbrace{-0.3,\ 1.0}_{\text{node 4}}\ ]^T.$$

3.7 The General Procedure of Generating Subelements for Multidimensional Finite Element Systems

The theorems and formulations presented in the previous sections can be easily generalized for finite element models in general, provided that their element stiffness matrix $\mathbf{K}^\alpha$ can be expressed as the contribution of subelement stiffness matrices $\mathbf{K}^\alpha_s$. However, the subelements generated from different element models will have different values of $\mathbf{E}^\alpha_s$ and $\mathbf{W}^\alpha_s$. In the following, a general procedure will be given for generating the subelements from any element model whose element stiffness matrix is expressed as $\mathbf{K}^\alpha=\int_\Omega \mathbf{B}^T\mathbf{M}\mathbf{B}d\Omega$, where Ω is the element volume and the elastic matrix **M** is symmetric and positive definite. By using the dimensionless local coordinates ξ,η,ζ [1], $\mathbf{K}^\alpha$ may be expressed as

$$K^{\alpha}=\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\mathbf{B}^{T}\mathbf{M}\mathbf{B}\det(\mathbf{J})d\xi d\eta d\zeta \qquad (3.43)$$

where det(J) is the determinant of Jacobian matrix **J**. The matrix $\mathbf{K}^{\alpha}$ can also be evaluated by Gauss quadrature with N points in the sum of several constant matrixes:

$$\mathbf{K}^{\alpha}=\sum_{m=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N}H_iH_jH_m\,(\,\mathbf{B}^{T}\mathbf{M}\mathbf{B}\det(\mathbf{J})\,)\,|_{\xi_i\eta_j\zeta_m} \qquad (3.44)$$

where H_i, H_j and H_m are the Gaussian weight coefficients and $(\;)\,|_{\xi_i\eta_j\zeta_m}$ indicates that the bracketed quantity is evaluated at the Gaussian point (ξ_i, η_j, ζ_m). For simplicity, let $\mathbf{k}\equiv\phi\mathbf{B}^{T}\mathbf{M}\mathbf{B}$ represent the general term of the constant matrices (evaluated at a Gaussian point) in Eq. (3.44), where ϕ stands for a scaler factor. Since **M** is symmetric and positive definite, there exists an orthogonal matrix **Q** of the same dimension as that of **M**, making $\mathbf{Q}^{T}\mathbf{M}\mathbf{Q}$ diagonal [14]. Therefore, one can rewrite the **k** as a product of **H** and diagonal matrix **W**:

$$\mathbf{k}=\phi\mathbf{B}^{T}\mathbf{M}\mathbf{B}=(\mathbf{B}^{T}\mathbf{Q}^{-T})(\phi\mathbf{Q}^{T}\mathbf{M}\mathbf{Q})(\mathbf{Q}^{-1}\mathbf{B})=\mathbf{H}\mathbf{W}\mathbf{H}^{T} \qquad (3.45)$$

where

$$\mathbf{H}\equiv\mathbf{B}^{T}\mathbf{Q}^{-T}\equiv[\,\mathbf{E}_1,\,\mathbf{E}_2,\cdots,\mathbf{E}_q\,] \qquad (3.46)$$

$$\mathbf{W}\equiv\phi\mathbf{Q}^{T}\mathbf{M}\mathbf{Q}\equiv\mathrm{diag}(\,W_1,\,W_2,\cdots,W_q\,) \qquad (3.47)$$

where q is the rank of **M**. Thus, the subelement stiffness matrix $\mathbf{k}_s$ is defined as

$$\mathbf{k}_s\equiv W_s\mathbf{E}_s(\mathbf{E}_s)^{T} \quad s=1,2,\cdots,q \qquad (3.48)$$

then, one has

$$\mathbf{k}=\sum_{s=1}^{q}\mathbf{k}_s \qquad (3.49)$$

where W_s is a diagonal element of **W**, serving as the subelement stiffness modulus, and $\mathbf{E}_s$ is a column vector of **H**, serving as the subelement vector. However, the expression (3.48) represents only one term in Eq. (3.44), corresponding to one Gaussian point.

More specifically, this term may be denoted as $(k_s^\alpha)_{ijm}$, where ijm corresponds to the $\xi_i\eta_j\zeta_m$. Consequently, the K^α of a finite element model can be expressed in terms of subelement stiffness matrices as

$$K^\alpha=\sum_{m=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{s=1}^{q}(k_s^\alpha)_{ijm}. \tag{3.50}$$

It should be noted that the matrix **Q** plays a very important role in constructing the subelements.

The matrix **Q** for an isotropic and homogeneous solid can be expressed as

$$\mathbf{Q}=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.51}$$

while the corresponding **M** is given as

$$\mathbf{M}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 1 & \gamma & 0 & 0 & 0 \\ \gamma & \gamma & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} \tag{3.52}$$

where $\gamma=\nu/(1-\nu)$ and $\delta=(1-2\nu)/(2(1-\nu))$.

Chapter 4

EXPLICIT FORMULATIONS FOR DESIGN SENSITIVITIES IN STATIC ANALYSIS†

4.1 Introduction

There are many publications, e.g., [16] and [17], on structural design sensitivity analysis. In those works, sensitivity derivations are obtained by solving linear simultaneous equations. This chapter, however, will derive a set of explicit formulations for the static design sensitivities of displacements, internal forces and stresses. There is no need to assemble and solve simultaneous equations in these formulations. These formulations are derived based upon a new theorem, the Gradient Orthogonality Theorem which will be proved in Section 2.4.

Assume that the finite element equation of a structural system is

$$\mathbf{K}(\mathbf{b})\mathbf{D}(\mathbf{b})=\mathbf{P}(\mathbf{b}) \tag{4.1}$$

and the performance function to be differentiated is given as

$$\psi=\psi(\ \mathbf{b},\ \mathbf{D}(\mathbf{b})\ ) \tag{4.2}$$

where $\mathbf{b}$ is the design variable vector, $\mathbf{K}$ the global stiffness matrix, $\mathbf{D}$ the nodal

† The contents of this chapter has been presented in [15].

displacement vector and **P** the external nodal load vector. The sensitivity of ψ to **b** is calculated by the chain rule as

$$\frac{d\psi}{d\mathbf{b}}=\frac{\partial\psi}{\partial\mathbf{b}}+\frac{\partial\psi}{\partial\mathbf{D}}\frac{d\mathbf{D}}{d\mathbf{b}}\,. \tag{4.3}$$

Note that in this dissertation the derivatives with respect to a vector are defined the same way as those in Appendix 1 of [16].

The derivatives **dD/db** in Eq. (4.3) can be obtained by the following equation, which is obtained by differentiating Eq. (4.1):

$$\mathbf{K}\frac{d\mathbf{D}}{d\mathbf{b}} = -\frac{d\mathbf{K}}{d\mathbf{b}}\mathbf{D}+\frac{d\mathbf{P}}{d\mathbf{b}}\,.$$

However, the new concepts and theorems of structural variations introduce an interesting intrinsic property of finite element systems, i.e., the Gradient Orthogonality Theorem of Basic Displacements, based on which the explicit formulations for sensitivity analysis can be derived. The beam element and the constant strain triangular element will be used as samples to facilitate the discussion and derivation. The resultant formulations, however, can be extended to other finite element models.

Assume that a finite element system of plane beams or constant strain triangular elements has n nodes and m elements. According to the theory of structural variations, each element α has three subelements, $(^{\alpha}_{s})$, s=1,2,3, and each subelement has a stiffness modulus, $\tilde{W}^{\alpha}_{s}$. To obtain the explicit formulation of **dD/db** in Eq. (4.3), one can define a vector **w** representing all the subelement stiffness moduli:

$$\mathbf{w}\equiv[\, W^1_1,\, W^1_2,\, \cdots,\, W^m_3\,]^T. \tag{4.4}$$

The design variable vector **b** may be taken as either the sizes or the material properties of the elements. In this case, the theory of structural variations has shown that **w** is a function of **b**. Thus, the sensitivity can be obtained through **w** as

$$\frac{d}{db} = \sum_{\alpha=1}^{m}\sum_{s=1}^{3} \frac{\partial}{\partial W_s^\alpha}\frac{dW_s^\alpha}{db} \tag{4.5}$$

where $\frac{dW_s^\alpha}{db}$, $\alpha=1, 2,\cdots,m$ and $s=1,2,3$, are known (see Eqs. (2.8) and (3.14), or Eq. (2-10) in [9]). Therefore, the sensitivity problem now focuses on how to obtain $\frac{\partial \mathbf{D}}{\partial W_s^\alpha}$.

Since the displacement vector **D** is expressed explicitly in terms of the basic displacement matrix **V** (see Eq. (2.35)):

$$\mathbf{D}=\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\mathbf{P} \tag{4.6}$$

one can explicitly formulate $\frac{\partial \mathbf{D}}{\partial W_s^\alpha}$, if $\frac{\partial \mathbf{V}}{\partial W_s^\alpha}$ is known explicitly. Consequently, the derivation will start with how to find $\frac{\partial \mathbf{V}}{\partial W_s^\alpha}$, which is the main objective of the following theorem.

4.2 The Gradient Orthogonality Theorem of Basic Displacements

For convenience, use matrix **C** to represent Kronecker δ:

$$\mathbf{C} \equiv [C_{sr}^{\alpha\beta}]_{3m\times 3m} \tag{4.7}$$

where $C_{sr}^{\alpha\beta}= 1$, if $\binom{\alpha}{s}=\binom{\beta}{r}$ or $C_{sr}^{\alpha\beta}= 0$, if $\binom{\alpha}{s}\neq\binom{\beta}{r}$, in which $\binom{\alpha}{s}$ and $\binom{\beta}{r}$ represent the subelements of elements α and β, respectively, s and $r=1,2,3$. An additional symbol $\mathbf{C}_{s\bullet}^{\alpha\bullet}$ denotes the 1x3m row vector corresponding to $\binom{\alpha}{s}$ in the matrix **C**,

$$\begin{aligned}\mathbf{C}_{s\bullet}^{\alpha\bullet} &\equiv [\ C_{s1}^{\alpha 1},\ C_{s2}^{\alpha 1},\ \cdots,\ C_{ss}^{\alpha\alpha},\ \cdots,\ C_{s3}^{\alpha m}\ ] \\ &=[\ 0,\ \ 0\ ,\cdots,\ \ 1\ ,\cdots,\ \ 0\ ].\end{aligned} \tag{4.8}$$

By using this notation, one can easily prove the following simple relationships :

$$\mathbf{C}^{\alpha\bullet}_{s\bullet}\mathbf{W}=W^{\alpha}_{s}\mathbf{C}^{\alpha\bullet}_{s\bullet} \tag{4.9a}$$

$$\mathbf{C}^{\alpha\bullet}_{s\bullet}\mathbf{W}^{-1}=\mathbf{C}^{\alpha\bullet}_{s\bullet}/W^{\alpha}_{s} \tag{4.9b}$$

$$\frac{\partial \mathbf{W}}{\partial W^{\alpha}_{s}}=(\mathbf{C}^{\alpha\bullet}_{s\bullet})^{T}\mathbf{C}^{\alpha\bullet}_{s\bullet} \tag{4.10a}$$

$$\frac{\partial \mathbf{W}^{-1}}{\partial W^{\alpha}_{s}}=-(\mathbf{C}^{\alpha\bullet}_{s\bullet})^{T}\mathbf{C}^{\alpha\bullet}_{s\bullet}/(W^{\alpha}_{s})^{2} \tag{4.10b}$$

$$\mathbf{V}^{T}(\mathbf{C}^{\alpha\bullet}_{s\bullet})^{T}=\mathbf{V}^{\alpha}_{s} \tag{4.11}$$

where the diagonal matrix **W** is the global stiffness modulus matrix. Note that the vector **w** in Eq. (4.4) is different from **W**. Furthermore, the vector $\mathbf{V}^{\alpha}_{s}$ in Eq. (4.11) is the basic displacement vector of subelement $\binom{\alpha}{s}$, which stands as a row vector in **V**, corresponding to $\binom{\alpha}{s}$ and defined in Eq. (2.28) as

$$\mathbf{V}^{\alpha}_{s}\equiv\mathbf{K}^{-1}\mathbf{P}^{\alpha}_{s} \tag{4.12}$$

where $\mathbf{P}^{\alpha}_{s}$ is the intrinsic load vector of subelement $\binom{\alpha}{s}$:

$$\mathbf{P}^{\alpha}_{s}\equiv W^{\alpha}_{s}\mathbf{E}^{\alpha}_{s} \tag{4.13}$$

and $\mathbf{E}^{\alpha}_{s}$ is the subelement vector of subelement $\binom{\alpha}{s}$. The detailed definitions of $\mathbf{V}^{\alpha}_{s}$ and $\mathbf{P}^{\alpha}_{s}$ have been given in Chapters 2 and 3.

Next, to find $\frac{\partial \mathbf{V}}{\partial W^{\alpha}_{s}}$, one can use the definition of derivatives and Theorem 4 given in Chapter 2 to obtain

$$\frac{\partial \mathbf{V}^{\beta}_{r}}{\partial W^{\alpha}_{s}}=[\,(\hat{\mathbf{V}}^{\beta}_{r}-\mathbf{V}^{\beta}_{r})/\Delta W^{\alpha}_{s}\,]\,|_{\Delta W^{\alpha}_{s}\to 0} \qquad \text{when } \binom{\beta}{r}\neq\binom{\alpha}{s}$$

$$=-\{\,\mathbf{V}^{\alpha}_{s}Z^{\alpha\beta}_{sr}m^{\alpha}_{s}/[\,(1+m^{\alpha}_{s}Z^{\alpha\alpha}_{ss})\Delta W^{\alpha}_{s}\,]\,\}\,|_{\Delta W^{\alpha}_{s}\to 0}$$

$$=-\mathbf{V}^{\alpha}_{s}Z^{\alpha\beta}_{sr}/W^{\alpha}_{s}$$

where $m^{\alpha}_{s}=\Delta W^{\alpha}_{s}/W^{\alpha}_{s}$ and goes to zero as ΔW^{α}_{s}, the variation of W^{α}_{s}, goes to zero; $\hat{\mathbf{V}}^{\beta}_{r}$ denotes the new basic displacement vector of subelement $({}^{\beta}_{r})$ of the modified structure whose subelement $({}^{\alpha}_{s})$ takes a new stiffness modulus as $W^{\alpha}_{s}+\Delta W^{\alpha}_{s}$, and $Z^{\alpha\beta}_{sr}$ is the Z-deformation of subelement $({}^{\alpha}_{s})$, induced by $\mathbf{V}^{\beta}_{r}$ as

$$Z^{\alpha\beta}_{sr}\equiv(\mathbf{E}^{\alpha}_{s})^{T}\mathbf{V}^{\beta}_{r}. \tag{4.14}$$

In terms of matrix **C**, the derivative of $\mathbf{V}^{\beta}_{r}$ can be rewritten as

$$\frac{\partial \mathbf{V}^{\beta}_{r}}{\partial W^{\alpha}_{s}}=\mathbf{V}^{\alpha}_{s}(C^{\alpha\beta}_{sr}-Z^{\alpha\beta}_{sr})/W^{\alpha}_{s}. \tag{4.15}$$

It is easy to verify that Eq. (4.15) is also valid for the case when $({}^{\beta}_{r})=({}^{\alpha}_{s})$ by repeating the same deriving procedure as has been done for the case when $({}^{\beta}_{r})\neq({}^{\alpha}_{s})$. Collecting the expressions of Eq. (4.15) for all the subelements $\beta=1,2,\cdots,m$ and $r=1,2,3$, yields

$$\frac{\partial \mathbf{V}^{T}}{\partial W^{\alpha}_{s}}=\mathbf{V}^{\alpha}_{s}(\mathbf{C}^{\alpha\bullet}_{s\bullet}-\mathbf{Z}^{\alpha\bullet}_{s\bullet})/W^{\alpha}_{s} \tag{4.16}$$

or

$$\frac{\partial \mathbf{V}}{\partial W^{\alpha}_{s}}=[(\mathbf{C}^{\alpha\bullet}_{s\bullet})^{T}-(\mathbf{Z}^{\alpha\bullet}_{s\bullet})^{T}](\mathbf{V}^{\alpha}_{s})^{T}/W^{\alpha}_{s} \tag{4.17}$$

where $\mathbf{Z}^{\alpha\bullet}_{s\bullet}$ represents the row vector of Z-deformations of subelement $({}^{\alpha}_{s})$, which is a product of $\mathbf{E}^{\alpha}_{s}$ and **V**:

$$\mathbf{Z}^{\alpha\bullet}_{s\bullet}\equiv[\, Z^{\alpha1}_{s1},\, Z^{\alpha1}_{s2},\, \cdots,\, Z^{\alpha m}_{s3}\,]\equiv(\mathbf{E}^{\alpha}_{s})^{T}\mathbf{V}^{T}. \tag{4.18}$$

The gradient of the basic displacements $\dfrac{\partial \mathbf{V}}{\partial W^{\alpha}_{s}}$ has an inherent property, being stated in the following theorem.

Theorem 6. *Gradient Orthogonality Theorem of Basic Displacements*:

The gradient of the basic displacement matrix V of a structural system with respect to any of its subelement stiffness modulus W^{α}_{s} is orthogonal to the matrix V

itself with respect to the inverse of its global stiffness modulus matrix W (diagonal), i.e.,

$$\frac{\partial \mathbf{V}^T}{\partial W_s^\alpha}\mathbf{W}^{-1}\mathbf{V} \equiv \mathbf{0}. \tag{4.19}$$

Proof.

From Eqs. (4.16), (4.18), (4.9b) and noting Eqs. (4.11)-(4.13), one has

$$\begin{aligned}
\frac{\partial \mathbf{V}^T}{\partial W_s^\alpha}\mathbf{W}^{-1}\mathbf{V} &= \mathbf{V}_s^\alpha(\mathbf{C}_{s\bullet}^{\alpha\bullet}-\mathbf{Z}_{s\bullet}^{\alpha\bullet})\mathbf{W}^{-1}\mathbf{V}/W_s^\alpha \\
&= \mathbf{V}_s^\alpha \mathbf{C}_{s\bullet}^{\alpha\bullet}\mathbf{V}/(W_s^\alpha)^2 - \mathbf{V}_s^\alpha(\mathbf{E}_s^\alpha)^T\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}/W_s^\alpha \\
&= \mathbf{V}_s^\alpha(\mathbf{V}_s^\alpha)^T/(W_s^\alpha)^2 - \mathbf{V}_s^\alpha(\mathbf{P}_s^\alpha)^T\mathbf{K}^{-1}/(W_s^\alpha)^2 \\
&= \mathbf{V}_s^\alpha(\mathbf{V}_s^\alpha)^T/(W_s^\alpha)^2 - \mathbf{V}_s^\alpha(\mathbf{V}_s^\alpha)^T/(W_s^\alpha)^2 \\
&\equiv \mathbf{0}
\end{aligned}$$

where Theorem 2 in Chapter 2 has been used, i.e.,

$$\mathbf{K}^{-1} = \mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}. \tag{4.20}$$

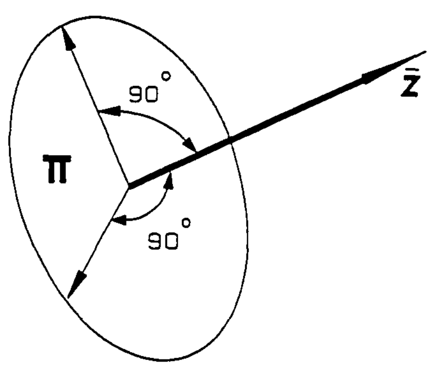

Figure 4.1 Geometrical Interpretation of the Gradient Orthogonality Theorem

By manipulating Eq. (4.19) one can have another form of the theorem, which interprets Eq. (4.19) in terms of forces and deformations.

The global Z-deformation vector is always normal to the hyperplane, Π, formed by the gradients of the basic internal forces of a structural system with respect to the design variables, i.e.,

$$(\frac{d\bar{F}^{\cdot \ell}_{\cdot r}}{db})^T \bar{Z}^{\cdot \ell}_{\cdot r} \equiv 0 \tag{4.21}$$

where $\bar{F}^{\cdot \ell}_{\cdot r}$ represents the global basic internal force vector induced by the unit-load vector $\bar{p}^{\ell}_{r}$ applied at the DOF ($^{\ell}_{r}$) and $\bar{Z}^{\cdot \ell}_{\cdot r}$ is the corresponding global Z-deformation vector. The theorem has a geometrical interpretation as shown in Fig. 4.1, where $\bar{Z}$ stands for the global Z-deformation vector.

Proof.

Premultiplying and postmultiplying Eq. (4.19) by $(\bar{P}^{\ell}_{r})^T$ and by $\bar{P}^{\ell}_{r}$, respectively, yield

$$\frac{\partial(V\bar{P}^{\ell}_{r})^T}{\partial W^{\alpha}_{s}} W^{-1} V\bar{P}^{\ell}_{r} = 0. \tag{4.22}$$

Since $\bar{P}^{\ell}_{r}$ is a unit-load vector acting at the degree of freedom ($^{\ell}_{r}$), the multiplication, $V\bar{P}^{\ell}_{r}$, gives a vector of the entire components of **V** at ($^{\ell}_{r}$). And according to Theorem 1 (see Chapter 2), this vector is the global basic internal force vector $\bar{F}^{\cdot \ell}_{\cdot r}$. The theory of structural variations has shown that the basic internal forces and the Z-deformations are related (see Eq. (2.33)) by

$$\bar{F}^{\cdot \ell}_{\cdot r} = W\bar{Z}^{\cdot \ell}_{\cdot r}. \tag{4.23}$$

Therefore, one has from Eq. (4.22)

$$(\frac{\partial\bar{F}^{\cdot \ell}_{\cdot r}}{\partial W^{\alpha}_{s}})^T \bar{Z}^{\cdot \ell}_{\cdot r} = 0. \tag{4.24}$$

Premultiplying the above equation by $(dw_s^\alpha/db)^T$ and summing it for all the subelements yield the conclusion, Eq. (4.21):

$$\sum_{\alpha=1}^{m}\sum_{s=1}^{3}\left(\frac{\partial \bar{F}^{\cdot\ell}_{\cdot r}}{\partial W_s^\alpha}\frac{dW_s^\alpha}{db}\right)^T \bar{Z}^{\cdot\ell}_{\cdot r} = \left(\frac{d\bar{F}^{\cdot\ell}_{\cdot r}}{db}\right)^T \bar{Z}^{\cdot\ell}_{\cdot r} = 0. \tag{4.25}$$

If an external load **P** is independent of the design variable vector **b**, the orthogonality relationship still holds true for **F** and **Z** induced by **P**, i.e.,

$$\left(\frac{dF}{db}\right)^T Z \equiv 0 \tag{4.26}$$

which can be proven by the same procedure as the one derived above.

4.3 Explicit Formulations for Design Sensitivities

Based on the Gradient Orthogonality Theorem, one can establish a set of explicit formulations for sensitivity analysis to be discussed below.

4.3.1 Explicit Formulation for Design Sensitivity of Inverse Matrix K^{-1}

The derivative of the inverse matrix K^{-1} of the global stiffness matrix K with respect to the stiffness modulus W_s^α of any subelement ($_s^\alpha$) is formulated as

$$\frac{\partial K^{-1}}{\partial W_s^\alpha} = -V_s^\alpha (V_s^\alpha)^T/(W_s^\alpha)^2. \tag{4.27}$$

Proof.

By taking derivative of Eq. (4.20) with respect to W_s^α and noting Eqs. (4.19), (4.10b) and (4.11), one has

$$\frac{\partial K^{-1}}{\partial W_s^\alpha} = \frac{\partial}{\partial W_s^\alpha}\left(V^T W^{-1} V \right)$$

$$=\frac{\partial \mathbf{V}^T}{\partial W_s^\alpha}\mathbf{W}^{-1}\mathbf{V}+\mathbf{V}^T\frac{\partial \mathbf{W}^{-1}}{\partial W_s^\alpha}\mathbf{V}+\mathbf{V}^T\mathbf{W}^{-1}\frac{\partial \mathbf{V}}{\partial W_s^\alpha}$$

$$=-\mathbf{V}^T(\mathbf{C}_{s\bullet}^{\alpha\bullet})^T\mathbf{C}_{s\bullet}^{\alpha\bullet}\mathbf{V}/(W_s^\alpha)^2$$

$$=-\mathbf{V}_s^\alpha(\mathbf{V}_s^\alpha)^T/(W_s^\alpha)^2.$$

4.3.2 Explicit Formulation for Design Sensitivity of Displacement Vector D

First, consider $\partial\mathbf{D}/\partial W_s^\alpha$. From Eqs. (4.1), (4.20) and (4.27), one has

$$\frac{\partial \mathbf{D}}{\partial W_s^\alpha}=\frac{\partial}{\partial W_s^\alpha}(\mathbf{K}^{-1}\mathbf{P})$$

$$=\frac{\partial \mathbf{K}^{-1}}{\partial W_s^\alpha}\mathbf{P}+\mathbf{K}^{-1}\frac{\partial \mathbf{P}}{\partial W_s^\alpha}$$

$$=-\mathbf{V}_s^\alpha(\mathbf{V}_s^\alpha)^T\mathbf{P}/(W_s^\alpha)^2+\mathbf{K}^{-1}\frac{\partial \mathbf{P}}{\partial W_s^\alpha}$$

$$=-\mathbf{V}_s^\alpha Z_s^\alpha/W_s^\alpha+\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha}$$

where Z_s^α is the Z-deformation induced by the external load **P**, i.e., $Z_s^\alpha=(\mathbf{V}_s^\alpha)^T\mathbf{P}/W_s^\alpha$. Using Eq. (4.5) and above expression yields

$$\frac{d\mathbf{D}}{db}=[-\sum_{\alpha=1}^{m}\sum_{s=1}^{3}(\mathbf{V}_s^\alpha Z_s^\alpha/W_s^\alpha+\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha})]\frac{dW_s^\alpha}{db}$$

or rewriting it in the matrix form, one has the final formulation :

$$\frac{d\mathbf{D}}{db}=\mathbf{V}^T\mathbf{W}^{-1}(-\mathbf{Z}^\#\frac{d\mathbf{W}}{db}+\mathbf{V}\frac{d\mathbf{P}}{db}) \tag{4.28}$$

where

$$\mathbf{Z}^\#\equiv\mathrm{diag}(Z_s^\alpha),\quad \binom{\alpha}{s}=\binom{1}{1},\binom{1}{2},\cdots,\binom{m}{3}\,. \tag{4.29}$$

4.3.3 Explicit Formulation for Design Sensitivity of Generalized Internal Force Vector F

In the theory of structural variations, the element internal forces or stresses are calculated via the generalized internal forces **F**, while **F** is explicitly formulated through **V** (Eq. (2.33)):

$$\mathbf{F}=\mathbf{VP}. \tag{4.30}$$

Note that for a skeletal structure, **F**, called the mid-section internal force vector, is the global internal force vector at the middle-span sections of the beam elements which was denoted by $\mathring{\mathbf{F}}$ in [9].

The derivative of **F** with respect to W_s^α can be derived from Eqs. (4.30) and (4.17) as

$$\begin{aligned}
\frac{\partial \mathbf{F}}{\partial W_s^\alpha} &= \frac{\partial}{\partial W_s^\alpha}(\mathbf{VP}) \\
&=\left(\frac{\partial \mathbf{V}}{\partial W_s^\alpha}\right)\mathbf{P} + \mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha} \\
&=[(\mathbf{C}_{s\bullet}^{\alpha\bullet})^T-(\mathbf{Z}_{s\bullet}^{\alpha\bullet})^T](\mathbf{V}_s^\alpha)^T\mathbf{P}/W_s^\alpha + \mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha} \\
&=[(\mathbf{C}_{s\bullet}^{\alpha\bullet})^T-(\mathbf{Z}_{s\bullet}^{\alpha\bullet})^T]Z_s^\alpha + \mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha} .
\end{aligned}$$

Using Eq. (4.5) yields

$$\frac{d\mathbf{F}}{db} = \sum_{\alpha=1}^{m}\sum_{s=1}^{3}[(\mathbf{C}_{s\bullet}^{\alpha\bullet})^T-(\mathbf{Z}_{s\bullet}^{\alpha\bullet})^T)Z_s^\alpha + \mathbf{V}\frac{\partial \mathbf{P}}{\partial W_s^\alpha}]\frac{dW_s^\alpha}{db}$$

or

$$\frac{d\mathbf{F}}{db} = [\mathbf{I} - \tilde{\mathbf{Z}}^T]\mathbf{Z}^{\#}\frac{dw}{db} + \mathbf{V}\frac{d\mathbf{P}}{db} \tag{4.31}$$

where **I**, 3mx3m, is a unit-matrix and $\tilde{\mathbf{Z}}$, 3mx3m, is the global Z-deformation matrix pertaining to all the subelements of the system:

$$\tilde{\mathbf{Z}} \equiv [Z_{sr}^{\alpha\beta}] \equiv \mathbf{H}^T\mathbf{V}^T \tag{4.32}$$

where

$$Z_{sr}^{\alpha\beta} \equiv (\mathbf{E}_s^{\alpha})^T \mathbf{V}_r^{\beta} \tag{4.33}$$

and **H** is the global transfer matrix of the system:

$$\mathbf{H} \equiv [\ \mathbf{E}_1^1,\ \mathbf{E}_2^1,\ \cdots,\ \mathbf{E}_3^m\ ]. \tag{4.34}$$

4.3.4 Explicit Formulation for Design Sensitivity of Stresses in a Triangular Element

Equation (4.31) is general and applicable to any finite element models. However, the relationship between **F** and σ differs from one element model to another one. Considering a constant strain triangular element of an isotropic material, Eqs. (3.24) and (2.32) have shown that

$$\sigma = \frac{1}{t}\mathbf{Q}^{-1}\mathbf{F}^{\alpha} \tag{4.35}$$

where t is the thickness of element α, and

$$\mathbf{F}^{\alpha} = \mathbf{V}^{\alpha}\mathbf{P} \tag{4.36}$$

where $\mathbf{F}^{\alpha}$, 3x1, is the generalized internal force vector of element α, i.e., a subset of **F**, and $(\mathbf{V}^{\alpha})_{3x2n}$ the basic displacement matrix of element α, a subset of **V**, while **Q** in this case is a matrix:

$$\mathbf{Q} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.37}$$

Thus, the sensitivity of element stresses σ can be obtained from Eqs. (4.35), (4.36) and (4.31):

$$\frac{d\sigma}{db} = \frac{d}{db}\left(\frac{1}{t}\, \mathbf{Q}^{-1}\mathbf{F}^{\alpha}\right)$$

$$= \frac{1}{t}\, \mathbf{Q}^{-1}\frac{d\mathbf{F}^{\alpha}}{db} - \frac{1}{t^2}\mathbf{Q}^{-1}\, \mathbf{F}^{\alpha}\frac{dt}{db}$$

$$= \frac{\mathbf{Q}^{-1}}{t}\left\{ [(\mathbf{C}_{\bullet\bullet}^{\bullet\alpha})^{T}-(\mathbf{Z}_{\bullet\bullet}^{\bullet\alpha})^{T}]\mathbf{Z}^{\#}\frac{d\mathbf{W}}{db} - \frac{\mathbf{F}^{\alpha}}{t}\frac{dt}{db} + \mathbf{V}^{\alpha}\frac{d\mathbf{P}}{db} \right\} \tag{4.38}$$

where $\mathbf{C}_{\bullet\bullet}^{\bullet\alpha}$, 3mx3, stands for a portion of **C** and $\mathbf{Z}_{\bullet\bullet}^{\bullet\alpha}$, 3mx3, a portion of $\tilde{\mathbf{Z}}$, which correspond to subelements $({}_{s}^{\alpha})$, s=1,2,3, of element α.

Thus, one can directly obtain the sensitivities of $\mathbf{K}^{-1}$, **D**, **F** and σ from the explicit formulations, Eqs. (4.27), (4.28), (4.31) and (4.38), respectively, provided that **V** is obtained by the SVM, which requires neither assembling nor solving simultaneous equations.

4.4 The Evaluation Theorem of Principal Z-Deformations in Static Systems

In Chapter 2, a question has been left open to be clarified: whether the term $(1+m_s^{\alpha}Z_{ss}^{\alpha\alpha})$ can become zero. If so, $\hat{V}_s^{\alpha}$ and $\hat{Z}_{ss}^{\alpha\alpha}$ could not be determined by Eqs. (2.37) and (2.38). Fortunately, it can be proven that the principal Z-deformation, Z_{ss}, has a finite value for all W_s^{α}, $0 \leq W_s^{\alpha} \leq \infty$. This observation represents an important feature of finite element systems, being stated as the following theorem.

Note: to simplify the notations, from now on the superscripts in Greek of any key symbol will be dropped, e.g., W_s^α *will be simplified as* W_s, $Z_{ss}^{\alpha\alpha}$ *as* Z_{ss} *and so on so forth, where the element number* α *is dropped and the subscript s is regarded as a subelement number in the global order. Nevertheless, when the element number becomes important, its superscript* α *in the notation will be restored as before.*

Theorem 7. *The Evaluation Theorem of Principal Z-Deformations* (static systems):

In a finite element system, the principal Z-deformation Z_{ss} of any subelement s is subject to

$$0 \leq Z_{ss} \leq 1 \tag{4.49}$$

and varies monotonously with W_s, i.e.,

$$\frac{dZ_{ss}}{dW_s} > 0, \quad \text{for all } W_s \geq 0\,. \tag{4.50}$$

Proof.

By the definition of Z-deformations and the Explicit Decomposition Theorem on the inverse of the global stiffness matrix one has

$$\begin{aligned} Z_{ss} &= (\mathbf{E}_s)^T\mathbf{V}_s \\ &= (\mathbf{E}_s)^T\mathbf{K}^{-1}\mathbf{P}_s \\ &= (\mathbf{E}_s)^T\mathbf{V}^T\mathbf{W}^{-1}\mathbf{V}\mathbf{E}_s\mathbf{W}_s \\ &= \sum_{r=1}^{p}(Z_{sr})^2 W_s/W_r \\ &= (Z_{ss})^2 + \sum_{r=1}^{p}(Z_{sr})^2 W_s/W_r, \quad (r \neq s) \\ &= (Z_{ss})^2 + S > 0 \end{aligned} \tag{4.51}$$

where p is the total number of subelements and S stands for the sum, i.e.,

$$S \equiv \sum_{r=1}^{p} (Z_{sr})^2 W_s/W_r > 0, \quad r \neq s. \tag{4.52}$$

Thus, from Eq. (4.51) one has

$$(Z_{ss})^2 - Z_{ss} + S = 0$$

from which one further has

$$Z_{ss} = \tfrac{1}{2}(1 \pm \sqrt{1-4S}). \tag{4.53}$$

Since Z_{ss} is a positive real number due to Eq. (4.51), S must be subjected to

$$0 \leq S \leq \frac{1}{4} \tag{4.54}$$

which implies $0 \leq Z_{ss} \leq 1$. Thus, with Eqs. (4.53) and (4.54) one arrives at the conclusion (4.49). To prove the second part of the theorem, Eq. (4.50), one should take advantage of Eq. (2.37) of Chapter 2, from which one has the varied Z-deformation due to ΔW_s as

$$\begin{aligned} \hat{Z}_{ss} &= Z_{ss}(W_s^{\alpha} + \Delta W_s) \\ &= (E_s)^T \hat{V}_s \\ &= Z_{ss} \frac{1+m_s}{1+m_s Z_{ss}}. \end{aligned} \tag{4.55}$$

By the definition of derivatives and Eq. (4.55), one has

$$\begin{aligned} \frac{dZ_{ss}}{dW_s} &= \lim (\hat{Z}_{ss} - Z_{ss}) \big|_{m_s \to 0} \\ &= \lim \left(\left(\frac{Z_{ss}(1+m_s)}{1+m_s Z_{ss}} - Z_{ss} \right)/(m_s W_s) \right) \big|_{m_s \to 0} \\ &= Z_{ss}(1-Z_{ss})/W_s . \end{aligned} \tag{4.56}$$

Thus, Eqs. (4.49) and (4.56) lead to the conclusion (4.50).

This theorem is illustrated in Fig. 4.2, in which Z and W represent any principal Z-deformation and the corresponding stiffness modulus, respectively. From Eq. (4.49) and Fig. 4.2, one can see that the principal Z-deformation, $Z^{\alpha\alpha}_{ss}$, of a real subelement $(^{\alpha}_{s})$, whose stiffness modulus is subject to $0 < W^{\alpha}_{s} < \infty$, must be limited to $0 < Z^{\alpha\alpha}_{ss} < 1$. And with the variation factor $m^{\alpha}_{s} > -1$, the term $(1+m^{\alpha}_{s}Z^{\alpha\alpha}_{ss})$ never becomes zero.

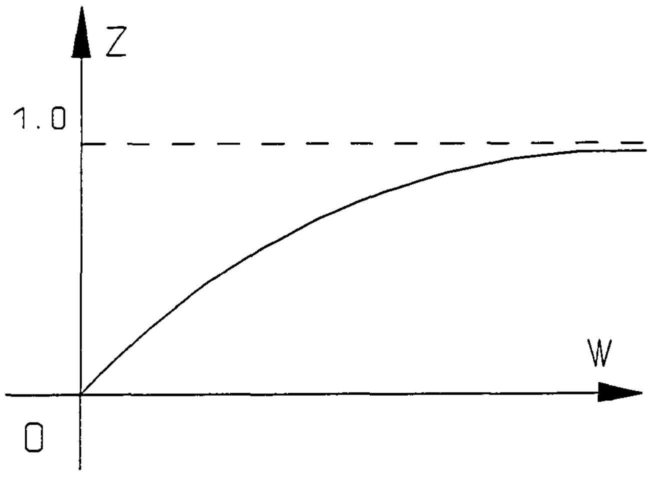

Figure 4.2 Limits and Monotonousness of Principal Z-deformations in Static Systems

4.5 Illustrative Example

A square plate with an edge length L=1.0, Young's modulus E=1.0, Poisson's ratio ν=0.3 and thickness t=1.0, is discretized into two triangular elements as shown in Fig. 4.3. It has been analyzed by using SVM in Chapter 3 and its **V**, **F** and **D** have been obtained as

$$\mathbf{F}=[\underbrace{0.5, -0.5, 0.0}_{\text{element 1}} \,\vdots\, \underbrace{0.5, -0.5, 0.0}_{\text{element 2}}]^T$$

$$\mathbf{D}=[\,-0.3,\ 0.0\ \vdots\ 0.0,\ 1.0\ \vdots\ -0.3,\ 1.0\,]$$

node 2 node 3 node 4

$$\mathbf{V}=\begin{bmatrix} .825, & 0 & \vdots & .65, & .825 & \vdots & .825, & .175 \\ 1.0, & 0 & \vdots & 0, & -1.0 & \vdots & 0, & 0 \\ .175, & 0 & \vdots & 1.35, & .175 & \vdots & 1.175, & -.175 \\ .175, & 0 & \vdots & -.65, & .175 & \vdots & .175, & .825 \\ 0, & 0 & \vdots & 0, & 0 & \vdots & 1.0, & -1.0 \\ -.175, & 0 & \vdots & .65, & -.175 & \vdots & .825, & .175 \end{bmatrix}.$$

node 2 node 3 node 4

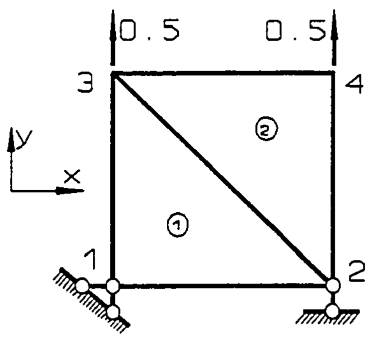

Figure 4.3 A Triangular Finite Element System

Now, find $\frac{d\mathbf{D}}{d\mathbf{b}}$ by using Eq. (4.28), where $\mathbf{b}=[\,t_1,\ t_2\,]^T$; t_1 and t_2 are the thicknesses of elements 1 and 2, respectively, and $t_1=t_2=1.0$ at the current design. From the same example given in Subsection 3.6 of Chapter 3, one has already had the initial data:

$$\mathbf{H}=\begin{bmatrix} .5 & .5 & 0 & 0 & 0 & -.5 \\ 0 & 0 & .5 & -.5 & .5 & 0 \\ 0 & 0 & .5 & -.5 & -.5 & 0 \\ .5 & -.5 & 0 & 0 & 0 & -.5 \\ 0 & 0 & 0 & .5 & .5 & .5 \\ 0 & 0 & 0 & .5 & -.5 & .5 \end{bmatrix}$$

where the components of **H** at node 1 have been dropped in the purpose of calculating $\mathbf{H}^T\mathbf{V}^T$.

$$W=\text{diag}\left[\frac{10}{7}\ \ \frac{10}{13}\ \ \frac{10}{13}\ \ \frac{10}{7}\ \ \frac{10}{13}\ \ \frac{10}{13}\right]$$

$$w=\left[\frac{10}{7}\ \ \frac{10}{13}\ \ \frac{10}{13}\ \ \frac{10}{7}\ \ \frac{10}{13}\ \ \frac{10}{13}\right]^T$$

$$\frac{dw}{db}=\begin{bmatrix} \frac{10}{7} & \frac{10}{13} & \frac{10}{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{10}{7} & \frac{10}{13} & \frac{10}{13} \end{bmatrix}^T$$

$$\frac{dP}{db}=0$$

$$Z=W^{-1}F=[\ .35,\ -.65,\ 0,\ .35,\ -.65,\ 0\ ]^T$$

$$Z^{\#}=\text{diag}(\ .35,\ -.65,\ 0,\ .35,\ -.65,\ 0\ ).$$

Substituting them into Eq. (4.28) yields

$$\frac{dD}{db}=-V^TW^{-1}Z^{\#}\frac{dw}{db}=\begin{bmatrix} .36125, & -.06125 \\ 0, & 0 \\ -.22750, & .22750 \\ -.93875, & -.06125 \\ -.28875, & .58875 \\ -.06125, & -.93875 \end{bmatrix}.$$

One can see that the result for $\frac{dD}{db}$ is the exact derivative for this simple example, without rounded error.

One may be interested in verification of the orthogonality, Eq. (4.26), by employing the above example. From the above obtained information, one has

$$\tilde{Z}=H^TV^T=\begin{bmatrix} .8255, & 0, & .175, & .175, & 0, & -.175 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ .325, & 0, & .675, & -.325, & 0, & .325 \\ .175, & 0, & -.175, & .825, & 0, & .175 \\ 0, & 0, & 0, & 0, & 1, & 0 \\ -.175, & 0, & .325, & .175, & 0, & .675 \end{bmatrix}.$$

Then, Eq. (4.31) gives

$$\frac{dF}{db}=[I-\tilde{Z}^T]Z^{\#}\frac{dw}{db}=\begin{bmatrix} .0875, & -.0875 \\ 0, & 0 \\ -.0875, & .0875 \\ -.0875, & .0875 \\ 0, & 0 \\ .0875, & -.0875 \end{bmatrix}.$$

Therefore, the product of $\frac{dF}{db}$ and $\mathbf{Z}$ is obtained as

$$\left(\frac{dF}{db}\right)^T Z=\begin{bmatrix} .0875, & 0, & -.0875, & -.0875, & 0, & .0875 \\ -.0875, & 0, & .0875, & .0875, & 0, & -.0875 \end{bmatrix}\begin{bmatrix} .35 \\ -.65 \\ 0 \\ .35 \\ -.65 \\ 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Chapter 5

VIBRATION ANALYSIS USING THE THEORY OF STRUCTURAL VARIATIONS*

5.1 Introduction

This chapter will present a new method based upon the theory of structural variations for calculating eigenvalues and eigenvectors (eigenpairs) of finite element systems in solid mechanics. For convenience, this method is called the **Z-deformation method.**

Although there exist a number of methods [19]-[21] for computing eigenpairs of finite element systems, some questions still remain to be investigated. For example, one of the most commonly used methods to solve a few lowest eigenpairs of structural systems is the inverse power iteration method. However, the convergence rate of the inverse power method [19] strongly depends on the closeness of the adjacent eigenvalues and the initial guess for the eigenvector. Even with the shifting technique, this method still performs very poorly in terms of accuracy and efficiency when the adjacent eigenvalues are very close.

* The contents of this chapter has been presented in [18].

The proposed Z-deformation method is not an iteration method but a procedure of successive advances. This method can provide as many eigenpairs as needed just like the inverse power method, however, without the shortcomings mentioned above.

The new method is based on an interesting and useful property of finite element systems, which is stated as **the Monotonousness Theorem of Principal Z-deformations** and will be proven in this chapter by using the theory of structural variations established in the previous chapters. The so-called Z-deformation is a technical term defined in Section 2.1 of Chapter 2 and Section 3.3 of Chapter 3, representing a sort of generalized deformations of an element. The Z-deformations discussed in the previous chapters are about the structural systems with positive stiffness moduli. However, to extend the initial theory of structural variations to include vibration analysis, the concept of the negative stiffness of subelements has to be introduced into the system. The subelement with negative stiffness will be called the **mass-subelement** which is related to the inertial properties of the system.

5.2 Mass-Subelements

Suppose that an eigensystem is described by

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{D}=\mathbf{0} \tag{5.1}$$

where **M** is the mass matrix (symmetric), **D** the nodal displacement vector and λ the parameter to be determined for eigenvalues λ_i, $i=1, 2, ..., N$, where N is the total number of eigenpairs of the system. For simplicity, **M** is assumed to be a lumped mass matrix and the eigenvalues are arranged in an ascendant order:

$$0 < \lambda_1 < \lambda_2 \ldots < \lambda_N \tag{5.2}$$

As discussed in Chapters 2 and 3, the global stiffness matrix is the matrix **K** which is composed of p number of subelements in a static system. In this chapter, however, the global matrix is the matrix (**K**-λ**M**) which includes the negative stiffness matrix, -λ**M**. The negative stiffness, $-\lambda M_k$ (the k-th non-zero diagonal element of -λ**M**), can be regarded as the contribution of a special subelement to the global stiffness matrix (**K**-λ**M**). This special subelement is called the **mass-subelement** and denoted by $\binom{M}{s}$, where the subscript s implies the subelement number of this mass-subelement and is always arranged after the p subelements with positive stiffness, i.e., for mass-subelements $\binom{M}{s}$, $p+1 \le s \le p+N$. Its subelement vector $\mathbf{E}_s$ and stiffness modulus W_s are defined as follows:

$$\mathbf{E}_s \equiv [\,-1,\ 1\,]^T \tag{5.3}$$

$$W_s \equiv -\lambda M_k, \quad s=p+k;\ k=1,2,\cdots,N \tag{5.4}$$

where the values -1 and 1 in $\mathbf{E}_s$ correspond to two degrees of freedom; one of them is associated with the mass-subelement $\binom{M}{s}$ and the other is the degree of freedom of the ground, respectively. One can see that the distinction of a mass-subelement from a typical subelement in static systems is that a mass-subelement may have a negative stiffness modulus depending on the value of the parameter λ, while in static systems every stiffness modulus should be positive (see Eqs. (2.8) and (3.14)).

Thus, an eigensystem is composed of p subelements with positive stiffness moduli and N mass-subelements with negative moduli; therefore, the total number of subelements of an eigensystem will be p+N. The global stiffness matrix (**K**-λ**M**) may or may not be non-singular, depending on the value of λ. However, they are all legitimate subelements and therefore all the formulas and theorems established in the TSV can also be applied here to the eigensystems, except for the case when λ takes some special values, i.e., eigenvalues $\lambda=\lambda_i$, $i=1,2,\cdots,N$, which make $(K-\lambda_i M)$ singular.

To find out these special values, i.e., the eigenvalues, a new computational procedure can be derived by taking advantage of some intrinsic properties of finite element systems, including those already established in the foregoing chapters and the new one to be proven below.

5.3 The Monotonousness Theorem of Principal Z-deformations for Eigensystems

The Evaluation Theorem of Principal Z-Deformations proved in Chapter 4 states that the principal Z-deformation, Z_{ss}, of any subelement $\binom{\alpha}{s}$ with its subelement stiffness modulus $W_s \geq 0$, $1 \leq s \leq p$, is always less than or equal to 1 and has the nature of monotonousness. In an eigensystem, the Z_{ss} of a mass-subelement $\binom{M}{s}$, $p+1 \leq s \leq p+N$, may encounter some W_s which may be negative, ranging from $-\infty$ to ∞, i.e., $-\infty < W_s < +\infty$, or $-\infty < \lambda < +\infty$. In this case, however, the nature of monotonousness still holds true almost everywhere as stated in the following theorem.

Theorem 8. *The Monotonousness Theorem of Principal Z-deformations*

(for eigensystems):

In an eigensystem, the derivative of the principal Z-deformation, Z_{ss}, of any mass-subelement with respect to its stiffness modulus W_s is always greater than zero, except at N singular points which correspond to the eigenvalues, i.e.,

$$\frac{dZ_{ss}}{dW_s} > 0 \text{ except at } \lambda=\lambda_i,\ i=1,2,\ \ldots,\ N; \quad p+1\leq s\leq p+N \tag{5.5}$$

$$\frac{dZ_{ss}}{dW_s}\Big|_{W_s\to\pm\infty} = 0 \tag{5.6}$$

and

$$Z_{ss}\Big|_{W_s\to\pm\infty} = 1. \tag{5.7}$$

Proof.

For convenience, use K_λ to denote $(K-\lambda M)$ with $\lambda\neq\lambda_i$, so, K_λ is non-singular and rewrite Eq. (5.4) as

$$W_s = \lambda\bar{W}_s, \qquad s=p+1,\ p+2,\ \cdots,\ p+N \tag{5.8}$$

where

$$\bar{W}_s \equiv -M_k, \qquad k= s - p. \tag{5.9}$$

Therefore, a differential dW_s can be given as

$$dW_s = \bar{W}_s d\lambda. \tag{5.10}$$

From Eqs. (2.28), (2.19), (2.29), (4.27) and (5.10), one has

$$\frac{dZ_{ss}}{dW_s} = \frac{d}{dW_s}\left((E_s)^T V_s \right)$$

$$= \frac{d}{dW_s}\left((E_s)^T (K_\lambda)^{-1} E_s W_s \right)$$

$$= (E_s)^T\left(\frac{dK_\lambda^{-1}}{d\lambda}\frac{d\lambda}{dW_s} \right)E_s W_s + (E_s)^T V^T W^{-1} V E_s$$

$$= (E_s)^T\left(\sum_{r=p+1}^{p+N} \frac{\partial K_\lambda^{-1}}{\partial W_r}\frac{dW_r}{d\lambda} \right) \frac{d\lambda}{dW_s}\, E_s W_s + \sum_{r=1}^{p+N} (Z_{sr})^2/W_r$$

$$= -(E_s)^T\left[\sum_{r=p+1}^{p+N} V_r (V_r)^T \bar{W}_r W_s / ((W_r)^2 \bar{W}_s) \right]E_s + \sum_{r=1}^{p} (Z_{sr})^2/W_r + \sum_{r=p+1}^{p+N} (Z_{sr})^2/W_r$$

$$= -\sum_{r=p+1}^{p+N} \left((E_s)^T V_r (V_r)^T E_s / W_r \right) + \sum_{r=1}^{p} (Z_{sr})^2/W_r + \sum_{r=p+1}^{p+N} (Z_{sr})^2/W_r$$

$$= -\sum_{r=p+1}^{p+N} (Z_{sr})^2/W_r + \sum_{r=1}^{p} (Z_{sr})^2/W_r + \sum_{r=p+1}^{p+N} (Z_{sr})^2/W_r$$

$$= \sum_{r=1}^{p} (Z_{sr})^2/W_r \; > 0.$$

The last equation has shown the conclusion, Eq. (5.5). To show the conclusions, Eqs. (5.6) and (5.7), one needs to recall Eqs. (4.55) and (4.56) given in Chapter 4. Since Eq. (4.55) holds for any real number of W_s, it leads to the conclusion, Eq. (5.7) for eigensystems. And therefore, Eq. (4.56) leads to Eq. (5.6), too. Then the theorem is proven.

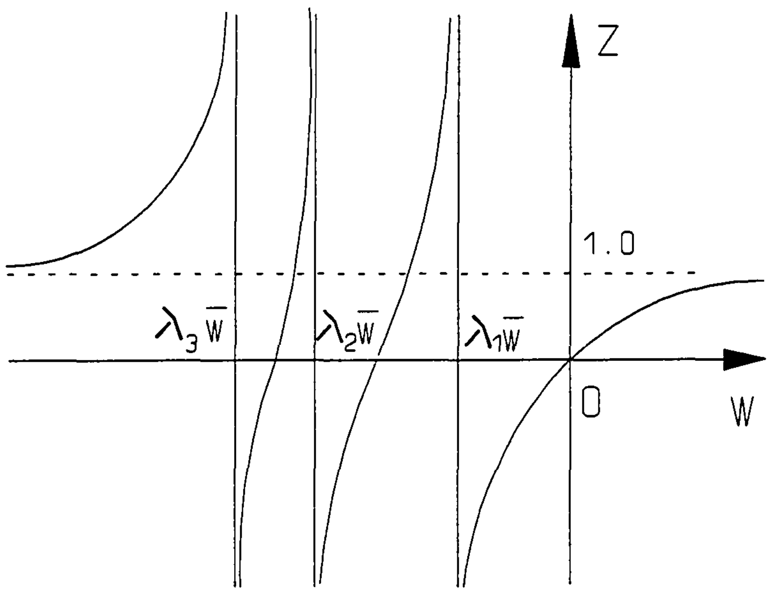

Figure 5.1 Monotonousness of Principal Z-Deformations in Eigensystems

Thus, one has a typical plot for Z_{ss} vs. W_s for eigensystems, as shown in Fig. 5.1, in which Z and W stand for the principal Z-deformation of any $\binom{M}{s}$ and the corresponding stiffness modulus, respectively. By comparing Fig. 4.2 to Fig. 5.1, it is an interesting observation that the former is a special case of the latter one when $W_s^\alpha \geq 0$, and therefore, a system with positive stiffness moduli is a special case of a general system with unbounded stiffness moduli.

5.4 The Z-Deformation Method for Vibration Analysis

A new method for calculating eigenpairs is provided here, based on the Monotonousness Theorem described in the previous section. In this method, the eigenvectors are identified as the basic displacement vectors. The procedures to calculate the eigenvalues and eigenvectors are discussed respectively in Subsections 5.4.1 and 5.4.2, while an equivalent eigensystem will be introduced for the calculation of higher order eigenpairs in Subsection 5.4.3.

5.4.1 Method for Finding the Fundamental Eigenvalue

The monotonousness of principal Z-deformations gives a hint to find the lowest eigenvalue λ_1 (Fig. 5.1) by using a simple successive approach, starting from $Z(\lambda=0)=0$ and stopping at $Z=-\infty$. This approach depends on neither the ratio λ_1/λ_2 nor the initial guess for the eigenvector. The Z-deformation corresponding to any value of λ can be computed by using the explicit formulations, Eqs. (2.37) and (4.55).

To implement this approach, one can devise a variety of recurrence formulas for λ to reach λ_1 in successive steps. One of such formulas is suggested below. For simplicity, in the following discussion one will use the letters Z and x to represent any principal Z-deformation and the corresponding λ, and the letter Z_k to represent the Z evaluated at x_k, where k is the step number in the calculation. Suppose that one has already known $Z_k=Z(x_k)$, $x_k<\lambda_1$ (see Fig. 5.2), then one can have an interpolation for Z(x) based upon the Z values at x_{k-2}, x_{k-1} and x_k as follows.

$$Z(x)=(A+Bx)/(1+\alpha x), \quad x_{k-2}\le x\le x_k \tag{5.11}$$

where the constants A, B and α are determined by a curve fitting $Z(x_k)$ through $Z_{k-2}(x_{k-2})$, $Z_{k-1}(x_{k-1})$ and $Z_k(x_k)$, resulting in Eq. (5.12). For simplicity, these three pairs of values are simply denoted as (Z_1, x_1), (Z_2, x_2) and (Z_3, x_3), respectively, in Fig. 5.2 and in the following equations.

$$\begin{bmatrix} A \\ B \\ \alpha \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{5.12}$$

where

$$g_{11}= x_2x_3(Z_3-Z_2)/\Delta$$

$$g_{21}= (x_2Z_2-x_3Z_3)/\Delta$$

$$g_{31}= (x_3-x_2)/\Delta$$

$$g_{12}= x_1x_3(Z_1-Z_3)/\Delta$$

$$g_{22}= (x_3Z_3-x_1Z_1)/\Delta$$

$$g_{32}= (x_1-x_3)/\Delta$$

$$g_{13}= x_1x_2(Z_2-Z_1)/\Delta$$

$$g_{23}= (x_1Z_1-x_2Z_2)/\Delta$$

$$g_{33}= (x_2-x_1)/\Delta$$

$$\Delta= x_1x_2(Z_2-Z_1)+x_1x_3(Z_1-Z_3)+x_2x_3(Z_3-Z_2).$$

By approximating $Z_4=\beta Z_3$ in Eq. (5.11), where $\beta>1$ is an arbitrary factor indicating the step length from x_3 to x_4, the next step, x_4 (Fig. 5.2), is determined from Eq. (5.11) as

$$x_4=(\beta Z_3-A)/(B+\alpha\beta Z_3) \tag{5.13}$$

which provides a new estimated value, x_4 , approaching towards λ_1. Next, one has to compute the true Z_4 from Eq. (4.55) with $\lambda=x_4$ to keep Z_4 on the true Z-λ curve. Then, one can use Z_2, Z_3 and Z_4 to obtain x_5, and so on so forth until $x_k-x_{k-1}<\epsilon$, where ϵ is a tolerance, e.g., 10^{-10}; therefore $\lambda_1= x_k\pm\epsilon$ will be achieved.

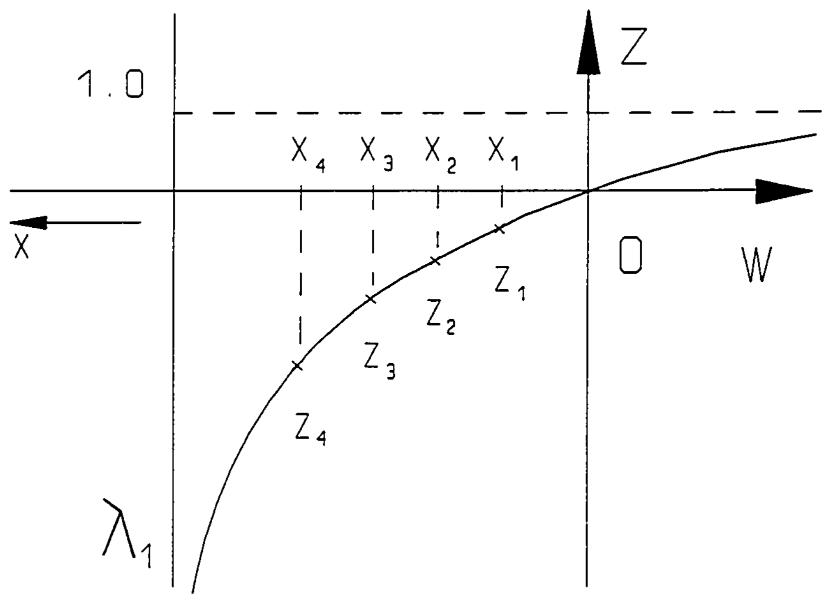

Figure 5.2 Advancing Steps towards λ_1

When x_k gets close to λ_1, Z_k grows very rapidly to $-\infty$. In this situation, a more efficient recurrence formula would rather be used:

$$x_{k+2} = x_{k+1} + \left(1-\frac{1}{\beta} \right) Z_k \frac{x_{k+1}-x_k}{Z_{k+1}-Z_k} \tag{5.14}$$

which comes from an approximation $Z(x)=A/(1-\alpha x)$ where A and α are certain new constants determined by requiring $Z(x_k)=Z_k$ for the last two steps. This new approximation gives a function with steeper slope compared to that in Eq. (5.13).

5.4.2 Proof of the Equivalence between BD Vectors and Eigenvectors

On obtaining λ_1, one can find the corresponding eigenvector $\mathbf{D}_1$ by taking advantage of another property of eigensystems, the equivalence between BD vectors and eigenvectors, being proven below.

Theorem 9. *The Equivalence Theorem of BD Vectors and Eigenvectors*:

In an eigensystem with the λ_i known, the BD vector, $\mathbf{V}_s$, of any mass-subelement $\binom{M}{s}$ with its $W_s \rightarrow \pm\infty$, $p+1 \leq s \leq p+N$, is just the eigenvector $\mathbf{D}_i$ corresponding to λ_i, i.e.,

$$\mathbf{D}_i = \mathbf{V}_s\big|_{W_s \rightarrow \pm\infty}, \qquad i=1,2,\cdots,N;\ \ p+1 \leq s \leq p+N. \tag{5.15}$$

Proof.

Let a single stiffness modulus, W_s of $\binom{M}{s}$, be increased by an increment $\Delta W_s=\xi W_s=\xi\lambda_i\bar{W}_s$, where ξ is an arbitrary parameter and λ_i is the eigenvalue. Let the basic displacement vector of the mass-subelement $\binom{M}{s}$ with $W_s+\Delta W_s$ be denoted by $\hat{\mathbf{V}}_s$. Then, its new stiffness modulus of the perturbed subelement is $W_s(1+\xi)=\lambda_i\bar{W}_s(1+\xi)=-\lambda_i M_k(1+\xi)$, where the subscript k is the DOF number to which the mass-subelement $\binom{M}{s}$ is attached. According to the theory of structural variations, $\hat{\mathbf{V}}_s$ must satisfy the following equation:

$$[\mathbf{K}-\lambda_i\mathbf{M}+\xi\lambda_i\bar{W}_s\mathbf{E}_s(\mathbf{E}_s)^T]\hat{\mathbf{V}}_s=\lambda_i\bar{W}_s\mathbf{E}_s(1+\xi)$$

or

$$[\mathbf{K}-\lambda_i\mathbf{M}]\hat{\mathbf{V}}_s + \xi\lambda_i\bar{W}_s\mathbf{E}_s(\mathbf{E}_s)^T\hat{\mathbf{V}}_s=\lambda_i\bar{W}_s\mathbf{E}_s(1+\xi). \tag{5.16}$$

Premultiplying Eq. (5.16) by $(\mathbf{D}_i)^T$ yields

$$\xi(\mathbf{D}_i)^T\mathbf{E}_s(\mathbf{E}_s)^T\hat{\mathbf{V}}_s=(\mathbf{D}_i)^T\mathbf{E}_s(1+\xi)$$

where $(\mathbf{K}-\lambda_i\mathbf{M})\mathbf{D}_i=\mathbf{0}$ has been used. If $(\mathbf{D}_i)^T\mathbf{E}_s\neq 0$, then one has from the above equation

$$(\mathbf{E}_s)^T\hat{\mathbf{V}}_s=\hat{Z}_{ss}=1+\frac{1}{\xi}, \qquad \xi\neq 0. \tag{5.17}$$

Substituting Eq. (5.17) into Eq. (5.16) gives

$$[\mathbf{K}-\lambda_i\mathbf{M}]\hat{\mathbf{V}}_s + \lambda_i\bar{W}_s\mathbf{E}_s(1+\xi)=\lambda_i\bar{W}_s\mathbf{E}_s(1+\xi)$$

from which one has

$$[\mathbf{K}-\lambda_i\mathbf{M}]\hat{\mathbf{V}}_s=\mathbf{0}. \tag{5.18}$$

Equation (5.18) shows that the BD vector $\hat{\mathbf{V}}_s$ of any mass-subelement (^M_s) with $W_s(1+\xi)$ is an eigenvector for all possible values of ξ except at $\xi=0$. To be specific, let ξ approach $\pm\infty$, i.e., $W_s\rightarrow\pm\infty$. In this situation, Eq. (5.17) leads to $\hat{Z}_{ss}=1$. Thus, the equivalence (5.15) is proven. Nevertheless, the $\mathbf{D}_i$ as an eigenvector obtained from Eq. (5.15) should be normalized by requiring $(\mathbf{D}_i)^T\mathbf{M}\mathbf{D}_i=1$ for its standard normalization.

In practical computations, $\mathbf{V}_s$ with $W_s=\lambda_i\bar{W}_s$ is only a symbol, because its components will theoretically approaches $\pm\infty$ (see Fig. 5.1). However, $\hat{\mathbf{V}}_s\,|_{W_s\rightarrow\pm\infty}$ has a limit (by the Evaluation Theorem and the Monotonousness Theorem). Therefore, with the λ_i and $\mathbf{V}_s$ known, the eigenvector $\mathbf{D}_i$ can be obtained by

$$\begin{aligned}\mathbf{D}_i&=\hat{\mathbf{V}}_s\,|_{W_s\rightarrow\pm\infty}\\ &=\mathbf{V}_s\frac{1+\xi}{1+\xi Z_{ss}}\,|_{\xi\rightarrow\pm\infty}\\ &=\mathbf{V}_s\,/\,Z_{ss}\end{aligned} \tag{5.19}$$

where Theorem 4 has been used. So, $\mathbf{D}_i$ is actually obtained simultaneously with λ_i.

As to the requirement $(\mathbf{D}_i)^T\mathbf{E}_s \neq 0$, one can see from Eq. (5.3) that $(\mathbf{D}_i)^T\mathbf{E}_s = (\mathbf{D}_i)_k$, the component of $\mathbf{D}_i$ at DOF k where $\binom{M}{s}$ resides. So, to meet this requirement one can take any mass-subelement, at which $\mathbf{D}_i$ has a non-zero component. As an example, for this purpose, one may choose a DOF at which the displacement vector $\mathbf{D}_{stat}$, produced by the weight of the mass, has its maximum component in magnitude. And this $\mathbf{D}_{stat}$ can be computed easily by using Eq. (2.35).

The actual procedure of calculating eigenpairs by the Z-deformation method starts with λ_1 and $\mathbf{D}_1$. On obtaining the first eigenpair, λ_1 and $\mathbf{D}_1$, one may need the higher order eigenpair. The higher order eigenpair may be obtained by constructing a new eigensystem, which considers λ_2 and $\mathbf{D}_2$ of the previous eigensystem as the lowest eigenpair of the new system, which is called the **equivalent eigensystem.** This equivalence will be proven in the following subsection.

5.4.3 Proof of the Equivalent Eigensystem for Next Eigenpairs

In an eigensystem with the first eigenpair λ_1 and $\mathbf{D}_1$ known, its second eigenpair λ_2 and $\mathbf{D}_2$ are the lowest eigenpair of a new eigensystem described by

$$(\mathbf{K} - \lambda\mathbf{M}^*)\mathbf{D} = \mathbf{0} \qquad (5.20)$$

where

$$\mathbf{M}^* \equiv \mathbf{M} - \mathbf{M}\mathbf{D}_1(\mathbf{D}_1)^T\mathbf{M}. \qquad (5.21)$$

Proof.

Suppose that λ_1 and $\mathbf{D}_1$ are the lowest eigenpair satisfying

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{D}_1 = \mathbf{0} \qquad (5.22)$$

where $\mathbf{D}_1$ has been normalized, i.e.,

$$(\mathbf{D}_1)^T\mathbf{M}\mathbf{D}_1=1 \tag{5.23}$$

and that λ_1^* and $\mathbf{D}_1^*$ are the lowest eigenpair of Eq. (5.20), i.e.,

$$(\ \mathbf{K} - \lambda_1^*\mathbf{M}^*\ )\mathbf{D}_1^*=\mathbf{0}. \tag{5.24}$$

Premultiplying Eq. (5.24) by $(\mathbf{D}_1)^T$ yields

$$(\mathbf{D}_1)^T\mathbf{K}\mathbf{D}_1^*=0 \tag{5.25}$$

where Eq. (5.23) has been used. Then, premultiplying Eq. (5.22) by $(\mathbf{D}_1^*)^T$ yields

$$(\mathbf{D}_1)^T\mathbf{M}\mathbf{D}_1^*=0 \tag{5.26}$$

where Eq. (5.25) has been used. Equation (5.26) indicates that $\mathbf{D}_1$ and $\mathbf{D}_1^*$ are M-orthogonal to each other. Use the symbol $\mathbf{Y}^*$ to denote the set of all admissible displacement vectors in Eq. (5.24) and $\mathbf{Y}^0$ the subset of $\mathbf{Y}^*$, whose members are all M-orthogonal to $\mathbf{D}_1$. According to Rayleigh's Quotient Theorem,

$$\lambda_1^* = \min \frac{(\mathbf{D}^*)^T\ \mathbf{K}\ \mathbf{D}^*}{(\mathbf{D}^*)^T\ \mathbf{M}^*\mathbf{D}^*} \qquad \text{for all } \mathbf{D}^*\in\mathbf{Y}^*. \tag{5.27}$$

Since λ_1^* and $\mathbf{D}_1^*$ are the solution of Eq. (5.24), they satisfy Eq. (5.27), then due to Eq. (5.26), one has $\mathbf{D}_1^*\in\mathbf{Y}^0$. Therefore, λ_1^* may also be expressed by

$$\lambda_1^* = \min \frac{(\mathbf{D}^0)^T\ \mathbf{K}\ \mathbf{D}^0}{(\mathbf{D}^0)^T\ \mathbf{M}^*\mathbf{D}^0} \qquad \text{for all } \mathbf{D}^0\in\mathbf{Y}^0. \tag{5.28}$$

However, due to $\mathbf{D}^0\in\mathbf{Y}^0$ and Eq. (5.21), one will see that the denominator in Eq. (5.28) can be rewritten as:

$$\begin{aligned}(\mathbf{D}^0)^T\mathbf{M}^*\mathbf{D}^0 &= (\mathbf{D}^0)^T\mathbf{M}\mathbf{D}^0-(\mathbf{D}^0)^T\mathbf{M}\mathbf{D}_1(\mathbf{D}_1)^T\mathbf{M}\mathbf{D}^0\\ &=(\mathbf{D}^0)^T\mathbf{M}\mathbf{D}^0.\end{aligned}$$

Thus, Eq. (5.28) becomes

$$\lambda_1^* = \min \frac{(\mathbf{D}^0)^T \mathbf{K} \mathbf{D}^0}{(\mathbf{D}^0)^T \mathbf{M} \mathbf{D}^0} \qquad \text{for all } \mathbf{D}^0 \in \mathbf{Y}^0$$
$$= \lambda_2 \tag{5.29}$$

where Rayleigh's Quotient Theorem has been used again for λ_2.

Thus, repeating the same procedure with the equivalent eigensystem of Eq. (5.20), as done for λ_1 and $\mathbf{D}_1$, will yield λ_2 and $\mathbf{D}_2$ of the original eigensystem, and so on so forth until the last one. According the concept of subelements introduced in Chapters 2 and 3, the additional mass-subelement pertaining to the additional term of $\mathbf{MD}_1(\mathbf{D}_1)^T\mathbf{M}$ in Eq. (5.21) in the equivalent eigensystem should have a subelement vector as $\mathbf{MD}_1$ and a subelement stiffness modulus as 1.

5.5 Computational Procedure

Based on the derivations given in the previous sections one can summarize the following steps for computing the eigenpairs by the Z-deformation method.

Step 1. Build up the basic displacement matrix of the given structural system with $\lambda=0$ by the structural variation method presented in Chapter 2.

Step 2. Choose one DOF of the system, say r, where a mass-subelement is located, for calculating the first eigenpair. To specify this DOF, find the D_{stat} produced by the weight of the mass as usually done in conventional methods [16] by using Eq. (2.35). And the DOF on which D_{stat} has its maximum component in magnitude should be taken as the DOF r.

Step 3. Take $x_1=0$, and two other arbitrary values for x_k, k=2,3 (see Fig. 5.2), which are near $\lambda=0$, then evaluate the corresponding Z_k by using Eqs. (2.37) and (4.54).

Step 4. Use Eq. (5.13) with a selected β, e.g., 10, to calculate x_4 (see Fig. 5.2), moving one step towards λ_1.

Step 5. Evaluate Z_4 at x_4 by using Eqs. (2.37) and (4.54) again to obtain the exact Z-deformation at x_4. This step guarantees the moving point (Z_k, x_k) to stay at the exact Z-λ curve (see Fig. 5.2).

Step 6. Use x_i and Z_i, i=2,3,4, to obtain x_5 to move one more step towards λ_1. Repeat steps 4 to 6 with a selected tolerance ϵ, e.g., 10^{-12}, until $x_k-x_{k-1}<\epsilon$ and $Z_k\rightarrow -\infty$, to arrive at $\lambda_1=x_k\pm\epsilon$.

Step 7. If Z_k is found to be a large positive number, then pull the corresponding x_k back for one step and switch to Eq. (5.14) to calculate x_4 and continue on Step 5.

Step 8. On obtaining λ_1, use Eq. (5.15) to obtain the corresponding eigenvector $\mathbf{D}_1$.

Step 9. After obtaining the first eigenpair λ_1 and $\mathbf{D}_1$, introduce a new mass-subelement into the eigensystem, whose subelement vector is $\mathbf{MD}_1$ and the corresponding modulus $\bar{W}$ is 1 (see Eq. (5.21)) to form an equivalent eigensystem, and repeat steps 3-8 on it to obtain the second eigenpair, and so on so forth until the desired one is obtained.

5.6 Illustrative Example

Following the procedure given in Section 5.5, the plane frame eigensystem shown in Fig. 5.3 has been analyzed by the Z-deformation method, and as a comparison, the inverse power method [16] has also been used to solve the same problem. The eigensystem is made of a lumped mass, M=1.0, and four beams which have identical properties, L=1.0, EA=100.0 and EI=0.01, except that the EI_2 of element 2 is different. Three cases will be discussed below. Case 1 has EI_2 equal to 0.011 which makes the first two eigenvalues very close, case 2 leads to a situation of a repeated eigenvalue with EI_2=EI, and case 3 with $EI_2=10.0 \gg$ EI gives a situation that the first two eigenvalues are quite different.

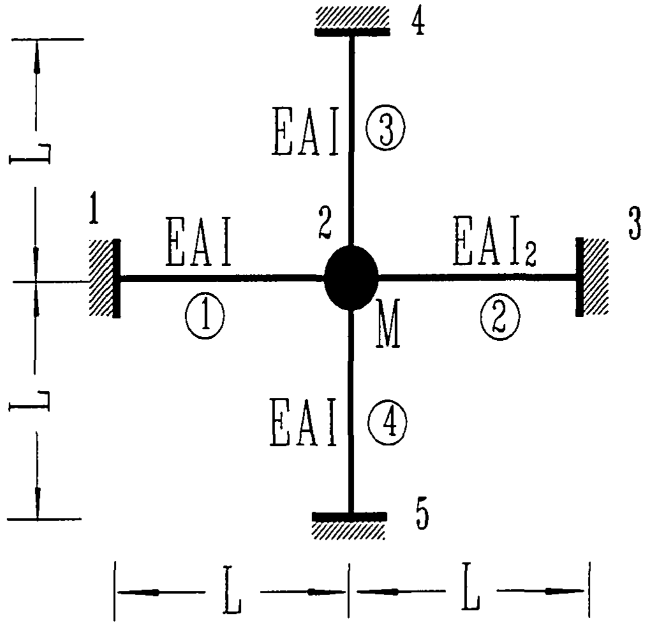

Figure 5.3 An Eigensystem

Case 1: EI_2=0.011.

In this case, the similarity in the structure produces a pair of close eigenvalues as follows (obtained by the Z-deformation method):

λ_1=200.240000000

$\mathbf{D}_1$=[1., 0., 0. $]^T$

λ_2=200.251780488

$\mathbf{D}_2$=[0., 1., -.036585365854 $]^T$.

Thus,

λ_1/λ_2=0.999529.

In this example, the DOF, r, in step 1 for the proposed procedure was chosen to be the vertical DOF at the mass M. To obtain the lowest eigenpair, λ_1 and $\mathbf{D}_1$ as listed above, the Z-deformation method took only 17 advancing steps with $\beta=10$ and $\epsilon= 10^{-12}$, while the power method ran 454452 iteration cycles with the initial $\mathbf{D}_0$=[1, 1, 1 $]^T$. The CPU time ratio T_Z/T_P was about 1:217, where T_Z stands for the time spent by the Z-deformation method and T_P by the inverse power method.

The intermediate values of the first eigensolution performed by the Z-deformation method for this case are tabulated in Table 5.1. Note that at step 14, the value of x_{14} calculated by Eq. (5.13) is a little bit larger than the true λ_1, which results in a large positive value of Z_{14}, i.e., 0.277715747031D+10. Consequently, as suggested by step 7 in Section 5.5, the value x_{14} recalculated based upon Eq. (5.14).

Table 5.1. Computations by Z-deformation Method in Case 1

No.	x_k	$\Delta x = x_k - x_{k-1}$	Z_k	$\Delta Z = Z_k - Z_{k-1}$
0	.0		.0	
1	.250000000000D+00	.250000000000D+00	-.125006250313D-002	-.125006250313D-002
2	.500000000000D+00	.250000000000D+00	-.250325423050D-002	-.125319172737D-002
3	.489010452281D+01	.439010452281D+01	-.250325423050D-001	-.225292880745D-001
4	.400896932810D+02	.351995887582D+02	-.250325423050D+00	-.225292880745D+00
5	.143081716066D+03	.102992022785D+03	-.250325423050D+01	-.225292880745D+01
6	.192548088943D+03	.494663728769D+02	-.250325423050D+02	-.225292880745D+02
7	.199443264049D+03	.689517510587D+01	-.250325423050D+03	-.225292880745D+03
8	.200160040067D+03	.716776018416D+00	-.250325423049D+04	-.225292880744D+04
9	.200232001132D+03	.719610648246D-001	-.250325423039D+05	-.225292880734D+05
10	.200239200084D+03	.719895243714D-002	-.250325426825D+06	-.225292884522D+06
11	.200239920008D+03	.719923582295D-003	-.250325050983D+07	-.225292508300D+07
12	.200239992001D+03	.719924740054D-004	-.250316006069D+08	-.225283500971D+08
13	.200239999201D+03	.720035743029D-005	-.250572273015D+09	-.225540672408D+09
14	.200240000072D+03	.871233226007D-006	.277715747031D+10	.302772974333D+10
15	.200239999920D+03	.719217614047D-006	-.250572148442D+10	-.225514921141D+10
16	.200239999992D+03	.719218011336D-007	-.250572357653D+11	-.225515142809D+11
17	.200239999999D+03	.719217344116D-008	-.250568872359D+12	-.225511636594D+12

Case 2: $EI_2 = EI = 0.01$.

In this case, the intermediate values of the eigensolutions performed by the Z-deformation method are tabulated in Table 5.2, from which the eigenpairs obtained by Z-deformation method are below.

$\lambda_1 = 200.239999999$

$\mathbf{D}_1 = [\ 1.,\ \ 0.,\ \ 0.\ ]^T$

$\lambda_2 = 200.239999999$

D_2=[0., -1., 0.]^T.

Therefore, one has a pair of repeated eigenvalues, i.e.,

$\lambda_1/\lambda_2=1.0.$

Table 5.2. Computations for λ_1 by Z-Deformation Method in Case 2

No.	x_k	$\Delta x=x_k-x_{k-1}$	Z_k	$\Delta Z=Z_k-Z_{k-1}$
0	0.0		.0	
1	0.250000000000D+00	.250000000000D+00	-.125006250313D-002	-.125006250313D-002
2	0.500000000000D+00	.250000000000D+00	-.250325423050D-002	-.125319172737D-002
3	0.489010452281D+01	.439010452281D+01	-.250325423050D-001	-.225292880745D-001
4	0.489010452281D+01	.439010452281D+01	-.250325423050D-001	-.225292880745D-001
5	0.400896932810D+02	.351995887582D+02	-.250325423050D+00	-.225292880745D+00
6	0.143081716066D+03	.102992022785D+03	-.250325423050D+01	-.225292880745D+01
7	0.192548088943D+03	.494663728769D+02	-.250325423050D+02	-.225292880745D+02
8	0.199443264049D+03	.689517510587D+01	-.250325423050D+03	-.225292880745D+03
9	0.200160040067D+03	.716776018416D+00	-.250325423050D+04	-.225292880745D+04
10	0.200232001132D+03	.719610648258D-001	-.250325423084D+05	-.225292880779D+05
11	0.200239200084D+03	.719895242628D-002	-.250325423869D+06	-.225292881560D+06
12	0.200239920008D+03	.719923715224D-003	-.250325437442D+07	-.225292895055D+07
13	0.200239992000D+03	.719916636172D-004	-.250294513883D+08	-.225261970139D+08
14	0.200239999194D+03	.719441371189D-005	-.248510315240D+09	-.223480863852D+09
15	0.200239999887D+03	.692954955639D-006	-.177507711466D+10	-.152656679942D+10
16	0.200239999325D+03	-.562109846669D-006	-.296688679875D+09	.147838843478D+10
17	0.200239999989D+03	.101525730210D-006	-.177507641592D+11	-.159756870445D+11
18	0.200239999999D+03	.101525774615D-007	-.177509740544D+12	-.159758976385D+12

The advancing procedure towards λ_1 is 18 steps, about the same as that in the case 1. However, the second eigenpair was obtained extremely easily. Actually, no advancing steps were performed for λ_2 and D_2, because the Z-deformation Z_0 at the initial step x_0 in the equivalent eigensystem becomes $-\infty$, satisfying the requirement of

Eq. (5.6) at once and implying that the first eigenvalue of the equivalent system $\lambda_1^*=x_0=\lambda_1$, i.e., λ_2 equals λ_1 of the original system. In the present case, the Z-deformation at the initial step, $x_0=\lambda_0+0$, is less than -1.0×10^{130} in the equivalent eigensystem derived from the known first eigenpair. Besides, the BD vector obtained by Eq. (5.19) is the corresponding eigenvector D_1^* of the equivalent system, i.e., the second eigenvector $\mathbf{D}_2$ of the original system.

The inverse power method gave the same repeated eigenvalue as that by Z-formation method, but with different eigenvectors:

$\mathbf{D}_1=[\ .624695047554,\ .780868094430,\ 0\ ]^T$

$\mathbf{D}_2=[-.780868094430,\ .624695047554,\ 0\ ]^T$.

The CPU time ratio, T_Z/T_P, was about 18:7.

Case 3: $EI_2=1000EI=10.0$.

In this case, the intermediate values of the first eigensolution performed by the Z-deformation method are tabulated in Table 5.3. The eigenpairs obtained by the Z-deformation method are

$\lambda_1=200.239999999$

$\mathbf{D}_1=[\ 1.,\ 0.,\ 0.\ ]^T$

$\lambda_2=230.5685643068$

$\mathbf{D}_2=[\ 0.,\ 1.,\ -1.49401794616\ ]^T$.

Therefore,

$\lambda_1/\lambda_2=0.8684618$.

The inverse power method took 191 iteration cycles to give the same results as those by the Z-deformation method, but spent much less time; the CPU time ratio, T_z/T_p, was about 18:1.

Table 5.3. Computations by Z-deformation Method in Case 3

No.	x_k	$\Delta x=x_k-x_{k-1}$	Z_k	$\Delta Z=Z_k-Z_{k-1}$
0	.0		.0	
1	0.250000000000D+00	.250000000000D+00	-.125006250313D-002	-.125006250313D-002
2	0.500000000000D+00	.250000000000D+00	-.250325423050D-002	-.125319172737D-002
3	0.489010452281D+01	.439010452281D+01	-.250325423050D-001	-.225292880745D-001
4	0.400896932810D+02	.351995887582D+02	-.250325423050D+00	-.225292880745D+00
5	0.143081716066D+03	.102992022785D+03	-.250325423050D+01	-.225292880745D+01
6	0.192548088943D+03	.494663728769D+02	-.250325423050D+02	-.225292880745D+02
7	0.199443264049D+03	.689517510587D+01	-.250325423050D+03	-.225292880745D+03
8	0.200160040067D+03	.716776018416D+00	-.250325423050D+04	-.225292880745D+04
9	0.200232001132D+03	.719610648241D-001	-.250325423027D+05	-.225292880722D+05
10	0.200239200084D+03	.719895242017D-002	-.250325421392D+06	-.225292879089D+06
11	0.200239920008D+03	.719923784544D-003	-.250325629594D+07	-.225293087455D+07
12	0.200239992001D+03	.719924366308D-004	-.250320622453D+08	-.225288059493D+08
13	0.200239999195D+03	.719443346497D-005	-.248774033176D+09	-.223741970931D+09
14	0.200239999989D+03	.794172592578D-006	-.186537753111D+11	-.184050012779D+11
15	0.200240001258D+03	.126913880649D-005	.159122160316D+09	.188128974714D+11
16	0.200239999999D+03	.966110049759D-008	-.186538525757D+12	-.167884750446D+12

From the above three example cases, it has been observed that the proposed method, Z-deformation method, is superior to the commonly used power method when the adjacent eigenvalues are close, and can easily handle the repeated eigenpairs. Nevertheless, when the ratio of the adjacent eigenvalues is small, the power iteration method is very efficient. Therefore, the combination of the two methods is expected to be the best choice for the vibration analysis of finite element systems.

Chapter 6

EXPLICIT FORMULATIONS FOR DESIGN SENSITIVITIES IN VIBRATION

Although there are quite a few publications available on the eigenpair design sensitivities of finite element systems [16, 17, 22], the methods presented in those publications require to solve a set of simultaneous equations; this chapter will derive a set of explicit formulations for the computation of eigenpair sensitivities with respect to sizes and masses of the elements of an eigensystem in terms of the BD vectors of the system. In the new method, the sensitivity calculations involve neither assembling nor solving any set of simultaneous equations.

The eigenvalue and eigenvector design sensitivities will be discussed in Sections 6.1 and 6.2, respectively. Each section first presents a general explicit formulation and then follows with two special cases in which a stiffness design variable and a mass design variable are considered, respectively. Numerical examples are given in Section 6.3.

6.1 Eigenvalue Design Sensitivities

6.1.1 General Explicit Formulation

Subsection 5.4.2 has proven that if an eigenvalue λ_i of the system is known, then the BD vector of any mass-subelement $\binom{M}{s}$ with its stiffness modulus $W_s = \infty$ is the corresponding eigenvector $\mathbf{D}_i$. Based upon this special feature, the explicit formulation of eigenvalue design sensitivities can be derived. For convenience, the regular symbol $\mathbf{V}_s$ is used to denote the BD vector pertaining to a mass-subelement (^M_s) with its stiffness modulus $W_s=\infty$. Then, according to Eq. (5.15), one has:

$$\mathbf{D}_i=\mathbf{V}_s. \tag{6.1}$$

However, the eigenvector given by Eq. (6.1) is not yet normalized. The corresponding normalized eigenvector, denoted by $\mathbf{Y}_i$, can be defined as

$$\mathbf{Y}_i \equiv \mathbf{V}_s G^{-1/2} \tag{6.2}$$

where

$$G \equiv (\mathbf{V}_s)^T \mathbf{M} \mathbf{V}_s. \tag{6.3}$$

It can be shown that

$$(\mathbf{Y}_i)^T \mathbf{M} \mathbf{Y}_i = 1. \tag{6.4}$$

Since $\mathbf{Y}_i$ is an eigenvector, it must satisfy Eq. (5.1), i.e.,

$$(\mathbf{K}-\lambda_i \mathbf{M})\mathbf{Y}_i = \mathbf{0}. \tag{6.5}$$

Taking the derivative of Eq. (6.5) with respect to a single design variable b gives

$$[\ \mathbf{K}_b'-\lambda_i \mathbf{M}_b'-(\lambda_i)_b' \mathbf{M}\ ]\mathbf{Y}_i + (\mathbf{K}-\lambda_i \mathbf{M})(\mathbf{Y}_i)_b' = \mathbf{0} \tag{6.6}$$

where $(\ )_b'$ stands for the derivative with respect to b. Premultiplying Eq. (6.6) by $(\mathbf{Y}_i)^T$ gives

$$(\mathbf{Y}_i)^T[\mathbf{K}_b'-\lambda_i \mathbf{M}_b'-(\lambda_i)_b' \mathbf{M}]\mathbf{Y}_i = 0 \tag{6.7}$$

which implies

$$(\lambda_i)_b' = (\mathbf{Y}_i)^T[\mathbf{K}_b'-\lambda_i \mathbf{M}_b']\mathbf{Y}_i \tag{6.8}$$

where Eqs. (6.4) and (6.5) have been used.

Since the term $[K-\lambda_i M]$, denoted as K_λ, is considered as the global stiffness matrix of the eigensystem in the context of the theory of structural variations, as discussed in Section 5.3, K_λ can be expressed in terms of the transfer matrix **H** and the stiffness matrix **W** (see Eq. (2.25)), i.e.,

$$\mathbf{K}_\lambda = \mathbf{H}\mathbf{W}\mathbf{H}^T \tag{6.9}$$

where **H** and **W** should include both the regular subelements and the mass-subelements. Then, Eq. (6.8) can be rewritten as

$$(\lambda_i)'_b = (\mathbf{Y}_i)^T \mathbf{H} \tilde{\mathbf{W}}'_b \mathbf{H}^T \mathbf{Y}_i$$

or

$$(\lambda_i)'_b = (\mathbf{Z}_{\bullet i})^T \tilde{\mathbf{W}}'_b \mathbf{Z}_{\bullet i} \tag{6.10}$$

where the symbol ~ over the letter W in Eq. (6.10) implies that λ_i keeps constant during the process of differentiation and $\mathbf{Z}_{\bullet i}$ is the vector of the Z-deformations of all subelements of the system from $\mathbf{Y}_i$, i.e.,

$$\mathbf{Z}_{\bullet i} = \mathbf{H}^T \mathbf{Y}_i. \tag{6.11}$$

Equation (6.10) is a general formulation for the eigenvalue sensitivity calculation. If the design variable b takes a specific design parameter, e.g., I_α, the moment of inertia of element α, or a certain lumped mass, M_k, the calculation of $(\lambda_i)'_b$ can be further simplified. The following two subsections will discuss these two special cases.

6.1.2 Explicit Formulation for Eigenvalue Sensitivities with Respect to a Stiffness Variable

In this case, the moment of inertia of an element α is considered as the only design variable b, i.e., $b=I_\alpha$. There are only two subelements of element α, which involve bending and shear, say k and k+1, whose subelement stiffness moduli are functions of I_α. The subelement numbers k and k+1 are in the global order and correspond to the subelements $\binom{\alpha}{2}$ and $\binom{\alpha}{3}$, respectively. Then, from Eq. (2.8), one has

$$(\bar{W}_r)_b'=0, \qquad r\neq k,\ k+1 \tag{6.12a}$$

$$(\bar{W}_k)_b'=12E/L^3=W_k/I_\alpha \tag{2.12b}$$

and

$$(\bar{W}_{k+1})_b'=E/L=W_{k+1}/I_\alpha. \tag{6.12c}$$

Thus, one has

$$\bar{W}_b'=\mathrm{diag}(\,0,\,\cdots,\,0,\,12E/L^3,\,E/L,\,0,\,\cdots,\,0\,). \tag{6.13}$$

Substituting Eq. (6.13) into Eq. (6.10) gives

$$\begin{aligned}(\lambda_i)_b' &= (Z_{\cdot i})^T\bar{W}_b'Z_{\cdot i}\\ &= (Z_{\cdot i})^T\mathrm{diag}(\,0,\,\cdots,\,0,\,12E/L^3,\,E/L,\,0,\,\cdots,\,0\,)Z_{\cdot i}\\ &= 12E(Z_{ki})^2/L^3 + E(Z_{(k+1)i})^2/L\end{aligned} \tag{6.14}$$

where Z_{ki} and $Z_{(k+1)i}$ are the Z-deformations of the two subelements $\binom{\alpha}{2}$ and $\binom{\alpha}{3}$ which can be obtained from Y_i, respectively.

6.1.3 Explicit Formulation for Eigenvalue Sensitivities with Respect to a Mass Variable

In this case, the lumped mass at a node r is considered as the only design variable, i.e., $b=M_r$. There are only two mass-subelements related to translations, say k and k+1, whose subelement moduli are functions of M_r. From Eq. (5.8), one has

$$\bar{W}_b'=-\mathrm{diag}(\,0,\cdots,0,1,1,0,\cdots,0\,)\lambda_i \tag{6.15}$$

$$(\lambda_i)_b'=(Z_{\cdot i})^T\bar{W}_b'Z_{\cdot i}$$

$$=-\lambda_i(\mathbf{Z}_{\bullet i})^T \mathrm{diag}(\,0,\cdots,0,1,1,0,\cdots,0\,)\mathbf{Z}_{\bullet i}$$

$$=-[\,(Z_{ki})^2+(Z_{(k+1)i})^2\,]\lambda_i \qquad (6.16)$$

where Z_{ki} and $Z_{(k+1)i}$ are the Z-deformations of the two mass-subelements k and k+1, respectively, which can be obtained from Y_i.

6.2 Explicit Formulation for Eigenvector Design Sensitivities

This section will derive the explicit formulation to calculate eigenvector design sensitivities using the theory of structural variations. The resultant sensitivity equation will eliminate the need of assembling and solving any set of simultaneous equations which is required by the commonly used sensitivity analysis techniques. In the following subsections, the general explicit formulation will be derived first and then two special cases which consider an element stiffness and a lumped mass as design variables will be discussed, as done in the previous section.

6.2.1 General Explicit Formulation

Note that in this section, the symbols $(\ )_r'$ and $(\ )_b'$ are used to denote the differentiations with respect to a subelement stiffness modulus W_r and a design variable b, respectively. Then, by taking derivative of Eq. (6.2), one has

$$(\mathbf{Y}_i)_b'=(\,G^{-1/2}\mathbf{V}_s\,)_b'$$

$$=G^{-1/2}(\mathbf{V}_s)_b' + \mathbf{V}_s(G^{-1/2})_b'$$

$$=G^{-1/2}(\mathbf{V}_s)_b' - \tfrac{1}{2}G^{-3/2}\mathbf{V}_s G_b'$$

$$=G^{-1/2}(\mathbf{V}_s)_b' - G^{-3/2}\mathbf{V}_s[\,(\mathbf{V}_s)^T\mathbf{M}(\mathbf{V}_s)_b' + \tfrac{1}{2}(\mathbf{V}_s)^T\mathbf{M}_b'\mathbf{V}_s\,]$$

$$=G^{-1/2}[\,(\mathbf{V}_s)_b' - G^{-1}\mathbf{V}_s(\mathbf{V}_s)^T\mathbf{M}(\mathbf{V}_s)_b'\,] - \tfrac{1}{2}G^{-3/2}\mathbf{V}_s(\mathbf{V}_s)^T\mathbf{M}_b'\mathbf{V}_s$$

$$=G^{-1/2}[\,I-Y_i(Y_i)^T M\,](V_s)'_b - \tfrac{1}{2}Y_i(Y_i)^T M'_b Y_i$$

$$=G^{-1/2}\{\sum_{r=1}^{p+N}[\,I - Y_i(Y_i)^T M\,](V_s)'_r(W_r)'_b\} - \tfrac{1}{2}Y_i(Y_i)^T M'_b Y_i$$

where **I** is an nxn unit-matrix, p is the total number of subelements with positive moduli, and N is the total number of mass-subelements of the eigensystem. Furthermore, the term $(V_s)'_r$ can be replaced by $V_r(C_{rs}-Z_{rs})(W_r)^{-1}$ as proved by Eq. (4.15). Therefore, the $(Y_i)'_b$ can be rewritten as

$$(Y_i)'_b=G^{-1/2}\{\,[\,I - Y_i(Y_i)^T M\,][\sum_{r=1}^{p+N}V_r(C_{rs}-Z_{rs})(W_r)^{-1}(W_r)'_b\,]\,\} - \tfrac{1}{2}Y_i(Y_i)^T M'_b Y_i$$

or alternatively,

$$(Y_i)'_b=G^{-1/2}[\,I-Y_i(Y_i)^T M\,]VW^{-1}(C^{\#}_{\cdot s}-Z^{\#}_{\cdot s})w'_b - \tfrac{1}{2}Y_i(Y_i)^T M'_b Y_i \tag{6.17}$$

where **w** has been defined by Eq. (4.4), and $C^{\#}_{\cdot s}$ and $Z^{\#}_{\cdot s}$ are two diagonal matrices defined as

$$C^{\#}_{\cdot s}\equiv\mathrm{diag}(\,C_{rs}\,),\quad r=1,2,\cdots,p+N \tag{6.18}$$

$$Z^{\#}_{\cdot s}\equiv\mathrm{diag}(\,Z_{rs}\,),\quad r=1,2,\cdots,p+N \tag{6.19}$$

where $C_{ss}=1$ and $C_{rs}=0$ for $r\neq s$ as indicated in Eq. (4.7)). Note that the vector **w** is a function of λ_i, as defined by Eq. (5.8). Therefore, w'_b involves $(\lambda_i)'_b$ which has been given by Eq. (6.10). Equation (6.17) gives a general formulation for eigenvector sensitivity calculations. The following two subsections will demonstrate its application.

6.2.2 Explicit Formulation for Eigenvector Derivatives with Respect to a Stiffness Variable

Let the moment of inertia of element α be considered as the only design variable, i.e., $b=I_\alpha$. In this case, there are only two subelements, say k and k+1, whose subelement stiffness moduli are functions of b. Then, one has

$$(W_r)_b'=0, \qquad r\neq k,\ k+1 \text{ and } r\leq p \tag{6.20a}$$

$$(W_k)_b'=12E/L^3=W_k/I_\alpha \tag{6.20b}$$

$$(W_{k+1})_b'=E/L=W_{k+1}/I_\alpha \tag{6.20c}$$

$$(W_r)_b'=-(\lambda_i)_b'M_{r-p}, \qquad p+1\leq r\leq p+N \tag{6.20d}$$

$$C_{ss}-Z_{ss}=0 \tag{6.21}$$

$$\mathbf{M}_b'=\mathbf{0} \tag{6.22}$$

where Eqs. (2.8), (5.4), (4.7) and (5.7) have been used for derivation. With the aid of Eqs. (6.20)-(6.22), one obtains

$$w_b'=[\ 0,\cdots,0,W_k/I_\alpha,W_{k+1}/I_\alpha,0,\cdots,0,-(\lambda_i)_b'M_1,\cdots,-(\lambda_i)_b'M_N\ ]^T. \tag{6.23}$$

Therefore, one has

$$\mathbf{VW}^{-1}(\mathbf{C}^{\#}_{\cdot s}-\mathbf{Z}^{\#}_{\cdot s})w_b'=-V_kZ_{ks}/I_\alpha-V_{k+1}Z_{(k+1)s}/I_\alpha -[\sum_{r=p+1}^{p+N}(V_rZ_{rs})\,][(\lambda_i)_b'/\lambda_i],\ r\neq s. \tag{6.24}$$

Substituting Eq. (6.24) into Eq. (6.17) gives the final formulation for the calculation of the desired eigenvector derivative

$$(Y_i)_b'=G^{-1/2}[Y_i(Y_i)^TM-I]\{V_kZ_{ks}/I_\alpha +V_{k+1}Z_{(k+1)s}/I_\alpha+[\sum_{r=p+1}^{p+N}(V_rZ_{rs})\,][(\lambda_i)_b'/\lambda_i]\},\ r\neq s. \tag{6.25}$$

6.2.3 Explicit Formulation for Eigenvector sensitivities with Respect to a Mass Variable

Let the lumped mass at node j be considered as the only design variable, i.e., $b=M_j$. In this case, two mass-subelements, say k and k+1, whose subelement stiffness moduli are functions of b. These two mass-subelements are located at two DOFs, say ℓ

and $\ell+1$, respectively. With this information, one can obtain the following relations from Eqs. (2.8) and (5.4) as

$$(W_r)_b'=0, \quad 1\le r\le p \tag{6.26a}$$

$$(W_r)_b'=-(\lambda_i)_b'M_{(r-p)}, \qquad p+1\le r\le p+N;\ r\ne p+k,\ p+k+1 \tag{6.26b}$$

$$(W_r)_b'=-(\lambda_i)_b'M_{(r-p)}\ -\lambda_i, \quad r=p+k,\ p+k+1. \tag{6.26c}$$

Consequently, Eq. (6.24) can be rewritten as

$$VW^{-1}(C^{\#}_{\cdot r}-Z^{\#}_{\cdot i})w_b'=\ -[\sum_{r=p+1}^{p+N}(V_rZ_{rs})\ ][(\lambda_i)_b'/\lambda_i]\ -\sum_{r=p+k}^{p+k+1}V_rZ_{rs}/M_{r-p}, \quad r\ne s. \tag{6.27}$$

Furthermore, it is easy to see that

$$M_b'=\text{diag}(\ 0,\ \cdots,\ 0,\ 1,\ 1,\ 0,\ \cdots,\ 0\ ) \tag{6.28}$$

where the two non-zero components correspond to the two degrees of freedom, ℓ and $\ell+1$. As a result, one has

$$(Y_i)^TM_b'Y_i=(Y_{i\ell})^2+(Y_{i(\ell+1)})^2 \tag{6.29}$$

where $Y_{i\ell}$ and $Y_{i(\ell+1)}$ are the components of Y_i at DOFs of ℓ and $\ell+1$, respectively. Substituting Eqs. (6.27) and (6.29) into Eq. (6.17) gives the final formulation for the calculation of the sensitivity of the normalized eigenvector, $(Y_i)_b'$.

$$(Y_i)_b'=G^{-1/2}[\ Y_i(Y_i)^TM\ -I\ ]\{\ [\ \sum_{r=p+1}^{p+N}(V_rZ_{rs})][(\lambda_i)_b'/\lambda_i\ ]\ -\sum_{r=p+k}^{p+k+1}V_rZ_{rs}/M_{r-p}\ \}$$
$$-\ \tfrac{1}{2}[\ (Y_{i\ell})^2+(Y_{i(\ell+1)})^2\ ],\quad r\ne s. \tag{6.30}$$

To use Eqs. (6.10), (6.14), (6.16), (6.25) and (6.30), one has to obtain the eigenpair λ_i and $\mathbf{D}_i$ first. The Z-deformation method given in Chapter 5 should be used for this purpose.

6.3 Illustrative Examples

Two examples are given here to verify the equations derived above. The first example is a simple frame eigensystem (Fig. 6.1). The second one is a building structure (Fig. 6.2a) modeled as a plane frame.

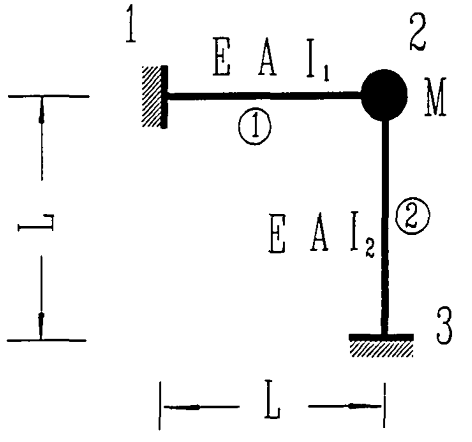

Figure 6.1 A Frame Eigensystem

Example 1

The plane frame eigensystem (Fig. 6.1) is made of a lumped mass M=1.0 and two beams which have identical properties except that their moments of inertia are different but quite close, A=2.0, E=1.0, L=1.0, $I_1=1.0\text{x}10^{-4}$ and $I_2=1.01\text{x}10^{-4}$. This similarity in the structure produces a pair of close eigenvalues.

The total number of subelements is p+N=8 where p=6 and N=2. The mass-subelement used for finding the first eigenpair is $\binom{M}{2}$, i.e., s=2 which is the 8-th subelement in the global order. The design variable b is I_2. The first eigenpair has been obtained by using the Z-deformation method given in Chapter 5 as

$\lambda_1=2.000301497511\pm10^{-12}$

$\mathbf{Y}_1=[0.705933135990, -0.708278481610, -1.06064996191]^T$

and the corresponding $\mathbf{V}_s$ is

$\mathbf{V}_s=[0.996688667408D+00, -.100000000000D+01, -.149750414484D+01]^T$.

Note that the zero components of any vector or matrix, e.g., $\mathbf{Y}_1$ and $\mathbf{V}_s$, at the fixed nodes 1 and 3 are ignored in this example for simplicity.

Now, find $(\lambda_1)'_b$ and $(\mathbf{Y}_1)'_b$, where the design variable $b=I_2$ involves two subelements, k=5 and k+1=6. From the information given above and the results of the first eigenpair obtained by the Z-deformation method, one has the initial data needed for using Eqs. (6.14) and (6.25) as

$\mathbf{M}=\text{diag}(1, 1, 0)$

$\mathbf{E}_k=[1.0,\ 0.0,\ 0.5]^T$

$\mathbf{E}_{k+1}=[0.0,\ 0.0,\ 1.0]^T$

$\mathbf{E}_7=[-1.0,\ 0.0,\ 0.0]^T$

$Z_{k1}=(\mathbf{E}_k)^T\mathbf{Y}_1=0.175608155954$

$Z_{(k+1)1}=(\mathbf{E}_{k+1})^T\mathbf{Y}_1=1.06064996192$

$Z_{ks}=(\mathbf{E}_k)^T\mathbf{V}_s=0.247936594990$

$Z_{(k+1)s}=(\mathbf{E}_{k+1})^T\mathbf{V}_s=1.49750414484$

$Z_{7s}=(\mathbf{E}_7)^T\mathbf{V}_s=-0.996688667408$

$\mathbf{V}_k=[.208832960639D+9, -.209526771802D+9, -.313767209729D+9]^T$

$\mathbf{V}_{k+1}=[.105110281036D+9, -.105459492323D+9, -.157926027118D+9]^T$

V_7=[.138551353074D+13, -.139011666495D+13, -.208170547090D+13]^T

$G=(V_s)^T M V_s$=1.9933882997405.

Substituting the above data into Eqs. (6.14) and (6.25) gives the desired sensitivities of the first eigenpair with respect to I_2 as

$(\lambda_1)_b' = 12E(Z_{k1})^2/L^3+E(Z_{(k+1)1})^2/L$

= 1.495037031077

$(Y_1)_b'=G^{-1/2}[Y_1(Y_1)^T M - I]\{V_k Z_{ks}/I_2+V_{k+1}Z_{(k+1)s}/I_2+V_7 Z_{7s}(\lambda_1)_b'/\lambda_1\}$

=[-.116882156351D+4, -.116495088904D+4, .203190847803D+2]^T.

Example 2

This example is a building structure modeled as a plane frame (Fig. 6.2a). The size and the material properties are given as

L_1=7.2; L_2=5.13; L_3=4.5

A_{beam}=0.2125; A_{column}=0.2025

$I_{beam}=9.15\times10^{-3}$; $I_{column}=3.24\times10^{-3}$

$E=2\times10^{11}$

$M_3=M_5=M_8=M_9=M_{11}=M_{12}=10^4$

$M_2=M_7=M_{10}=2.0\times10^4$

$M_{14}=M_{15}=1.35\times10^4$

and

$M_{13}=2.7\times10^4$

where M_i is the lumped mass at the node i.

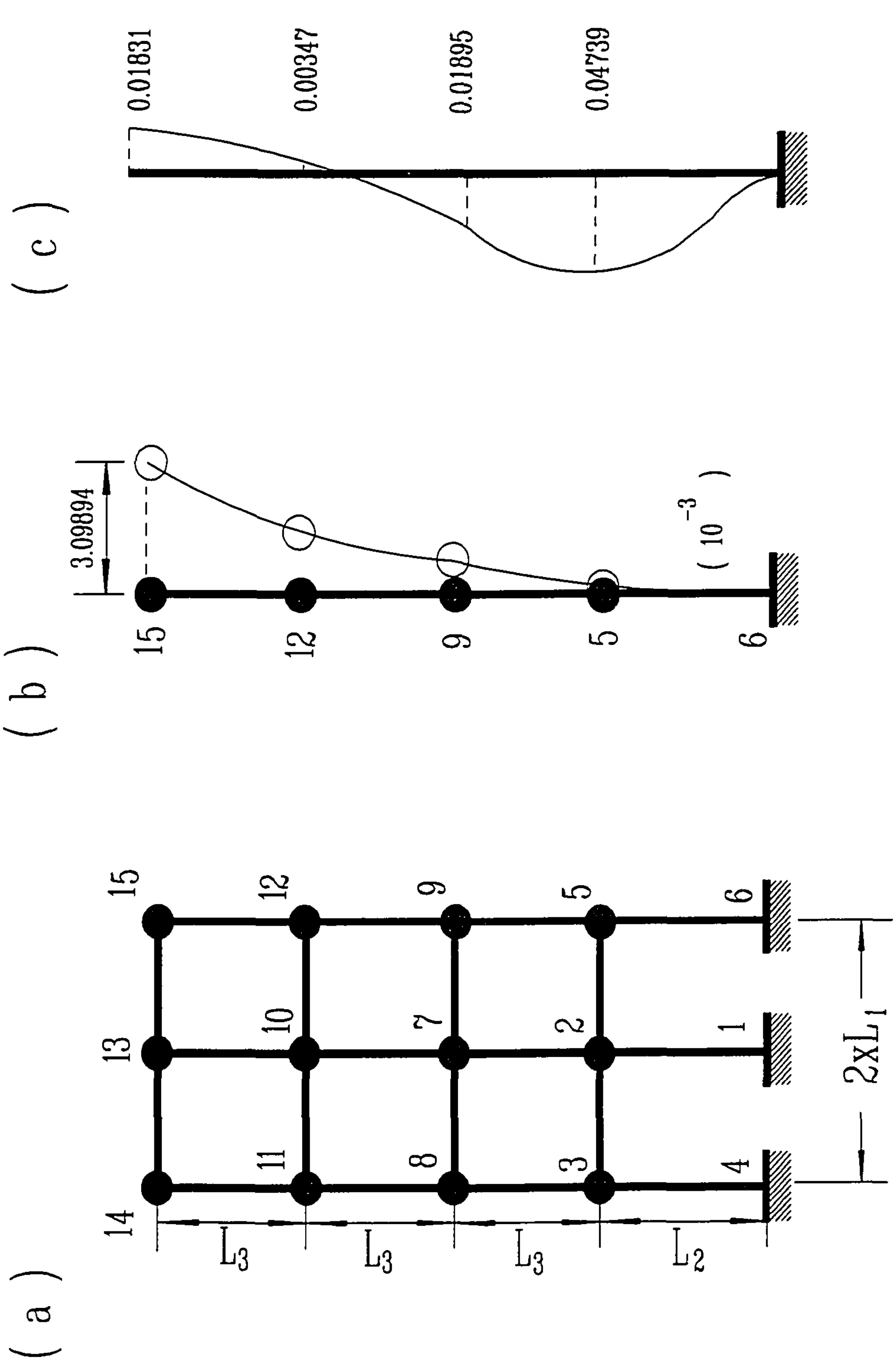

Figure 6.2 (a) An Eigensystem; (b) Eigenvector; (c) Eigenvector Sensitivity

This eigensystem has been analyzed by using the Z-deformation method combined with Eqs. (6.14) and (6.25). The first eigenvalue λ_1 and its sensitivity with respect to the moment of inertia of the element between nodes 5 and 6 have been obtained as

$\lambda_1 = 340.559363722$

$(\lambda_1)_b' = 9790.32969696.$

The first eigenvector $\mathbf{Y}_1$ and its sensitivity $(\mathbf{Y}_1)_b'$ with respect to the moment of inertia of the element between the nodes 5 and 6 are given in Fig. 6.2(b) (for $\mathbf{Y}_1$) and Fig. 6.2(c) (for $(\mathbf{Y}_1)_b'$), respectively, but only the horizontal components of $\mathbf{Y}_1$ and $(\mathbf{Y}_1)_b'$ at the nodes 6, 5, 9, 12 and 15 are shown in the figures.

Chapter 7

CONCLUSIONS AND REMARKS

7.1 Conclusions

(1) The dissertation has extended the theory and the method of structural variations established in [9], 1985, from skeletal structures to general finite element systems and from static analysis to vibration analysis and design sensitivity analysis.

(2) It suggests a new direction of research in finite element problems, treating finite elements from a new point of view, i.e., subelements.

(3) The new analysis tool, the structural variation method, developed in [9] and extended in this dissertation, has distinct features; it eliminates the need of matrix assembly and inversion which are indispensable in the commonly used FEM. This feature makes the structural variation method a favorable choice for structural modifications and sensitivity calculations in many analysis and design processes, such as those in structural optimization, structural reliability analysis, elastic-plastic analysis, contact problems, propagation of cracks in solids, etc. For instance, the solution of a discontinuous structure with a crack as shown in Fig. 7.1 can be obtained from that of the original structure without the crack by removing a constraint-subelement $\binom{R}{t}$, which

holds the structure in contact by connecting nodes R and R' together. This task can be done easily by using Eq. (3.39).

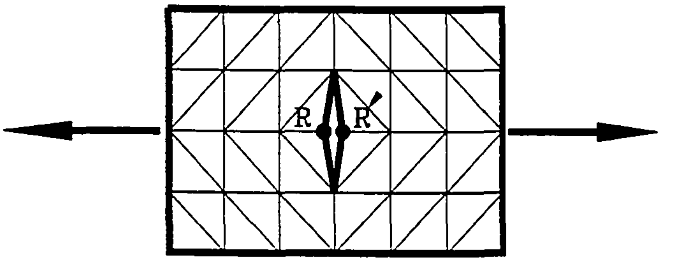

Figure 7.1 A Crack Formed by Removing a Constraint-Subelement

(4) The structural variation method is inherently suitable for parallel computations. The basic displacement matrix **V** of the final structural system shown in Fig. 7.2(b) can be built up by parts. First, the basic displacement matrices $\mathbf{V}^1$, $\mathbf{V}^2$, $\mathbf{V}^3$ ··· of individual parts are built separately and parallelly, as shown in Fig. 7.2(a), then these parts are assembled together by using constraint-subelements to obtain **V** for the final system shown in Fig. 7.2(b), i.e., $\mathbf{V}_{final} = \mathbf{V}^1 \dotplus \mathbf{V}^2 \dotplus \mathbf{V}^3 \dotplus \cdots$, where the symbol

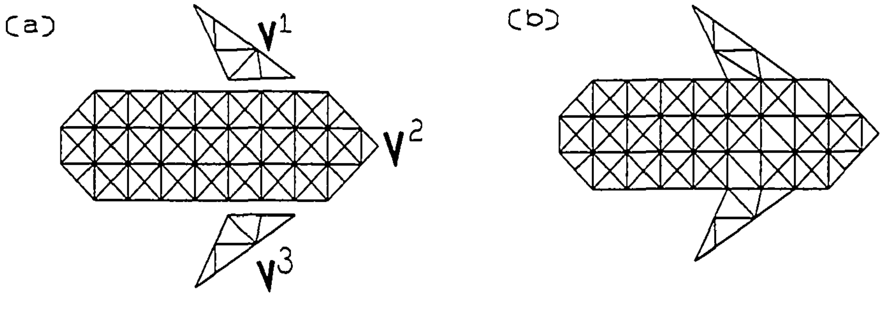

Figure 7.2 Building Basic Displacements by Parts

$\dotplus$ is used to imply the topological " addition ". It is interesting to note that $\mathbf{V}^1$, $\mathbf{V}^2$, $\mathbf{V}^3$ ... for the individual parts may be stored in a computer separately for reference.

(5) The dissertation has established a set of explicit formulations for design sensitivity analysis for static finite element systems, which can be used in the engineering areas where design sensitivity analysis is needed.

(6) This dissertation has revealed the following interesting properties of eigensystems.

(a) The Monotonousness Theorem of Principal Z-Deformations for eigensystems, i.e., Eqs. (5.5)-(5.7). This theorem provides a mathematical foundation for using the Z-deformation method for eigenpair analysis.

(b) The equivalence between the BD vectors and the eigenvectors, i.e., Eq. (5.15). This observation gives a convenient way to find an eigenvector when the corresponding eigenvalue is known; actually, it is obtained simultaneously with the eigenvalue if the Z-deformation method is used.

(c) The equivalent eigensystem for finding the higher order eigenpairs. This equivalence permits the higher order eigenpair to be found by repeating the same computation procedure as that for the previous one.

Based on these properties, the dissertation has established a new numerical method, the Z-Deformation method, for calculating eigenpairs. This method is a procedure of successive advances, whose performance does not depend on the closeness of the adjacent eigenvalues. The theory and the examples given in this dissertation have shown that the Z-deformation method is superior to the inverse power iteration method when adjacent eigenvalues are close.

(7) This dissertation has derived explicit formulations for eigenpair design sensitivities of eigensystems in terms of basic displacement vectors, which can be used in combination with the Z-deformation method for eigenvector design sensitivity computations.

7.2 Remarks

Simple examples for analysis including sensitivity analysis have been given in the dissertation to validate the theory of structural variations. However, this dissertation does not suggest using SVM for a simple analysis of an unchanging structure. This is because the matrix **V** constructed by SVM is actually the Green's functions of all the internal forces of the structure of interest (as Theorem 1 implies). Therefore, **V** gives much more information than what is required by a simple analysis, and hence needs more computer space for data storage and more efforts for computation. It is not suitable to compare SVM to the conventional displacement method based upon a simple analysis because they have different purposes and capabilities.

The structural variation method represents a new structural analysis tool, more efficiently than the conventional displacement method, to handle engineering problems which require structural variations and repeated analyses. Such engineering problems include design sensitivity analysis, structural optimization, vibration analysis, plastic-elastic analysis, structural stability analysis, structural reliability analysis, contact problems, crack propagation in solids, etc. Each specific application requires research efforts to establish explicit expressions based on SVM. Chapters 4-6 in this dissertation represent the development of SVM for vibration analysis and design sensitivity analysis. Some other applications of SVM have also been presented, e.g., [18, 23-25]. However, these applications are only part of the potential applications of TSV and SVM. Further efforts are needed to extend TSV and SVM to more broad engineering applications.

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APPENDIX

This appendix gives short proofs of the 5 theorems used in Subsections 2.2.1-2.2.3, which have been given in [9] in Chinese. The notations already defined in previous sections will not be restated herein.

A.1 Theorem 1. Suppose that a $\bar{\mathbf{P}}_r^\ell$ is applied at $\binom{\ell}{r}$, then the corresponding displacement vector, denoted by $\bar{\mathbf{D}}_r^\ell$, is determined by $\bar{\mathbf{D}}_r^\ell=\mathbf{K}^{-1}\bar{\mathbf{P}}_r^\ell$. According to Eqs. (2.18), (2.19) and (2.28), one has

$$
\begin{aligned}
\bar{F}_{sr}^{\alpha\ell} &= W_s^\alpha(\mathbf{E}_s^\alpha)^T\bar{\mathbf{D}}_r^\ell \\
&= (\mathbf{P}_s^\alpha)^T\mathbf{K}^{-1}\bar{\mathbf{P}}_r^\ell \\
&= (\mathbf{K}^{-1}\mathbf{P}_s^\alpha)^T\bar{\mathbf{P}}_r^\ell \\
&= (\mathbf{V}_s^\alpha)^T\bar{\mathbf{P}}_r^\ell \\
&= V_{sr}^{\alpha\ell}
\end{aligned}
\qquad (A.1)
$$

which is just Eq. (2.31).

A.2 Theorem 2. It has been shown in Eq. (2.25) that

$$\mathbf{K}=\mathbf{H}\mathbf{W}\mathbf{H}^T \qquad (A.2)$$

where $\mathbf{H}$ may involve constraint- or support-subelements and

$$\mathbf{H}=[\ \mathbf{H}^1, \mathbf{H}^2, \cdots, \mathbf{H}^m\ ]. \qquad (A.3)$$

Then, one has the conclusion for Theorem 2 by the following derivation.

$$
\begin{aligned}
\mathbf{K}^{-1} &= \mathbf{K}^{-1}\mathbf{K}\mathbf{K}^{-1} \\
&= \mathbf{K}^{-1}\mathbf{H}\mathbf{W}\mathbf{H}^T\mathbf{K}^{-1}
\end{aligned}
$$

$$=\mathbf{K}^{-1}\mathbf{H}\mathbf{W}\mathbf{W}^{-1}\mathbf{W}\mathbf{H}^{T}\mathbf{K}^{-1}$$

$$=(\mathbf{K}^{-1}\mathbf{H}\mathbf{W})\mathbf{W}^{-1}(\mathbf{K}^{-1}\mathbf{H}\mathbf{W})^{T}$$

$$=\mathbf{V}^{T}\mathbf{W}^{-1}\mathbf{V}.$$

A.3 Theorem 3. According to the definitions (2.29) (2.28) and (2.19), one has the conclusion:

$$Z_{sr}^{\alpha\beta}=(\mathbf{E}_s^{\alpha})^{T}\mathbf{V}_r^{\beta}$$

$$=(\mathbf{E}_s^{\alpha})^{T}\mathbf{K}^{-1}\mathbf{E}_r^{\beta}W_r^{\beta}$$

$$=W_s^{\alpha}(\mathbf{E}_s^{\alpha})^{T}\mathbf{K}^{-1}\mathbf{E}_r^{\beta}W_r^{\beta}/W_s^{\alpha}$$

$$=(\mathbf{K}^{-1}\mathbf{P}_s^{\alpha})^{T}\mathbf{E}_r^{\beta}W_r^{\beta}/W_s^{\alpha}$$

$$=(\mathbf{V}_s^{\alpha})^{T}\mathbf{E}_r^{\beta}W_r^{\beta}/W_s^{\alpha}$$

$$=Z_{rs}^{\beta\alpha}W_r^{\beta}/W_s^{\alpha}.$$

A.4 Theorem 4. Suppose W_s^{α} is changed into $\hat{W}=W_s^{\alpha}+\Delta W_s^{\alpha}$ where ΔW_s^{α} stands for any increment of W_s^{α}, then, from the definition of the BD vector, the new one, $\hat{\mathbf{V}}_s^{\alpha}$, must satisfy the following equations

$$(\mathbf{K}+\Delta\mathbf{K})\hat{\mathbf{V}}_s^{\alpha}=\mathbf{P}_s^{\alpha}+\Delta\mathbf{P}_s^{\alpha}$$

$$=\mathbf{E}_s^{\alpha}(W_s^{\alpha}+\Delta W_s^{\alpha})$$

$$=\mathbf{P}_s^{\alpha}(1+m_s^{\alpha}).$$

However, due to the variation of a single W_s^{α}, one has

$$\Delta\mathbf{K}=\Delta\mathbf{K}^{\alpha}=\mathbf{E}_s^{\alpha}(\mathbf{E}_s^{\alpha})^{T}\Delta W_s^{\alpha}=\mathbf{P}_s^{\alpha}m_s^{\alpha}(\mathbf{E}_s^{\alpha})^{T}.$$

Therefore, one has

$$(\mathbf{K}+\mathbf{P}_s^{\alpha}m_s^{\alpha}(\mathbf{E}_s^{\alpha})^{T})\hat{\mathbf{V}}_s^{\alpha}=\mathbf{P}_s^{\alpha}(1+m_s^{\alpha}).$$

Premultiplying the last equation by $\mathbf{K}^{-1}$ yields

$$\hat{V}_s^\alpha = -K^{-1}P_s^\alpha m_s^\alpha (E_s^\alpha)^T)\hat{V}_s^\alpha + K^{-1}P_s^\alpha(1+m_s^\alpha)$$

$$= -V_s^\alpha m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha} + V_s^\alpha(1+m_s^\alpha)$$

$$= V_s^\alpha(1+m_s^\alpha - m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha}). \tag{A.4}$$

Premultiplying the above equation by $(E_s^\alpha)^T$ yields

$$\hat{Z}_{ss}^{\alpha\alpha} = Z_{ss}^{\alpha\alpha}(1+m_s^\alpha - m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha})$$

from which, one has

$$\hat{Z}_{ss}^{\alpha\alpha} = Z_{ss}^{\alpha\alpha}/(1+m_s^\alpha Z_{ss}^{\alpha\alpha}). \tag{A.5}$$

Substituting Eq. (A.5) into Eq. (A.4) yields Eq. (2.37); and repeating the same procedure and noting $\Delta \mathbf{P}_r^\beta = \mathbf{0}$ when W_s^α varies will give Eq. (2.38).

A.5 Proof of equations (2.45) and (2.46) (part of Theorem 5).

Let the new connecting subelement ($_s^\alpha$) have its W_s^α, E_s^α and $P_s^\alpha = W_s^\alpha E_s^\alpha$, then, through the similar procedure as has been done for Eq. (A.4), one has

$$\hat{K}\hat{V}_s^\alpha = P_s^\alpha$$

or

$$(K + P_s^\alpha (E_s^\alpha)^T)\hat{V}_s^\alpha = P_s^\alpha$$

or

$$K\hat{V}_s^\alpha = P_s^\alpha(1-\hat{Z}_{ss}^{\alpha\alpha}).$$

Therefore, one has

$$\hat{V}_s^\alpha = K^{-1}P_s^\alpha(1-\hat{Z}_{ss}^{\alpha\alpha})$$

$$= (1-\hat{Z}_{ss}^{\alpha\alpha})\mathring{V}_s^\alpha \tag{A.6}$$

where $\mathring{V}_s^\alpha$ is the auxiliary basic displacement vector which can be obtained directly from Theorem 2 as

$$\mathring{V}_s^\alpha \equiv K^{-1}P_s^\alpha = V^T W^{-1} V P_s^\alpha. \tag{A.7}$$

Premultiplying Eq. (A.6) by $(E_s^\alpha)^T$ yields

$$\hat{Z}_{ss}^{\alpha\alpha} = \mathring{Z}_{ss}^{\alpha\alpha}/(1+\mathring{Z}_{ss}^{\alpha\alpha}) \tag{A.8}$$

where

$$\mathring{Z}_{ss}^{\alpha\alpha} \equiv (E_s^\alpha)^T \mathring{V}_s^\alpha \tag{A.9}$$

Substituting Eq. (A.8) into Eq. (A.6) yields Eq. (2.45), i.e.,

$$\hat{V}_s^\alpha = \mathring{V}_s^\alpha/(1+\mathring{Z}_{ss}^{\alpha\alpha}) \tag{A.10}$$

and going through the similar procedure gives Eq. (2.46).

A.6 Proof of equation (2.49a) (another part of Theorem 5).

A constraint-subelement or support-subelement $\binom{R}{t}$ is a special case of a connecting beam subelement with $W_t^R = \infty$ (Fig. 2.3). Actually, one can treat it as $W_t^R \to \infty$. So, before it becomes ∞, Eq. (2.46) can apply to the case of adding $\binom{R}{t}$ with $W_t^R < \infty$. Thus, one has

$$\hat{V}_r^\beta = [V_r^\beta - Z_{tr}^{R\beta}(\mathring{V}_t^R)^*/(1+(\mathring{Z}_{tt}^{RR})^*)]\,|_{W_t^R \to \infty} \tag{A.11}$$

where

$$(\mathring{V}_t^R)^* \equiv K^{-1}E_t^R W_t^R = \mathring{V}_t^R W_t^R$$

$$\mathring{V}_t^R \equiv K^{-1}E_t^R$$

$$\begin{aligned}(\mathring{Z}_{tt}^{RR})^* &\equiv (E_t^R)^T(\mathring{V}_t^R)^* \\ &= (E_t^R)^T \mathring{V}_t^R W_t^R \\ &= \mathring{Z}_{tt}^{RR} W_t^R.\end{aligned}$$

Thus, one has

$$(\mathring{V}_t^R)^*/(1+(\mathring{Z}_{tt}^{RR})^*) = \mathring{V}_t^R W_t^R/(1+\mathring{Z}_{tt}^{RR} W_t^R)$$

$$[(\mathring{V}^{R}_{t})^{*}/(1+(\mathring{Z}^{RR}_{tt})^{*})]\,|_{W^{R}_{t}\to\infty}=\mathring{V}^{R}_{t}/\mathring{Z}^{RR}_{tt}. \tag{A.12}$$

Substituting Eq. (A.12) back into Eq. (A.11) yields Eq. (2.49a).

A.7 Proof of equation (2.51)/(3.39).

Again, let the support-subelement $\binom{R}{t}$ (Fig. 2.3) be treated as $W^{R}_{t}\to\infty$. Then, before W^{R}_{t} becomes ∞, Eq. (2.38) can apply to the removal of $\binom{R}{t}$ by setting $m^{R}_{t}=-1$. Therefore, one has

$$\hat{V}^{\alpha}_{s}=V^{\alpha}_{s}+V^{R}_{t}Z^{R\alpha}_{ts}/(1-Z^{RR}_{tt}). \tag{A.13}$$

Using Theorem 3 to substitute $Z^{\alpha R}_{st}W^{\alpha}_{s}/W^{R}_{t}$ for $Z^{R\alpha}_{ts}$ in Eq. (A.13), one has

$$\hat{V}^{\alpha}_{s}=V^{\alpha}_{s}+V^{R}_{t}W^{\alpha}_{s}Z^{\alpha R}_{st}D^{R}_{t} \tag{A.14}$$

where

$$D^{R}_{t}\equiv 1/((1-Z^{RR}_{tt})W^{R}_{t}) \tag{A.15}$$

or in the component form for any DOF $\binom{\ell}{r}$,

$$\hat{V}^{\alpha\ell}_{sr}=V^{\alpha\ell}_{sr}+V^{R\ell}_{tr}W^{\alpha}_{s}Z^{\alpha R}_{st}D^{R}_{t} \tag{A.16}$$

or

$$\hat{\mathbf{V}}^{\alpha\ell}_{\bullet r}=\mathbf{V}^{\alpha\ell}_{\bullet r}+V^{R\ell}_{tr}\mathbf{W}^{\alpha}\mathbf{Z}^{\alpha R}_{\bullet t}D^{R}_{t})^{R}_{t} \tag{A.17}$$

where $\hat{\mathbf{V}}^{\alpha\ell}_{\bullet r}$, $\mathbf{V}^{\alpha\ell}_{\bullet r}$ and $\mathbf{Z}^{\alpha R}_{\bullet t}$ have been defined in Subsection 2.1.5.

Suppose there are in total q elements numbered $\beta=1,2,\cdots,$ q around the node R where the support-subelement $\binom{R}{t}$ is to be removed (Fig. 2.3). From Eq. (2.20), one has the nodal force vector $\mathbf{f}^{\beta}$ expressed in terms of $\mathbf{F}^{\beta}$ as

$$\mathbf{f}^{\beta}=\mathbf{h}^{\beta}\mathbf{F}^{\beta} \tag{A.18a}$$

or in global coordinates, denoting the counterpart of $\mathbf{f}^{\beta}$ by $\mathbf{G}^{\beta}$, the above equation can be rewritten as

$$\mathbf{G}^{\beta}=\mathbf{H}^{\beta}\mathbf{F}^{\beta}. \tag{A.18b}$$

Then, the nodal force vector $\mathbf{G}^{\beta}$ due to a unit-load vector $\bar{\mathbf{P}}^{\ell}_{r}$ applied at any DOF $\binom{\ell}{r}$ should be expressed by Eq. (A.18b) in the notations defined in Subsection 2.1.4, as

$$\bar{\mathbf{G}}^{\beta}=\mathbf{H}^{\beta}\bar{\mathbf{F}}^{\beta\ell}_{\bullet r}. \tag{A.19}$$

Thus, the force vector at node R (part of $\bar{\mathbf{G}}^{\beta}$), denoted by $\bar{\mathbf{G}}^{\beta}_{R}$, is obtained by partitioning according to the node R as

$$\bar{\mathbf{G}}^{\beta}_{R}=\mathbf{H}^{\beta}_{R}\bar{\mathbf{F}}^{\beta\ell}_{\bullet r}. \tag{A.20}$$

Projecting $\bar{\mathbf{G}}^{\beta}_{R}$ onto the direction $\binom{R}{t}$ by using $\mathbf{R}^{R}_{t}$ and $\mathbf{T}^{\beta}_{t}$ gives the force component in this direction as

$$\begin{aligned}(\mathbf{R}^{R}_{t})^{T}\bar{\mathbf{G}}^{\beta}_{R}&=(\mathbf{R}^{R}_{t})^{T}\mathbf{H}^{\beta}_{R}\bar{\mathbf{F}}^{\beta\ell}_{\bullet r}\\ &=-(\mathbf{T}^{\beta}_{t})^{T}\bar{\mathbf{F}}^{\beta\ell}_{\bullet r}.\end{aligned} \tag{A.21}$$

Nevertheless, according to Theorem 1, Eq. (A.21) can be rewritten as

$$(\mathbf{R}^{R}_{t})^{T}\bar{\mathbf{G}}^{\beta}_{R}=-(\mathbf{T}^{\beta}_{t})^{T}\mathbf{V}^{\beta\ell}_{\bullet r}. \tag{A.22}$$

Equation (A.22) is a general expression for the nodal force component associated with element β in a given direction $\binom{R}{t}$, valid for the node R either with the support-subelement $\binom{R}{t}$ or without it. Applying Eq. (A.22) to the node R after removing $\binom{R}{t}$, then the total of these components from all elements connected to it must be balanced, i.e.,

$$\sum_{\beta=1}^{q}(\mathbf{R}^{R}_{t})^{T}\bar{\mathbf{G}}^{\beta}_{R}=-\sum_{\beta=1}^{q}(\mathbf{T}^{\beta}_{t})^{T}\hat{\mathbf{V}}^{\beta\ell}_{\bullet r}=0 \tag{A.23}$$

and using Eq. (A.17), one has

$$[\,-\sum_{\beta=1}^{q}(\mathbf{T}^{\beta}_{t})^{T}\mathbf{V}^{\beta\ell}_{\bullet r}\,]-[\,\sum_{\beta=1}^{q}(\mathbf{T}^{\beta}_{t})^{T}\mathbf{W}^{\beta}\mathbf{Z}^{\beta\ell}_{\bullet r}\,]\mathrm{V}^{R\ell}_{tr}\mathrm{D}^{R}_{t}=0. \tag{A.24}$$

Nevertheless, according to Eq. (A.22), the first part of Eq. (A.24) is the total force component from all the elements around R before removing the support-subelement $\binom{R}{t}$, it should be balanced with the basic internal force of $\binom{R}{t}$; therefore, from Theorem 1, it must be equal to $V^{R\ell}_{tr}$, i.e.,

$$-\sum_{\beta=1}^{q}(T^{\beta}_{t})^{T}V^{\beta\ell}_{\bullet r}=V^{R\ell}_{tr}. \tag{A.25}$$

Thus, from Eqs. (A.25) and (A.24) one has

$$[\,1-\sum_{\beta=1}^{q}(T^{\beta}_{t})^{T}W^{\beta}Z^{\beta\ell}_{\bullet r}D^{R}_{t}\,]V^{R\ell}_{tr}=0$$

from which comes

$$D^{R}_{t}=1/(\sum_{\beta=1}^{q}(T^{\beta}_{t})^{T}W^{\beta}Z^{\beta\ell}_{\bullet r}). \tag{A.26}$$

Substituting Eq. (A.26) back into Eq. (A.14) and letting $W^{R}_{t}\rightarrow\infty$, one has

$$\begin{aligned}\hat{V}^{\alpha}_{s}&=V^{\alpha}_{s}+V^{R}_{t}W^{\alpha}_{s}Z^{\alpha R}_{st}/(\sum_{\beta=1}^{q}(T^{\beta}_{t})^{T}W^{\beta}Z^{\beta\ell}_{\bullet r})\\&=V^{\alpha}_{s}+V^{R}_{t}\eta^{R\alpha}_{ts}\end{aligned} \tag{A.27}$$

which is just Eq. (2.51). If $\mathbf{T}^{\beta}_{t}$ and $\mathbf{R}^{R}_{t}$ take their definitions given in Subsection 3.5 instead of those given in Subsection 2.2.3, the above proving procedure will give the same result as Eq. (A.27) for 2-D triangular element systems.