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Tidally driven residual circulation in shallow estuaries with lateral depth variation

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Abstract. Tidally driven residual circulation in shallow estuaries with lateral depth variation has been studied analytically using a two-dimensional, depth-averaged model. The solution is presented for a v-shaped channel. Exchange flow is found to be correlated with the topography. The magnitude of this exchange flow depends mainly on four parameters: the ratio between the minimum depth on the shoal and the maximum depth in the channel, the ratio between the tidal amplitude at the mouth and the mean depth, the ratio between the length of the estuary and the tidal wave length, and the ratio between the tidal timescale and the decay timescale due to friction. Generally, a net landward flow occurs over the shoals and is balanced by a return flow in the channel. The along-channel residual velocity changes nearly linearly across the estuary. The residual velocity decreases monotonically toward the head of the estuary. The transverse residual velocity is convergent in the channel and divergent on the shoals. The residual velocity is highly dependent on the length of the estuary. When the length of the estuary is much smaller than a quarter of the wave length, the residual velocity is relatively small. When the length of the estuary is about a quarter of the wave length, the magnitude of the residual velocity reaches its maximum. When the length of the estuary is much larger than a quarter of the wave length, the residual circulation approaches the pattern found in an infinite length estuary. On the basis of the analysis of the solution, we conclude that exchange flow generated by tides in a shallow estuary is the result of competition among several processes. An inward flux is caused by local nonlinearity, both in the bottom friction and from propagation of the tidal wave of finite amplitude. This inward flux is larger on the shoals and smaller in the channel owing to larger nonlinearity in the shallow water. The residual inward flow creates a setup (residual surface elevation) of water at the head of the estuary that produces a pressure gradient. This residual pressure gradient is approximately uniform across the estuary and drives an outward flow that is larger in the channel than on the shoals. In all of the parameter space the competition results in a net inflow on the shoals and a net outflow in the channel.

1. Introduction

The residual circulation in estuaries determines the net exchange of salt, water, and other biologically and chemically important materials [Kjerfve *et al.*, 1981] on timescales longer than one tidal cycle. The obvious need to understand the mechanisms controlling the residual motion has prompted considerable research. In analytic studies of both the first-order tide and the residual circulation, it has been common practice to assume that lateral variations in the flow are comparatively small

and to study models of estuaries with rectangular cross section [Johns, 1970; Ianniello, 1977a, b, 1979; McCarthy, 1991; Jay and Smith, 1990a, b]. With similar assumptions, models of the laterally averaged circulation have also been developed [Parker, 1984].

Some researchers [Dyer, 1977] have argued that the laterally homogenous assumption is not realistic since the magnitude of the circulation may be unevenly distributed across the estuary. The flow may have a spiral structure [Dyer, 1977], which can be caused by the cross-channel variation of depth producing unevenly distributed vertical mixing and, consequently, significant lateral and vertical circulation [Valle-Levinson and O'Donnell, 1996]. The longitudinal change of cross-sectional form and bends of the channel can also cause

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lateral flow structure [Dyer, 1977]. Although Ianniello [1977a] studied only models with rectangular cross sections, he pointed out that the assumption that the channel had a rectangular cross section was very likely the limiting factor in applying the theory to real estuaries.

In recent years, increased efforts have been made to resolve the lateral structure of estuarine circulation through observations as well as through model studies. Observations have indeed revealed lateral variability of salinity and current, not only for the first-order tide [Nunes and Simpson, 1985; Huzzey, 1988; Huzzey and Brubaker, 1988; Simpson and Turrell, 1986; Friedrichs and Hamrick, 1996], but also for the subtidal residual motion [Dyer, 1974; Murray and Siripong, 1978; Schroeder, 1977; Kjerfve, 1978, 1986; Kjerfve and Proehl, 1979; Uncles et al., 1986; Wong, 1994; Valle-Levinson and Lwiza, 1995; Wang and Chao, 1996; Friedrichs and Hamrick, 1996]. Differences in the results of these studies highlight the variety and complexity of mechanisms that result in residual circulation. These mechanisms include river discharge and gravitational circulation [Pritchard, 1952, 1956; Hansen and Rattray, 1965; Wilson, 1976], atmospheric pressure and wind forcing [Kjerfve et al., 1978; Wilson et al., 1985; Wilson and Filadelfo, 1986; Wong, 1994], river inflow, coastal sea level fluctuation [Wilson and Filadelfo, 1986], channel bends [Dyer, 1977; Boicourt, 1982; West and Mangat, 1986], the rotation of the Earth [Pritchard, 1952; Doyle and Wilson, 1978; Fischer et al., 1979], combined effect of wind stress and cross-channel bathymetry [Fischer, 1976; Engelund, 1986; Fischer et al., 1979], and nonlinear effects of the tide [Ianniello, 1977a, b, 1979]. It is likely that the particular characteristics of an estuary dictate the relative importance of these processes. The well-known theories of Pritchard [1952] and Hansen and Rattray [1965] describe estuarine circulation as freshened seawater flowing out in a near-surface layer and seawater flowing landward below. When the lateral variation of bathymetry is taken into account, the vertical stratification may be replaced, or partially replaced, by a lateral stratification along with a lateral variation of velocity field, which may show a landward flow of salt water in the deep channel and seaward flow of fresher water on the shoals. This mode of circulation was first suggested by Fischer [1972] and further explained and observed by Hamrick [1979] and Wong [1994].

However, observations do not always reveal this mode of circulation. Robinson [1960] indicated that in narrow estuaries there is often a dominant ebb channel. By conducting drifter experiments in the lower Humber Estuary in 1961, 1962, and 1963, Robinson [1965] concluded that the main axial channel of the Humber was of the ebb type, which was confirmed by the Admiralty Tidal Stream Table. Observation by Zimmerman [1974] at the tidal watershed in the Dutch Wadden Sea found net ebb directed flow everywhere in the deeper area, from which he concluded that a compensating net flood-directed flow should exist on the adjacent shoals. Studies for the Tay Estuary [Charlton et al., 1975] also showed that the ebb-directed flow tended to concen-

trate in the main channel, with flood-directed flow on the shoals. Observations in the Lower Hudson Estuary (K. Lwiza et al., personal communication, 1996) appeared to have similar results. In a study based on 9 days of observations at North Inlet, South Carolina [Kjerfve, 1978], the tidally averaged transverse velocity structure was also found to be the opposite of Fischer's [1972] mode. The residual flow was observed to be flood directed at all depths in shallow areas and ebb directed in the deeper areas of the same cross section. Two other similar experiments conducted at North Inlet by Kjerfve [Kjerfve and Proehl, 1979; Kjerfve, 1986], which covered 3 and 32 tidal cycles, respectively, including both neap and spring tides, showed the same pattern of residual circulation. The results also showed that the maximum residual varied with the amplitude of the semidiurnal tide between the spring and the neap. This indicates that the exchange flow may be tidally induced.

In this paper an analytic theory is presented for the tidally driven residual circulation in a channel with lateral depth variations. All other mechanisms that can cause residual motion are excluded to clarify the role of the combined effect of the tide and the lateral depth variation. Other mechanisms of residual current generation, including baroclinic pressure gradients [Pritchard, 1952, 1956; Hansen and Rattray, 1965], wind stress [Fischer, 1976], Coriolis effect modulated flows [Fischer et al., 1979], and other low-frequency motions [Wilson et al., 1985; Wilson and Filadelfo, 1986], have been studied elsewhere.

The two-dimensional, depth-averaged, long wave equations are adopted as a model of the dynamics that is most appropriate in well-mixed estuaries. As in all previous analytic models [Johns, 1970; Ianniello, 1977a, b, 1979; Hamrick, 1979; Sarabun, 1980; Parker, 1984; McCarthy, 1991; Friedrichs and Aubrey, 1994; Wong, 1994], no tributary is included in the model and only simple lateral boundaries are chosen. As a result, the residual flow caused by promontories [Pingree, 1978] is also excluded. In addition, the lateral friction is neglected since it has been shown to be a weak effect in most estuaries [Ianniello, 1977a].

The model estuary is assumed to be straight, with constant width and a lateral depth variation, and to have a solid boundary at the head. A semidiurnal tide is assumed to force the circulation at the mouth. These simplifications make it possible to solve the problem analytically using a perturbation method. The solution for the tidally induced residual circulation exhibits similar characteristics to the observations of Robinson [1960, 1965], Zimmerman [1974], Charlton et al. [1975], Kjerfve [1978, 1986], and Kjerfve and Proehl [1979]. It contains five parameters that allow us to explore the characteristics and importance of the physical processes they represent.

2. Model

The model geometry (Figure 1) is chosen to be symmetric about an axis along the deep channel, with only lateral depth variation. The x axis is taken to lie along

Figure 1. The model estuary. The length, width, shoal depth, and channel depth of the model estuary are L , $2D$, $h_{\min}$, and $h_{\max}$, respectively. The origin of the coordinate is chosen at the mouth on the right side. The x axis points toward the head, and the y axis points to the channel of the estuary, forming a right-hand coordinate system. Most of the results presented are for only half of the model estuary owing to symmetry.

the boundary and points toward the head of the estuary. The y axis lies along the open boundary at $x = 0$. A single-frequency, semidiurnal tide is imposed at the mouth of the estuary. Both the amplitude and phase of sea level variation are assumed to be uniform across the estuary and are specified. Since the Coriolis effect is neglected in this model, the flow is symmetric about the central axis of the estuary ($y = D$). It is therefore only necessary to discuss the problem within half of the estuary ($0 \leq y \leq D$).

The depth-averaged, shallow water momentum and continuity equations are used

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -g \frac{\partial \zeta}{\partial x} - \frac{C_D u \sqrt{u^2 + v^2}}{h + \zeta} \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -g \frac{\partial \zeta}{\partial y} - \frac{C_D v \sqrt{u^2 + v^2}}{h + \zeta} \\ \frac{\partial \zeta}{\partial t} + \frac{\partial(h + \zeta)u}{\partial x} + \frac{\partial(h + \zeta)v}{\partial y} &= 0 \end{aligned} \quad (1)$$

where u , v , ζ , h , x , y , t , C_D , and g are longitudinal velocity, lateral velocity, elevation, water depth, longitudinal coordinate, lateral coordinate, time, drag coefficient, and the gravitational constant, respectively.

The head of the estuary and the side boundaries are rigid, so the longitudinal velocity at the head and the lateral velocity at the side boundaries are zero. Symmetry also requires that there be no flux of water across the axis unless the tidal amplitude at the mouth is a function of y , a possibility we exclude at the moment. The boundary conditions for the estuary can be expressed therefore as

$$u|_{x=L} = 0, \quad v|_{y=0,D} = 0, \quad \zeta|_{x=0} = \text{Re}\{\zeta_0 e^{j\sigma t}\} \quad (2)$$

where the constants σ and ζ_0 are the tidal frequency and amplitude at the mouth, respectively, and $j = \sqrt{-1}$.

A Fourier decomposition [Parker, 1984] is applied to the quadratic friction to simplify the equations. Since the ratio of the magnitudes of the horizontal velocity components is $|v|/|u| = \mathcal{O}(D/L)$, which is small in many estuaries, the quadratic friction terms can be approximated, to second order in D/L , as

$$\frac{C_D u \sqrt{u^2 + v^2}}{h + \zeta} \sim \frac{C_D u |u|}{h + \zeta}, \quad \frac{C_D v \sqrt{u^2 + v^2}}{h + \zeta} \sim \frac{C_D v |u|}{h + \zeta} \quad (3)$$

The Fourier decomposition of $u|u|$ is well established [Proudman, 1953; Parker, 1984]. For a one-dimensional problem with a single astronomical tidal constituent plus its higher harmonics and a mean flow, Parker [1984] has shown that, correct to the third order,

$$u|u| = \frac{8}{3\pi} U u \quad (4)$$

in which U is the amplitude of the first-order velocity u .

Because $1/(1 + \zeta/h)$ can be expressed as

$$\frac{1}{1 + \zeta/h} = 1 - \frac{\zeta}{h} + \mathcal{O}\left[\left(\frac{\zeta}{h}\right)^2\right] \quad (5)$$

Combining (4) and (5), the quadratic bottom friction is then expressed, correct to the second order, as

$$\frac{C_D u |u|}{h + \zeta} = \frac{\beta}{h} u - \frac{\beta}{h^2} u \zeta \quad (6)$$

in which β has a dimension of velocity and is defined as

$$\beta = \frac{8C_D U}{3\pi} \quad (7)$$

Similarly, correct to the second order

$$\frac{C_D v |u|}{h + \zeta} = \frac{\beta}{h} v - \frac{\beta}{h^2} v \zeta \quad (8)$$

We see that if β is a known constant, then the bottom friction is linear at first order. Although U and therefore β may be a function of both x and y and are unknown in general, β is often assumed constant for the sake of simplicity [Pite, 1973]. Since U at the solid boundary ($x = L$) should be zero, β is overestimated by this approximation. On the other hand, the longitudinal friction is proportional to both β and u as shown by (6). At the head ($x = L$), $u \rightarrow 0$, the magnitude of friction term has the correct behavior, despite the fact that β is overestimated there. This simplification has therefore little impact on the solution. We expect that in most estuaries the magnitude of velocity is in the range of 0.5 to 1 m/s, which implies that β ranges from 1.06×10^{-3} to 2.12×10^{-3} m/s, assuming $C_D = 0.0025$.

Li [1996] examined quantitatively the effect of the $\beta = \text{const}$ approximation by computing the results of a numerical, finite difference solution to (1) to an approximate solution similar to the one developed in this arti-

cle. The results showed small differences in the residual circulation magnitudes near the closed end of the channel but essentially no difference in the pattern.

Applying the result of the Fourier decomposition for the quadratic friction to (1), the following equations, correct to the second order, are obtained

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -g \frac{\partial \zeta}{\partial x} - \beta \frac{u}{h} + \frac{\beta}{h^2} u \zeta \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -g \frac{\partial \zeta}{\partial y} - \beta \frac{v}{h} + \frac{\beta}{h^2} v \zeta \\ \frac{\partial \zeta}{\partial t} + \frac{\partial(h + \zeta)u}{\partial x} + \frac{\partial(h + \zeta)v}{\partial y} &= 0 \end{aligned} \quad (9)$$

In (9) the quadratic friction has been decomposed into a linear part and a higher-order term. The advantage of (9) over the original shallow water equations is that the perturbation method can be readily applied. A perturbation method, however, often results in clumsy notation that obscures the fundamental physics. In this paper therefore we establish the theory and obtain simple analytic solutions using an ad hoc method. We then use the solution to identify the fundamental dynamics that determine the structure of the residual flow. A formal perturbation solution will be presented in a subsequent paper in which we will stress the dependence of the structure of the residual circulation on the magnitudes of the parameters as well as the effect of the lateral variation of the first-order tide on the residual flow.

3. Solution

To obtain an analytic solution to this problem, we write each velocity component as well as the sea level as the sum of two parts: $u = u_p + u'$, $v = v_p + v'$ and $\zeta = \zeta_p + \zeta'$, where the subscript p is the solution when the bottom is flat and the dashed variables represent the influence of nonlinearities and lateral variation in depth. Since we have defined the geometry so that the coastline is straight and the boundary forcing to be independent of y , we must have $v_p = 0$. We now restrict the lateral depth variations to be small so that $O(u'/u_p) = \zeta_0/h_0$. It is then simple to show that

$$\begin{aligned} \frac{\partial u_p}{\partial t} &= -g \frac{\partial \zeta_p}{\partial x} - \beta \frac{u_p}{h_0} \\ v_p &= 0 \\ \frac{\partial \zeta_p}{\partial t} + h_0 \frac{\partial u_p}{\partial x} &= 0 \end{aligned} \quad (10)$$

which are the familiar linear long wave equations with friction.

The effect of the lateral depth variation is to make the bottom friction more significant in shallow water, resulting in a lateral shear of longitudinal velocity. A lateral pressure gradient can then be generated to drive a small lateral velocity. These effects and those resulting from the nonlinear terms in the momentum and continuity equations now appear in the higher-order equations:

$$\begin{aligned} \frac{\partial u'}{\partial t} + u_p \frac{\partial u_p}{\partial x} &= -g \frac{\partial \zeta'}{\partial x} - \frac{\beta u'}{h} + \frac{\beta u_p \zeta_p}{h^2} \\ \frac{\partial v'}{\partial t} &= -g \frac{\partial \zeta'}{\partial y} - \frac{\beta v'}{h} \\ \frac{\partial \zeta'}{\partial t} + h \frac{\partial u'}{\partial x} + \frac{\partial \zeta_p u_p}{\partial x} + \frac{\partial h v'}{\partial y} &= 0 \end{aligned} \quad (11)$$

The inhomogeneous terms in these equations are determined from the solution to the linearized problem and arise from the advection and the quadratic friction terms in the momentum balance and the third term in the continuity equation. Notice that the last term on the right-hand side of the x momentum equation, $\beta u_p \zeta_p / h^2$, can be either a source or a sink of momentum, depending on the phase difference between the depth-averaged velocity and the surface elevation. This term results from the nonlinear bottom friction term, which, in contrast, is always a sink of momentum. Of course, this term does not indicate that there can be a real physical source of energy. Rather, it reflects the fact that in the momentum equation in (10) the bottom friction is either overestimated or underestimated by using h_0 , instead of $h + \zeta$, in the denominator. A higher-order error is thus corrected in (11). The continuity equation in (11) has two more terms than in (10): (1) $\partial(\zeta_p u_p)/\partial x$, which is the "longitudinal divergence" of the transport above the trough in the solution to (10); and (2) $\partial(h v')/\partial y$, which is the "transverse divergence" of the depth-integrated transport.

The solution of problems like (11) may be separated into two parts [Ianniello, 1977a]: (1) a double-frequency oscillation that is a function of time and position and (2) a steady state or the residual circulation that is only a function of position. Our interest is in the latter. For this reason, we average the above equations over one tidal cycle to eliminate the time dependent component and only seek for the residual circulation solution. The averaged equations are as follows:

$$\begin{aligned} \overline{u_p \frac{\partial u_p}{\partial x}} &= -g \frac{\partial \overline{\zeta'}}{\partial x} - \frac{\beta \overline{u'}}{h} + \frac{\beta \overline{u_p \zeta_p}}{h^2} \\ 0 &= -g \frac{\partial \overline{\zeta'}}{\partial y} - \frac{\beta \overline{v'}}{h} \\ h \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{\zeta_p u_p}}{\partial x} + \frac{\partial h \overline{v'}}{\partial y} &= 0 \end{aligned} \quad (12)$$

with the boundary conditions

$$\overline{u'}|_{x=L} = 0, \quad \overline{v'}|_{y=0,D} = 0, \quad \overline{\zeta'}|_{x=0} = 0 \quad (13)$$

The solution for the first-order problem may be written as

$$\begin{aligned} u_p &= \text{Re} \{ U e^{j\sigma t} \}, \quad v_p = 0, \quad \zeta_p = \text{Re} \{ A e^{j\sigma t} \} \\ A &= \frac{h_0 \alpha}{\sigma} (C_2 e^{-j\alpha x} - C_1 e^{j\alpha x}) \\ U &= C_1 e^{j\alpha x} + C_2 e^{-j\alpha x} \end{aligned} \quad (14)$$

in which

$$\begin{aligned}\alpha &= \sqrt{\frac{\sigma}{gh_0}} \left(\sigma - j \frac{\beta}{h_0} \right) \\ C_1 &= -\frac{\zeta_0 \sigma}{h_0 \alpha} \frac{e^{-2j\alpha L}}{1 + e^{-2j\alpha L}} \\ C_2 &= \frac{\zeta_0 \sigma}{h_0 \alpha} \frac{1}{1 + e^{-2j\alpha L}}\end{aligned}\quad (15)$$

Or in a more concise, nondimensional form

$$\begin{aligned}\frac{U}{\sqrt{gh_0}} &= \frac{\zeta_0}{h_0} \frac{j \sin[\phi(1-x/L)]}{\Phi \cos(\phi)} \\ V_0 &= 0 \\ \frac{A}{h_0} &= \frac{\zeta_0}{h_0} \frac{\cos[\phi(1-x/L)]}{\cos(\phi)}\end{aligned}\quad (16)$$

in which

$$\phi = 2\pi \frac{L}{\lambda} \sqrt{1 - j \frac{\beta}{\sigma h_0}}, \quad \lambda = \frac{2\pi \sqrt{gh_0}}{\sigma}, \quad \Phi = \sqrt{1 - j \frac{\beta}{\sigma h_0}} \quad (17)$$

This solution is clearly a superposition of the damped incident and reflected waves with complex wave number α . Since the properties of this model are well established, e.g., *Ianniello* [1977a], we comment only briefly on some aspects of relevance to the residual circulation. As is evident from (16) and (17), the solution depends on L/λ and $\beta/\sigma h_0$. The ζ_0/h_0 is the magnitude of the tide and thus controls the magnitude of the tidally induced residual [*Ianniello*, 1977a]. The length ratio L/λ also affects the magnitude of the tide. For a length ratio of one quarter the estuary would resonate if not for the dissipation of energy by friction. Friction keeps the magnitude of the forced response finite and shifts the peak of the response. Since $1/\sigma$ is proportional to the tidal period and $(\beta/h_0)^{-1}$ is a decay timescale for the velocity, $\beta/\sigma h_0$ is a ratio of these two timescales. These parameters should then influence the magnitude of the tidally induced residual circulation.

To obtain the tidally averaged residual circulation, we must now solve the second-order equations (12). An exact solution of (12) for a v-shaped depth function (Figure 2) is presented in the appendix. A simpler approximate solution can be obtained for a narrow estuary by assuming that the lateral variation of elevation is small [*Friedrichs and Hamrick*, 1996; *Li*, 1996]. This assumption can be justified by a scaling analysis [*Li*, 1996, chapter 6], by examining the exact solution (see appendix [*Li*, 1996, Figures 4.5 and 4.6]), or by comparing the approximate solution with a numerical solution [*Li*, 1996, Figure 5.4].

Since we are interested in the pattern of the volume flux in the estuary, it is useful at this point to define the transport velocity components u_T and v_T as those required to transport the total volume flux through water of mean depth h . As is argued by *Robinson* [1983],

$$u_T = \bar{u}' + \frac{\bar{\zeta}_p u_p}{h}, \quad v_T = \bar{v}' + \frac{\bar{\zeta}_p v_p}{h} \quad (18)$$

or the total Eulerian residual transport is the Eulerian

Figure 2. Depth as a function of y and the range of depth variation. The depth function is exponential on either side of the estuary and is symmetric about the axis of the estuary. Different depth profiles are used to calculate the solution. (a) Shoal depth is fixed. By varying the channel depth, different depth profiles are obtained. (b) Channel depth is fixed. By varying the shoal depth, different depth profiles are obtained.

residual transport over the mean depth plus the Stokes flux. The integral across the estuary for u_T should be zero owing to mass conservation requirement, i.e.,

$$\int_0^D h u_T dy = 0 \quad (19)$$

From (12) we write the residual velocity $\bar{u}'$ as a function of the first-order solution and the residual pressure gradient,

$$\bar{u}' = \frac{\bar{\zeta}_p u_p}{h} - \frac{h}{\beta} \left(u_p \frac{\partial u_p}{\partial x} + g \frac{\partial \bar{\zeta}'}{\partial x} \right) \quad (20)$$

Substituting (20) and (18) into (19) and recalling the assumption that the residual pressure gradient is only a function of x (or laterally uniform), the residual pressure gradient can be solved as follows,

$$\begin{aligned}\frac{\partial \bar{\zeta}'}{\partial x} &= \frac{\beta}{g \int_0^D h^2 dy} \int_0^D \left(2\bar{\zeta}_p u_p - \frac{h^2}{\beta} u_p \frac{\partial u_p}{\partial x} \right) dy \\ &= \frac{1}{g \int_0^D h^2 dy} \left(2D\beta\bar{\zeta}_p u_p - u_p \frac{\partial u_p}{\partial x} \int_0^D h^2 dy \right)\end{aligned}\quad (21)$$

The right-hand side of (21) can be calculated once the depth function is defined and the first-order solution is obtained. The residual pressure gradient is readily computed. The transport velocity is then solved by

$$u_T = 2 \frac{\bar{\zeta}_p u_p}{h} - \frac{h}{\beta} \left(u_p \frac{\partial u_p}{\partial x} + g \frac{\partial \bar{\zeta}'}{\partial x} \right) \quad (22)$$

$$v_T = -\frac{1}{h} \frac{\partial}{\partial x} \int_0^D h u_T dy$$

Notice that this solution does not restrict the depth function explicitly. The assumption that the depth is symmetric about an along-channel axis is made for a simpler presentation and to compare with the exact solution in the appendix. It is important to remember, however, that the first-order solution is based on a flat bottom channel with the mean depth. To ensure that the solution for the tide is valid at the first order throughout the estuary, the variability of the depth should be smaller than the mean depth so that $\Delta h/h_0 < 1$, where Δh is the maximum variation of the depth and h_0 is the mean depth.

4. Structure of the Residual Circulation

The solution in section 3 showed that the first-order tide depends on ζ_0/h_0 , L/λ , and $\beta/(\sigma h_0)$. The tidally induced residual circulation also depends on these parameters. In addition, the appendix shows that the residual circulation depends on two more parameters: $h_{\min}/h_{\max}$ and D/L . Since the interest here is the effect due to lateral variation of depth, a v-shaped depth function defined by (A3) is chosen for the calculation. This particular depth function is chosen so that we can also compare with the exact solution of the appendix to examine the assumption that the lateral variation of the elevation is small in a narrow estuary.

Calculations for the solution are made for various sets of parameters in the following range: $\zeta_0/h_0 \sim 0.03 -$

Figure 3. Residual transport velocity, with only half of the estuary shown. The shoal depth, channel depth, tidal amplitude, and drag coefficient at the mouth are 5 m, 10 m, 1 m, and 0.0025, respectively. The length of the estuary is ~ 0.8 of a quarter of the wave length. The maximum longitudinal and lateral components of the residual transport velocity are ~ 11 cm/s and 1.6 mm/s, respectively. The magnitude of the lateral component is exaggerated to better visualize the structure. The net flow is flood directed on the shoal and ebb directed in the channel, as has been observed in shallow estuaries [Robinson, 1960, 1965; Zimmerman, 1974; Charlton et al., 1975; Kjerfve, 1978, 1986].

Figure 4. The mean sea level and pressure gradient along the estuary. Parameters are the same as in Figure 3. The maximum mean sea level is ~ 10 cm at the head. The mean pressure gradient resulting from the mean sea level is the largest at the mouth ($\sim 4 \times 10^{-5}$ m/s²) and decreases monotonically toward the interior until it reaches zero at the head. The scale for the residual pressure gradient is shown at right.

0.38, $L/\lambda \sim 0.014 - 1$, and $h_{\min}/h_{\max} \sim 0.5 - 0.98$. The depth on the shoals, $h_{\min}$, is chosen varying from 5 m to $h_{\max}$, which determines the mean depth h_0 if the depth ratio, $h_{\min}/h_{\max}$, is given. This in turn determines $\beta/(\sigma h_0)$. The width to length ratio, D/L , turns out to have little effect on the residual circulation structure and the maximum residual velocity, consistent with the analysis in the appendix. We use $D = 2$ km in all calculations presented below. Note that D is important in the calculation of the total cross-channel exchange rate.

Figure 3 shows a distorted vector map of the residual transport velocity (u_T, v_T) for $h_{\min} = 5$ m, $h_{\max} = 10$ m, $D = 2$ km, $\zeta_0 = 1$ m, and $L = 75$ km. The estuary length is ~ 0.8 times one quarter of the tidal wave length. In this example the maximum magnitude of the longitudinal component ($u_T \sim 11$ cm/s) is 2 orders of magnitude larger than the lateral component ($v_T \sim 1.6$ mm/s). Figure 3 is produced by enlarging the lateral component, in order to better visualize the residual gyre. The magnitude of the depth-averaged residual current is $\sim 10\%$ of the maximum tidal current. Notice that although Figure 3 is a result of a model estuary with a length close to a quarter of the wave length, the structure of the residual flow is representative of all parameter space. Specifically, the following characteristics are typical of our solutions: (1) the residual transport velocity magnitude decreases monotonically from the mouth toward the interior; (2) there is an inward flux on the shoals and an outward flux in the channel; and (3) the maximum magnitudes of the longitudinal component appear at the edge and center of the channel. The residual elevation is displayed in Figure 4 as

Figure 5. Effects of estuary length. Parameter values are the same as in Figures 1-4, except the length of the estuary varies from 5 to 390 km, corresponding to ~ 0.05 to 4.3 of a quarter of the wave length. (a) Residual pressure gradient and mean sea level. The scale for the mean sea level is shown at right. (b) Longitudinal component of the maximum residual transport velocity, both on the shoal and in the channel at the mouth, and the lateral component of the maximum residual transport velocity at the mouth. (c) M_2 tidal amplitude at the mouth as a function of the length of the estuary.

a function of x . As expected, $\bar{\zeta}'$ shows that there is a setup of water at the head of the estuary, resulting in a mean seaward pressure gradient force. This pressure gradient is largest at the mouth and decreases toward the head, where it becomes zero (Figure 4).

By varying the length of the estuary and keeping the other parameters the same, one obtains the maximum residual transport velocity, the maximum mean elevation, and the maximum residual pressure gradient, displayed in Figures 5a and 5b. Figure 5c shows the first-order (M_2) tidal amplitude at the mouth as a function of the length of the estuary. The maximum residual transport velocity and the mean pressure gradient show similar variation with the length of the estuary. When

the length of the estuary is much smaller than the wave length, the values are all close to zero. When the length of the estuary is close to the resonant length (a quarter of the wave length), u_T reaches its maximum value. In keeping with the behavior of the first-order solution (Figure 5c), the peaks are shifted to the left of the resonant length by friction.

When the length of the estuary is much larger than a quarter of the wave length, the maximum residual velocities, which occur at the mouth, become independent of the length of the estuary. For length larger than a quarter of the wave length, the maximum mean sea level slightly decreases with the increase of the length before it increases again to a value larger than the maximum

Figure 6. Effects of estuary length on residence time and flux at the mouth. Results are obtained for estuaries of different lengths (normalized by a quarter of wave length). Parameter values are the same as in Figure 5. The scale for the flux at the mouth is shown at right.

value at near a quarter of the wave length. For the maximum lateral residual velocity at the mouth, the peak shift from the quarter of the wave length is larger than those of the longitudinal velocities on the shoal and in the channel.

To compare the importance of the tidally induced residual flux to the exchange in estuaries of differing size, we define the exchange flux (volume transport per unit time) due to the residual circulation as the integral of the inward transport across the estuary at the mouth. This, of course, must equal the integral of the outward transport in the present case of no river flow. As shown in Figure 6, the exchange flux variation with the length of the estuary is similar to that of the transport velocity, with a maximum near the resonant length. The flushing or residence time for the estuary is obtained by dividing the volume of the estuary by the flux. Figure 6 shows that when the length of the estuary is much smaller than a quarter of the wave length, the tidally induced residual exchange flow is insignificant, so that the flux is very small and the residence time is extremely large. As the length of the estuary approaches the resonant length, the flux increases sharply and the residence time decreases. In very long estuaries the residence time increases with L/λ since, though the exchange flux is constant, the volume increases.

The effect of the channel depth variation is displayed in Figure 7 which shows the maximum residual velocity components on the shoal and the channel as a function of the depth of the channel $h_{\max}$ (Figure 7a) and a function of the depth of the shoal $h_{\min}$ (Figure 7b). The other parameters are unchanged. When the depth of the channel is very close to that of the shoals, the residual velocity is very small. As the depth of the channel increases and the magnitude of the across-channel

depth gradient increases, the magnitudes of residual velocity components increase (Figure 7a). Similarly, Figure 7b shows the maximum residual velocity components as functions of the depth of the shoals with the channel depth and other parameters unchanged. When $h_{\min}$ is minimum and the across-channel depth gradi-

Figure 7. Effects of lateral depth gradient on maximum residual velocities at the mouth as functions of either the shoal depth or the channel depth. The parameters are the same as in Figures 1-6, except that the (a) depth of the channel $h_{\max}$ varies from 5 to 10 m (with $h_{\min} = 5$ m) or (b) depth of the shoal $h_{\min}$ varies from 5 to 10 m (with $h_{\max} = 10$ m). The scale for the lateral component of the residual transport velocity at the mouth is shown at right.

Figure 8. Effects of drag coefficient which varies from 0.5×10^{-3} to 3.8×10^{-3} . (a) Maximum residual transport velocities at the mouth. (b) Residence time and flux at the mouth.

ent is largest, the residual velocity is maximum. As the depth of the shoal increases, the magnitudes of residual velocity components decrease.

The choice of the bottom drag coefficient has a significant influence on the magnitude of the predicted residual circulation but does not influence the structure. We adopted $C_D = 2.5 \times 10^{-3}$ for the standard case following Proudman [1953]. Figure 8 shows sensitivity of the maximum residual velocities on the channel and shoals (Figure 8a) and the residence time and exchange flux (Figure 8b) to the drag coefficient. The calculations show that a factor of 1.6 increase in C_D reduces the magnitude of the residual circulation by $\sim 12\%$. A decrease of a factor of 2 increases the residual flow by $\sim 55\%$. Accordingly, the exchange flux decreases and the residence time increases with the drag coefficient.

Figure 9. Effects of tidal forcing. Parameter values are the same as in Figures 1-8, except that the tidal amplitude at the mouth varies from 0.2 to 2.8 m. (a) Maximum residual transport velocities at the mouth. (b) Residence time and flux at the mouth. U_s , U_d , and V have the same meaning as in Figures 1-8.

The sensitivity of the solution with the standard parameter choices to the amplitude of the tidal forcing at the mouth is displayed in Figures 9a and 9b. Since the structure of the residual motion is not affected by ζ_0 , we display the maximum residual velocities on the channel and shoals (Figure 9a) and the residence time and exchange flux (Figure 9b) as functions of ζ_0 . The calculations show that the residual velocity magnitude and exchange flux increase with tidal elevation at the mouth and that the residence time decreases. We also find that doubling ζ_0 leads to a doubling of the residual velocity.

5. Discussion

5.1. Momentum Balance

An explanation for the residual circulation pattern can be formulated by analyzing the solution directly. From the first equation in (12) and the definition for the transport velocity in (18), the longitudinal transport velocity can be expressed as

$$u_T = 2 \frac{\zeta_p \overline{u_p}}{h} - \frac{h}{\beta} \left(\overline{u_p} \frac{\partial u_p}{\partial x} + g \frac{\partial \zeta'}{\partial x} \right) \quad (23)$$

The transport velocity can therefore be viewed as the sum of three contributions, which we term $u_{T1} = 2\zeta_p \overline{u_p}/h$ ("Stokes"), $u_{T2} = -h\overline{u_p} \partial u_p / \partial x / \beta$ ("advection"), and $u_{T3} = -gh \partial \zeta' / \partial x / \beta$ ("pressure").

The first component, u_{T1} , is 2 times the Stokes flux of a finite amplitude wave. This is because the nonlinear term in the representation of bottom friction arising from the Fourier decomposition of the quadratic parameterization happens to have the same effect as that of the Stokes flux. This term brings a net inward trans-

Figure 10. Residual transport velocity and its components derived from the momentum equation. These include the three effects of Stokes flux due to surface elevation and nonlinear bottom friction, advection, and residual pressure gradient. The advection has only a minor contribution.

port of water (Figure 10). This inward transport is larger at the mouth than inside the estuary and larger on the shoal than in the channel. The reason of an inward Stokes flux is that the wave is partially progressive, with only a weaker reflected wave due to bottom friction. Therefore the phase difference between the elevation and velocity is smaller at the mouth than in the interior and increases inward to 90° at the head, where it is almost a standing wave. For this reason, the closer it is to the mouth, the weaker the reflected wave is, compared to the incident wave at the same location. The larger values on the shoal are due to the fact that u_{T1} is inversely proportional to the water depth. Figure 10 shows that this component can generate as much as 20 cm/s residual velocity at the mouth.

The inward transport of water mass generates a setup at the head that is shown by an increasing residual mean elevation toward the head. Associated with this residual elevation is a seaward residual pressure gradient (Figure 4). This residual pressure gradient produces a seaward return flow u_{T3} (Figure 10). Since the residual pressure gradient is the largest at the mouth, this return flow has its maximum at the mouth. The third term in (23), u_{T3} , is proportional to the product of h and the residual pressure gradient. Since the residual pressure gradient is almost laterally independent, u_{T3} has a larger magnitude in deeper water than in shallower water. The return flow in the channel in Figure 10 reaches 22 cm/s, which is on the same order of magnitude as that of u_{T1} . However, the maximum for u_{T3} is in the channel rather than on the shoal.

The advection-induced transport u_{T2} is positive but much smaller in magnitude than either u_{T1} or u_{T3} . Figure 10 shows that within half of the estuary of the open end, this effect does not change with the longitudinal position. It has an almost constant lateral gradient too, with a slightly larger value in the channel than on the shoal. It decreases slowly toward the head thereafter. In Figure 10 the maximum contribution of this term is only ~ 3 cm/s, a significantly smaller value compared with that from the other two terms.

Because the advection effect is small, as previously recognized [LeBlond, 1978; Parker, 1991], the principle balance is between the inward flux due to u_{T1} and return flow due to u_{T3} . Furthermore, since the inward flux is larger on the shoal and the return flow is larger in the channel, the net result is an inward flux on the shoal balanced by a return flow in the channel (Figure 10).

It is important to note that although the present model is a simple one with a straight channel, (23) is valid, regardless of the bathymetry. The three components, Stokes, advection, and pressure, are present in real estuaries as well. The mechanisms identified here are therefore applicable to a general problem. The spatial distribution of the magnitude of each term, however, must be determined for each case.

5.2. Vorticity Balance

In this paper we have shown quantitatively that the lateral variation of depth of a shallow rectangular tidal

estuary results in lateral shear in the tidally induced residual circulation. The residual flow is toward the head of the estuary on the shoal and toward the mouth in the channel. Note that this structure is opposite to that predicted by the qualitative argument of Zimmerman [1981] and deduced in the analytic model of Yasuda [1980].

In the model of Yasuda [1980] the residual vorticity was generated by sidewall friction. In contrast, the present model neglects lateral friction since it has been convincingly argued to be a small effect in estuaries. Instead, we include bottom friction and lateral depth variations. Since these effects might qualitatively be expected to produce a similar distribution of vorticity, the main novelties of the vorticity dynamics in this model are the explicit inclusion of the free surface and the nonlinearities associated with bottom friction. In conjunction with the lateral depth gradient, these will be demonstrated below to produce sources of residual vorticity that compete with those considered in the models of Yasuda [1980] and Zimmerman [1981].

To most clearly explain the vorticity balance in the present model, we first discuss a simple experiment [Fischer, 1976] that models the wind-driven circulation in a lake with depth variations in the direction orthogonal to the wind vector since both laboratory observations and a theoretical explanation exist for that problem. We then interpret the predictions of the present model in terms of the residual vorticity balance and contrast them with those of Zimmerman [1981] and Yasuda [1980].

In the experiment described by Fischer [1976], a uniform and steady wind was blown along the axis of a tank of water in which the bottom depth varied across the tank. Observations of the motion field after transients had damped showed that there was a net drift downwind in the shallow side of the basin and upwind in the deeper area. A qualitative explanation of this phenomenon is that in the steady state the downwind motion resulting from the surface stress created a surface setup that was approximately independent of the across-channel coordinate. This barotropic pressure gradient then resisted the wind stress.

However, since the wind stress in deep areas had a smaller effect on the depth-integrated momentum balance than it did in the shallow areas, the pressure gradient overwhelmed the wind stress in deep areas and drove the water upwind. The induced bottom stress then acted in cooperation with the wind stress to balance the pressure gradient. Over the shoals, in contrast, the wind stress overwhelmed the pressure gradient and water moved downwind, so that the pressure gradient and bottom stress together balanced the wind stress. Of course, the magnitudes of the response depend upon the details of the geometry, but the important point, as far as we are concerned here, is that the uniform wind stress resulted in a flow with nonzero vorticity owing to depth variation.

To understand the vorticity balance, it is helpful to adopt a simple, linear drag law to model bottom fric-

tion, and representing the surface wind stress as τ_w , the steady state, along-channel, vertically integrated momentum balance can be written as

$$0 = -g \frac{\partial \zeta}{\partial x} + \frac{\tau_w - \beta u(y)}{h(y)} \quad (24)$$

where the advection of momentum has been neglected for simplicity. Assuming that τ_w is independent of y , then the y derivative of (24) multiplied by $h(y)$ yields

$$0 = g \frac{\partial h}{\partial y} \frac{\partial \zeta}{\partial x} + gh \frac{\partial^2 \zeta}{\partial x \partial y} + \beta \frac{\partial u}{\partial y} \quad (25)$$

Assuming that the across-channel variation in the pressure gradient is small compared with the across-channel variation in depth, then

$$-\frac{\partial u}{\partial y} = \frac{g}{\beta} \frac{\partial \zeta}{\partial x} \frac{\partial h}{\partial y} \quad (26)$$

Since, as is easily shown by a scaling argument, the along-channel gradient in the across-channel velocity component makes a negligible contribution to the vorticity, (26) is a vorticity balance. The term on the right can be thought to represent the vorticity tendency resulting from the shear of the depth-integrated, along-channel pressure gradient, $\partial(gh\partial\zeta/\partial x)/\partial y$, which tends to cause the fluid to rotate. This tendency is countered by the spatial variation of the bottom stress on the depth-integrated flow. The vorticity balance in (26) is then between the tendency resulting from the integrated pressure gradient (IPGT) and the rate of removal by the bottom stress (BST).

Note that for $\tau_w > 0$ and, consequently, $\partial\zeta/\partial x > 0$, the vorticity, $-\partial u/\partial y$, has the same sign as the depth gradient. If h increases with y across a tank or lake, then the vorticity generated should be positive and the current should decrease as y increases.

In the present theory the residual dynamic balance in the along-channel direction is presented in (12). Since the y dependence of $\bar{\zeta}'$ is weak, the y derivative of the along-channel momentum equation multiplied by h can be written as

$$-\frac{\partial \bar{u}'}{\partial y} = \frac{g}{\beta} \frac{\partial h}{\partial y} \frac{\partial \bar{\zeta}'}{\partial x} + \frac{\partial h}{\partial y} \left(\frac{\bar{u}_p \bar{\zeta}_p}{h^2} + \frac{1}{\beta} \bar{u}_p \frac{\partial \bar{u}_p}{\partial x} \right) \quad (27)$$

The first two terms in (27) represent the mechanisms that change vorticity in the wind-driven lake circulation problem, i.e., the IPGT and the BST. The difference here is that the forcing is not wind but, rather, nonlinear tidal stress. Though the third term is similar in form to the surface Stokes transport, it represents the nonlinear effect of the bottom stress distributed over the water column. It is therefore sensitive to the phase difference between the current and the elevation. We refer to the vorticity tendency resulting from this mechanism as the nonlinear bottom stress tendency (NLBST) (see Figure 11). The fourth term describes the vorticity tendency resulting from the advection of momentum (ADT), which is 1 order of magnitude smaller than the

Figure 11. Residual transport vorticity and its components (the vorticity tendencies (VTs), which include the nonlinear bottom stress tendency (NLBST), the net advective flux of vorticity or the advective vorticity tendency (ADVT), and the integrated pressure gradient tendency (IPGD) or a “torque” caused by along-channel gradient of the mean pressure force (integrated over the vertical) and the across-channel bottom slope).

other components (Figure 11). Note that though the magnitude of the NLBST varies slightly across the estuary, the tendency for water columns to rotate is due mainly to the nonuniform distribution of vertically integrated mass (Figure 11), corresponding to the variation of the bottom topography. In the models of *Yasuda* [1980] and *Zimmerman* [1981], only the effects of advection (ADT) were considered. Our result shows that this mechanism generates a vorticity tendency that is opposite of both NLBST and IPGT, but since ADT has a magnitude smaller than either of the other contributions (Figure 11), the total vorticity is determined by NLBST and IPGT rather than ADT. This explains the discrepancy between the present theory and that of *Zimmerman* [1981].

5.3. Additional Comments

The solution of section 3 does not allow the first-order tide (u_p, ζ_p) to vary in amplitude and phase across the estuary. *Friedrichs and Hamrick* [1996] showed that the tidal current in the James River Estuary does vary across the estuary. In deep channel the velocity amplitude is larger than that over the shoals, which can be explained from the longitudinal momentum balance [*Li*, 1996]. The lateral variation of amplitude and phase of the first-order tide will change the Stokes (u_{T1}), which will alter the solution. The quantitative effect cannot be assessed without resolving the lateral structure of the first-order tide. Detailed solutions are given and discussed by *Li* [1996].

The value of a dynamically simple theory such as that presented in this paper is that it allows the isolation of the dominant mechanisms and reveals the consequences of their interaction on observable properties of the flow. This demonstrates how the setup created by the Stokes' transport and friction in an estuary can drive a residual gyre that is closely linked to the bathymetry.

Much is left out of simple models, however. The effect of the Coriolis force is completely neglected. This is clearly justified in models of the barotropic circulation in shallow and narrow estuaries where the Rossby deformation radius is very large compared to the width of the channel. Note that, in the present problem, the solution for the longitudinal component of the residual transport velocity was obtained without using the lateral momentum equation. Since the longitudinal velocity is much larger than the lateral velocity, the Coriolis force will only alter the lateral momentum balance but not the longitudinal momentum balance or the longitudinal exchange flow. This approximation is permitted as long as the lateral variation of the mean elevation is negligible.

The structure of the flow driven by the baroclinic pressure field should be more sensitive to the Coriolis force since the width of larger estuaries might be of the order of the internal deformation radius. This effect has been studied using numerical models by *Friedrichs and Hamrick* [1996] and *Valle-Levinson and O'Donnell* [1996]. Both models show that in tidally dominated estuaries the effect of rotation in narrow estuaries is modest.

6. Summary

An analytical solution for tidally induced residual circulation in an estuary with lateral depth gradient has been obtained. The magnitude of the exchange flow depends mainly on the following parameters: $h_{\min}/h_{\max}$, the ratio between the minimum depth on the shoal and the maximum depth in the channel; ζ_0/h_0 , the ratio between the tidal amplitude at the mouth and the mean depth; L/λ , the ratio between the length of the estuary and the tidal wave length; and $\beta/\sigma h_0$, the ratio between the tidal timescale and the decay timescale due to friction. The maximum residual velocity is hardly dependent on the width-length ratio D/L , although the wider the estuary is, the larger the total flux rate will be. Generally, a net landward flow occurs over the shoals and is balanced by a return flow in the channel. The exchange flow decreases monotonically toward the head of the estuary. The along-channel residual velocity changes nearly linearly from a maximum positive value (into the estuary) on the shoal to a minimum negative value (out of the estuary) in the center of the channel. The transverse residual velocity is convergent at the center of the channel. The mean elevation shows a setup at the head. This drives the outward residual flow in the channel. The maximum longitudinal residual velocity is highly dependent on the length of the estuary. When the length of the estuary is very small compared to a quarter of the wave length, the residual velocity is very small. When the length of the estuary is about a quarter of the wave length, the magnitude of the residual velocity reaches its maximum. When the length of the estuary is much larger than a quarter of the wave length, the residual circulation approaches the pattern found in an infinite length estuary. For an estuary with $h_{\min} = 5$ m on the shoal, $h_{\max} = 10$ m in the channel, and $\zeta_0 = 1$

m at the mouth, the maximum values for the residual quantities are as follows: the longitudinal component of the residual transport velocity $u_T \sim 10$ cm/s; the lateral component of the residual transport velocity $v_T \sim 1$ mm/s; the mean sea level at the head $\bar{\zeta} \sim 10$ cm; the mean pressure gradient $g\partial\bar{\zeta}/\partial x \sim 4 \times 10^{-5}$ m/s²; and the residual transport vorticity is $\sim 10^{-4}$ s⁻¹.

On the basis of this solution and subsequent analysis we conclude that exchange flow can be generated in a shallow estuary by several processes. The inward flux is caused by local nonlinearity, both in the bottom friction and from propagation of the tidal wave of finite amplitude u_{T1} . This inward flux is larger on the shoals and smaller in the channel owing to higher nonlinearity at shallower water than at deeper water (Figure 10). The residual inward flow creates a setup (residual surface elevation) of water at the head of the estuary, which produces a pressure gradient to drive an outward flow. Because the residual pressure gradient is relatively uniform across the estuary, the outward flow is larger in the channel than on the shoals (Figure 10). The combined effect results in a net inflow on the shoals and net outflow in the channel.

A minor contribution from the advection of velocity shear, which is generated by cross-channel variation of bottom friction and depth variation, creates an opposite sense of circulation with a smaller intensity. Thus this contribution cannot reverse the overall pattern of the residual circulation. The apparent contradiction between the previous theory and observations, which has already been recognized [*Zimmerman*, 1981] is due to the exclusion of nonlinear effects that are much more important in shallow estuaries.

Finally, the present theory solves only the depth-averaged residual velocity, which underestimates the maximum residual velocity along the vertical and, consequently, overestimates the residence time. A three-dimensional model is needed for an accurate prediction of the magnitude of the residual velocity and the residence time.

Appendix: A Solution Without Simplification

Here we give an exact solution for the residual velocity and elevation for a symmetric, v-shaped depth function (Figure 2). First, notice that (12) may be written as

$$\begin{aligned} \frac{\partial^2 \bar{\zeta}'}{\partial x^2} + \frac{\partial^2 \bar{\zeta}'}{\partial y^2} + \frac{2}{h} \frac{dh}{dy} \frac{\partial \bar{\zeta}'}{\partial y} &= Q(x, y) \\ \bar{u}' &= -\frac{h}{\beta} \left[\left(u_p \frac{\partial u_p}{\partial x} - \beta \frac{u_p \zeta_p}{h^2} \right) + g \frac{\partial \bar{\zeta}'}{\partial x} \right] \\ \bar{v}' &= -\frac{gh}{\beta} \frac{\partial \bar{\zeta}'}{\partial y} \end{aligned} \quad (A1)$$

in which the inhomogeneous term $Q(x, y)$ is

$$Q(x, y) = -\frac{1}{g} \frac{\partial}{\partial x} \left(u_p \frac{\partial u_p}{\partial x} \right) + \frac{2\beta}{gh^2} \frac{\partial (u_p \zeta_p)}{\partial x} \quad (A2)$$

and is determined by the solution for u_p and ζ_p . Considerable simplification can be achieved by taking the lateral depth variation to exponential, i.e. (Figure 2),

$$h = ae^{by} \quad (A3)$$

where a, b are constants and

$$a = h_{\min}, \quad b = \ln(h_{\max}/h_{\min})/D \quad (A4)$$

for half estuary in $0 \leq y \leq D$. The depth of the other half in $D \leq y \leq 2D$ is defined by symmetry. The depth profiles are shown in Figure 2 for various $h_{\min}$ and $h_{\max}$. The first equation in (A1) is then the constant coefficient differential equation

$$\frac{\partial^2 \bar{\zeta}'}{\partial x^2} + \frac{\partial^2 \bar{\zeta}'}{\partial y^2} + 2b \frac{\partial \bar{\zeta}'}{\partial y} = Q(x, y) \quad (A5)$$

A Fourier series expansion can be applied to represent the mean elevation $\bar{\zeta}'$

$$\bar{\zeta}' = \sum_{n=1}^{\infty} B_n \sin(\lambda_n x), \quad \lambda_n = \frac{2n-1}{2L} \pi, \quad n = 1, 2, \dots \quad (A6)$$

This expansion satisfies the boundary conditions that $\bar{\zeta}'|_{x=0} = 0$ and the x derivative of $\bar{\zeta}'$ at the head is zero, which is equivalent to $\bar{u}'|_{x=L} = 0$ (see (13)). Using the other boundary condition in (13), $\bar{v}'|_{y=0,D} = 0$, the solution for B_n is

$$B_n(y) = e^{k_2 y} K_n(y) + c_{1n} e^{k_1 y} + c_{2n} e^{k_2 y} \quad (A7)$$

where

$$\begin{aligned} k_1 &= -b - \sqrt{b^2 + \lambda_n^2}, \quad k_2 = -b + \sqrt{b^2 + \lambda_n^2} \\ K_n(y) &= \int_0^y e^{(-k_2 + k_1)\tilde{y}} I_n(\tilde{y}) d\tilde{y} \\ I_n(y) &= \int_0^y e^{-k_1 \tilde{y}} q_n(\tilde{y}) d\tilde{y} \\ q_n(y) &= \frac{2}{L} \int_0^L Q(x, y) \sin(\lambda_n x) dx \\ c_{1n} &= \frac{1}{k_1(e^{k_2 D} - e^{k_1 D})} [k_2 e^{k_2 D} K_n(D) + e^{k_1 D} I_n(D)] \\ c_{2n} &= -\frac{k_1}{k_2} c_{1n} \end{aligned}$$

The residual velocity depends on the dimensionless parameters in the first-order solution: ζ_0/h_0 , L/λ , and $\beta/(\sigma h_0)$. In addition, they also depend on D/L and $h_{\min}/h_{\max}$. This can be seen more clearly when the exponents in c_{1n} and c_{2n} are written as

$$\begin{aligned} k_{1,2} D &= -bD \pm \sqrt{(bD)^2 + (\lambda_n D)^2} = \ln \left(\frac{h_{\min}}{h_{\max}} \right) \\ &\pm \sqrt{\left[\ln \left(\frac{h_{\min}}{h_{\max}} \right) \right]^2 + \left[\frac{(2n-1)\pi D}{2L} \right]^2} \end{aligned} \quad (A8)$$

The effect of the aspect ratio, D/L , on the second-order solution is small because the aspect ratio itself is usually

very small. Note that, although λ_n increases with n , it can be shown that both $K_n(y)$ and $I_n(y)$ and therefore c_{1n} and c_{2n} vanish as $n \rightarrow \infty$. Further, comparison of computations with 10 and 50 terms in the Fourier series shows negligible differences, indicating that the error introduced by truncating the expansion at $n = 10$ is small.

Calculations show little difference between the solution in this appendix and the simplified solution of section 3. This indicates that the approximation of (21), based on the assumption that the lateral variation of the elevation in a narrow estuary is small, is quite accurate for the present problem. However, (21) is valid for arbitrary depth functions with the only condition that the variability of the depth should be smaller than the mean depth so that the first-order solution is valid uniformly throughout the region.

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