Finite Element Model of a Timoshenko Beam with Structural Damping

Louis Ablen Roussos

Old Dominion University, gandlroussos@juno.com

Follow this and additional works at: https://digitalcommons.odu.edu/mae_etds

Part of the Acoustics, Dynamics, and Controls Commons, and the Applied Mechanics Commons

Recommended Citation

Roussos, Louis A.. "Finite Element Model of a Timoshenko Beam with Structural Damping" (1980). Master of Science (MS), Thesis, Mechanical & Aerospace Engineering, Old Dominion University, DOI: 10.25777/772g-h942

https://digitalcommons.odu.edu/mae_etds/302

This Thesis is brought to you for free and open access by the Mechanical & Aerospace Engineering at ODU Digital Commons. It has been accepted for inclusion in Mechanical & Aerospace Engineering Theses & Dissertations by an authorized administrator of ODU Digital Commons. For more information, please contact digitalcommons@odu.edu.
FINITE ELEMENT MODEL OF A TIMOSHENKO BEAM
WITH STRUCTURAL DAMPING

Louis Ablen Roussos
B.S. May 1977, Old Dominion University

A Thesis Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF ENGINEERING

ENGINEERING MECHANICS

OLD DOMINION UNIVERSITY

Approved by:

Earl A. Thornton (Director)
ABSTRACT

FINITE ELEMENT MODEL OF A TIMOSHENKO BEAM WITH STRUCTURAL DAMPING

Louis Ablen Roussos
Old Dominion University, 1980
Director: Earl A. Thornton

A numerical integration technique, a modified version of the Newmark method, is applied to transient motion problems of systems with mass, stiffness, and small nonlinear damping. The nonlinearity is cast as a pseudo-force to avoid repeated recalculation and decomposition of the effective stiffness matrix; thus, the solution technique is dubbed the "pseudo-force Newmark method." Comparisons with exact and perturbation solutions in single-degree-of-freedom problems and with a Gear-method numerical solution in a cantilevered Timoshenko beam finite element problem show the solution technique to be efficient, accurate, and, thus, feasible provided the nonlinear damping is small. As a preliminary step into the investigation of the active control of large space structures, a problem involving a free-free Timoshenko beam with nonlinear structural damping is solved. As expected, small damping is shown to be of little importance in the prediction of low-frequency vibrations while being of utmost importance in the prediction of high-frequency vibrations.
DEDICATION

This thesis is dedicated to my father, Christos C. Roussos, and
my mother, Sylvia M. Roussos. Because both lived through some very
tough times, the Great Depression and World War II, especially diffi­
cult for my father being an immigrant in a strange land with a strange
language, they both realized the great importance of insuring that
their children have the opportunity for as high an education as each
desired. Thus, through their constant encouragement, guidance, support,
and hard work, all seven of their children earned college degrees
(4 bachelor's, 2 master's, 1 Ph.D.). Both of my parents are very
bright but never had the opportunity to earn a college degree; however,
they have been able to fulfill their dreams seven-fold through me and
the rest of their children.

Lest I forget, the singlemost important reason for this
Dedication is a two-way street called "love" through which the diffi­
cult task of raising a large family became a most stunning success.
ACKNOWLEDGEMENTS

It is of paramount importance to acknowledge that most of the guidance for this research came from Dr. Michael W. Hyer, formerly of Old Dominion University and now an Associate Professor at Virginia Polytechnic Institute & State University in Blacksburg, Virginia. While Dr. Hyer and I were at O.D.U., he was my committee chairman; and while I was at V.P.I. & S.U. with him, he was my main advisor. Unfortunately, he was removed from my committee upon my return to O.D.U. to complete my thesis writing and defense. All of the numerical results presented in this thesis were calculated on V.P.I. & S.U.'s IBM-370 computer, and I am grateful to the Engineering Science and Mechanics Department there for their cooperation in using their computer facilities.

I am also grateful to the Mechanical Engineering and Mechanics Department at Old Dominion University for the fellowship provided for me, prior to my transfer to V.P.I. & S.U., through the Aeronautics Research Participation Program, NASA grant NGR 47-003-052, which included invaluable accessibility to research facilities at NASA Langley Research Center in Hampton, Virginia.

The typing of this thesis was a very demanding task, requiring patience, expediency, and especially above average intelligence. Becky M. Alden of Norfolk, Virginia was equal to the challenge.

Much of the graphics on the figures was prepared by Jerry Williams of Progressive Graphics in Virginia Beach, Virginia.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>List of Tables</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>vii</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>List of Figures</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ix</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>List of Symbols</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>xi</td>
</tr>
</tbody>
</table>

## Chapter

I. **INTRODUCTION**

- Background .................................. 1
- Objectives .................................. 1
- Procedure .................................. 2

II. **SOLUTION TECHNIQUE**

- Solution Technique for Linear Problems  .................................. 4
  - Conventional Newmark Method Algorithm  .................. 7
  - Solution Technique for Nonlinear Problems  .......... 9
    - Pseudo-force Newmark Method Algorithm  ............. 14

III. **STRUCTURAL DAMPING**

- Coulomb Damping ................................ 18
- Viscous Damping ................................ 19
- Hysteretic Damping .............................. 20
  - Literature Review ............................ 21
  - Summary ..................................... 24
- Reid's Model (Type I Damping) .................. 25
- Standard Linear Solid .......................... 26
- Quadratic Damping (Type II Damping) .......... 26
Chapter | Page
--- | ---
III. (Continued) | 
Reed's Model Number One | 27
Reed's Model Number Two (Type III Damping) | 28
Summary Table | 29

IV. TIMOSHENKO BEAM FINITE ELEMENT MODEL | 
Formulation of Element Matrix Integrals | 30
Element Stiffness Matrix | 31
Element Mass Matrix | 36
Element Viscous Damping Matrix | 37
Element Structural Damping Matrix | 38
Numerical Integration of Structural Damping Matrix | 39
Exact Integration with Type II Damping | 41

V. RESULTS AND DISCUSSION FOR SDOF PROBLEMS | 
Accuracy Discussion | 49
Viscous Damping Problem | 49
Problems with Damping Types I, II, and III | 51
Convergence Characteristics | 53

VI. RESULTS AND DISCUSSION FOR TIMOSHENKO BEAM FINITE ELEMENT PROBLEM | 
Cantilevered Beam Problem | 58
Free-free Beam Problem | 58

VII. CONCLUSIONS | 
SDOF Problems | 70
Timoshenko Beam Finite Element Problem | 71

BIBLIOGRAPHY | 73

TABLES | 80
### LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Effect of Time Step $\Delta t$ on Accuracy of Conventional Newmark Solution for Viscously Damped SDOF Problem</td>
<td>80</td>
</tr>
<tr>
<td>II. Displacement Time History Comparison of Pseudo-force Newmark Solution with Conventional Newmark and Exact Solutions for Viscously Damped SDOF Problem</td>
<td>81</td>
</tr>
<tr>
<td>III. Displacement Time History Comparison of Pseudo-force Newmark Solution with Exact Solution for Type I-Damped SDOF Problem</td>
<td>82</td>
</tr>
<tr>
<td>IV. Peak-Amplitude Comparison of Pseudo-force Newmark Solution with Exact Solution for Type I-Damped SDOF Problem</td>
<td>83</td>
</tr>
<tr>
<td>V. Zero-Crossing Comparison of Pseudo-force Newmark Solution with Exact Solution for Type I-Damped SDOF Problem</td>
<td>84</td>
</tr>
<tr>
<td>VI. Peak-Amplitude Comparison of Pseudo-force Newmark Solution with Perturbation Solution for Type II-Damped SDOF Problem</td>
<td>85</td>
</tr>
<tr>
<td>VII. Zero-Crossing Comparison of Pseudo-force Newmark Solution with Perturbation Solution for Type II-Damped SDOF Problem</td>
<td>86</td>
</tr>
<tr>
<td>VIII. Convergence Characteristics for Type I Damping in SDOF System</td>
<td>87</td>
</tr>
<tr>
<td>IX. Convergence Characteristics for Type II Damping in SDOF System</td>
<td>88</td>
</tr>
<tr>
<td>X. Convergence Characteristics for Type III Damping in SDOF System</td>
<td>90</td>
</tr>
<tr>
<td>XI. Endpoint Displacement Time History Comparison of Pseudo-force Newmark, Conventional Newmark, and DVOGER Solutions for Cantilevered Beam Problem with Viscous Damping</td>
<td>92</td>
</tr>
</tbody>
</table>
Table

XII. Zero-Crossing Comparison of Pseudo-force Newmark, Conventional Newmark, and DVOGER Solutions for Endpoint Displacement of Cantilevered Beam with Viscous Damping .......................... 93

XIII. Effect of Time Step $\Delta t$ on Zero-Crossing Comparison of Conventional Newmark and DVOGER Solutions for Endpoint Displacement of Cantilevered Beam With No Damping .......................... 93

XIV. Endpoint Displacement Time History Comparison for Pseudo-force Newmark and DVOGER Solutions for Cantilevered Beam Problem with Type II Damping .......................... 94

XV. Zero-Crossing Comparison of Pseudo-force Newmark and DVOGER Solutions for Endpoint Displacement of Cantilevered Beam with Type II Damping .......................... 95

XVI. Effect of Type II Damping on the Low-Frequency Motion of Free-Free Beam Resulting from an Initial Condition of $r_D = 0.5$ ........................................ 96

XVII. Effect of Type II Damping on the High-Frequency Motion of Free-Free Beam Resulting From An Initial Condition of $r_D = 0.1$ ........................................ 98
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Degrees of freedom for planar motion finite element model used in thesis</td>
<td>99</td>
</tr>
<tr>
<td>2.</td>
<td>Free-body diagram and geometry for differential element of beam</td>
<td>100</td>
</tr>
<tr>
<td>3.</td>
<td>SDOF system studied</td>
<td>101</td>
</tr>
<tr>
<td>4.</td>
<td>Comparison of pseudo-force Newmark solution with exact solution for type I damping in SDOF system</td>
<td>102</td>
</tr>
<tr>
<td>5.</td>
<td>Comparison of pseudo-force Newmark solution with perturbation solution for type II damping in SDOF system</td>
<td>103</td>
</tr>
<tr>
<td>6.</td>
<td>Initial condition sketches for cantilevered and free-free beam problems</td>
<td>104</td>
</tr>
<tr>
<td>7.</td>
<td>Effect of Δt on comparison of conventional Newmark and DVOGER solutions for endpoint displacement of cantilevered beam with no damping</td>
<td>106</td>
</tr>
<tr>
<td>8.</td>
<td>Comparison of pseudo-force Newmark and DVOGER solutions for endpoint displacement of cantilevered beam with type II damping. Conventional Newmark undamped solution is also plotted to show damping effect</td>
<td>107</td>
</tr>
<tr>
<td>9.</td>
<td>Undamped and type II-damped time histories for low-frequency motion of free-free beam resulting from initial condition of $r_0 = 0.5$</td>
<td>108</td>
</tr>
<tr>
<td>10.</td>
<td>Undamped and type II damped beamshape plots for low-frequency motion of free-free beam resulting from initial condition of $r_0 = 0.5$</td>
<td>109</td>
</tr>
<tr>
<td>11.</td>
<td>Undamped and type II-damped time histories for high-frequency motion of free-free beam resulting from initial condition of $r_0 = 0.1$</td>
<td>111</td>
</tr>
</tbody>
</table>
Figure 12. Undamped and type II-damped beamshape plots for high-frequency motion of free-free beam resulting from initial condition of $r_D = 0.1$. . . . . . 112
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>cross sectional area, in$^2$</td>
</tr>
<tr>
<td>$A_m$</td>
<td>amplitude of displacement, in.</td>
</tr>
<tr>
<td>$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$</td>
<td>constants used in Newmark method algorithm (see eqs. (5))</td>
</tr>
<tr>
<td>$b$</td>
<td>width of beam cross section, in.</td>
</tr>
<tr>
<td>$b$</td>
<td>hysteretic damping coefficient, lb/in</td>
</tr>
<tr>
<td>$C$</td>
<td>viscous damping coefficient, lb-sec/in</td>
</tr>
<tr>
<td>$C_{I}$</td>
<td>damping coefficient for type I damping in SDOF problems, lb/in</td>
</tr>
<tr>
<td>$C_{II}$</td>
<td>damping coefficient for type II damping in SDOF problems, lb-sec/in$^2$</td>
</tr>
<tr>
<td>$C_{III}$</td>
<td>damping coefficient for type III damping in SDOF problems, lb-sec/in$^3$</td>
</tr>
<tr>
<td>$C_D$</td>
<td>Kimball-Lovell constant of proportionality, lb/in</td>
</tr>
<tr>
<td>$[C_L]$</td>
<td>linear (viscous) damping matrix</td>
</tr>
<tr>
<td>$[C_{NL}]$</td>
<td>nonlinear (structural) damping matrix, which is a function of ${U(t)}$ and/or ${\dot{U}(t)}$</td>
</tr>
<tr>
<td>$D_E$</td>
<td>dissipated energy per cycle, lb/in</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus, lb/in$^2$</td>
</tr>
<tr>
<td>$E$</td>
<td>modulus relating viscous damping extensional stress to strain rate, lb-sec/in$^2$ (see eq. (21))</td>
</tr>
<tr>
<td>$E_{c}, e(\varepsilon_{xx}, \dot{\varepsilon}_{xx})$</td>
<td>constant part and functional part, respectively, of modulus relating structural damping extensional stress to strain rate (see eq. (21))</td>
</tr>
</tbody>
</table>
for quadratic damping, $\epsilon(\epsilon_{xx}, \dot{\epsilon}_{xx}) = |\dot{\epsilon}_{xx}|$

and $E^\kappa$ has units $1b-sec^2/in^2$

iteration error tolerance

$d_F$ damping force for SDOF system, lb

$G$ shear modulus, $1b/in^2$

$\tilde{G}$ modulus relating viscous damping shear stress to strain rate, $1b-sec/in^2$ (see eq. (22))

$G^\kappa$, $G(\epsilon_{xz}, \dot{\epsilon}_{xz})$ constant part and functional part, respectively, of modulus relating structural damping shear stress to strain rate (see eq. (22))

for quadratic damping, $g(\epsilon_{xz}, \dot{\epsilon}_{xz}) = |\dot{\epsilon}_{xz}|$

and $G^\kappa$ has units $1b-sec^2/in^2$

$H$ height of beam, in.

$I$ cross-sectional area moment of inertia, $in^4$

$k$ SDOF stiffness, lb/in

$k_s$ cross-section shear coefficient

$[K]$ stiffness matrix

$\hat{[K]}$ effective stiffness matrix of the Newmark method

$\tilde{[K]}$ linear part of the effective stiffness matrix in the pseudo-force Newmark method

$L$ beam element length, in.

$l$ beam total length, in.

$m$ SDOF mass, $1b-sec^2/in$

$[M]$ mass matrix

$n$ number of degrees of freedom

$N_L$ limit on number of iterations
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>radius of gyration of cross-section area, in.</td>
</tr>
<tr>
<td>$r_D$</td>
<td>parameter equal to ratio of initial displacement length to beam length for the free-free beam example problems</td>
</tr>
<tr>
<td>${R(t)}$</td>
<td>vector of externally applied loads, lb</td>
</tr>
<tr>
<td>$\hat{{R}}$</td>
<td>effective load vector of the Newmark method, lb</td>
</tr>
<tr>
<td>$\tilde{{R}}$</td>
<td>linear part of the effective load vector in the pseudo-force Newmark method, lb</td>
</tr>
<tr>
<td>$T$</td>
<td>period of undamped SDOF system, sec</td>
</tr>
<tr>
<td>$T_{1,E}$</td>
<td>period of Bernoulli-Euler beam's first bending mode, sec</td>
</tr>
<tr>
<td>$t$</td>
<td>time, sec</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>displacement components, in.</td>
</tr>
<tr>
<td>$w_1, w_2$</td>
<td>modal displacements of beam element, in.</td>
</tr>
<tr>
<td>${U(t)}$</td>
<td>displacement and rotation vector</td>
</tr>
<tr>
<td>${U(t)}_j$</td>
<td>$j^{th}$ estimate of ${U(t)}$ in iteration loop of pseudo-force Newmark method; $j \leq N_L$</td>
</tr>
<tr>
<td>$X, Y, Z$</td>
<td>Lagrangian cartesian coordinate axes</td>
</tr>
<tr>
<td>$X, Y, Z$</td>
<td>distances along the length, width, and height of beam, respectively, in.</td>
</tr>
<tr>
<td>$a, \delta$</td>
<td>parameters of the Newmark method</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>shear deformation angle about Y-axis of beam cross section, rad (fig. 2)</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>integration time step, sec</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
<td>elements of Lagrangian strain tensor $(i = x,y,z; j = x, y, z)$, in/in</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>viscous damping ratio for SDOF problems</td>
</tr>
<tr>
<td>$\zeta_E, \zeta_G$</td>
<td>$\bar{E}/E$ and $\bar{G}/G$, respectively, for viscous damping, sec</td>
</tr>
<tr>
<td>$\zeta^{<em>}_E, \zeta^{</em>}_G$</td>
<td>$E^{<em>}/E$ and $G^{</em>}/G$, respectively, for structural damping, sec$^2$</td>
</tr>
</tbody>
</table>
\( \theta \)  
**cross section rotation, rad**

\( \theta_1, \theta_2 \)  
**nodal rotations of beam element**

\( \xi \)  
**\( z/H \)**

\( \rho \)  
**mass density, \( 1 \text{b}-\text{sec}^2/\text{in}^4 \)**

\( \sigma_{ij} \)  
**elements of stress tensor \((i = x,y,z; j = x,y,z), \text{lb/in}^2 \)**

\( \phi \)  
**beam element parameter defined in equation (15)**

\( x \)  
**\( x/L \)**

\( \omega_n \)  
**SDOF natural frequency, rad/sec**

\( \bar{\omega} \)  
**forcing frequency, rad/sec**
Chapter I

INTRODUCTION

Background

The control of vibrations in aerospace, marine, and ground-based dynamic systems is one of the most urgent and complex problems facing today's engineering analysts and designers (ref. 1). Some systems that are often subject to vibration effects are thin shells, pressure vessels, nuclear reactors, impacting structures, equipment mounts, shock-excited underwater and underground structures, transmission wires, airplane propellers, turbine blades, and engine crankshafts (refs. 2 and 3). Development of advanced concepts such as large space structures, honeycomb panels, and fiber-reinforced composites has prompted concern that they be modelled realistically, i.e., including nonlinear effects (ref. 3). Though much research has been done in the area of nonlinear stiffness effects (ref. 4, for example), nonlinear damping research has lagged far behind (ref. 5). In addition, new damping materials are constantly being developed (ref. 6), thus increasing the gap between analytical damping research and state-of-the-art damper manufacturing expertise.

Objectives

This thesis had two objectives:

1. Apply a numerical integration technique, the Newmark method, to transient motion problems of systems with mass stiffness,
and small nonlinear damping and determine the technique's feasibility.

2. Determine the importance of structural damping in predicting the low- and high-frequency vibratory motion of a free-free Timoshenko beam.

**Procedure**

After much research of integration techniques, the Newmark method was chosen to be investigated in this thesis. Because the Newmark method was originally proposed as a solution technique for linear differential equations (ref. 7), an iteration loop had to be added to the method so that it could be applied to the nonlinear problems of this thesis. Also, in order to avoid repeated recalculation and decomposition (triangularization by Gaussian elimination (ref. 8)) of the effective stiffness matrix due to the presence of nonlinear damping, the nonlinearity has been cast as a pseudo-force. The conventional Newmark method used for linear problems and the "pseudo-force Newmark method" used for nonlinear problems are described in detail in Chapter II.

The next step in the research was to compile a thorough structural damping review describing the models in current use with special consideration to each model's realism and context of application. This review is given in Chapter III. The three nonviscous models (piecewise linear, quadratic, and cubic functions) that were retained for the single-degree-of-freedom problems, and the single model (quadratic damping) retained for the finite element Timoshenko beam problems are noted in Chapter III.
The next step was to use the finite element formulation to incorporate nonlinear structural damping in a Timoshenko beam. Chapter IV describes the formulation of the Timoshenko beam material matrices using the finite element formulation with particular detail given to the structural damping matrix for both a general damping model and for the model used in the beam problems of the thesis.

The next step in the investigation was the solution of initial-condition single-degree-of-freedom (SDOF) problems (with the three nonviscous damping models singled out in Chapter III) so that the accuracy of the pseudo-force Newmark solution could be examined by comparison with exact and perturbation solutions, and so that the convergence characteristics could be easily studied for each of the three types of nonviscous damping. Based on the SDOF results, one model of damping, quadratic damping, was retained for the finite element Timoshenko beam problems. To check the accuracy of the pseudo-force Newmark method in the Timoshenko beam problems, a cantilevered beam problem was solved and checked against a Gear-method numerical solution (see appendix A) obtained using an International Mathematical and Statistical Library (IMSL) subroutine.

Finally, as a preliminary step into the investigation of the active control of large space structures, a free-free Timoshenko beam problem is solved with and without structural damping for two initial conditions, one resulting in low-frequency motion and one resulting in high-frequency motion. The purpose of this problem was to determine the importance of considering damping in predicting vibrations that will be actively controlled.
Chapter II

SOLUTION TECHNIQUE

This thesis involves the numerical solution of systems of ordinary differential equations arising from discrete physical models. The discrete models contain the material properties of stiffness, mass, and damping (viscous and/or structural) and are thus governed by the following nonlinear equation.

\[
[M]{\ddot{U}(t)} + \left[[C_L] + [C_{NL}]\right]{\dot{U}(t)} + [K]{U(t)} = \{R(t)\}
\]  

(1)

where 

\[C_{NL} = \begin{bmatrix} C_{NL}(\{U(t)\}, \{\dot{U}(t)\}) \end{bmatrix} \] = structural damping matrix

\[C_L = \text{viscous damping matrix}\]

\[M = \text{mass matrix}\]

\[K = \text{stiffness matrix}\]

\[R(t) = \text{external load vector}\]

and 

\[U(t) = \text{displacement vector}\]

In order to understand the solution technique for solving the nonlinear set of differential equations (1), the solution technique for solving linear problems \(([C_{NL}] = 0)\) must first be understood.

Solution Technique for Linear Problems

Though it is possible to solve the set of linear differential equations exactly; for more than a few simultaneous equations, the
solution can quickly become very complicated. Because the morass of algebra resulting from the exact solution of most practical finite element problems makes obtainment of the exact solution prohibitive, numerical techniques have been developed which are easy to apply and give adequately accurate results (ref. 9). In terms of accuracy, stability and ease of use, the two techniques most widely recommended are the Wilson-Ω method and the Newmark method (ref. 9). This thesis is concerned with the Newmark method.

The Newmark method is an unconditionally stable integration scheme for linear problems and is based on the assumption that the average acceleration over a small time increment is a constant. Because the method is unconditionally stable, integration errors and round-off errors will not grow without bound as integration marches forward. That is, the error from time t to t + Δt will be of the same order as that from t + Δt to t + 2Δt. The total error, of course, does increase as integration continues. Since stability is unconditional, the integration time step Δt is chosen solely for the desired accuracy; the smaller Δt, the more accurate the integration.

Error sources of an integration scheme are usually divided into two effects: amplitude decay and period elongation. The simplest example that can be used to demonstrate these effects is an SDOF spring-mass system of period \( T = \frac{2\pi}{\sqrt{\frac{m}{k}}} \) (m = mass and k = spring constant). If the integration scheme reveals a period \( > 2\pi\sqrt{\frac{m}{k}} \), then period elongation is present which will accumulate as integration continues in time. If the mass is given an initial displacement \( u(0) \) and after one period the displacement is \( < u(0) \), then amplitude
decay is present which will accumulate as integration moves forward in time (ref. 9).

The Newmark method has no amplitude decay (amplitude decay may be introduced into Newmark integration if it is desired), but it does exhibit period elongation. The basic Newmark equations (proposed by Newmark in ref. 7) are

\[
\{\dot{U}(t)\} = \{\dot{U}(t-\Delta t)\} + [(1-\delta)\{\ddot{U}(t-\Delta t)\} + \delta\{\dddot{U}(t)\}] \Delta t \tag{2}
\]

and

\[
\{U(t)\} = \{U(t-\Delta t)\} + \{\dot{U}(t-\Delta t)\} \Delta t + [(1-2\alpha)\{\ddot{U}(t-\Delta t)\} + 2\alpha\{\dddot{U}(t)\}] \Delta t^2 \tag{3}
\]

The parameters \(\delta\) and \(\alpha\) indicate how much of the acceleration at time \(t\) enters into the relations for velocity and displacement at time \(t\) (ref. 7). Newmark originally proposed \(\alpha = 0.25\) and \(\delta = 0.50\) for an unconditionally stable linear acceleration method with no amplitude decay. These values of \(\alpha\) and \(\delta\) are used throughout this thesis.

Using equations (2) and (3) in conjunction with equation (1) results in three equations and three unknowns: \(\{U(t)\}\), \(\{\dot{U}(t)\}\), and \(\{\ddot{U}(t)\}\).

Solving these three equations for \(\{U(t)\}\) we get an equation of the form

\[
[\tilde{K}]\{U(t)\} = \{\hat{R}\} \tag{4}
\]

where \([\tilde{K}]\) is the effective stiffness matrix which is a constant for all time, and \(\{\hat{R}\}\) is the effective load vector which is a function of information at time \(t - \Delta t\), which of course is all known information at time \(t\). The expressions for \([\tilde{K}]\) and \(\{\hat{R}\}\) are
The algorithm most commonly used (and used in this thesis) for implementing the conventional Newmark method comes from equation (4) and is shown as follows:

**Conventional Newmark Method Algorithm**

1. Initial calculations
   (a) Form stiffness, mass, and viscous (linear) damping matrices from given material data
   (b) Specify initial displacements and velocities and calculate initial accelerations by using equation (1)
   (c) Specify time increment and Newmark convergence parameters \( \delta \) and \( \alpha \)
   (d) Calculate Newmark integration constants
      \[
      a_0 = 1/[\alpha(\Delta t)^2]
      \]
      \[
      a_1 = \delta/\alpha \Delta t
      \]
      \[
      a_2 = 1/(\alpha \Delta t)
      \]
      \[
      a_3 = -1 + 1/(2\alpha)
      \]
      \[
      a_4 = -1 + \delta/\alpha
      \]
      \[
      a_5 = \Delta t(-2 + \delta/\alpha)/2
      \]

\[
\hat{[K]} = [K] + \frac{1}{\alpha(\Delta t)^2} [M] + \frac{\delta}{\alpha \Delta t} [C_L]
\]

\[
\{\hat{R}\} = \{R(t)\} + [M]\left\{\frac{\delta}{\alpha \Delta t}\{U(t-\Delta t)\} + \frac{1}{\alpha \Delta t}\{\ddot{U}(t-\Delta t)\}\right\}
\]

\[
+ (-1 + \frac{1}{2 \alpha}) \{\ddot{U}(t-\Delta t)\}]
\]

\[
+ [C_L]\left\{\frac{\delta}{\alpha \Delta t}\{U(t-\Delta t)\} + (-1 + \frac{\delta}{\alpha}) \{\ddot{U}(t-\Delta t)\}\right\}
\]

\[
+ \frac{\Delta t}{2} (-2 + \frac{\delta}{\alpha}) \{\ddot{U}(t-\Delta t)\}\}
\]
\[ a_6 = \Delta t(1-\delta) \]
\[ a_7 = \delta \Delta t \] (5)

(e) Calculate effective stiffness matrix \([\tilde{K}]\)

\[ [\tilde{K}] = [K] + a_0[N] + a_1[C_L] \] (6)

(f) Decompose \([\tilde{K}]\) into an upper triangular matrix in preparation for solving for \([U(t)]\) by backward substitution

2. Calculations for each time step

(a) Calculate the effective load vector \([\tilde{R}]\)

\[ \{\tilde{R}\} = \{R(t)\} + [M]\{R_1(t-\Delta t)\} + \]
\[ + [C_L]\{R_2(t-\Delta t)\} \] (7)

where \([R(t)]\) = externally applied forcing function

\[ \{R_1(t-\Delta t)\} = a_0\{U(t-\Delta t)\} + a_2\{\ddot{U}(t-\Delta t)\} \]
\[ + a_3\{\dot{U}(t-\Delta t)\} \]
\[ \{R_2(t-\Delta t)\} = a_1\{U(t-\Delta t)\} + a_4\{\ddot{U}(t-\Delta t)\} \]
\[ + a_5\{\dot{U}(t-\Delta t)\} \] (8)

(b) Decompose \([\tilde{R}]\) in the same manner \([\tilde{K}]\) was decomposed

(c) Solve for \([U(t)]\) by using back substitution on equation (4)

(d) Calculate \([\ddot{U}(t)]\) and \([\dot{U}(t)]\) from the following equations,
\[ U(t) = \alpha_0\{U(t) - U(t-\Delta t)\} - \alpha_2\{\dot{U}(t-\Delta t)\} \]
\[ - \alpha_3\{\ddot{U}(t-\Delta t)\} \]
\[ \dot{U}(t) = \{\dot{U}(t-\Delta t)\} + \alpha_6\{\ddot{U}(t-\Delta t)\} + \alpha_7\{\ddot{U}(t)\} \]

(e) Now return to 2(a) and use \( \{U(t)\} \), \( \{\dot{U}(t)\} \), and \( \{\ddot{U}(t)\} \) to solve for \( \{U(t+\Delta t)\} \), \( \{\dot{U}(t+\Delta t)\} \), and \( \{\ddot{U}(t+\Delta t)\} \) by advancing time by \( \Delta t \) in the above formulas. Continue integrating until desired final time is reached.

The above algorithm is presented in reference 9.

The equilibrium equation (eq. (4)) will become the focus of the following discussion on applying the Newmark method to nonlinear problems.

**Solution Technique for Nonlinear Problems**

While numerical integration schemes for linear finite element problems have been developed for at least 25 years, the application of the finite element method to nonlinear problems has been under development for only about fifteen years (ref. 10). In recent years a number of general purpose finite element computer programs have been developed for nonlinear elastic and inelastic problems (refs. 11 to 14). Several of these have included application of the Newmark method. Still, most of today’s structural analyses are done with linear techniques (refs. 3, 15, and 16) despite the fact that many structural problems are nonlinear (refs. 10 and 17). Though almost all nonlinear structural problems can now be accurately solved by nonlinear finite element analysis, the reason nonlinear analysis is not used is the prohibitive computer cost (refs. 16 and 18). With respect to reduction of cost,
the two most important areas of research concern accuracy and stability and are considered to be (1) analyzing solution techniques mathematically and (2) matching problem classes to solution techniques (see refs. 11, 12, 14, 19, and 20).

While mathematical stability analyses of linear solution techniques are well developed (including an unconditional stability proof for the conventional Newmark method (ref. 21)); for nonlinear solution techniques, stability analysis has become a controversial matter. Several different notions of stability criteria have been proposed; and while a solution technique may be found stable by one notion, the same technique may be found unstable by another notion (see refs. 19, 22, and 23). Thus, much more investigation is needed in the area of mathematical stability analyses of nonlinear solution techniques (refs. 11, 12, and 14).

The research area of matching problem classes to solution techniques has proven much more fruitful and useful to investigators. This thesis contributes to this research area as the nonlinearity studied herein, structural damping, has had little past work. Most nonlinear finite element analyses have been concerned with statics problems, and almost all of the nonlinear problems studied (static or transient) have dealt with nonlinearities due to large displacement, large strain, and nonlinear stress-strain laws (refs. 3, 11, 13, and 24). Nonlinear relations between stress and strain rate at small strains have had little investigation in nonlinear finite element applications (using the method of averaging a highly restricted Bernoulli-Euler beam problem with a nonlinear stress-strain-strain rate law was solved in ref. 25). Most analyses of any kind concerning
nonlinear relations between stress and strain rate have been confined to creep and creep-rate analyses, usually with constant strain rate, (see refs. 3, 11, 24, and 26 to 31) while the present study is concerned with elastic deformation and vibratory (or cyclic) strain rate. In spite of these limitations, past work involving matching problem classes to solution techniques has been very helpful in guiding the present study as conclusions drawn in static analysis may be used as a guide in transient analysis and conclusions drawn on nonlinear stress-strain laws may be used as a guide for problems involving nonlinear relations between stress and strain rate.

In applying the Newmark method to the nonlinear set of differential equations (1), the equations for \( [K] \) and \( \{R\} \) (equations (6) and (7), respectively) must be rewritten to include terms with \( [C_{NL}] \) by changing the damping matrix from \( [C_L] \) to \( [C_L] + [C_{NL}(\{U\}, \{U\})] \). The new equation for \( [\ddot{K}] \) is

\[
[\ddot{K}] = [K] + a_0 [M] + a_1 [C_L] + [C_{NL}]
\]

And the new equation for \( \{\ddot{R}\} \) is

\[
\{\ddot{R}\} = \{R(t)\} + [M]\{R_1(t-\Delta t)\} + [\{C_L\}] + [C_{NL}]\{R_2(t-\Delta t)\}
\]

where \( \{R_1(t-\Delta t)\} \) and \( \{R_2(t-\Delta t)\} \) have the same definitions as previously given in equation (8).

Letting

\[
[\ddot{K}] = [K] + a_0 [M] + a_1 [C_L]
\]

and

\[
\{\ddot{R}\} = \{R(t)\} + [M]\{R_1(t-\Delta t)\} + [C_L]\{R_2(t-\Delta t)\}
\]
the equilibrium equation (eq. (4)) can be rewritten as

$$[(\tilde{K} + a_1 [C_{NL}]) (U(t))] = (\tilde{R}) + [C_{NL}] \{R_2(t-\Delta t)\} \tag{9}$$

(Note that $[\tilde{K}]$ and $\{\tilde{R}\}$ are the same as $[\hat{K}]$ and $\{\hat{R}\}$ for linear problems.)

Since equation (9) is nonlinear, in general, it can be solved only approximately by iterating until the error between successive iterations is at or below a reasonable tolerance. And because the effective stiffness $[\hat{K}]$ is nonlinear, it will have to be repeatedly updated and decomposed when the Newmark method is applied; and such repeated decomposition is very expensive. To avoid this repeated decomposition, $a_1 [C_{NL}]$, the nonlinear part of $[\hat{K}]$, may be factored out by multiplying it by $\{U(t)\}$ and shifting the resulting vector to the right hand side. The resulting equilibrium equation is then

$$[\tilde{K}] (U(t)) = (\tilde{R}) + [C_{NL}] (\{U(t)\}, \{\dot{U}(t)\}) ([R_2(t-\Delta t))$$

$$- a_1 \{U(t)\}] \tag{10}$$

To put the question of whether to use equation (9) or (10) into the proper perspective, the following paragraphs discuss solution approaches to nonlinear problems.

In general, there are two approaches to setting up nonlinear equations involving matrix methods of structural analysis. One approach is known as the "tangent stiffness" method (eq. (9) is in this form) and the other as the "pseudo-force" method (eq. (10) is in this form) (see refs. 3, 18, 24, 32, and 33).
The general form of the equation of concern is

\[ [K] \{u\} = \{R\} \quad (11) \]

If equation (11) is in tangent stiffness form, then all the nonlinearities are lumped in the stiffness matrix \([K]\). If equation (11) is in the pseudo-force form, then all the nonlinearities are lumped in the vector \([R]\).

Solution methods for statics problems involving material and/or geometrically nonlinear problems are many and varied and are superbly discussed in reference 24 by Sticklin, Haisler, and Von Riesemann. In this reference the authors report that the pseudo-force method is completely inappropriate for problems involving geometric nonlinearities as it is invariably unstable. However, for material nonlinearities (nonlinear stress-strain laws) they concluded that the pseudo-force method was very attractive because of the immense computational saving of the single stiffness matrix inversion while requiring only a few more convergence iterations than the tangent stiffness method. In a private communication (ref. 34) to the author of this thesis, Professor Klaus-Jürgen Bathe (Massachusetts Institute of Technology), in direct response to a query on this very problem, lauded the pseudo-force method in equation (10) as being a "very effective" technique providing the nonlinearities are small in comparison with the other linear mechanisms present. Professor Bathe also remarked that severe convergence problems would occur if large nonlinearities are modelled in this manner.

Since structural damping is a small (refs. 15 and 35 to 37) material nonlinearity, the pseudo-force method of equation (10) is employed in this thesis because of its computational efficiency without loss of stability for small nonlinearities.
Equation (10) is the equilibrium equation on which the following algorithm for nonlinear problems is based.

**Pseudo-force Newmark Method Algorithm**

1. **Initial calculations**
   
   (a) Form stiffness, mass, and viscous (linear) damping matrices from given material data

   (b) Specify initial displacements and velocities and calculate initial accelerations by using equation (1)

   (c) Specify time increment, convergence error tolerance $\epsilon_t$, limit on number of iterations $N_L$, and the Newmark convergence parameters $a$ and $\delta$

   (d) Calculate Newmark integration constants $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$, and $a_7$ (see eqs. (5))

   (e) Calculate linearized effective stiffness matrix $[\tilde{K}]$

   $$[\tilde{K}] = [K] + a_0 [M] + a_1 [C_L]$$

   (f) Decompose $[\tilde{K}]$ into an upper triangular matrix in preparation for solving for $\{U(t)\}$ with backward substitutions in the iteration loop

2. **Calculations for each time step.**

   (a) Calculate $\{\tilde{R}\}$ part of the effective load vector

   $$\{\tilde{R}\} = \{R(t)\} + [N] \{R_1(t-\Delta t)\}$$

   $$+ [C_L] \{R_2(t-\Delta t)\}$$

   where $\{R(t)\}$ = externally applied forcing function and $\{R_1(t-\Delta t)\}$ and $\{R_2(t-\Delta t)\}$ are defined by equation (8)
(b) Calculate first estimate of \( \{U(t)\} \), \( \{\dot{U}(t)\} \), and \( \{\ddot{U}(t)\} \), i.e.,

\[
\{\ddot{U}(t)\}^1 = \{U(t-\Delta t)\}
\]

\[
\{\dot{U}(t)\}^1 = \{\ddot{U}(t-\Delta t)\} \Delta t + \{U(t-\Delta t)\}
\]

\[
\{U(t)\}^1 = \{\dddot{U}(t-\Delta t)\} \frac{\Delta t^2}{2} + \{\dot{U}(t-\Delta t)\} \Delta t + \{U(t-\Delta t)\}
\]

(c) Beginning of the iteration loop

\( j = \) iteration number = 1, 2, 3, . . . , \( N_L \)

(i) Using the latest \( (j^{th}) \) estimate of the displacement and velocity vectors calculate the latest \( (j^{th}) \) estimate of the nonlinear damping matrix.

(ii) Calculate effective load vector \( \{\hat{R}\} \) using the latest \( (j^{th}) \) estimates

\[
\{\hat{R}\} = \{\hat{R}\} + [C_{NL} (\{U(t)\}_j, \{\dot{U}(t)\}_j)] [\{R_2(t-\Delta t)\}]
\]

\[
- a_1 \{U(t)\}_j
\]

(iii) Decompose \( \{\hat{R}\} \) in same manner \( [\hat{R}] \) was decomposed.

(iv) Solve for \( (j+1)^{th} \) estimate of \( \{U(t)\} \) by using backward substitution on equation (1).

(v) Calculate \( \{\dot{U}(t)\}^{j+1} \) and \( \{\ddot{U}(t)\}^{j+1} \) from the following,

\[
\{\ddot{U}(t)\}^{j+1} = a_0 \{\{U(t)\}^{j+1} - (U(t-\Delta t))\}
\]

\[
- a_2 \{\dot{U}(t-\Delta t)\} - a_3 \{\dddot{U}(t-\Delta t)\}
\]
\[
\{\dot{U}(t)\}_{j+1} = \{\dot{U}(t-\Delta t)\} + a_6 \{\ddot{U}(t-\Delta t)\} + a_7 \{\dddot{U}(t)\}_{j+1}
\]

(vi) Check for iteration convergence between \{U(t)\}_j and \{U(t)\}_{j+1} relative to \{U(t)\}_j.

The iteration convergence error C.E. is the magnitude of the vector \{U(t)\}_{j+1} - \{U(t)\}_j divided by the magnitude of the vector \{U(t)\}_j. Thus,

\[
C.E. = \frac{\sum_{k=1}^{N} (U_{k,j+1} - U_{k,j})^2}{\sqrt{\sum_{k=1}^{N} (U_{k,j})^2}}
\]

The iteration convergence error C.E. is then checked against the error tolerance \(e_t\) as follows

<table>
<thead>
<tr>
<th>C.E. - (e_t) (\leq 0)</th>
<th>C.E. - (e_t) (&gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>convergence at or below tolerance has occurred; go to step 2(d)</td>
<td>let (j = j+1) and continue iterating until convergence or until limit on number of iterations in which case go to step 2(d)</td>
</tr>
</tbody>
</table>

(d) Now return to step 2(a) and use the final converged values of \{U(t)\}, \{\dot{U}(t)\}, and \{\ddot{U}(t)\} to solve for \{U(t+\Delta t)\}, \{\dot{U}(t+\Delta t)\}, and \{\ddot{U}(t+\Delta t)\} by advancing \(t\) by \(\Delta t\) in the above formulas. Continue until desired final time is reached.
Chapter III

STRUCTURAL DAMPING

This section surveys several analytical models of structural damping. Structural damping (also known as internal damping, material damping, and internal friction) is defined as the energy dissipation mechanism within the volume of a vibrating solid that is independent of any dissipation capacity at the boundaries of the solid (refs. 2, 38, and 39). A literature search leaves little doubt that structural damping is basically a nonlinear phenomenon. Linear procedures and assumptions are appropriate for only linear materials (such as, polymers, elastomers, many glasses, plastics, rubbers, and viscoelastic organic compounds at low stress levels) and are generally unrealistic for structural materials at stress levels of interest in structural mechanics (refs. 2, 5, 6, 17, 36, 38, and 40 to 43). Structural damping is also small in size; that is, the structural damping forces and stresses are small in comparison with the linear elastic and inertial forces and stresses (refs. 15 and 35 to 37).

A mathematical model for the physical behavior of structural damping, indeed, of any material property, is necessary to obtain realistic results from mathematical analyses of structural systems (refs. 1, 5, and 43). While most material properties have been inter-related and defined by simple expressions and parameters such as Hooke’s law, Young’s modulus, and Poisson’s ratio; structural damping
has not been so well interrelated nor defined due to conflicting experimental results and in spite of over 2,500 publications on damping as of 1965 (refs. 2 and 17). The underlying physical properties of damping are only qualitatively or superficially understood (ref. 44). Thus, there is a continuing search for structural damping models which are simple and still accurately represent the behavior of actual materials (ref. 43). Though structural damping is decidedly nonlinear, viscous damping and hysteretic damping (linear models) are the most commonly employed (ref. 31). In fact, hysteretic damping has been used very successfully in flutter analysis (refs. 37, 41, 43, and 44). In general, however, as previously noted, linear models are inappropriate in structural mechanics and the primary reason linear models are so commonly used is because the cost of nonlinear analysis is often prohibitive (refs. 16 and 18). When applying a linear model, great care, physical sense, and vigilance is needed as the use of any model, linear or nonlinear, out of context should be avoided. In the rest of this chapter some common structural damping models, linear and nonlinear, are reviewed to determine which ones will be used in formulating $[C_{NL}]$. Particular attention is given to context and/or limitations of usage.

**Coulomb Damping**

Coulomb damping (also known as slip damping and dry friction damping) results from the relative motion of two surfaces sliding one upon the other with a constant normal force holding them together. In equation form the damping force $F_D$ is

$$F_D = \mu_k F_n \text{sgn}(\ddot{u}(t))$$
where $\mu_k = \text{kinetic coefficient of friction}$

$F_N = \text{normal force}$

and $\dot{u}(t) = \text{velocity}$

This idealized model of damping is not a good approximation of nonlinear internal damping.

**Viscous Damping**

The simplest form of linear damping is viscous damping where the damping force is assumed to be proportional to velocity and in phase with velocity. The damping force equation is $F_D = Cu(t)$ where $C$ is a constant called the viscous damping coefficient. In single-degree-of-freedom analysis, this damping is a reasonably good structural damping model only when the motion is dominated by a single frequency. The viscous damping coefficient is then chosen to yield an energy loss per cycle that is equivalent to that of the more complicated system being modelled (see refs. 36 and 44). In multi-degree-of-freedom structural analysis, viscous damping may be used effectively if a modal analysis is done and the appropriate amount of viscous damping is assigned to each mode. Since this thesis is concerned with general transient motion problems (especially finite element problems) where many dominant frequencies may occur and because almost no real materials behave as though they are viscously damped (refs. 43 and 46), this model was considered unacceptable for use as a structural damping model. (Viscous damping has had success in modelling friction created by low Reynolds number flow (refs. 45 and 47)).
Hysteretic Damping

Investigations of materials undergoing sinusoidally-forced motion have led to a linear damping model called hysteretic damping. With hysteretic damping the damping force is proportional to displacement and in phase with velocity (ref. 43). Hysteretic damping is usually expressed in one of two ways: either as a frequency-dependent viscous damper or as complex stiffness. With the frequency-dependent viscous damper, the SDOF equation of motion is

\[ m \ddot{u}(t) + \frac{b}{\omega} \dot{u}(t) + ku(t) = Fe^{i\omega t} \tag{12} \]

where \( F \) is the amplitude of the forcing function and \( b \) is a constant independent of the forcing frequency \( \omega \). With complex stiffness the SDOF equation of motion is

\[ m \ddot{u}(t) + (k + ib)u(t) = Fe^{i\omega t} \tag{13} \]

Since \( \dot{u}(t) = i\omega u(t) \) for sinusoidally-forced motion, it is easily seen that the damping force is proportional to displacement and in phase with velocity, it is also readily seen how equation (13) is obtained from equation (12).

Hysteretic damping is the most widely used model of structural damping (refs. 15, 31, 37, and 48 to 51). It has been used extensively in aircraft structural dynamics and has met much success in flutter analysis (refs. 37, 41, 43, and 44). However, the use of hysteretic damping has become a controversial and sometimes confusing subject. The following literature review is an attempt to present in a clear manner the conflicting opinions regarding this subject.
Those readers not intensely interested in this discussion are advised to skip to the summary paragraph for hysteretic damping.

**Literature Review**

In 1927 Kimball and Lovell observed for sinusoidally-forced vibrations that many engineering materials exhibit an energy loss (dissipated energy) $D_E$ per cycle proportional to the amplitude of the displacement squared, $A_m^2$, but independent of the forcing frequency $\bar{\omega}$ (refs. 43 and 52). In equation form the Kimball-Lovell observation is

$$D_E = C_D A_m^2$$

where $C_D'$ is a constant independent of $\bar{\omega}$. In 1935 Wegel and Walther (ref. 53) also observed for small strains that $D_E$ was proportional to $A_m^2$ and that $C_D'$ was practically independent of $\bar{\omega}$.

Using the earlier notation for hysteretic damping it is seen that $D_E = b \pi A_m^2$; therefore, $b = C_D'/\pi$. So, since $C_D'$ is assumed independent of $\bar{\omega}$, $b$ is also independent of $\bar{\omega}$ which was stated earlier in defining hysteretic damping.

In 1969, however, V. D. Naylor stated that he regarded the experimental evidence supporting the conclusion that dissipation is independent of forcing frequency as being "very shaky and open to criticism" (ref. 54). Moreover, he derived the formula for $D_E$ using hysteretic damping (with $F \sin(\bar{\omega}t)$ forcing function) and obtained

$$D_E = \frac{\pi F^2 b}{(k - m \bar{\omega}^2)^2 + b^2}$$

which is clearly a function of $\bar{\omega}$. The above formula can be derived from $D_E = b \pi A_m^2$ simply by realizing that $A_m$ is a function of $\bar{\omega}$.
such that

\[ A_m = \frac{F}{(k - m\omega^2)^2 + b^2} \]

For an \( e^{i\omega t} \) forcing function as in equation (13),

\[ A_m = \frac{F}{[(k - m\omega^2)^2 + b]^2} \] and \( D_E \) is still readily seen to be a function of \( \omega \).

C. W. Bert (ref. 43) referenced a 1970 paper by Scanlan where Bert says Scanlan reveals an error in Naylor's reasoning. Scanlan's paper (ref. 44) does not directly refer to Naylor; however, Scanlan does state that the damping coefficient used in hysteretic damping is only good over a small restricted frequency range, i.e., it is a "local constant" that is different for different frequency ranges. Furthermore, Crandall (ref. 36) points out that in letting the damping coefficient \( b \) vary with \( \omega \), certain restrictions are required. The damping coefficient must be an even function of \( \omega \), must be real, and must be nonnegative. But stating that \( b \) is a function of \( \omega \) is confusing in that it contradicts the experimental evidence on which hysteretic damping has been based. As stated previously, the experimental evidence indicated the hysteretic damping coefficient was, at most, a weak function of \( \omega \) and was probably, for all practical purposes, independent of \( \omega \). The author of this thesis is ready and willing to adopt Scanlan's sensible observations, but is also confused as to why current literature still refers to the hysteretic damping coefficient as being independent of \( \omega \).

Another topic of dispute concerning the experimental conclusions which form the basis of hysteretic damping is the observation that cyclic loss \( D_E \) is proportional to the amplitude squared, \( A_m^2 \).
In 1946 Robertson and Yorgiadis (ref. 2) reported with strong conviction that their structural damping experiments showed that $D_E$ is proportional to $A_m^3$. In 1952 Pian and Hallowell (ref. 42) also reported that their experiments showed $D_E$ proportional to $A_m^3$. In 1968 when Lazan reported that $D_E$ was proportional to $A_m^2$ for low stress, he also stated that $D_E$ was proportional to $A_m^n$, $n \geq 3$, for intermediate and high stress, i.e., for stresses of interest in structural mechanics (ref. 17). This topic of conflict is equivalent to arguing whether structural damping is linear or nonlinear as $D_E$ proportional to $A_m^2$ is a result of the damping force being some linear or piecewise linear function of $u(t)$ or $\dot{u}(t)$ and $D_E$ proportional to $A_m^3$ is a result of the damping force being some quadratic function of $u(t)$ and/or $\dot{u}(t)$. Since there is little doubt that structural damping is generally nonlinear, $D_E$ being proportional to $A_m^2$ should be good only under certain restricted circumstances, such as small stress as suggested by Lazan, or in only certain special cases, such as flutter analysis.

All discussion on hysteretic damping up to this point has been concerned with sinusoidally-forced vibrations. All experimental evidence referenced thus far has been based on sinusoidally-forced vibrations; indeed, hysteretic damping inherently came from studying sinusoidally-forced vibrations. The present study, however, is concerned with general transient motion, including free vibrations. Being the most widely used form of structural damping, hysteretic damping has often been applied to general transient motion, including free vibrations. Recall that the damping force for the frequency-dependent viscous damper is $F_D = \frac{b}{\omega} \dot{u}(t)$. But, how is $\ddot{u}$ interpreted
for, say, free vibrations? Milne (ref. 54) has suggested that \( \bar{\omega} \) is associated with the imaginary part of the pair of complex roots; however, he went on to say that no clear justification exists for this interpretation. In general transient vibrations, neither the frequency-dependent viscous damper nor the complex stiffness gives a damping force that is proportional to displacement and in phase with velocity (ref. 43) which was, perhaps, the most important characteristic of hysteretic damping for sinusoidally-forced vibrations.

The biggest problem of all in applying hysteretic damping to general transient vibrations is one noted by Scanlan and Mendelson in 1963 (ref. 56): hysteretic damping represents a physically unrealizable system for the case of non-sinusoidally-forced vibrations as the system response may be a function of the response in the future. In 1970 Scalan reiterated this message in stating that for the case of free vibrations and simple initial conditions, hysteretic damping "cannot properly describe the physical case of decaying oscillations ... (and) must also be confined to the context of motion representable by the force \( e^{i\omega t} \)" (ref. 44).

**Summary.** Hysteretic damping is appropriate only for sinusoidally-forced vibrations and, since it is a linear model, only within certain restrictions or for special cases. Since general transient motion is the concern of the present study, hysteretic damping was deemed unacceptable as a structural damping model herein. In 1956, however, Reid (ref. 57) developed a structural damping model based on the concept of hysteretic damping that can be applied to general transient vibrations; Reid’s model is the topic of the following section.
Reid's Model (Type I Damping)

Noting for sinusoidally-forced vibrations that hysteretic damping produces a force proportional to displacement but in phase with velocity, Reid decided to make this the basic definition for a general transient vibration analogy of hysteretic damping. Thus, he postulated the following equation for damping force \( F_D \) (ref. 57),

\[
F_D = (C_I |u(t)|) \text{sgn}(\dot{u}(t)) = C_I \dot{u}(t) \frac{|u(t)|}{|\dot{u}(t)|}
\]

where \( C_I \) is a damping coefficient.

Reed (ref. 5) gives the first-order equation (obtained by using an approximate analytical solution technique called "the method of Klotter" (ref. 58)) for the cyclic dissipation with sinusoidal forcing to be

\[
D_E = 2C_I A_m^2
\]

Reid's model reduces to the hysteretic damping model for the case of sinusoidally-forced vibrations. For the SDOF case, Reid's model is piecewise linear: for \( \frac{|u(t)|}{|\dot{u}(t)|} > 0 \), \( F_D = C_I u(t) \) and for \( \frac{|u(t)|}{|\dot{u}(t)|} < 0 \), \( F_D = -C_I u(t) \).

Because hysteretic damping has been used extensively in aero-dynamic structural analysis, has met much success in flutter analysis, and is the most widely used form of structural damping; Reid's model, an adaptation of hysteretic damping to general transient vibrations, was chosen as one of three models to be explored in SDOF example problems. For the remainder of this thesis, Reid's model has been designated type I damping.
Standard Linear Solid

The standard linear solid assumes a linear relation between stress and strain and between stress and strain rate and allows for both viscous and hysteretic damping (refs. 43, 59, and 60). Since both viscous and hysteretic damping have already been discussed, no further discussion of this model is necessary.

Quadratic Damping (Type II damping)

The equation for the damping force for quadratic damping is

\[ F_D = C_{II} |\dot{u}(t)| \ddot{u}(t) = C_{II} \frac{\dot{u}(t)^3}{|\ddot{u}(t)|} = C_{II} \dot{u}(t)^2 \text{sgn}(\dot{u}(t)) \]

where \( C_{II} \) is a damping coefficient. Reed (ref. 5) gives the first-order equation for sinusoidally-forced cyclic dissipation to be

\[ D_E = \frac{8}{3} C_{II} A_m^3 \omega^2 \]

One situation where this type of damping is commonly used as a model is for a body immersed in a high Reynolds number flow (refs. 1, 45, and 61 to 63). Quadratic damping has sometimes been called a correction to viscous damping. This may be because viscous damping is a model used with low Reynolds number flow (refs. 45 and 47).

Since \( D_E \) for quadratic damping is proportional to \( A_m^3 \), it is readily evident that this model might also be applicable for structural damping. As no definitive nonlinear models for structural damping exist, it was decided that one nonlinear model would be just as good as another for demonstrating the use and effects of a nonlinear structural damping model in finite element analysis. Thus quadratic
damping was another model chosen to be explored in SDOF example problems. For the remainder of this thesis, quadratic damping has been designated as type II damping. This damping model is the one used in the finite element Timoshenko beam problems.

Reed's Model Number One

In 1966 Reed (see ref. 5) proposed the following SDOF model for structural damping force,

\[ F_D = C^* |u(t)| \dot{u}(t) \]

where \( C^* \) is a damping coefficient.

Reed gives the first-order equation for sinusoidal-forced cyclic dissipation to be

\[ D_E = \frac{4}{3} C^* A_m^3 \bar{\omega} \]

Reed indicated in reference 5 that this model was based on matching analytical hysteretic loops with experimental ones. In 1969 Brammeier (ref. 64) investigated this model along with several others but could not come to any conclusions as to whether this model (or any of the others) was a realistic model except for his stating that a combination of two models may give the best results. Except for Reed no other investigations could be found which could substantiate the realism of this model. As with quadratic damping, this model had \( D_E \) proportional to \( A_m^3 \) which in itself indicates that this model may be applicable to structural damping. Due to the limited scope of this thesis this model was not explored. This model certainly
warrants more investigation relative to both its realism and its analytical feasibility and behavior.

Reed's Model Number Two (Type III Damping)

Another model for structural damping force proposed by Reed in 1966 was

\[ F_D = C_{III} u(t)^2 \dot{u}(t) \]

where \( C_{III} \) is a damping coefficient.

Reed gave the first-order equation for sinusoidal-forced cyclic dissipation to be

\[ D_e = \frac{\pi}{4} A_m \tilde{\omega} C_{III} \]

(see ref. 5). Actually, Reed proposed a model which included both environmental and structural damping and had a force \( F_D \) equal to

\[ C_{III} (\beta + u(t)^2) \dot{u}(t) \] where \( \beta \) is an arbitrary constant. Since \( C_{III} \beta \dot{u}(t) \) is considered to be environment-induced (viscous) damping, then \( C_{III} u(t)^2 \dot{u}(t) \) is the structural damping. As with the previously discussed model of Reed's, no other investigation other than Reed's could be found to substantiate the experimental basis for this model.

With a piecewise linear model (Reid's model) and a quadratically nonlinear model (quadratic damping) so far chosen as models to be explored in SDOF example problems, Reed's model number two, a cubically nonlinear model, was chosen as the third model to be studied. For the remainder of this thesis, Reed's model number two has been designated as type III damping.
Summary Table

In summary, a table is shown below describing the three types of damping chosen to be explored in SDOF problems.

<table>
<thead>
<tr>
<th>Damping Type</th>
<th>Reference</th>
<th>Damping force, $F_D$</th>
<th>Functional Character</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>57</td>
<td>$C_I</td>
<td>u(t)</td>
</tr>
<tr>
<td>II</td>
<td>5</td>
<td>$C_{II} \dot{u}(t)^2 \text{sgn}(\dot{u}(t))$</td>
<td>quadratic</td>
</tr>
<tr>
<td>III</td>
<td>5</td>
<td>$C_{III} u(t)^2 \ddot{u}(t)$</td>
<td>cubic</td>
</tr>
</tbody>
</table>
Chapter IV

TIMOSHENKO BEAM FINITE ELEMENT MODEL

Formulation of Element Matrix Integrals

The finite element method is a numerical technique for solving partial differential equations. With a finer and finer mesh of elements, the discrete finite element model's potential, kinetic, and dissipative mechanisms will converge to those of the continuous system being modelled. For transient analysis the finite element method leads to a system of ordinary differential equations of the form of equation (1). For a more detailed description of the finite element method, the reader is referred to reference 65, one of many finite element textbooks.

The finite element of concern in this paper is a Timoshenko beam element. Timoshenko beam theory is a theory that includes the effects of rotatory inertia and cross sectional shear deformation, effects that are ignored in elementary Bernoulli-Euler beam theory. For a brief description of Timoshenko beam theory see appendix B, or, for a more detailed description, see references 66 and 67.

In 1966 J. S. Przemieniecki (ref. 68) developed a Timoshenko beam finite element model with six degrees of freedom at each node: a rotation and displacement in three orthogonal directions. The rotation angle includes both rotation due to bending and rotation due to shear deformation. A two-dimensional version of Przemieniecki's model,
shown in figure 1, is employed in this thesis; it allows for just two degrees of freedom, a transverse displacement and a rotation at each node, i.e., planar motion.

The assumed displacement and shearing strain fields (ref. 68) of the element are

\[ w(x,t) = \frac{1}{1+\phi} \left[ (1 - 3 \frac{x^2}{L^2} + 2 \frac{x^3}{L^3} + (1 - \frac{x}{L})\phi \right] w_1(t) \]

\[ + \left[ \frac{3}{L} - 2 \frac{x^2}{L^2} + \frac{x^3}{L^3} + \frac{1}{2} (\frac{x}{L} - \frac{x^2}{L^2})\phi \right] L \theta_1(t) \]

\[ + \left[ 3 \frac{x^2}{L^2} - 2 \frac{x^3}{L^3} + \frac{x}{L} \phi \right] w_2(t) \]

\[ + \left[ - \frac{x^2}{L^2} + \frac{x^3}{L^3} - \frac{1}{2} (\frac{x}{L} - \frac{x^2}{L^2})\phi \right] L \theta_2(t) \] (14)

and

\[ \gamma(t) = \frac{1}{1+\phi} \left[ \frac{1}{L} (w_2(t) - w_1(t)) - \frac{1}{2} (\theta_2(t) + \theta_1(t)) \right] \] (15)

where

- \( x = \) distance along beam length
- \( w = \) transverse displacement (z direction)
- \( w_1, w_2 = \) nodal transverse displacements
- \( \gamma = \) cross-sectional shear deformation about Y axis
- \( \theta_1, \theta_2 = \) nodal cross-sectional rotations
- \( L = \) element length

and

\[ \phi = \frac{12 EI}{GkA L^2} \] (16)

The cross-sectional rotation about the Y axis is given by

\[ \theta(x,t) = \frac{\partial w(x,t)}{\partial x} - \gamma(t) \] (see fig. 2). Thus
\[ \varepsilon(x, t) = -\frac{1}{L^3} \left( \left[ \frac{6}{L} \left( \frac{x}{L} - \frac{x^2}{L^2} \right) \right. \right. \\
\left. \left. + [-1 + 4 \frac{x}{L} - 3 \frac{x^2}{L^2} - (1 - \frac{x}{L})^2] \theta_1 (t) \right\right) \varepsilon_1 (t) \]
\[ + \left[ \frac{6}{L} \left( \frac{x^2}{L^2} - \frac{x}{L} \right) \right] \varepsilon_2 (t) \]
\[ + \left[ 2 \frac{x}{L} - 3 \frac{x^2}{L^2} - \frac{x}{L} \theta_2 (t) \right\right) \varepsilon_2 (t) \]

Equations (14), (15), and (17) are used in the equations for strain and strain rate which are, in turn, used in the Principle of Virtual Work formulation (see ref. 69) to obtain the mass, stiffness, and damping matrices.

The Lagrangian strain tensor is given in standard notation by

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad i = x, y, z \]

where \( u_x = u, u_y = v, u_z = w, a_x = x, a_y = y, \) and \( a_z = z. \) The symbols \( u, v, \) and \( w \) denote the displacements in the \( x, y, \) and \( z \) directions, respectively, and are given by

\[ u = -z \varepsilon(x, t) \]
\[ v = 0 \]
\[ w = w(x, t) \]

where \( z \) is the distance along the beam height; \( -\frac{H}{2} \leq z \leq \frac{H}{2} \) where \( H \) is the beam height.
The components of the strain tensor are, thus, calculated to be

\[ \varepsilon_{xx} = -z \frac{\partial \theta(x,t)}{\partial x} \]  
(18)

\[ \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2} [ -\theta(x,t) + \frac{\partial w(x,t)}{\partial x} ] = \frac{1}{2} \gamma(t) \]  
(19)

\[ \varepsilon_{zz} = \varepsilon_{zy} = \varepsilon_{yz} = \varepsilon_{xy} = \varepsilon_{yx} = \varepsilon_{yy} = 0 \]

Using standard finite element vector-matrix formulation, the strain equations are rewritten as

\[ \{ \varepsilon \} = [B] \{ D \} \]  
(20)

where \( \{ \varepsilon \} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{xz} \\ \varepsilon_{zx} \end{bmatrix} \)

\[ [B] = \frac{z}{1+\phi} \begin{bmatrix} 6 \left( \frac{1}{L} - \frac{2x}{L^2} \right) & \left( \frac{4}{L} - \frac{6x}{L^2} + \frac{\phi}{L} \right) & \frac{6}{L} \left( \frac{2x}{L^2} - \frac{1}{L} \right) \\\ -\frac{\phi}{2zL} & -\frac{\phi}{4z} & \frac{\phi}{2zL} & -\frac{\phi}{4z} \\\ -\frac{\phi}{2zL} & -\frac{\phi}{4z} & \frac{\phi}{2zL} & -\frac{\phi}{4z} \end{bmatrix} \]

and

\[ \{ D \} = \begin{bmatrix} w_1(t) \\ \theta_1(t) \\ w_2(t) \\ \theta_2(t) \end{bmatrix} \]
And the displacement equations are rewritten as

\[ \{f\} = [N]\{D\} \]

where

\[ \{f\} = \begin{bmatrix} u(x,t) \\ w(x,t) \end{bmatrix} \]

and

\[ [N] = \frac{1}{1+\phi} \left[ \begin{array}{c} \frac{6z(x^2/L^2) - x^2}{L^3} \\ -1 + 4\frac{x}{L} - 3\frac{x^2}{L^2} - (1 - \frac{x}{L}) \phi \\ \frac{6z(x^2/L^2) - x}{L} \\ z[2\frac{x}{L} - 3\frac{x^2}{L^2} - \frac{x}{L} \phi] \end{array} \right] \]

The Principle of Virtual Work equation for planar motion of the Timoshenko beam is

\[ \iiint_{x,y,z} \left[ \sigma_{xx} \sigma_{xz} \sigma_{zx} \sigma_{zz} \right] \left[ \begin{array}{c} \delta \varepsilon_{xx} \\ \delta \varepsilon_{xz} \\ \delta \varepsilon_{zx} \end{array} \right] dV
\]

\[ + \iiint_{x,y,z} \left[ F_x F_z \right] \left[ \begin{array}{c} \delta u \\ \delta w \end{array} \right] dV = 0 \]

where \( dV = dx \, dy \, dz \)
\[\sigma_{xx} = E\varepsilon_{xx} + E^* e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) \dot{\varepsilon}_{xx} + \bar{E} \ddot{\varepsilon}_{xx}\]  
(21)

\[\sigma_{xz} = \sigma_{zx} = 2G\varepsilon_{xz} + 2G^* g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) \dot{\varepsilon}_{xz} + 2\bar{G} \ddot{\varepsilon}_{xz}\]  
(22)

\[F_x = \rho \ddot{u} = -\rho z \ddot{u}(x,t)\]

\[F_z = \rho \ddot{w}(x,t)\]

e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) = \text{functional part of modulus relating structural damping extensional stress to strain rate}

= |\dot{\varepsilon}_{xx}| \text{ for quadratic (type II) damping}

g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) = \text{functional part of modulus relating structural damping shear stress to strain rate}

= |\dot{\varepsilon}_{xz}| \text{ for quadratic (type II) damping}

E^* = \text{constant part of modulus relating structural damping extensional stress to strain rate}

G^* = \text{constant part of modulus relating structural damping shear stress to strain rate}

\[E = \text{modulus relating viscous damping extensional stress to strain rate}\]

\[G = \text{modulus relating viscous damping shear stress to strain rate}\]

and \(k_s = \text{cross-section shear coefficient}\)

The following integral relations which then arise from the Principle of Virtual Work formulation are used to obtain the mass, stiffness, viscous damping, and structural damping matrices,
\[ [K] = \int \int \int [B]^T [E][B] \, dV \]

\[ [C_L] = \int \int \int [B]^T [\bar{E}][B] \, dV \]

\[ [C_{NL}] = \int \int \int [B]^T [E''\ell][B] \, dV \]

\[ [N] = \int \int \int \rho[N]^T [N] \, dV \]

where

\[ [E] = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & G \end{bmatrix} \]

and

\[ [\bar{E}] = \begin{bmatrix} E \bar{e}(\varepsilon_{xx}, \varepsilon_{xx}) \\ 0 & G \bar{g}(\varepsilon_{xz}, \varepsilon_{xz}) \\ 0 & 0 & G \bar{g}(\varepsilon_{xz}, \varepsilon_{xz}) \end{bmatrix} \]

and

\[ [E''\ell] = \begin{bmatrix} E''\ell(e_{xx}, e_{xx}) \\ 0 & G''\ell(e_{xz}, e_{xz}) \\ 0 & 0 & G''\ell(e_{xz}, e_{xz}) \end{bmatrix} \]

Note that from equation (19) \( \varepsilon_{xz} \) and \( \dot{e}_{xz} \) are not functions of the spatial coordinates \( x, y, \) and \( z \) which greatly simplifies part of the integration needed to form the matrices. Carrying out the above integrations yields the following element matrices.

**Element Stiffness Matrix**

\[
[K] = \frac{E}{L^3(1+\phi)} \begin{bmatrix}
12 & L^2(4+\phi) \\
6L & 12 -6L \\
-12 & 12 \\
6L & L^2(2-\phi) & -6L & L^2(4+\phi)
\end{bmatrix} \text{ Symmetric}
\]
Element Mass Matrix

\[ [M] = \frac{\rho \Delta L}{(1+\phi)^2} \begin{bmatrix} m_1 & \text{Symmetric} \\ m_2 & m_5 \\ m_3 & -m_4 & m_1 \\ m_4 & m_6 & m_2 & m_5 \end{bmatrix} + \frac{\rho \Delta L}{(1+\phi)^2} \left( \frac{\phi}{L} \right)^2 \begin{bmatrix} m_7 & \text{Symmetric} \\ m_8 & m_9 \\ -m_7 & -m_8 & m_7 \\ m_8 & m_{10} & m_8 & m_9 \end{bmatrix} \]

where

\[ m_1 = \frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{3} \]
\[ m_2 = \left( \frac{11}{210} + \frac{11\phi}{120} + \frac{\phi^2}{24} \right) L \]
\[ m_3 = \frac{9}{70} + \frac{3\phi}{10} + \frac{\phi^2}{6} \]
\[ m_4 = -\left( \frac{13}{420} + \frac{3\phi}{40} + \frac{\phi^2}{24} \right) L \]
\[ m_5 = \left( \frac{1}{105} + \frac{\phi}{60} + \frac{\phi^2}{120} \right) L^2 \]
\[ m_6 = -\left( \frac{1}{140} + \frac{\phi}{60} + \frac{\phi^2}{120} \right) L^2 \]
\[ m_7 = \frac{6}{5} \]
\[ m_8 = \left( \frac{1}{10} - \frac{\phi}{2} \right) L \]
\[ m_9 = \left( \frac{2}{15} + \frac{\phi}{6} + \frac{\phi^2}{3} \right) L^2 \]
\[ m_{10} = \left( - \frac{1}{30} - \frac{\phi}{6} + \frac{\phi^2}{6} \right) L^2 \]

and \( r \) = radius of gyration.
Element Viscous Damping Matrix

For $\frac{E}{G} = \frac{E}{G}$

$$[c_L] = \lambda (1 + \phi)$$

$$\begin{bmatrix}
12 & & \quad \text{Symmetric} \\
6L & L^2(4+\phi) & \\
-12 & -6L & 12 \\
6L & L^2(2-\phi) & -6L & L^2(4+\phi)
\end{bmatrix}$$

For $\frac{E}{G} \neq \frac{E}{G}$

$$[c_L] = \lambda (1 + \phi)$$

$$\begin{bmatrix}
12(1+\frac{\phi^2}{\phi}) & & \quad \text{Symmetric} \\
6L(1+\frac{\phi^2}{\phi}) & L^2[(4+\phi)(1+\phi) + 3\phi(\phi/\phi - 1)] & \\
-12(1+\frac{\phi^2}{\phi}) & -6L(1+\frac{\phi^2}{\phi}) & 12(1+\frac{\phi^2}{\phi}) \\
6L(1+\frac{\phi^2}{\phi}) & L^2[(2-\phi)(1+\phi) + 3\phi(\phi/\phi - 1)] & -6L(1+\frac{\phi^2}{\phi}) & L^2[(4+\phi)(1+\phi) + 3\phi(\phi/\phi - 1)]
\end{bmatrix}$$

where $\phi = \frac{12 EI}{G k_s A L^2}$

and $\lambda = \frac{E I}{(1+\phi)^2} \frac{L^3}{L^3}$
Element Structural Damping Matrix

Shown below are the general integral equations for each term of the structural damping element matrix prior to assigning particular functions to $e(\varepsilon_{xx}, \dot{\varepsilon}_{xx})$ and $g(\varepsilon_{xz}, \dot{\varepsilon}_{xz})$. Nondimensional length and height distances, $\chi = x/L$ and $\xi = z/H$, respectively, have been employed. Note that $y$ integration over the beam width $B$ has already been done as no variables were a function of $y$.

\[
C_{NL,11} = \frac{BE^3 H^3}{(1 + \epsilon)^2 L^2} \int_{x=0}^{x=1} (6 - 12\chi)^2 \left[ \int_{\xi=-1/2}^{\xi=1/2} e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) \xi^2 d\xi \right] d\chi
\]

\[+ G^\infty g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) \frac{k_A}{L} \left( \frac{\phi}{1 + \phi} \right)^2 \]

Letting $J_1 = \int_{\xi=-1/2}^{\xi=1/2} e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) \xi^2 d\xi$

\[J_2 = \frac{G^\infty g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) k_A}{L} \left( \frac{\phi}{1 + \phi} \right)^2 \]

and $J_3 = \frac{BE^3 H^3}{(1 + \epsilon)^2 L^2}$

the remaining terms are

\[
C_{NL,12} = L J_3 \int_{\chi=0}^{\chi=1} (6 - 12\chi)(4 - 6\chi + \phi) J_1 d\chi + \frac{L}{2} J_2
\]
\[ C_{NL,13} = -C_{NL,11} \]
\[ C_{NL,14} = L J_3 \int_{x=0}^{x=1} (6 - 12x)(2 - 6x - \phi) \, J_1 \, dx + \frac{L^2}{4} \, J_2 \]
\[ C_{NL,22} = L^2 J_3 \int_{x=0}^{x=1} (4 - 6x + \phi)^2 \, J_1 \, dx + \frac{L^2}{4} \, J_2 \]
\[ C_{NL,23} = -C_{NL,12} \]
\[ C_{NL,24} = L^2 J_3 \int_{x=0}^{x=1} (4 - 6x + \phi)(2 - 6x - \phi) \, J_1 \, dx + \frac{L^2}{4} \, J_2 \]
\[ C_{NL,33} = C_{NL,11} \]
\[ C_{NL,34} = -C_{NL,14} \]
\[ C_{NL,44} = L^2 J_3 \int_{x=0}^{x=1} (2 - 6x - \phi)^2 \, J_1 \, dx + \frac{L^2}{4} \, J_2 \]

Since \( \{U\} = [w_1, \theta_1, \omega_2, \theta_2]^T \) and since \( w_1, \theta_1, \omega_2, \theta_2 \) and their derivatives are in \( \varepsilon_{xx}, \varepsilon_{xx}', \varepsilon_{xz}, \) and \( \varepsilon_{xz} \) (see eq. (20)), then the reason for using the notation \( [C_{NL}(\{U(t)\}, \{\dot{U}(t)\})] \) earlier is clearly seen. Finally, it is important to keep in mind that the
strains and strain rates are functions of length along the element, \( x \), as well as distance through the element height, \( z \).

The element structural damping matrix (like each of the other element matrices) is symmetric.

**Numerical Integration of Structural Damping Matrix**

The importance of utilizing efficient numerical methods in calculating the structural damping matrix cannot be overemphasized as each integral must be evaluated for every element, each iteration of each time step in the solution process. Exact closed-form integration is obviously always preferred over numerical integration. However, if \( e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) \) and \( g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) \) are not readily integrable, the programmer has to use numerical integration to calculate the structural damping matrix. Shown below is the use of a numerical integration technique, Simpson's 1/3 rule (see ref. 70), in calculating the terms of the element structural damping matrix for any general functions \( e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) \) and \( g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) \).

For a function \( f(x) \) between \( x = x_1 \) and \( x = x_n \), Simpson's 1/3 rule for numerical integration is

\[
\int_{x_1}^{x_n} f(x) \, dx = \frac{\Delta x}{3} \left[ f(x_1) + 4 f(x_2) + 2 f(x_3) + 4 f(x_4) + 2 f(x_5) + \ldots + 2 f(x_{n-2}) + 4 f(x_{n-1}) + f(x_n) \right]
\]
where $\Delta X = X_2 - X_1 = X_3 - X_2$, etc.; $n$ = number of evaluation points; the local error is of the order of $(\Delta X)^5$, and the global error is of the order of $(\Delta X)^4$. Simpson's 1/3 rule requires an odd number of evaluation points.

Simpson's 1/3 rule is first applied to the inner ($\xi$) integral (e.g., see eq. (23)) whose integrand is a function of $X$ and $\xi$. The results of integrating the inner integral will be a function of $X$. Simpson's 1/3 rule is then applied over the $X$ integral whose integrand is now a function only of $X$. This integration process is demonstrated below using $\Delta \xi = 0.1$, $\Delta X = 0.1$, and $n = 11$. All the functions of $X$ and/or $\xi$ that are discussed here are also functions of time through $\omega_1$, $\theta_1$, $\omega_2$, $\theta_2$, and their derivatives. To simplify the notation, this dependence is not shown as spatial integration is the primary concern here.

The $\xi$-integral integrand, some function of $X$ and $\xi$, is the same for each term, i.e., $e(\varepsilon_{xx}, \varepsilon_{xx}) \xi^2$, and for simplification is designated $F(X, \xi)$. Applying Simpson's 1/3 rule to the $\xi$ integral results in

$$
\int_{\xi_1 = -1/2}^{\xi_1 = 1/2} F(X, \xi) \, d\xi = \frac{\Delta \xi}{3} [F(X, \xi_1) + 4F(X, \xi_2) + 2F(X, \xi_3) + 4F(X, \xi_4) + 2F(X, \xi_5) + \ldots + 2F(X, \xi_9) + 4F(X, \xi_{10}) + F(X, \xi_{11}) = n(X)
$$
The above result is some function of $\chi$ which is designated, for simplicity, $\eta(\chi)$. Simpson's $1/3$ rule is now applied to the $X$ integrals. To simplify the writing of each integral the following substitutions are made,

$$n_1(x) = (6 - 12x)^2 \eta(x)$$

$$n_2(x) = (6 - 12x)(4 - 6x + \phi) \eta(x)$$

$$n_3(x) = (6 - 12x)(2 - 6x - \phi) \eta(x)$$

$$n_4(x) = (4 - 6x + \phi)^2 \eta(x)$$

$$n_5(x) = (4 - 6x + \phi)(2 - 6x - \phi) \eta(x)$$

$$n_6(x) = (2 - 6x - \phi)^2 \eta(x)$$

$$\int_{i} = n_1(x_1) + 4n_1(x_2) + 2n_1(x_3) + 4n_1(x_4) + 2n_1(x_5)$$

$$+ \ldots + 2n_1(x_9) + 4n_1(x_{10}) + n_1(x_{11}); \ i = 1 \text{ to } 6$$

$$P = \frac{BE_h}{(1 + \phi)^2} \frac{H}{L} \frac{3}{3} \frac{\Delta x}{3}$$

and

$$Q = G^* g(c_{xz}, \dot{c}_{xz}) \frac{k_{sA}}{L} \frac{\phi}{(1 + \phi)^2}$$

Thus, the terms of the element structural damping matrix are found to be

$$C_{NL,11} = P \int_1 + Q$$

$$C_{NL,12} = PL \int_2 + \frac{QL}{2}$$
\[ C_{NL,13} = -C_{NL,11} \]
\[ C_{NL,14} = PL f_3 + \frac{QL}{2} \]
\[ C_{NL,22} = PL^2 f_4 + \frac{QL^2}{4} \]
\[ C_{NL,23} = -C_{NL,12} \]
\[ C_{NL,24} = PL^2 f_5 + \frac{QL^2}{4} \]
\[ C_{NL,33} = C_{NL,11} \]
\[ C_{NL,34} = -C_{NL,14} \]
\[ C_{NL,44} = PL^2 f_6 + \frac{QL^2}{4} \]

**Exact Integration with Type II Damping**

Numerical integration is very time consuming and was found in thesis problems on rare occasions to cause convergence failure by itself because of its inaccuracies and round-off errors. After much effort, type II (quadratic) damping was fortunately found to be amenable to exact closed-form integration by judicious partitioning of the \( x \) and \( z \) integrals in each term of the damping matrix. A detailed description of how this exact integration is accomplished is given below.

For type II damping, \( e(\varepsilon_{xx}, \dot{\varepsilon}_{xx}) = |\dot{\varepsilon}_{xx}| \) and \( g(\varepsilon_{xz}, \dot{\varepsilon}_{xz}) = |\dot{\varepsilon}_{xz}| \).

The quantity \( |\dot{\varepsilon}_{xx}| \) which must be integrated over \( z \) and \( x \) is the one of concern. First \( |\dot{\varepsilon}_{xx}| \) is written as follows (from eq. (20)),
\[ |\dot{\epsilon}_{xx}| = \frac{1}{1+\phi} |z| \left[ \frac{6x}{L^2} \left( \frac{2}{L} \hat{\omega}_2 - \frac{L}{2} \hat{\omega}_1 - \hat{\theta}_2 - \hat{\theta}_1 \right) \right. \\
\left. + \frac{6}{L^2} (\hat{\omega}_1 - \hat{\omega}_2) + (\frac{4+\phi}{L}) \hat{\theta}_1 + (\frac{2-\phi}{L}) \hat{\theta}_2 \right] \]

The term \( 1/1+\phi \) is always positive. So, \( |\dot{\epsilon}_{xx}| \) is written above as a constant, \( 1/1+\phi \), times a function of \( z, |z| \), times a function of \( x \) (the term in brackets). For each damping matrix term, the constant will come outside the \( x \) integral, the function of \( x \) will be under the \( x \) integral only, and the function of \( z \) under the \( z \) integral will come outside the \( x \) integral once it is integrated.

The problem with the \( z \) integrand is integrating the quantity \( |z| \). This problem is solved by recognizing that \( |z| = -z \) from \( z = \frac{-H}{2} \) to \( 0 \) and \( |z| = z \) from \( z = 0 \) to \( \frac{H}{2} \). Thus, the \( z \) integral is split in two and integrated as follows (reintroduce \( \xi = z/H \)),

\[
\int_{\xi = \frac{-H}{2}}^{\xi = \frac{H}{2}} |\xi| \xi^2 \, d\xi = \int_{\xi = 0}^{\xi = \frac{H}{2}} -\xi^3 \, d\xi + \int_{\xi = 0}^{\xi = \frac{-H}{2}} \xi^3 \, d\xi = \frac{1}{32}
\]

The solution to the \( x \) integral is more complicated. The \( x \)-function part of \( |\dot{\epsilon}_{xx}| \) is of the form \( y = mx + b \), i.e.,

\[
y = \left[ \frac{6}{L^2} \left( \frac{2}{L} \hat{\omega}_2 - \frac{L}{2} \hat{\omega}_1 - \hat{\theta}_2 - \hat{\theta}_1 \right) \right] x \\
\left. + \frac{6}{L^2} (\hat{\omega}_1 - \hat{\omega}_2) + (\frac{4+\phi}{L}) \hat{\theta}_1 + (\frac{2-\phi}{L}) \hat{\theta}_2 \right]
\]
The quantity $y$ is readily recognized as being linear in $x$ with a slope of

$$\left[ \frac{6}{L^2} \left( \frac{2}{L} \dot{w}_2 - \frac{2}{L} \dot{w}_1 - \dot{\theta}_2 - \dot{\theta}_1 \right) \right]$$

and an intercept of

$$\left[ \frac{6}{L^2} (\dot{w}_1 - \dot{w}_2) + \left( \frac{4+\phi}{L} \right) \dot{\theta}_1 + \left( \frac{2-\phi}{L} \right) \dot{\theta}_2 \right].$$

At $x = L$ the function as a value of

$$\left[ \frac{6}{L^2} \dot{w}_2 - \frac{6}{L^2} \dot{w}_1 + \frac{-4-\phi}{L} \dot{\theta}_2 + \frac{-2+\phi}{L} \dot{\theta}_1 \right].$$

The method used to decide if and when $y$, the $x$-function part of $|\hat{\varepsilon}_{xx}|$, changed signs over the $x$ interval of an element was as follows,

1. Multiply $y(0)$ by $y(L)$

   (a) If the product is positive or zero, then $y$ does not change sign over the $x$ interval of the element.

   For $y(\frac{L}{2}) > 0$, $|y| = y$ for the entire element. For $y(\frac{L}{2}) < 0$, $|y| = -y$ for the entire element.

   (b) If the product is negative, then $y$ changes sign over the $x$ interval of the element at $x^* = -y(0)/(slope)$.

   Split up the integral according to $x^*$ and determine sign of $y$ on either side of $x^*$. Where $y < 0$ use $|y| = -y$; where $y > 0$, use $|y| = y$. 
2. Complete calculation of \( x \) integrand by multiplying \( y \) (or \(-y\), as the case may be) by the other \( x \)-integrand function \( ((6 - 12 \frac{x}{L})^2, \) for example, for \( C_{NL,11} \)).

3. Integrate in closed form.

As an example, a possible case involving \( C_{NL,11} \) is set up below. As previously done, \( x = x/L \) and \( \xi = z/H \) are employed. Note that \( \xi \) integration as previously described has already been done.

\[
C_{NL,11} = \frac{BE^p}{(1+\phi)^3} \left( \frac{H}{L} \right)^3 \int_{x=0}^{x=1} (6 - 12x)^2 |y(x)| \, dx + G^p \int_{\xi=x}^{\xi=L} \frac{k_x}{\left( 1 + \psi \right)^2} \, d\xi
\]

If, for instance, \( y(0) \) times \( y(L) \) is negative with \( y = 0 \) at \( x = 0.4 \), and \( y(0) \) is negative, then the \( x \) integral for \( C_{NL,11} \) becomes

\[
\int_{x=0}^{x=0.4} -(6 - 12x)^2 \, y(x) \, dx + \int_{x=0.4}^{x=1.0} (6 - 12x)^2 \, y(x) \, dx
\]

The remainder is straightforward integration. The process is repeated as often as necessary for each unique member of the element structural damping matrix.

As a check to ensure that the subroutine was correctly calculating the damping matrix by this exact integration (there was much more algebra involved in programming the exact integration than in programming the numerical integration), subroutines using numerical double integration and single integration (\( \xi \) integration done exactly) were also written. Agreement between the subroutines
was, as expected, excellent, which was important to develop confidence that no algebraic or programming errors had occurred in spite of meticulous checking. Even though the exact integration had more algebra, it enabled the program to run about three times faster than when using the numerical single integration subroutine due to the elimination of the double-nested do-loop required with numerical single integration.
Chapter V

RESULTS AND DISCUSSION FOR SDOF PROBLEMS

Linear and nonlinear initial-condition problems involving an SDOF system (see fig. 3) are discussed in this chapter. The first section of this chapter is concerned with checking the accuracy of the pseudo-force Newmark method by comparing its solution with: (1) the conventional Newmark method and the exact solutions for a viscously damped problem, (2) the exact solution of a type I-damped problem, (3) a perturbation solution of a type II-damped problem, and (4) a perturbation solution of a type III-damped problem. After this exhaustive accuracy check; the convergence characteristics of the pseudo-force Newmark method are studied for damping types I, II, and III to show why type II (quadratic) damping was chosen to be used in the finite element Timoshenko beam finite element problems. The effect of changes in initial velocity, error tolerance, and size of damping constant are discussed. Finally, a summary of the convergence characteristics is given.

Accuracy Discussion

Viscous Damping Problem. The parameters for the viscous damping problem discussed here are stiffness $k = 4.0 \pi^2 \text{ lb/in}$, mass $m = 1.0 \text{ lb-sec}^2/\text{in}$, damping ratio $\zeta = 0.5$, initial displacement $u(0) = 10.0000 \text{ in}$., initial velocity $\dot{u}(0) = 0 \text{ in/sec}$, and undamped
period $T = 1.0$ sec. In order to determine the value of the integration time step $\Delta t$ to use with the SDOF problems, the effect of $\Delta t$ on the accuracy of the conventional Newmark method for a viscously damped problem was first studied. This was done by comparing the exact and conventional Newmark solutions, for several time steps, of a viscously underdamped system. The results for time steps of $T/40$, $T/20$, and $T/10$ are shown in table I. Because of the excellent results obtained with $\Delta t = T/40$ (less than 1.5 percent error after a full period of motion), this time step was chosen as a basis for the SDOF problems. As shown in table I, lowering $\Delta t$ expectedly increases the accuracy of the conventional Newmark method.

As the next step in the checking procedure the same viscous damping problem (with $\Delta t = T/40$) was also solved by the pseudo-force Newmark (viscous damping contained in effective load vector as pseudo-force) method to ensure that the excellent accuracy of the conventional Newmark method was retained in the pseudo-force representation. The results are tabulated in table II. The "best" pseudo-force Newmark solution is shown, i.e., iteration continued within each time step until the computer rounded off the error between iterations to be zero. The displayed percent error between the Newmark and exact solutions at each time step was always reached in two or three iterations although four or five iterations were allowed to occur as a check. The results indicate that the accuracy of the conventional Newmark method is retained in the pseudo-force representation and that the slight iteration error due to computer round-off actually resulted in slightly improved accuracy for the problem solved.
For the remainder of the SDOF results in tables and figures, the following parameters were common to all unless otherwise noted:

\[ k = 4.0 \text{ lb/in}, \quad m = 4.0 \text{ lb-sec}^2/\text{in}, \quad u(0) = 1.000 \text{ in}, \quad T = 2\pi \text{ sec}, \quad \text{and} \quad \Delta t = T/40. \]

Problems with Damping Types I, II, and III. As a third step, the pseudo-force Newmark method solution \((\varepsilon_t = 0.0001\) and initial velocity of 1.000 in/sec) is compared with the exact solution for a problem with type I damping \((C_I = 0.4 \text{ lb/in})\). Appendix C presents this exact solution. The results are shown in tables III, IV, and V and in figure 4 and agreement is seen to be excellent. Peak amplitudes were predicted with less than one percent error relative to the exact solution for the first 11 peaks; the difference in times of zero crossing was less than one percent of the undamped period for the first 12 zero crossings.

As the fourth and fifth steps, the pseudo-force Newmark method solution was calculated for problems with damping types II and III. For these damping types, first-order perturbation solutions were calculated by the method of multiple scales (ref. 71) for comparison. These solutions are presented in appendix D. Because they were first-order approximations, the perturbation solutions accounted only for amplitude decay while maintaining the undamped period of motion. The validity of each perturbation solution was contingent upon the damping force being an order of magnitude less than the inertial and elastic forces. For type II damping with an initial velocity of 1.000 in/sec, \(C_{II}\) was chosen to be 0.2 lb-sec\(^2\)/in\(^2\) to ensure this condition. This \(C_{II}\) value resulted in the inertial and elastic forces being 10 or more times greater than the damping force for 70 percent of the time from
From t = 0 to t = T and averaging over 4 times greater for the remaining 30 percent of the time. For type III damping a $C_{III}$ value of 0.4 lb-sec/in$^3$ was used (again with initial velocity of 1.000 in/sec) which resulted in the inertial and elastic forces being more than 10 times greater than the damping force for 100 percent of the time from $t = 0$ to $t = 0.5T$.

The graphic results for type II damping are shown in figure 5 and agreement between the perturbation and pseudo-force Newmark ($\varepsilon = 0.0001$) solutions is seen to be excellent. The numerical results for type II damping are shown in tables VI and VII. The peak amplitude results reveal that for the first 12 pseudo-force Newmark and perturbation peaks there is less than a 2 percent difference relative to the perturbation peaks. The difference in times of zero crossing is seen to grow with each zero crossing, yet the difference at the 11th zero crossing was still less than 1.5 percent of the undamped period. This zero-crossing agreement is amazing considering that the Newmark method has numerical period elongation (ref. 9) in addition to the natural lengthening of the period due to damping, while the perturbation solution uses the undamped period (see appendix D).

The results for type III damping, not graphed or tabulated in this thesis, showed excellent agreement between the pseudo-force Newmark ($\varepsilon = 0.001$) and perturbation solutions. The data showed that the first pseudo-force Newmark peak had a percent different of 0.07 relative to the perturbation peak. The difference in times of the first zero crossing was 0.31 percent of the undamped period.
The results in figures 4 and 5 and tables III to VII provide ample verification of the ability of the pseudo-force Newmark method to solve accurately SDOF problems having different types of nonlinear damping models with small damping forces relative to the inertial and elastic forces.

**Convergence Characteristics**

The convergence characteristics induced by each damping model were the deciding factors in determining which model would be retained for the finite element beam problems. This final section of the investigation of SDOF systems is designed to show the effects of the initial velocity \( u(0) \), the iteration error tolerance \( e_t \), and large damping on the number of iterations needed for convergence for problems with damping types I, II, and III. The numerical results showing the effects of initial velocity and prescribed iteration error tolerance for the three types of damping are presented in tables VIII, IX, and X. Error tolerances of 0.005 and 0.001 and initial velocities of 1.000, 10.00, and 100.0 in/sec were investigated for each damping model. The damping constants corresponding to these results were \( C_I = 0.4 \text{ lb/in} \), \( C_{II} = 0.4 \text{ lb-sec}^2/\text{in}^2 \), and \( C_{III} = 0.4 \text{ lb-sec/in}^3 \). For all cases, numerical decreases in error tolerance and increases in initial velocity are seen to cause the number of iterations to increase. Except for type I damping which is quite well behaved, the higher the initial velocity was, the greater the effect of decreasing tolerance. Increases in the degree of the nonlinearity of the damping are also seen to cause the number of iterations to increase, depending on the initial velocity. For an
initial velocity of 1.000 in/sec, very little, if any, discernable difference in number of iterations occurred between the three damping types. For an initial velocity of 10.00 in/sec, type III damping required noticeably more iterations than damping types I and II which were about the same in required iterations. For an initial velocity of 100.0 in/sec (also for 50.00 in/sec), type III damping does not even converge (using $\Delta t = T/40$ for the comparison since this $\Delta t$ was the one used with damping types I and II) while type II damping required considerably more iterations than type I damping. Even with $\Delta t = T/80$ for an initial velocity of 100.0 in/sec, type III damping still caused more convergence problems than damping types I and II.

In addition to the solutions in tables VIII to X, other solutions, not tabulated in this thesis, were attempted with damping constants ten times larger than the ones in these tables.

Solutions with type I damping showed almost no effect on convergence iterations; more than one iteration was rarely needed for the full initial velocity range (initial velocities of 1.000, 10.00, and 100.0 in/sec). Solutions with type II damping showed a dramatic effect as convergence failed for an initial velocity of 10.00 in/sec. Lowering $\Delta t$ to $T/80$ gave convergence again for this initial velocity, but convergence then failed at an initial velocity of 50.00 in/sec.

Solutions with type III damping for a higher damping constant were not acquired because of extreme convergence difficulties.

Although the solutions for type I damping showed almost no effect when damping was increased ten times, a large enough damping constant would have undoubtably caused divergence simply because a
pseudo-force representation was being used. In fact, even for viscous
damping in the pseudo-force representation, a damping ratio 20 times
that previously used (unrealistically large for demonstrative purposes)
caused divergence when a $\Delta t$ of $T/40$ was used. Lowering $\Delta t$ to
$T/80$ gave convergence, but this $\Delta t$ was clearly lower than that
needed for accurate integration as $\Delta t = T/40$ resulted in less than
0.01 percent error when the conventional Newmark method was used.
The need to use a $\Delta t$ smaller than that for reasonable accuracy was
a sign that the damping was too large for the pseudo-force representa-
tion. Thus, large damping is certainly not feasible for the
pseudo-force Newmark method especially for damping types II and III.

The SDOF results show that the higher the nonlinearity of the
damping function, the more convergence problems, i.e., the greater
the sensitivity of the solution to increases in values of damping
constant and initial velocity and to decreases in values of error
tolerance. These effects are traceable to the reasoning that for
velocities and displacements much greater than unity, the higher the
nonlinearity of the damping function, the greater the damping force.
And, of course, the higher the damping force is with respect to the
inertial and elastic forces, the greater the convergence problems.

Overall, the following comments can be made about the conver-
gence characteristics of each damping model.

1. Type I damping solutions were very well behaved for both
small and large damping constants and rarely needed more than one
iteration for convergence. The only difficulty that was encountered
was when the velocity changed signs where, for some problems, the
solution would oscillate between two answers. However, since the
velocity was near zero, the displacement was hardly changing value and little error was introduced into the solution.

2. Type II damping solutions were very well behaved for small damping and initial velocities of 50.00 in/sec or less. For an initial velocity of 100.0 in/sec and error tolerance of 0.0001, the solution was still fairly well behaved, converging usually in two or three iterations with convergence difficulties only initially. Solutions were unstable for large damping with initial velocities of 10.00 in/sec or greater. Relative to type I damping, type II damping resulted in solutions that were more sensitive to changes in the magnitude of the damping constant, initial velocity, and error tolerance.

3. Type III damping solutions were very well behaved for small damping and initial velocities of 10.00 in/sec or less. However, for an initial velocity of even 50.00 in/sec, convergence failed. Solutions were highly unstable at high velocities and large damping. Type III damping resulted in solutions that were the most sensitive to changes in the magnitude of the damping constant, initial velocity, and error tolerance.

Because use of type III damping was the most restrictive and unpredictable so far as stability is concerned, type III damping was not retained for the finite element beam problems. Because structural damping is known to be nonlinear, and because type I damping's behavior seemed quite well behaved, type II damping was chosen for the finite element beam problems. Also, it was felt that type II damping's quadratic nonlinearity would be a sterner test of the solution technique.
Clearly, from the results discussed, the following factors all have an influence on convergence characteristics:

1. type of damping (piecewise linear, quadratic, cubic)
2. magnitude of damping constant
3. initial velocity
4. prescribed iteration error tolerance

and 5. size of integration time step.
Chapter VI

RESULTS AND DISCUSSION FOR TIMOSHENKO BEAM
FINITE ELEMENT PROBLEMS

Linear and nonlinear problems involving cantilevered and free-free Timoshenko beams (see fig. 6) are discussed in this chapter. In the first section, initial-condition two-element cantilevered beam problems are solved to check the accuracy and efficiency of the conventional and pseudo-force Newmark methods for multi-degree-of-freedom problems by comparing their solutions with Gear-method numerical solutions. In the second section, the importance of damping in the prediction of low- and high-frequency vibratory motion of a free-free beam is investigated as a preliminary step into the investigation of the active control of large space structures.

Cantilevered Beam Problems

This section is an investigation of the accuracy and efficiency of the conventional and pseudo-force Newmark methods in solving two-element (4 degrees of freedom) cantilevered beam problems with no damping, viscous damping, and type II damping. The initial condition used for all the problems was a tip displacement of 1.000 in. with the midpoint displacement and tip and midpoint cross-section rotations calculated using Bernoulli-Euler static beam theory. Thus, with the tip displaced 1.000 in., the midpoint displacement is 0.3125 in.; and the tip and midpoint cross-section rotations are 0.025 rad and
0.01875 rad, respectively. All initial velocities were set equal to zero. Figure 6(a) shows a sketch of the initial condition. The physical parameters that were common to all the cantilevered beam problems solved were beam length \( l = 60.00 \text{ in.} \), beam height \( H = 2.00 \text{ in.} \), beam width \( B = 2.00 \text{ in.} \), Young's modulus \( E = 30 \times 10^6 \text{ lb/in}^2 \), shear modulus \( G = 12 \times 10^6 \text{ lb/in}^2 \), mass density \( \rho = 0.00073 \text{ lb-sec}^2/\text{in}^4 \), and \( T_{1,E} = 0.055 \text{ sec} \) where \( T_{1,E} \) is the period of the Bernoulli-Euler beam's first bending mode.

As a verification and support of the results obtained using the conventional and pseudo-force Newmark methods, a Gear-method numerical solution obtained using the IMSL subroutine DVOGER is also shown for each problem solved. A description of the subroutine DVOGER as given in The IMSL Library Reference Manual (ref. 72) is shown in appendix A. Also shown in appendix A are the DVOGER parameters of each problem for which a DVOGER solution was calculated. (Note: the integration time step used by DVOGER was continuously updated by the subroutine as it solved a problem.)

In deciding what iteration error tolerance \( e_t \) to use for the pseudo-force Newmark solutions, much tracking and checking of iteration errors through many computer runs was done. For problems with realistically light structural damping, it was found that convergence to 0 percent error occurred after two or three iterations for most time steps, and that convergence, at worst, to extremely low errors (e.g., 0.00005 percent) occurred after four iterations for the remaining time steps. Even for a difficult problem of unrealistically heavy damping which required many iterations for most time steps, the iterative solutions within each time step were found to be in
repetitive order such that the final solution (often reached after 50 iterations with 0.03 percent error or less) was no more accurate than the solution after 4 iterations.

So, for the cantilevered beam problems of this section, an error tolerance of 0.0000001 was used along with a maximum number of allowable iterations of four. Since extremely low percent errors invariably occurred by four iterations, the extremely low error tolerance had the effect of forcing the solution to go to four iterations in order to get the utmost accuracy.

A comparison of pseudo-force Newmark ($\Delta t = 5 \times 10^{-5}$ sec) solution with conventional Newmark ($\Delta t = 5 \times 10^{-5}$ sec) and DVOGER (average $\Delta t = 11 \times 10^{-5}$ sec) solutions for the case of viscous damping (damping constants $\zeta_E = \zeta_G = 0.000040$ sec) is shown in tables XI and XII (shear coefficient $k_s = 1.00$). These results serve as a verification that the accuracy of the conventional Newmark method is retained in the pseudo-force representation for multi-degree-of-freedom problems. The DVOGER solution is shown as a check on the Newmark solution. Agreement is seen to be excellent between the two Newmark solutions (times of zero crossing differed by less than 0.15 percent of $T_{1,E}$ after four zero crossings). Agreement is seen to be reasonably good between either Newmark solution and the DVOGER solution (times of zero crossing differed by less than 5.6 percent of $T_{1,E}$ between either Newmark solution and the DVOGER solution after four zero crossings).

The effect of $\Delta t$ on the comparison of the conventional Newmark and DVOGER solutions for no damping is shown in table XIII and figure 7 ($k_s = 1.00$). The graph in figure 7 clearly shows excellent agreement between the less-accurate Newmark solution ($\Delta t = 25 \times 10^{-5}$ sec)
and the DVOGER solution, but only good agreement between the more-accurate Newmark solution \((\Delta t = 5 \times 10^{-5} \text{ sec})\) and the DVOGER solution. For the second zero crossing, the difference in times of zero crossing between the DVOGER and more-accurate Newmark solutions was about 1.7 percent of \(T_{1,E}\), while the difference between the DVOGER and the less-accurate Newmark solution was less than 0.1 percent of \(T_{1,E}\). The central processing unit (c.p.u.) times for the more-accurate Newmark, less-accurate Newmark, and DVOGER solutions were about 27, 8, and 43 sec, respectively. The integration time step used by DVOGER averaged about \(5 \times 10^{-5} \text{ sec}\). From the results of this study it can be concluded:

1. the less-accurate Newmark solution compared very well with the DVOGER solution thus indicating that the less-accurate Newmark solution obtained the same accuracy as the DVOGER solution in about one-fifth the c.p.u. time while using an integration time step about five times bigger, and

2. the more-accurate Newmark solution which used about the same integration time step as DVOGER did not compare as well with DVOGER as the less-accurate Newmark solution thus indicating that the more-accurate Newmark solution was more accurate than the DVOGER solution while using about 35 percent less c.p.u. time.

Clearly, the conventional Newmark method is a more efficient time-integration technique than DVOGER for the problem just discussed. The data of figure 7 and table XIII demonstrate that the DVOGER solution is indeed an effective, though inefficient, check for the Newmark solutions.

A comparison of pseudo-force Newmark \((\Delta t = 10 \times 10^{-5} \text{ sec})\) and DVOGER \((\text{average} \Delta t = 44 \times 10^{-5} \text{ sec})\) solutions for a problem with
type II damping \((\zeta_E = \zeta_G = 0.002 \text{ sec}^2)\) is shown in tables XIV and XV and in figure 8 \((k_s = 0.822)\). Also plotted in figure 3 is the conventional Newmark undamped solution to show the damping effect. The DVOGER solution serves as a check of the pseudo-force Newmark solution with good agreement between the two (difference in times of zero crossing was less than 1.7 percent of \(T_{1,E}\) after four zero crossings). Comparison of the damped and undamped Newmark solutions reveals a one-cycle damping of about 5.0 percent. Comparison of the damped DVOGER solution with the undamped Newmark solution also reveals a one-cycle percent damping of about 5.0 percent. The c.p.u. times for the pseudo-force Newmark and DVOGER solutions were about 16 and 230 sec, respectively. Judging by the comparative sizes of the integration time steps and by the results presented for table XIII, the pseudo-force Newmark solution for the type II-damped problem was probably more accurate than the DVOGER solution and was obtained in less than one-tenth the c.p.u. time.

The results presented in this section certainly demonstrate that the pseudo-force Newmark method is an efficient, accurate, and, thereby, feasible solution technique for cantilevered beam problems with small damping.

However, increasing the time step and damping constants much beyond the values presented here will result in serious convergence problems and perhaps convergence failure. For instance, for the type II damping problem presented in this section a time step of \(10 \times 10^{-5} \text{ sec}\) was used with \(\zeta_E = \zeta_G = 0.002 \text{ sec}^2\). However, when \(\zeta_E = \zeta_G = 0.005 \text{ sec}^2\) was used, the time step had to be lowered to \(5 \times 10^{-5} \text{ sec}\) to obtain convergence. The many computer runs that were
made clearly indicated that the lower the damping employed, the higher the time step that could be used. The analysis of the cantilevered beam problems again reiterated that the pseudo-force representation should be reserved for small damping.

**Free-free Beam Problem**

As a preliminary step into the investigation of the active control of large space structures, this section presents results of an investigation into the importance of type II damping in predicting the low- and high-frequency vibratory motion of a free-free Timoshenko beam. The motion resulted from an initial displacement condition; initial velocities were set equal to zero, and there were no external forces. The initial displacement consisted of a cosine-shaped disturbance of height 1.000 in. centered about the beam mid span. If $l_D$ is the disturbance length and $x$ is measured from the beam mid space, then the equation for the initial displacement is given by

$$\text{initial displacement} = \begin{cases} \frac{1}{2} \{1 + \cos \left[ \frac{2\pi}{l_D} (x - \frac{l}{2}) \right] \} ; & \frac{l}{2} < x < \frac{l_D}{2} \\ 0 ; & \text{all other } x \end{cases}$$

$$\text{initial rotation} = -\frac{\pi x}{2} \arcsin \left( \frac{2\pi x}{l} \right)$$

where $r_D$ is the ratio of disturbance length $l_D$ to beam length $l$.

Two values of $r_D$ are investigated here, 0.5 and 0.1. The initial condition is illustrated in figure 6(b). The initial cross-section rotations were defined so that they were perpendicular to the neutral axis.

Based on the strong support given in the results of the cantilevered beam problems, an iteration error tolerance of 0.0000001 and...
a maximum number of allowed iterations of four were used. As will be seen, the results obtained for this section support this choice of parameters.

For the problems solved in this section, the beam parameters and material properties were \( l = 480.00 \text{ in.}, \ H = 2.00 \text{ in.}, \ B = 2.00 \text{ in.}, \ E = 30 \times 10^6 \text{ lb/in}^2, \ G = 12 \times 10^6 \text{ lb/in}^2, \rho = 0.00073 \text{ lb-sec}^2/\text{in}^4, \ k_s = 0.822, \) and (for the type II-damped problems) \( \xi_E \equiv \xi_G = 0.002 \text{ sec}^2. \) Note that the beam material properties for the free-free beam problems are the same as for the cantilevered beam problem of table XIV. In fact, one of the reasons for solving the cantilevered beam problem was to decide what values of the damping constants \( \xi_E \) and \( \xi_G \) were to be used in the free-free beam problems based on a reasonable attenuation in the cantilevered beam problem.

The first Euler mode period \( T_{1,E} \) of the cantilevered beam was 0.055 sec while \( T_{1,E} = 0.55 \text{ sec} \) for the free-free beam. The difference is solely due to the free-free beam being much longer than the cantilevered beam.

With all the beam properties and dimensions chosen, the following steps were then followed for both the \( r_D = 0.5 \) and \( r_D = 0.1 \) problems:

1. For the undamped problem using any single \( \Delta t \), the minimum number of elements to give consistent results with greater numbers of elements was determined.

2. For the undamped problem using the number of elements decided on in step 1, the largest (most efficient) \( \Delta t \)'s to give consistent results with smaller \( \Delta t \)'s was determined.
3. Using the number of elements decided on in step 1, the $\Delta t$ decided on in step 2 was checked to see if it was small enough for convergence in the type II-damped problem. If convergence failed, $\Delta t$ was decreased until the largest $\Delta t$ to give convergence was found. This $\Delta t$ was then checked for consistent results with even smaller $\Delta t$'s.

4. Using the number of elements decided on in step 1 and the $\Delta t$ decided on in steps 2 and 3, the conventional Newmark solution of the undamped problem was obtained to generate beamshape and time history data and to use as a basis for comparison with the damped solution.

5. Using the same number of elements and $\Delta t$ as in step 4, the pseudo-force Newmark solution of the type II-damped problem was obtained to generate beamshape and time history data for investigation of the effects of type II damping on the beam response.

$r_D = 0.5$. In step 1 using $\Delta t = 5 \times 10^{-4}$ sec, 8 elements were found to yield the same results as 16 elements, and the number of elements to be used in the $r_D = 0.5$ problems was thus decided on to be 8.

In step 2 using 8 elements, $\Delta t = 25 \times 10^{-4}$ sec was found to yield the same results as $\Delta t = 5 \times 10^{-4}$ sec. Increasing $\Delta t$ further to $50 \times 10^{-4}$ sec was found to result in significant discrepancies in comparison with $\Delta t = 25 \times 10^{-4}$ sec, so $25 \times 10^{-4}$ sec was tentatively decided on as the $\Delta t$ to be used in the $r_D = 0.5$ problems.

In step 3, $\Delta t = 25 \times 10^{-4}$ sec did give convergence in the damped case thus finally affirming it as the $\Delta t$ to be used.
The results of steps 4 and 5 are displayed in table XVI and in figures 9 and 10. Figure 9 shows graphs of the endpoint, quarterpoint, and midpoint time histories for both no damping and type II damping; the numerical data graphed in figure 9 is shown in table XVI. Figure 10 shows beamshape plots at eight selected times for both no damping and type II damping.

The beam midpoint time history was used to determine the period of the motion on which to base the degree of damping. Based on observation of this time history, \( t = 0.62 \text{ sec} \) was chosen as the time period to base the degree of damping on (note: \( T_{1,E} = 0.55 \text{ sec} \)). At \( t = 0.62 \text{ sec} \) the damped displacement was 0.8595 in. which is about 5.3 percent damping relative to the undamped displacement, 0.9121 in.

Quite clearly from figures 9 and 10, structural damping had little effect on the low-frequency beam motion displayed with \( r_D = 0.5 \) which indicates that structural damping is of little importance in predicting the low-frequency vibratory motion relative to the active control of such motion. The c.p.u. times for the undamped and damped solutions were about 11 and 20 sec, respectively. Thus, both the conventional Newmark and pseudo-force Newmark methods had little difficulty solving a low-frequency motion beam problem.

\( r_D = 0.1 \). The same beam material properties and parameters as used with \( r_D = 0.5 \) were used with \( r_D = 0.1 \). The first three steps followed in the investigation were again completed in full; however, in order to conserve computer time, determination of a period on which to base the degree of damping as done with \( r_D = 0.5 \) in steps 4 and 5 was not accomplished. Fortunately, the results show
that determination of a period was not needed to determine the effect of structural damping.

In step 1 using $\Delta t = 50 \times 10^{-5}$ sec, numbers of elements of 12, 16, 20, and 24 were investigated. By comparing midpoint time histories, the results using 16 elements were seen to differ greatly from those using 12 elements. And the results using 20 elements were seen to differ significantly from those using 16 elements. However, the results using 24 elements differed only slightly from those using 20 elements. So, the minimum number of elements that could be used was decided on to be 20.

In step 2 using 20 elements, $\Delta t = 20 \times 10^{-5}$ sec was found to give about the same results as $\Delta t = 10 \times 10^{-5}$ sec and $\Delta t = 5 \times 10^{-5}$ sec. Then $\Delta t = 25 \times 10^{-5}$ sec was found to give significantly different results; so, $\Delta t = 20 \times 10^{-5}$ sec was tentatively decided on as the $\Delta t$ to be used in the $r_D = 0.1$ problems.

In step 3 using 20 elements, convergence failed in the type II-damped problem when $\Delta t = 20 \times 10^{-5}$ sec was used. Investigation of the damped problem soon revealed that $\Delta t = 5 \times 10^{-6}$ sec was the largest $\Delta t$ that could be used and still have convergence. Further investigation showed that $\Delta t = 4 \times 10^{-6}$ sec gave a slightly faster solution than $\Delta t = 5 \times 10^{-6}$ sec thus indicating that $\Delta t = 5 \times 10^{-6}$ sec resulted in only marginal convergence. Investigation of $\Delta t = 2.5 \times 10^{-6}$ and $1 \times 10^{-6}$ sec revealed the expected increases in c.p.u. time with results about the same as with $\Delta t = 4 \times 10^{-6}$ sec. Thus, $4 \times 10^{-6}$ sec was decided on as the $\Delta t$ to be used in the $r_D = 0.1$ problems.

The results of steps 4 and 5 are displayed in table XVII and in figures 11 and 12. Figure 11 shows graphs of the endpoint,
quarterpoint, and midpoint time histories for both no damping and type II damping; the numerical data graphed in figure 11 is shown in table XVII. Figure 12 shows beamshape plots of four selected times for both no damping and type II damping.

As seen in figures 11 and 12, structural damping, as expected, greatly affected the high-frequency beam motion displayed with \( r_D = 0.1 \).

In the undamped solution the initial displacement appears to disperse as it moves outward from the center of the beam. The undamped initial displacement dispersed so much of its energy outward that by the time the initial displacement inverted at about \( t = 0.0054 \) sec, its midpoint displacement was only about \(-0.51\) in. (a perfectly inverted initial displacement would have had a midpoint displacement of \(-1.000\) in.). During the motion shown, the vibration amplitudes were least and increasing outside the initial displacement area and greatest and decreasing in the center.

In the damped solution the dispersive vibratory motion that was displayed in the undamped solution has been almost totally damped. At the final time investigated, figure 11 shows that the endpoint had not yet noticeably moved while the quarterpoint had just begun to move. In effect, with damping, the ends of the beam do not even know that the center of the beam had an initial displacement while with no damping, the ends do eventually feel the effect of the initial displacement in the center of the beam. The midpoint vibratory motion has also been significantly damped; the negative peak value for the undamped solution was \(-0.5052\) in. while the negative peak value for the damped solution was \(-0.4176\) in. Thus, the inclusion of structural damping in vibration analysis, even damping so small as to have little
effect on low-frequency beam motion, appears to be of utmost importance in predicting the high-frequency motion of the beam, especially if that motion is to be automatically controlled. The c.p.u. times for undamped and damped solutions were about 100 and 300 sec, respectively. With high-frequency motion, the need for more finite elements and lower time steps caused significant increases in c.p.u. times in comparison with the demands of low-frequency motion. With such problems care must be taken to use the most efficient parameters.

Finally, in support of the choices of error tolerance (0.0000001) and maximum number of iterations (4), the results of the problems of this section revealed

1. for \( r_D = 0.5 \), 90 percent of the time steps had iteration errors less than or equal to the error tolerance while the other 10 percent had extremely low iteration errors (e.g., 0.00006 percent), and

2. for \( r_D = 0.1 \), 75 percent of the time steps had iteration errors less than or equal to the error tolerance while the other 25 percent had extremely low iteration errors (e.g., 0.00006 percent).
Chapter VII

CONCLUSIONS

A numerical integration technique, a modified version of the Newmark method, has been applied to transient motion problems of systems with mass, stiffness, and small nonlinear damping. The nonlinearity has been cast as a pseudo-force to avoid repeated recalculation and decomposition of the effective stiffness matrix; thus, the solution technique used has been dubbed the "pseudo-force Newmark method." The solution technique's accuracy and efficiency has been studied in single-degree-of-freedom (SDOF) problems for three types of nonviscous damping (piecewise linear, quadratic, and cubic damping functions) and in cantilevered Timoshenko beam finite element problems for a quadratic damping function.

The importance of small nonlinear structural damping (quadratic) in predicting the low- and high-frequency vibratory motion of a free-free Timoshenko beam has been studied in a preliminary step into the investigation of the active control of large space structure.

SDOF Problems

By comparison with the conventional Newmark method solution for a viscously damped problem, it has been concluded that the pseudo-force Newmark method retains the excellent accuracy associated with the conventional Newmark method for SDOF problems.
By studying the convergence characteristics of the pseudo-force Newmark method for piecewise linear, quadratic, and cubic damping functions, it has been concluded that increases in degree of nonlinearity of damping, magnitude of damping constant, and initial velocity and decreases in error tolerance all result in slower convergence and, if carried to a great enough degree, convergence failure.

By comparing the pseudo-force Newmark method solution with exact and first-order perturbation solutions for piecewise linear, quadratic, and cubic damping functions, it has been concluded that the pseudo-force Newmark method is an accurate solution technique for SDOF problems provided the nonlinear damping is small.

**Timoshenko Beam Finite Element Problems**

By comparison with the conventional Newmark method solution for a viscously damped problem, it has been concluded that the pseudo-force Newmark method retains the accuracy associated with the conventional Newmark method for multi-degree-of-freedom problems.

By comparison with a Gear-method numerical solution obtained by using a subroutine from the International Mathematical and Statistical Library (IMSL), it has been concluded in conjunction with the SDOF results, that, overall, the pseudo-force Newmark method is an efficient, accurate, and, thus, feasible, solution technique for transient motion problems with small nonlinear damping.

By studying the effect of small nonlinear structural damping on the low- and high-frequency motion of a free-free Timoshenko beam, it has been concluded that, as expected, the inclusion of structural
damping in vibration analysis even that so small as to have little
effect on low-frequency motion, is of utmost importance in the predic-
tion of high-frequency motion that is to be automatically controlled.
REFERENCES


TABLE I

EFFECT OF TIME STEP $\Delta t$ ON ACCURACY OF CONVENTIONAL NEWMARK
SOLUTION FOR VISCOSLY DAMPED SDOF PROBLEM

<table>
<thead>
<tr>
<th>$N$ (a)</th>
<th>$\Delta t = T/40$</th>
<th>$\Delta t = T/20$</th>
<th>$\Delta t = T/10$</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8.4432</td>
<td>0.13</td>
<td>8.4755</td>
<td>0.51</td>
</tr>
<tr>
<td>8</td>
<td>5.2176</td>
<td>0.26</td>
<td>5.3579</td>
<td>1.04</td>
</tr>
<tr>
<td>12</td>
<td>2.0137</td>
<td>0.41</td>
<td>2.0373</td>
<td>1.58</td>
</tr>
<tr>
<td>16</td>
<td>-2.2707</td>
<td>0.35</td>
<td>-2.2746</td>
<td>1.80</td>
</tr>
<tr>
<td>20</td>
<td>-1.4158</td>
<td>0.62</td>
<td>-1.4430</td>
<td>2.56</td>
</tr>
<tr>
<td>24</td>
<td>-1.6270</td>
<td>0.77</td>
<td>-1.6648</td>
<td>3.11</td>
</tr>
<tr>
<td>28</td>
<td>-1.2788</td>
<td>0.90</td>
<td>-1.3138</td>
<td>3.66</td>
</tr>
<tr>
<td>32</td>
<td>-1.7301</td>
<td>1.03</td>
<td>-1.7532</td>
<td>4.23</td>
</tr>
<tr>
<td>36</td>
<td>-2.2297</td>
<td>1.16</td>
<td>-2.381</td>
<td>4.88</td>
</tr>
<tr>
<td>40</td>
<td>.1031</td>
<td>1.30</td>
<td>.1068</td>
<td>4.97</td>
</tr>
<tr>
<td>44</td>
<td>.2519</td>
<td>1.43</td>
<td>.2625</td>
<td>5.73</td>
</tr>
<tr>
<td>48</td>
<td>.2600</td>
<td>1.56</td>
<td>.2723</td>
<td>6.33</td>
</tr>
<tr>
<td>52</td>
<td>.1907</td>
<td>1.69</td>
<td>.2005</td>
<td>6.91</td>
</tr>
</tbody>
</table>

$\Delta t = N/40$
### TABLE II

**DISPLACEMENT TIME HISTORY COMPARISON OF PSEUDO-FORCE NEWMARK SOLUTION WITH CONVENTIONAL NEWMARK AND EXACT SOLUTIONS FOR VISCOUSLY DAMPED SDOF PROBLEM**

<table>
<thead>
<tr>
<th>N (a)</th>
<th>Conventional Newmark Solution</th>
<th>Pseudo-force Newmark Solution</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement, in.</td>
<td>Percent Error</td>
<td>Displacement, in.</td>
</tr>
<tr>
<td>1</td>
<td>9.8863</td>
<td>.03</td>
<td>9.8863</td>
</tr>
<tr>
<td>2</td>
<td>9.5641</td>
<td>.06</td>
<td>9.5641</td>
</tr>
<tr>
<td>3</td>
<td>9.0711</td>
<td>.10</td>
<td>9.0710</td>
</tr>
<tr>
<td>4</td>
<td>8.4432</td>
<td>.13</td>
<td>8.4430</td>
</tr>
<tr>
<td>5</td>
<td>7.7141</td>
<td>.16</td>
<td>7.7139</td>
</tr>
<tr>
<td>6</td>
<td>6.9151</td>
<td>.20</td>
<td>6.9149</td>
</tr>
<tr>
<td>7</td>
<td>6.0746</td>
<td>.23</td>
<td>6.0743</td>
</tr>
<tr>
<td>8</td>
<td>5.2176</td>
<td>.26</td>
<td>5.2173</td>
</tr>
<tr>
<td>9</td>
<td>4.3661</td>
<td>.30</td>
<td>4.3657</td>
</tr>
<tr>
<td>10</td>
<td>3.5385</td>
<td>.33</td>
<td>3.5381</td>
</tr>
<tr>
<td>11</td>
<td>2.7503</td>
<td>.37</td>
<td>2.7499</td>
</tr>
<tr>
<td>12</td>
<td>2.0137</td>
<td>.41</td>
<td>2.0133</td>
</tr>
</tbody>
</table>

\[\Delta t = \frac{(N)(2\pi)}{40} = N \Delta t\]
## TABLE III

DISPLACEMENT TIME HISTORY COMPARISON OF PSEUDO-FORCE NEWMARK SOLUTION WITH EXACT SOLUTION FOR TYPE I-DAMPED SDOF PROBLEM

<table>
<thead>
<tr>
<th>N (a)</th>
<th>Displacement Time History, in.</th>
<th>Pseudo-force Newmark</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.143</td>
<td>1.143</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.254</td>
<td>1.255</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.332</td>
<td>1.333</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.374</td>
<td>1.374</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.381</td>
<td>1.380</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.355</td>
<td>1.353</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.300</td>
<td>1.296</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1.216</td>
<td>1.210</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.105</td>
<td>1.098</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.9692</td>
<td>.9613</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>.8123</td>
<td>.8033</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>.6376</td>
<td>.6275</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>.4487</td>
<td>.4378</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>.2499</td>
<td>.2384</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>.04565</td>
<td>.03369</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>-.1595</td>
<td>-.1717</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>-.3602</td>
<td>-.3724</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>-.5512</td>
<td>-.5630</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>-.7274</td>
<td>-.7384</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>-.8840</td>
<td>-.8937</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>-1.017</td>
<td>-1.025</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>-1.122</td>
<td>-1.128</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>-1.197</td>
<td>-1.201</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>-1.240</td>
<td>-1.242</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>-1.251</td>
<td>-1.249</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>-1.233</td>
<td>-1.226</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>-1.187</td>
<td>-1.177</td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>-1.115</td>
<td>-1.102</td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>-1.019</td>
<td>-1.002</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>-.9002</td>
<td>-.8796</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>-.7615</td>
<td>-.7380</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>-.6059</td>
<td>-.5801</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>-.4370</td>
<td>-.4093</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>-.2584</td>
<td>-.2295</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>-.07416</td>
<td>-.04454</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>.1116</td>
<td>.1413</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>.2943</td>
<td>.3234</td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>.4690</td>
<td>.4966</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>.6311</td>
<td>.6565</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>.7762</td>
<td>.7985</td>
<td></td>
</tr>
</tbody>
</table>

\[
\Delta t = \frac{(N)(2\pi)}{40} = N \Delta t
\]
TABLE IV
PEAK-AMPLITUDE COMPARISON OF PSEUDO-FORCE NEWMARK SOLUTION WITH EXACT SOLUTION FOR TYPE I-DAMPED SDOF PROBLEM

<table>
<thead>
<tr>
<th>Peak Number</th>
<th>Peak Amplitude, in.</th>
<th>Percent Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
<td>Exact (a)</td>
</tr>
<tr>
<td>1</td>
<td>1.381</td>
<td>1.380</td>
</tr>
<tr>
<td>2</td>
<td>-1.251</td>
<td>-1.249</td>
</tr>
<tr>
<td>3</td>
<td>1.131</td>
<td>1.130</td>
</tr>
<tr>
<td>4</td>
<td>-1.024</td>
<td>-1.022</td>
</tr>
<tr>
<td>5</td>
<td>.9271</td>
<td>.9249</td>
</tr>
<tr>
<td>6</td>
<td>-.8394</td>
<td>-.8366</td>
</tr>
<tr>
<td>7</td>
<td>.7600</td>
<td>.7567</td>
</tr>
<tr>
<td>8</td>
<td>-.6881</td>
<td>-.6843</td>
</tr>
<tr>
<td>9</td>
<td>.6239</td>
<td>.6187</td>
</tr>
<tr>
<td>10</td>
<td>-.5634</td>
<td>-.5594</td>
</tr>
<tr>
<td>11</td>
<td>.5103</td>
<td>.5056</td>
</tr>
<tr>
<td>12</td>
<td>-.4617</td>
<td>-.4569</td>
</tr>
</tbody>
</table>

\(^a\)Exact solution value corresponds to time at which Pseudo-force Newmark approximate peak occurred.
### TABLE V

ZERO-CROSSING COMPARISON OF PSEUDO-FORCE NEWMARK SOLUTION WITH EXACT SOLUTION FOR TYPE I-DAMPED SDOF PROBLEM

<table>
<thead>
<tr>
<th>Zero-crossing Number</th>
<th>Time of Zero Crossing, sec</th>
<th>Percent Error (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
<td>Exact</td>
</tr>
<tr>
<td>1</td>
<td>2.39</td>
<td>2.38</td>
</tr>
<tr>
<td>2</td>
<td>5.56</td>
<td>5.54</td>
</tr>
<tr>
<td>3</td>
<td>8.70</td>
<td>8.69</td>
</tr>
<tr>
<td>4</td>
<td>11.84</td>
<td>11.84</td>
</tr>
<tr>
<td>5</td>
<td>14.98</td>
<td>15.00</td>
</tr>
<tr>
<td>6</td>
<td>18.12</td>
<td>18.15</td>
</tr>
<tr>
<td>7</td>
<td>21.26</td>
<td>21.30</td>
</tr>
<tr>
<td>8</td>
<td>24.40</td>
<td>24.46</td>
</tr>
<tr>
<td>9</td>
<td>27.58</td>
<td>27.61</td>
</tr>
<tr>
<td>10</td>
<td>30.72</td>
<td>30.76</td>
</tr>
<tr>
<td>11</td>
<td>33.87</td>
<td>33.92</td>
</tr>
<tr>
<td>12</td>
<td>37.02</td>
<td>37.07</td>
</tr>
</tbody>
</table>

(a) Time difference error as percentage of period $T$ ($T = 2\pi$ sec).
TABLE VI
PEAK-AMPLITUDE COMPARISON OF PSEUDO-FORCE NEWMARK SOLUTION WITH PERTURBATION SOLUTION FOR TYPE II-DAMPED SDOF PROBLEM

<table>
<thead>
<tr>
<th>Peak Number</th>
<th>Peak Amplitude, in.</th>
<th>Percent Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
<td>Perturbation</td>
</tr>
<tr>
<td>1</td>
<td>1.407</td>
<td>1.382</td>
</tr>
<tr>
<td>2</td>
<td>-1.286</td>
<td>-1.265</td>
</tr>
<tr>
<td>3</td>
<td>1.185</td>
<td>1.167</td>
</tr>
<tr>
<td>4</td>
<td>-1.098</td>
<td>-1.083</td>
</tr>
<tr>
<td>5</td>
<td>1.024</td>
<td>1.010</td>
</tr>
<tr>
<td>6</td>
<td>-.9581</td>
<td>-.9460</td>
</tr>
<tr>
<td>7</td>
<td>.9005</td>
<td>.8899</td>
</tr>
<tr>
<td>8</td>
<td>-.8494</td>
<td>-.8400</td>
</tr>
<tr>
<td>9</td>
<td>.8037</td>
<td>.7955</td>
</tr>
<tr>
<td>10</td>
<td>-.7626</td>
<td>-.7554</td>
</tr>
<tr>
<td>11</td>
<td>.7254</td>
<td>.7192</td>
</tr>
<tr>
<td>12</td>
<td>-.6917</td>
<td>-.6863</td>
</tr>
</tbody>
</table>
### TABLE VII

Zero-crossing comparison of pseudo-force Newmark solution with perturbation solution for Type II-damped SDOF problem

<table>
<thead>
<tr>
<th>Zero-crossing Number</th>
<th>Time of Zero Crossing, sec</th>
<th>Percent Difference (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
<td>Perturbation</td>
</tr>
<tr>
<td>1</td>
<td>2.38</td>
<td>2.36</td>
</tr>
<tr>
<td>2</td>
<td>5.52</td>
<td>5.50</td>
</tr>
<tr>
<td>3</td>
<td>8.67</td>
<td>8.64</td>
</tr>
<tr>
<td>4</td>
<td>11.82</td>
<td>11.78</td>
</tr>
<tr>
<td>5</td>
<td>14.97</td>
<td>14.92</td>
</tr>
<tr>
<td>6</td>
<td>18.12</td>
<td>18.06</td>
</tr>
<tr>
<td>7</td>
<td>21.27</td>
<td>21.21</td>
</tr>
<tr>
<td>8</td>
<td>24.41</td>
<td>24.35</td>
</tr>
<tr>
<td>9</td>
<td>27.56</td>
<td>27.49</td>
</tr>
<tr>
<td>10</td>
<td>30.71</td>
<td>30.63</td>
</tr>
<tr>
<td>11</td>
<td>33.86</td>
<td>33.77</td>
</tr>
<tr>
<td>12</td>
<td>37.01</td>
<td>36.91</td>
</tr>
</tbody>
</table>

*Note: Percent difference as percentage of period T (T = 2π sec).*
<table>
<thead>
<tr>
<th>N (a)</th>
<th>$e_t = 0.005$</th>
<th>$e_t = 0.001$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>1</td>
<td>1.143</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.254</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.332</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.374</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.381</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>..9691</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>..04557</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>..-8841</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>..-9001</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>..7764</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\dot{u}(o) = 1.000$ in/sec</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.547</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4.025</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5.394</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6.618</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>7.664</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>9.493</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>6.033</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>..-6.106</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>..-8.597</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>..5115</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$\dot{u}(o) = 10.00$ in/sec</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>16.59</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>31.73</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>46.02</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>59.06</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>70.51</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>95.13</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>65.11</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>..7016</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>..-86.21</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>..-2.407</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ e_t = \frac{(N)(2\pi)}{40} = N \Delta t \]
TABLE IX
CONVERGENCE CHARACTERISTICS FOR TYPE II DAMPING IN SDOF SYSTEM

<table>
<thead>
<tr>
<th>N (a)</th>
<th>$e_t = 0.005$ Displacement, in.</th>
<th>$e_t = 0.001$ Displacement, in.</th>
<th>$e_t = 0.0001$ Displacement, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Iterations</td>
<td>Number of Iterations</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
<td>------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>1</td>
<td>1.143</td>
<td>0</td>
<td>1.143</td>
</tr>
<tr>
<td>2</td>
<td>1.256</td>
<td>0</td>
<td>1.256</td>
</tr>
<tr>
<td>3</td>
<td>1.337</td>
<td>0</td>
<td>1.337</td>
</tr>
<tr>
<td>4</td>
<td>1.385</td>
<td>0</td>
<td>1.385</td>
</tr>
<tr>
<td>5</td>
<td>1.399</td>
<td>0</td>
<td>1.399</td>
</tr>
<tr>
<td>10</td>
<td>.9823</td>
<td>1</td>
<td>.9823</td>
</tr>
<tr>
<td>15</td>
<td>.04714</td>
<td>1</td>
<td>.04714</td>
</tr>
<tr>
<td>20</td>
<td>-.8286</td>
<td>0</td>
<td>-.8286</td>
</tr>
<tr>
<td>30</td>
<td>-.8375</td>
<td>0</td>
<td>-.8375</td>
</tr>
<tr>
<td>40</td>
<td>.7056</td>
<td>0</td>
<td>.7056</td>
</tr>
</tbody>
</table>

$\dot{u}(o) = 1.000\ \text{in/sec}$

<table>
<thead>
<tr>
<th></th>
<th>$e_t = 0.005$ Displacement, in.</th>
<th>$e_t = 0.001$ Displacement, in.</th>
<th>$e_t = 0.0001$ Displacement, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Iterations</td>
<td>Number of Iterations</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>-------</td>
<td>------------------</td>
<td>------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>1</td>
<td>2.446</td>
<td>0</td>
<td>2.444</td>
</tr>
<tr>
<td>2</td>
<td>3.657</td>
<td>0</td>
<td>3.651</td>
</tr>
<tr>
<td>3</td>
<td>4.657</td>
<td>0</td>
<td>4.645</td>
</tr>
<tr>
<td>4</td>
<td>5.461</td>
<td>0</td>
<td>5.445</td>
</tr>
<tr>
<td>5</td>
<td>6.081</td>
<td>0</td>
<td>6.060</td>
</tr>
<tr>
<td>10</td>
<td>6.576</td>
<td>0</td>
<td>6.547</td>
</tr>
<tr>
<td>15</td>
<td>3.601</td>
<td>0</td>
<td>3.570</td>
</tr>
<tr>
<td>20</td>
<td>-.1933</td>
<td>1</td>
<td>-.2394</td>
</tr>
<tr>
<td>30</td>
<td>-3.466</td>
<td>0</td>
<td>-3.465</td>
</tr>
<tr>
<td>40</td>
<td>-.1092</td>
<td>1</td>
<td>-.1165</td>
</tr>
</tbody>
</table>

$\dot{u}(o) = 10.00\ \text{in/sec}$
TABLE IX - Continued

<table>
<thead>
<tr>
<th>N (a)</th>
<th>$e_t = 0.005$</th>
<th></th>
<th>$e_t = 0.001$</th>
<th></th>
<th>$e_t = 0.0001$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>1</td>
<td>10.27</td>
<td>5</td>
<td>10.27</td>
<td>6</td>
<td>10.27</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>12.64</td>
<td>1</td>
<td>12.67</td>
<td>2</td>
<td>12.66</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>14.28</td>
<td>0</td>
<td>14.32</td>
<td>1</td>
<td>14.31</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>15.37</td>
<td>0</td>
<td>15.42</td>
<td>1</td>
<td>15.41</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>16.01</td>
<td>0</td>
<td>16.05</td>
<td>0</td>
<td>16.04</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>13.52</td>
<td>0</td>
<td>13.51</td>
<td>1</td>
<td>13.49</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>6.445</td>
<td>0</td>
<td>6.324</td>
<td>1</td>
<td>6.312</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>.2101</td>
<td>1</td>
<td>.04910</td>
<td>2</td>
<td>.05282</td>
<td>3</td>
</tr>
<tr>
<td>30</td>
<td>-4.643</td>
<td>0</td>
<td>-4.638</td>
<td>1</td>
<td>-4.638</td>
<td>1</td>
</tr>
<tr>
<td>40</td>
<td>-.1045</td>
<td>2</td>
<td>-.1026</td>
<td>2</td>
<td>-.1026</td>
<td>2</td>
</tr>
</tbody>
</table>

$\dot{u}(o) = 100.0 \text{ in/sec}$

\[ a_t = (N)(2\pi)/40 = N \Delta t \]
### TABLE X

CONVERGENCE CHARACTERISTICS FOR TYPE III DAMPING IN SDOF SYSTEM

(a) $\Delta t = 2\pi/40$

<table>
<thead>
<tr>
<th>$N$ (a)</th>
<th>$e_t = 0.005$</th>
<th>$e_t = 0.001$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>$\dot{u}(o) = 1.000 \text{ in/sec}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.143</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.255</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.334</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.378</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.388</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1.364</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1.222</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>.9690</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>.02592</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>-.9194</td>
<td>0</td>
</tr>
<tr>
<td>$\dot{u}(o) = 10.00 \text{ in/sec}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.508</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3.812</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4.785</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>5.382</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5.658</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>5.710</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>5.473</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>5.038</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>3.547</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>1.559</td>
<td>0</td>
</tr>
</tbody>
</table>

$a_t = N \Delta t$
TABLE X (Continued)

(b) $\Delta t = 2\pi/80$

<table>
<thead>
<tr>
<th>N (a)</th>
<th>$u(0) = 50.00 \text{ in/sec}$</th>
<th>$u(0) = 100.0 \text{ in/sec}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e_t = 0.005$</td>
<td>$e_t = 0.001$</td>
</tr>
<tr>
<td></td>
<td>Displacement, in.</td>
<td>Number of Iterations</td>
</tr>
<tr>
<td>-------</td>
<td>---------------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>1</td>
<td>7.830</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>10.75</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>11.16</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>11.04</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>10.91</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>10.76</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>10.47</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>10.17</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>9.369</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>8.502</td>
<td>0</td>
</tr>
</tbody>
</table>

$e_t = N \Delta t$
## TABLE XI

**Endpoint Displacement Time History Comparison of Pseudo-Force Newmark, Conventional Newmark, and DVOGER Solutions for Cantilevered Beam Problem with Viscous Damping**

<table>
<thead>
<tr>
<th>Time, sec</th>
<th>Endpoint Displacement Time History, in.</th>
<th>Conventional Newmark</th>
<th>Pseudo-Force Newmark</th>
<th>DVOGER</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td></td>
</tr>
<tr>
<td>.003</td>
<td>.9008</td>
<td>.9023</td>
<td>.9025</td>
<td></td>
</tr>
<tr>
<td>.006</td>
<td>.7308</td>
<td>.7331</td>
<td>.7419</td>
<td></td>
</tr>
<tr>
<td>.009</td>
<td>.4977</td>
<td>.4982</td>
<td>.5230</td>
<td></td>
</tr>
<tr>
<td>.012</td>
<td>.1351</td>
<td>.1366</td>
<td>.1738</td>
<td></td>
</tr>
<tr>
<td>.015</td>
<td>-.1967</td>
<td>-.1965</td>
<td>-.1518</td>
<td></td>
</tr>
<tr>
<td>.018</td>
<td>-.4921</td>
<td>-.4941</td>
<td>-.4366</td>
<td></td>
</tr>
<tr>
<td>.021</td>
<td>-.7799</td>
<td>-.7792</td>
<td>-.7267</td>
<td></td>
</tr>
<tr>
<td>.024</td>
<td>-.9192</td>
<td>-.9187</td>
<td>-.8895</td>
<td></td>
</tr>
<tr>
<td>.027</td>
<td>-.9449</td>
<td>-.9457</td>
<td>-.9475</td>
<td></td>
</tr>
<tr>
<td>.030</td>
<td>-.9028</td>
<td>-.8992</td>
<td>-.9405</td>
<td></td>
</tr>
<tr>
<td>.033</td>
<td>-.6988</td>
<td>-.6971</td>
<td>-.7753</td>
<td></td>
</tr>
<tr>
<td>.036</td>
<td>-.4167</td>
<td>-.4161</td>
<td>-.5289</td>
<td></td>
</tr>
<tr>
<td>.039</td>
<td>-.1171</td>
<td>-.1134</td>
<td>-.2560</td>
<td></td>
</tr>
<tr>
<td>.042</td>
<td>.2441</td>
<td>.2448</td>
<td>.0957</td>
<td></td>
</tr>
<tr>
<td>.045</td>
<td>.5564</td>
<td>.5552</td>
<td>.4091</td>
<td></td>
</tr>
<tr>
<td>.048</td>
<td>.7706</td>
<td>.7726</td>
<td>.6543</td>
<td></td>
</tr>
<tr>
<td>.051</td>
<td>.9322</td>
<td>.9295</td>
<td>.8680</td>
<td></td>
</tr>
<tr>
<td>.054</td>
<td>.9551</td>
<td>.9511</td>
<td>.9579</td>
<td></td>
</tr>
<tr>
<td>.057</td>
<td>.8455</td>
<td>.8460</td>
<td>.9142</td>
<td></td>
</tr>
<tr>
<td>.060</td>
<td>.6715</td>
<td>.6669</td>
<td>.8099</td>
<td></td>
</tr>
<tr>
<td>.063</td>
<td>.3843</td>
<td>.3883</td>
<td>.5844</td>
<td></td>
</tr>
<tr>
<td>.066</td>
<td>.0402</td>
<td>.0455</td>
<td>.2763</td>
<td></td>
</tr>
<tr>
<td>.069</td>
<td>-.2729</td>
<td>-.2709</td>
<td>-.0416</td>
<td></td>
</tr>
<tr>
<td>.072</td>
<td>-.5814</td>
<td>-.5736</td>
<td>-.3462</td>
<td></td>
</tr>
<tr>
<td>.075</td>
<td>-.8139</td>
<td>-.8011</td>
<td>-.6405</td>
<td></td>
</tr>
<tr>
<td>.078</td>
<td>-.9147</td>
<td>-.9078</td>
<td>-.8196</td>
<td></td>
</tr>
<tr>
<td>.081</td>
<td>-.9335</td>
<td>-.9227</td>
<td>-.9393</td>
<td></td>
</tr>
<tr>
<td>.084</td>
<td>-.8331</td>
<td>-.8220</td>
<td>-.9417</td>
<td></td>
</tr>
<tr>
<td>.087</td>
<td>-.6040</td>
<td>-.6074</td>
<td>-.8154</td>
<td></td>
</tr>
<tr>
<td>.090</td>
<td>-.3280</td>
<td>-.3312</td>
<td>-.6177</td>
<td></td>
</tr>
<tr>
<td>.093</td>
<td>-.0009</td>
<td>-.0087</td>
<td>-.3446</td>
<td></td>
</tr>
<tr>
<td>.096</td>
<td>.3446</td>
<td>.3219</td>
<td>-.0086</td>
<td></td>
</tr>
<tr>
<td>.099</td>
<td>.6155</td>
<td>.5973</td>
<td>.2992</td>
<td></td>
</tr>
<tr>
<td>.102</td>
<td>.8210</td>
<td>.8016</td>
<td>.5766</td>
<td></td>
</tr>
<tr>
<td>.105</td>
<td>.9351</td>
<td>.9089</td>
<td>.8114</td>
<td></td>
</tr>
<tr>
<td>.108</td>
<td>.9074</td>
<td>.8959</td>
<td>.9164</td>
<td></td>
</tr>
<tr>
<td>.111</td>
<td>.7861</td>
<td>.7799</td>
<td>.9335</td>
<td></td>
</tr>
<tr>
<td>.114</td>
<td>.5730</td>
<td>.5695</td>
<td>.8449</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE XII
ZERO-CROSSING COMPARISON OF PSEUDO-FORCE NEWMARK, CONVENTIONAL NEWMARK, AND DVOGER SOLUTIONS FOR ENDPOINT DISPLACEMENT OF CANTILEVERED BEAM WITH VISCOUS DAMPING

<table>
<thead>
<tr>
<th>Zero-crossing Number</th>
<th>Time of Zero Crossing, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
</tr>
<tr>
<td>1</td>
<td>.01323</td>
</tr>
<tr>
<td>2</td>
<td>.03995</td>
</tr>
<tr>
<td>3</td>
<td>.06638</td>
</tr>
<tr>
<td>4</td>
<td>.09301</td>
</tr>
</tbody>
</table>

### TABLE XIII
EFFECT OF TIME STEP Δt ON ZERO-CROSSING COMPARISON OF CONVENTIONAL NEWMARK AND DVOGER SOLUTIONS FOR ENDPOINT DISPLACEMENT OF CANTILEVERED BEAM WITH NO DAMPING

<table>
<thead>
<tr>
<th>Zero-crossing Number</th>
<th>Time of Zero Crossing, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Conventional Newmark</td>
</tr>
<tr>
<td></td>
<td>Δt = 5 x 10^-5 sec</td>
</tr>
<tr>
<td>1</td>
<td>.01332</td>
</tr>
<tr>
<td>2</td>
<td>.04031</td>
</tr>
</tbody>
</table>
**TABLE XIV**

ENDPOINT DISPLACEMENT TIME HISTORY COMPARISON FOR PSEUDO-FORCE
NEWMARK AND DVOGER SOLUTIONS FOR CANTILEVERED BEAM
PROBLEM WITH TYPE II DAMPING

<table>
<thead>
<tr>
<th>Time, sec</th>
<th>Endpoint Displacement Time History, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
</tr>
<tr>
<td>0</td>
<td>1.0000</td>
</tr>
<tr>
<td>.003</td>
<td>.9021</td>
</tr>
<tr>
<td>.006</td>
<td>.7385</td>
</tr>
<tr>
<td>.009</td>
<td>.5139</td>
</tr>
<tr>
<td>.012</td>
<td>.1729</td>
</tr>
<tr>
<td>.015</td>
<td>-.1512</td>
</tr>
<tr>
<td>.018</td>
<td>-.4446</td>
</tr>
<tr>
<td>.021</td>
<td>-.7242</td>
</tr>
<tr>
<td>.024</td>
<td>-.8855</td>
</tr>
<tr>
<td>.027</td>
<td>-.9389</td>
</tr>
<tr>
<td>.030</td>
<td>-.9115</td>
</tr>
<tr>
<td>.033</td>
<td>-.7454</td>
</tr>
<tr>
<td>.036</td>
<td>-.4953</td>
</tr>
<tr>
<td>.039</td>
<td>-.2105</td>
</tr>
<tr>
<td>.042</td>
<td>-.1245</td>
</tr>
<tr>
<td>.045</td>
<td>.4346</td>
</tr>
<tr>
<td>.048</td>
<td>.6792</td>
</tr>
<tr>
<td>.051</td>
<td>.8623</td>
</tr>
<tr>
<td>.054</td>
<td>.9337</td>
</tr>
<tr>
<td>.057</td>
<td>.8863</td>
</tr>
<tr>
<td>.060</td>
<td>.7504</td>
</tr>
<tr>
<td>.063</td>
<td>.5154</td>
</tr>
<tr>
<td>.066</td>
<td>.2168</td>
</tr>
<tr>
<td>.069</td>
<td>-.0942</td>
</tr>
<tr>
<td>.072</td>
<td>-.4010</td>
</tr>
<tr>
<td>.075</td>
<td>-.6609</td>
</tr>
<tr>
<td>.078</td>
<td>-.8328</td>
</tr>
<tr>
<td>.081</td>
<td>-.9129</td>
</tr>
<tr>
<td>.084</td>
<td>-.8851</td>
</tr>
<tr>
<td>.087</td>
<td>-.7439</td>
</tr>
<tr>
<td>.090</td>
<td>-.5230</td>
</tr>
<tr>
<td>.093</td>
<td>-.2402</td>
</tr>
<tr>
<td>.096</td>
<td>.0761</td>
</tr>
<tr>
<td>.099</td>
<td>.3744</td>
</tr>
<tr>
<td>.102</td>
<td>.6301</td>
</tr>
<tr>
<td>.105</td>
<td>.8140</td>
</tr>
<tr>
<td>.108</td>
<td>.8945</td>
</tr>
<tr>
<td>.111</td>
<td>.8724</td>
</tr>
<tr>
<td>.114</td>
<td>.7482</td>
</tr>
<tr>
<td>.117</td>
<td>.3302</td>
</tr>
<tr>
<td>.120</td>
<td>.2549</td>
</tr>
</tbody>
</table>
TABLE XV

ZERO-CROSSING COMPARISON OF PSEUDO-FORCE NEWMARK AND DVQGER SOLUTIONS FOR ENDPOINT DISPLACEMENT OF CANTILEVERED BEAM WITH TYPE II DAMPING

<table>
<thead>
<tr>
<th>Zero-crossing Number</th>
<th>Time of Zero Crossing, sec</th>
<th>Difference, percent of $T_{1,E}$ ($T_{1,E} = 0.055$ sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pseudo-force Newmark</td>
<td>DVQGER</td>
</tr>
<tr>
<td>1</td>
<td>.01360</td>
<td>.01370</td>
</tr>
<tr>
<td>2</td>
<td>.04089</td>
<td>.04094</td>
</tr>
<tr>
<td>3</td>
<td>.06809</td>
<td>.06867</td>
</tr>
<tr>
<td>4</td>
<td>.09528</td>
<td>.09618</td>
</tr>
</tbody>
</table>
TABLE XVI

EFFECT OF TYPE II DAMPING ON THE LOW-FREQUENCY MOTION OF FREE-FREE BEAM
RESULTING FROM AN INITIAL CONDITION OF $r_0 = 0.5$

<table>
<thead>
<tr>
<th>Time, sec</th>
<th>Pseudo-force Newmark Type II-Damped Displacement, in. (a)</th>
<th>Conventional Newmark Undamped Displacement, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Endpoint</td>
<td>Quarterpoint</td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
<td>--------------</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.02</td>
<td>0.03358</td>
<td>0.2742</td>
</tr>
<tr>
<td>0.04</td>
<td>-0.7142</td>
<td>0.4551</td>
</tr>
<tr>
<td>0.06</td>
<td>-0.3890</td>
<td>0.5550</td>
</tr>
<tr>
<td>0.08</td>
<td>-0.1590</td>
<td>0.1831</td>
</tr>
<tr>
<td>0.10</td>
<td>0.5731</td>
<td>0.04498</td>
</tr>
<tr>
<td>0.12</td>
<td>0.2062</td>
<td>0.1088</td>
</tr>
<tr>
<td>0.14</td>
<td>0.1055</td>
<td>0.4613</td>
</tr>
<tr>
<td>0.16</td>
<td>-0.09627</td>
<td>0.4633</td>
</tr>
<tr>
<td>0.18</td>
<td>0.6081</td>
<td>0.2388</td>
</tr>
<tr>
<td>0.20</td>
<td>0.9311</td>
<td>-0.04908</td>
</tr>
<tr>
<td>0.22</td>
<td>0.9617</td>
<td>0.06244</td>
</tr>
<tr>
<td>0.24</td>
<td>0.4744</td>
<td>0.3334</td>
</tr>
<tr>
<td>0.26</td>
<td>0.3728</td>
<td>0.4510</td>
</tr>
<tr>
<td>0.28</td>
<td>0.7666</td>
<td>0.2253</td>
</tr>
<tr>
<td>0.30</td>
<td>1.124</td>
<td>-0.03186</td>
</tr>
<tr>
<td>0.32</td>
<td>1.071</td>
<td>0.04310</td>
</tr>
<tr>
<td>0.34</td>
<td>0.4197</td>
<td>0.3143</td>
</tr>
<tr>
<td>0.36</td>
<td>0.2229</td>
<td>0.4967</td>
</tr>
<tr>
<td>0.38</td>
<td>0.2735</td>
<td>0.2636</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7751</td>
<td>0.05290</td>
</tr>
<tr>
<td>0.42</td>
<td>0.4451</td>
<td>0.02041</td>
</tr>
</tbody>
</table>
### TABLE XVI (Continued)

<table>
<thead>
<tr>
<th>Time, sec</th>
<th>Pseudo-force Newmark Type II-Damped Displacement, in. (a)</th>
<th>Conventional Newmark Undamped Displacement, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Endpoint</td>
<td>Quarterpoint</td>
</tr>
<tr>
<td>.44</td>
<td>.09407</td>
<td>.3843</td>
</tr>
<tr>
<td>.46</td>
<td>-.5317</td>
<td>.5074</td>
</tr>
<tr>
<td>.48</td>
<td>-.1222</td>
<td>.4295</td>
</tr>
<tr>
<td>.50</td>
<td>-.00550</td>
<td>.07308</td>
</tr>
<tr>
<td>.52</td>
<td>.2522</td>
<td>.1202</td>
</tr>
<tr>
<td>.54</td>
<td>-.4840</td>
<td>.3090</td>
</tr>
<tr>
<td>.56</td>
<td>-.5464</td>
<td>.5950</td>
</tr>
<tr>
<td>.58</td>
<td>-.5926</td>
<td>.3983</td>
</tr>
<tr>
<td>.60</td>
<td>.1981</td>
<td>.1784</td>
</tr>
<tr>
<td>.62</td>
<td>.1388</td>
<td>.03280</td>
</tr>
<tr>
<td>.64</td>
<td>.07904</td>
<td>.3316</td>
</tr>
<tr>
<td>.66</td>
<td>-.4119</td>
<td>.4927</td>
</tr>
</tbody>
</table>

\[ a\tau_E = \tau_c = 0.002 \text{ sec}^2 \]
TABLE XVII

EFFECT OF TYPE II DAMPING ON THE HIGH-FREQUENCY MOTION OF FREE-FREE BEAM
RESULTING FROM AN INITIAL CONDITION OF $r_D = 0.1$

<table>
<thead>
<tr>
<th>Time, sec</th>
<th>Pseudo-force Newmark Type II-Damped Displacement, in.</th>
<th>Conventional Newmark Undamped Displacement, in.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Endpoint</td>
<td>Quarterpoint</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.0004</td>
<td>$0.21 \times 10^{-5}$</td>
<td>$-0.82 \times 10^{-4}$</td>
</tr>
<tr>
<td>.0008</td>
<td>$0.37 \times 10^{-4}$</td>
<td>$0.4 \times 10^{-4}$</td>
</tr>
<tr>
<td>.0012</td>
<td>$0.32 \times 10^{-4}$</td>
<td>$0.0002287$</td>
</tr>
<tr>
<td>.0016</td>
<td>$-0.33 \times 10^{-4}$</td>
<td>$-0.0004696$</td>
</tr>
<tr>
<td>.0020</td>
<td>$0.59 \times 10^{-4}$</td>
<td>$-0.001331$</td>
</tr>
<tr>
<td>.0024</td>
<td>$-0.38 \times 10^{-4}$</td>
<td>$-0.0008921$</td>
</tr>
<tr>
<td>.0028</td>
<td>$-0.0001351$</td>
<td>$0.001260$</td>
</tr>
<tr>
<td>.0032</td>
<td>$0.0003037$</td>
<td>$0.004349$</td>
</tr>
<tr>
<td>.0036</td>
<td>$0.0001595$</td>
<td>$0.007037$</td>
</tr>
<tr>
<td>.0040</td>
<td>$-0.0006371$</td>
<td>$0.008249$</td>
</tr>
<tr>
<td>.0044</td>
<td>$-0.0006581$</td>
<td>$0.007020$</td>
</tr>
<tr>
<td>.0048</td>
<td>$0.0005079$</td>
<td>$0.003192$</td>
</tr>
<tr>
<td>.0052</td>
<td>$0.001697$</td>
<td>$-0.002691$</td>
</tr>
<tr>
<td>.0056</td>
<td>$0.001562$</td>
<td>$-0.009280$</td>
</tr>
<tr>
<td>.0060</td>
<td>$-0.0002867$</td>
<td>$-0.01541$</td>
</tr>
<tr>
<td>.0064</td>
<td>$-0.003013$</td>
<td>$-0.02000$</td>
</tr>
<tr>
<td>.0068</td>
<td>$-0.004859$</td>
<td>$-0.02150$</td>
</tr>
<tr>
<td>.0072</td>
<td>$-0.004670$</td>
<td>$-0.01991$</td>
</tr>
<tr>
<td>.0076</td>
<td>$-0.001905$</td>
<td>$-0.01592$</td>
</tr>
<tr>
<td>.0080</td>
<td>$0.002938$</td>
<td>$-0.009640$</td>
</tr>
</tbody>
</table>

\[ a_{r_E} = r_G = 0.002 \text{ sec}^2 \]
Fig. 1 - Degrees of freedom for planar motion finite element model used in thesis.
\[ M = \text{Internal bending moment about Y axis} \]

\[ Q = \text{Internal shear force in z direction} \]

\[ \theta = \frac{\partial w}{\partial x} + \gamma \]

Fig. 2 - Free-body diagram and geometry for differential element of beam.
Damping Constants

Viscous - $C$, $\frac{\text{lb-sec}}{\text{in}}$

Type I - $C_I$, $\frac{\text{lb}}{\text{in}}$

Type II - $C_{II}$, $\frac{\text{lb-sec}^2}{\text{in}^2}$

Type III - $C_{III}$, $\frac{\text{lb-sec}^3}{\text{in}^3}$

Fig. 3 - SDOF system studied.
Fig. 4 - Comparison of pseudo-force Newmark solution with exact solution for type I damping in SDOF system.
Fig. 5 - Comparison of pseudo-force Newmark solution with perturbation solution for type II damping in SDOF system.
Fig. 6 - Initial condition sketches for cantilevered and free-free beam problems.

\[ U_1 = 0.3125 \text{ in.} \]
\[ U_2 = 0.01875 \text{ rad} \]
\[ U_3 = 1.000 \text{ in.} \]
\[ U_4 = 0.025 \text{ rad} \]

(a) Cantilevered beam
Initial displacement length
Beam length = 480.00 in.

<table>
<thead>
<tr>
<th>Initial displacement length</th>
<th>0.5</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of elements</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

(b) Free-free beam

Fig. 6 - Concluded.
Fig. 7 - Effect of $\Delta t$ on comparison of conventional Newmark and DVOGER solutions for endpoint displacement of cantilevered beam with no damping.
Fig. 8 - Comparison of pseudo-force Newmark and DVOGER solutions for endpoint displacement of cantilevered beam with type II damping. Conventional Newmark undamped solution is also plotted to show damping effect.
Fig. 9 - Undamped and type II-damped time histories for low-frequency motion of free-free beam resulting from initial condition of $r_D = 0.5$. 

Undamped
Damped
Fig. 10 - Undamped and type II-damped beamshape plots for low-frequency motion of free-free beam resulting from initial condition of $r_D = 0.5$. 
Fig. 10 - Concluded.
Fig. 11 - Undamped and type II-damped time histories for high-frequency motion of free-free beam resulting from initial condition of $r_0 = 0.1$. 
Fig. 12 - Undamped and type II-damped beamshape plots for high-frequency motion of free-free beam resulting from initial condition of $r_D = 0.1$. 

(a) $t = 0$ sec

(b) $t = 0.0020$ sec

(c) $t = 0.0040$ sec

(d) $t = 0.0064$ sec

Longitudinal distance from midpoint, in.  Longitudinal distance from midpoint, in.
APPENDIX A

DESCRIPTION OF IMSL SUBROUTINE DVOGER AND LIST OF PARAMETER VALUES FOR DVOGER SOLUTIONS

The following four pages give the description of the IMSL subroutine DVOGER as found in the IMSL Library Reference Manual, edition 6 (ref. 72). Following those four pages is a table giving the values of the most pertinent of the DVOGER parameters for the DVOGER solutions discussed in Chapter VI of the thesis.
**C FUNCTION**  
- FIRST ORDER DIFFERENTIAL EQUATION SOLVER—

**USAGF**  
- CALL DVGFRT(DFUN,Y,T,N,MTH,MAXDER,JSTART,H,HMIN,HMAX,EP,S,DFU,GERROR,WK,IER)

**PARAMETERS** DFUN — USER SUPPLIED EXTERNAL SUBROUTINE, DFUN(YP,TP,N,M,YP,DY,IND), WHERE

*YP CONTAINS THE PRESENT (I.E. AT TP)*

SOLUTION VECTOR AS X(1)=YP(1),

X(2)=YP(1)+TP, etc.

*TP IS THE PRESENT TIME,* AND

*M IS THE ORDER OF THE JACOBIAN.*

IF IND=0, DFUN MUST COMPUTE THE N-VECTOR

F(YP,TP) AND STORE THE VALUES IN DY.

IF IND=1, DFUN MUST COMPUTE THE JACOBIAN OF

F EVALUATED AT (YP,TP) AND STORE THE RESULT IN THE M BY M MATRIX PW.

THE JACOBIAN IS COMPUTED ONLY FOR CALLS WITH MTH=1.

**YP IS AN 8 BY N ARRAY. THE SOLUTION COMPONENTS ARE Y(1,1), Y(1,2), ..., Y(1,N).**

**Y** — Y IS A TWO DIMENSIONAL ARRAY (8 BY N) CONTAINING THE DEPENDENT VARIABLES AND THEIR SCALED DERIVATIVES. THE SOLUTION COMPONENTS ARE X(1)=Y(1),

X(2)=Y(1)+TP, etc.

X(N)=Y(1+N) AND Y(J+1,I) CONTAINS THE J-TH DERIVATIVE OF X(I) SCAL ED BY H/J/FAC(TORIAL). HEPE H IS THE CURRENT STEP SIZE.

ONLY Y(1,1), I=1,2,...,N NEED BE PROVIDED BY THE CALLING PROGRAM ON THE FIRST CALL TO DVGER, I.E. WITH JSTART = 0.

**T** — T IS THE INDEPENDENT VARIABLE. ON INPUT, T SHOULD CONTAIN THE INITIAL VALUE OF THE INDEPENDENT VARIABLE. ON OUTPUT, T CONTAINS THE UPDATED VALUE OF THE INDEPENDENT VARIABLE.

**N** — N IS THE NUMBER OF FIRST ORDER DIFFERENTIAL EQUATIONS.

**MTH** — MTH IS THE METHOD INDICATOR. THE USER MAY SELECT ONE OF THE FOLLOWING:

*MTH=0* INDICATES A PREDICTOR-CORRECTOR (ADAMS) METHOD.
MAXDFR - MAXDFR MUST BE SET BY THE USER TO THE MAXIMUM
ORDER TO BE USED IN THE APPROXIMATION.
IT MUST BE LESS THAN 8 FOR THE ADAMS METHOD
AND LESS THEN 7 FOR THE STIFF METHODS.

JSTART - ON INPUT, JSTART HAS THE FOLLOWING MEANINGS...
-1 - REPEAT THE LAST STEP WITH A NEW VALUE OF H.
0 - INITIALIZE THE INTEGRATION. THE FIRST CALL TO DVOGER MUST BE DONE WITH THIS VALUE OF JSTART.
+1 - TAKE A NEW STEP CONTINUING FROM THE LAST.
ON OUTPUT, JSTART IS SET TO 10, THE CURRENT ORDER OF THE METHOD. JSTART IS ALSO THE ORDER OF THE MAXIMUM DERIVATIVE AVAILABLE INTERFACE.

H - ON INPUT, H CONTAINS THE STEP SIZE TO BE ATTEMPTED ON THE NEXT STEP. IF THIS STEP SIZE DOES NOT CAUSE A LARGER ERROR THAN REQUESTED, IT WILL BE USED. OTHERWISE, SEE PARAMETER IEP, THE USER IS ADVISED TO USE A FAIRLY SMALL STEP IN THE FIRST CALL TO DVOGER.
ON OUTPUT, H CONTAINS A SUGGESTED STEP SIZE FOR THE NEXT STEP IN ORDER TO ACHIEVE AN ECONOMICAL INTEGRATION.

HMIN - HMIN MUST BE SET TO THE SMALLEST STEP SIZE ALLOWABLE IN THIS INTEGRATION. HMIN SHOULD BE MUCH SMALLER THAN THE EXPECTED AVERAGE STEP SIZE FOR THE FIRST CALL SINCE A FIRST ORDER METHOD IS USED INITIALLY.

HMAX - HMAX MUST BE SET TO THE LARGEST STEP SIZE ALLOWABLE IN THIS INTEGRATION.

EPS - EPS IS USED TO SPECIFY THE MAXIMUM ERROR CRITERION. THE STEP SIZE AND/OR THE ORDER IS ADJUSTED SO THAT THE SINGLE STEP ERROR ESTIMATE DIVIDED BY YMAX(1) ARE LESS THAN EPS IN THE EUCLIDEAN NORM.

YMAX - YMAX IS AN N-VECTOR WHICH CONTAINS THE MAXIMUM ABSOLUTE VALUE OF EACH COMPONENT OF X CALCULATED SO FAR. THE COMPONENTS OF YMAX SHOULD NORMALLY BE SET TO 1. BEFORE THE FIRST CALL TO DVOGER.

ERROR - ERROR IS AN N-VECTOR WHICH CONTAINS THE ESTIMATED ONE STEP ERROR IN EACH COMPONENT.

WK - WORK AREA OF DIMENSION 17*N IF MTH = 0
N*(N+17) OTHERWISE

IFP - ERROR PARAMETER
WARNING ERROR = 32 + N
N = 1 INDICATES THE STEP WAS TAKEN WITH H=HMIN, BUT THE REQUESTED ERROR WAS NOT ACHIEVED
N = 2 INDICATES CONVERGENCE COULD NOT BE ACHIEVED FOR H GREATER THAN HMIN
N = 3 INDICATES THE REQUESTED ERROR IS SMALLER THAN CAN BE HANDLED FOR THIS PROBLEM
WARNING ERROR (WITH FIX ) = 64 + N
N = 4 INDICATES THE MAXIMUM ORDER SPECIFIED
CALL DVGER (DFUN,Y,T,N,MTH,MAXDER,JSTART,H,HMIN,HMAX,EPS,YMAX,ERROR,WK,IER)

**Purpose**

This routine provides one step in the integration of a system of 1-st order differential equations. In particular, we solve the system $X' = f(X,T)$ for $X(T+H)$ where $X(T)$ is given. Here $X$ is an $N$-vector of solutions, $T$ is the independent variable, and $f$ is a (possibly nonlinear) function supplied by the user through the subroutine DFUN.

**Algorithm**

DVGER is a modification of the Gear subroutine DIFSUB. The algorithm features a switch (MTH) which allows efficient solutions to both stiff and nonstiff systems. For nonstiff equations, the routine uses a predictor corrector method.


**Programming Notes**

1. The input parameter MTH is the method indicator. For nonstiff systems, MTH should be set to zero. For stiff systems, MTH can be set to either one or two. If MTH=1, the user must be prepared to supply the Jacobian ($J = \frac{\partial f_i(X,T)}{\partial x_j}$, where $f_i$ and $x_j$ are components $i$ and $j$ of $f$ and $X$ respectively) through the subroutine DFUN. If MTH=2, the Jacobian is computed via numerical differences.

2. The routine is much more efficient if an analytic expression is available for the Jacobian.

3. The subroutine DFUN(YP,TP,M,DY,PW,IND) must be supplied by the user and must be defined by an EXTERNAL specification statement in the calling program. The parameter YP (dimensioned $8 \times N$) contains the current value of the solution (i.e. at TP), in its first row. That is, the solution at time TP is $X(1)=Y(1,1)$, $X(2)=Y(1,2)$, ..., $X(N)=Y(1,N)$.

   IND is an input parameter. If IND=0, DFUN must compute $f(X,T)$, i.e. $DY(I) = f_I(Y(1,1), \ldots, Y(1,N),TP)$ for $I=1,2,\ldots,N$. If IND=1, DFUN must compute the $M$ by $M$ Jacobian matrix, i.e. $PW(I,J) = \frac{\partial f_I(Y(1,1), \ldots, Y(1,N),TP)}{\partial Y(I,J)}$ for $I,J=1,2,\ldots,M$. (PW is assumed to have dimension $(M,N)$). Note that in calling the function subprogram DFUN, IND will usually be set to zero. DVGER will only call DFUN with IND=1 when the user sets MTH=1.

   In calling DFUN, DVGER will usually use $N$ as the calling value for $M$, but not always. The user should dimension PW as $(N,1)$ (or PW(N,N) if the user prefers) in the subroutine DFUN.

4. A step size of length $H$ can be forced by setting $HMAX=HMIN=H$. This procedure is particularly valuable when computing a table of solutions at specified points. Note, however, that the step size must be small enough to fulfill the accuracy requirement.
Accuracy

Let $x_i^n$ and $x_i^{n+1}$ be the element $i$ of the solution $X$ at step $n$ and $(n+1)$, respectively.

Assume that both $x_i^n$ and $x_i^{n+1}$ are exact. Let $\tilde{x}_i^{n+1}$ be the computed approximation to $x_i^{n+1}$ using $x_i^n (i=1,2, \ldots, N)$ as a starting value. Then

$$\sum_{i=1}^{N} \left( \frac{\text{ERROR}(I)}{\text{YMAX}(I)} \right)^2 \leq E$$

for all steps taken. Here $\text{ERROR}(I)$ is an estimate of $|\tilde{x}_i^{n+1} - x_i^{n+1}|$, $E=(\text{EPS}^2 \cdot C)$, and $C$ depends on the method being used. See reference. The global error will depend upon both EPS and the number of steps taken.
The following table gives the values of the DVOGER parameters for the DVOGER solutions discussed in Chapter VI of the thesis and shown in the figures and tables as noted below.

<table>
<thead>
<tr>
<th>DVOGER Parameters</th>
<th>Tables XI and XII</th>
<th>Figure 7 and Table XIII</th>
<th>Figure 8 and Tables XIV and XV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>8</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>( \text{EPS} )</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.001</td>
</tr>
<tr>
<td>( \text{MAXDER} )</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( \text{HMIN} )</td>
<td>( 1 \times 10^{-13} )</td>
<td>( 1 \times 10^{-13} )</td>
<td>( 1 \times 10^{-13} )</td>
</tr>
<tr>
<td>( \text{HMAX} )</td>
<td>0.003</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>( \text{NTH} )</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
APPENDIX B

DESCRIPTION OF TIMOSHENKO BEAM THEORY

The Timoshenko beam theory accounts for both rotatory inertia and shear deformation, effects which are neglected in Bernoulli-Euler beam theory.

The free-body diagram and geometry for a differential element of a beam are shown in figure 2.

The two elastic equations for the beam are (refs. 66 and 73):

\[ Q = k_s GA \theta = k_s GA \left( \theta - \frac{\partial w}{\partial x} \right) \]  (B1)

and

\[ M = EI \frac{\partial^2 \theta}{\partial x^2} \]  (B2)

where \( k_s \) = a cross-section shear coefficient which depends on both the shape of the cross section and the frequency of vibration (ref. 66); \( M = \) internal bending moment about \( Y \) axis; and \( Q = \) internal shear force in \( Z \) direction.

The two free-vibration equations of motion are (ref. 73).

Sum of moments about \( Y \) axis = \( 3M/3x - Q = \rho I \ddot{\theta} \)  (B3)

Sum of forces in \( Z \) direction = \(-3Q/3x = \rho A \ddot{w} \)  (B4)

Substituting equations (B1) and (B2) into equations (B3) and (B4) results in (refs. 66 and 73)

\[ \frac{3}{3x} (EI \frac{3\theta}{3x}) + k_s A G (\theta - \frac{3w}{3x}) - \rho I \ddot{\theta} = 0 \]  (B5)
\[ \rho A \dddot{w} - \frac{\partial}{\partial x} \left[ k_s A G \left( \theta - \frac{\partial w}{\partial x} \right) \right] = 0 \quad (B6) \]

Eliminating \( \theta \) between equations (B5) and (B6) and assuming constant cross-sectional properties results in a single equation of motion for \( w \) as follows,

\[ EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I(1 + \frac{E}{k_s G}) \frac{\partial^4 w}{\partial x^4 \partial t^2} + \frac{\rho I}{k_s G} \frac{\partial^4 w}{\partial t^4} = 0 \quad (B7) \]

which is the most common version of Timoshenko's free-vibration equation for beam deflection.

If shear deformation and rotatory inertia are ignored, then equation (B7) reduces to the Bernoulli-Euler free-vibration beam equation, i.e.,

\[ EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \]

For Bernoulli-Euler beam theory, equation (B1) becomes \( Q = 0 \) since Bernoulli-Euler beam theory neglects shear deformation, i.e.,

\[ \theta = \frac{\partial w}{\partial x} \]

In Timoshenko beam theory, determination of the shear coefficient \( k_s \) has been the subject of much discussion ever since Timoshenko beam theory was introduced (see refs. 66, 67, and 74 to 76). Mindlin and Deresiewicz (ref. 67) stated in 1955 that \( k_s \) "... dependends both on the shape of the (cross) section and the frequency of vibration" and that, in general, using a constant value of \( k_s \) will be good only for motion dominated by one frequency. However, they
also stated that using $k_s = \text{constant}$ does give "satisfactory" results for some special cases, such as, "the low modes of motion of slender beams." Of particular interest to this thesis is Mindlin's and Deresiewicz's finding that "... for a rectangular section ... ($k_s = \pi^2/12 \approx 0.822$) gives good results for both low and high frequencies of beams with free ends." Thus, this value of $k_s$ (0.822) was deemed as the most appropriate for the Timoshenko beam problems of this thesis.
APPENDIX C

EXACT SOLUTION TO SINGLE-DEGREE-OF-FREEDOM (SDOF) PROBLEM WITH TYPE I DAMPING

The SDOF differential equation with type I damping is

\[ \ddot{u} + \frac{C_I}{m} \dot{u} + \frac{k}{m} u = 0 \]  \hspace{1cm} (C1)

where \( u \) = displacement = \( u(t) \) \( (t = \text{time}) \)

\( C_I \) = type I damping coefficient

\( k \) = stiffness

and \( m \) = mass

Equation (C1) is piecewise linear. For

\[ \ddot{u} > 0, \quad |\frac{u}{\dot{u}}| = \frac{\ddot{u}}{\dot{u}} \]

and equation (C1) becomes

\[ \ddot{u} + \left( \frac{k + C_I}{m} \right) u = 0 \]  \hspace{1cm} (C2)

For

\[ \ddot{u} < 0, \quad |\frac{u}{\dot{u}}| = -\frac{\ddot{u}}{\dot{u}} \]

and equation (C1) becomes

\[ \ddot{u} + \left( \frac{k - C_I}{m} \right) u = 0 \]  \hspace{1cm} (C3)
The exact solution to equation (C2) for any initial displacement \( u(0) \) and initial velocity \( \dot{u}(0) \) is

\[
u(t) = u(0) \cos \sqrt{\frac{k + C}{m}} t + \frac{\dot{u}(0)}{\sqrt{\frac{k + C}{m}}} \sin \sqrt{\frac{k + C}{m}} t \quad (C4)
\]

Similarly, the exact solution to equation (C3) (since solution is for small damping assume \( k > C_I \)) is

\[
u(t) = u(0) \cos \sqrt{\frac{k - C}{m}} t + \frac{\dot{u}(0)}{\sqrt{\frac{k - C}{m}}} \sin \sqrt{\frac{k - C}{m}} t \quad (C5)
\]

The period of equation (C4) is

\[
2\pi \sqrt{\frac{m}{k + C_I}}
\]

while the period of equation (C5) is

\[
2\pi \sqrt{\frac{m}{k - C_I}}.
\]

Each solution is calculated for alternating quarters of each's period. For example, if \( u(0) = 1 \) and \( \dot{u}(0) = 0 \), then, from \( t = 0 \) to

\[
t = \frac{\pi}{2} \sqrt{\frac{m}{k - C_I}},
\]

equation (C5) is used. Then from

\[
t = \frac{\pi}{2} \sqrt{\frac{m}{k - C_I}}.
\]
to

\[ t = \frac{\pi}{2} \sqrt{\frac{m}{k - C_I}} + \frac{\pi}{2} \sqrt{\frac{m}{k + C_I}}, \]

equation (C4) is used, etc.
APPENDIX D

FIRST-ORDER PERTURBATION SOLUTION TO SINGLE-DEGREE-OF-FREEDOM (SDOF) PROBLEMS WITH DAMPING TYPES II AND III

**Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a(T_1) )</td>
<td>displacement amplitude function for solution, in.</td>
</tr>
<tr>
<td>( C_{II} )</td>
<td>type II damping coefficient, lb-sec(^2)/in(^2)</td>
</tr>
<tr>
<td>( C_{III} )</td>
<td>type III damping coefficient, lb-sec/in(^3)</td>
</tr>
<tr>
<td>( \bar{C}_{II} )</td>
<td>( C_{II}/\varepsilon )</td>
</tr>
<tr>
<td>( \bar{C}_{III} )</td>
<td>( C_{III}/\varepsilon )</td>
</tr>
<tr>
<td>( D(T_1) )</td>
<td>complex function used in a solution form of ( u_0 ), in.</td>
</tr>
<tr>
<td>( D^*(T_1) )</td>
<td>complex conjugate of ( D )</td>
</tr>
<tr>
<td>( f(u, \ddot{u}) )</td>
<td>damping force function, lb</td>
</tr>
<tr>
<td>( i )</td>
<td>( \sqrt{-1} )</td>
</tr>
<tr>
<td>( k )</td>
<td>stiffness, lb/in</td>
</tr>
<tr>
<td>( m )</td>
<td>mass, lb-sec(^2)/in</td>
</tr>
<tr>
<td>( O( \ ) )</td>
<td>means &quot;of the order of&quot; the quantity in parentheses</td>
</tr>
<tr>
<td>( O(\varepsilon^0) )</td>
<td>first order</td>
</tr>
<tr>
<td>( O(\varepsilon^1) )</td>
<td>second order</td>
</tr>
<tr>
<td>( T_n )</td>
<td>( \varepsilon^n t ); multiple time scales used to replace single time scale ( t ) in method of multiple scales, sec</td>
</tr>
<tr>
<td>( t )</td>
<td>time, sec</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>displacement function, in.</td>
</tr>
</tbody>
</table>
\( u_n \)  
- \( n \)th-order term in perturbation expansion  
- solution of \( u(t) \), in.  
\( n = 0, 1, 2, \ldots \)  

\( v(T, t) \)  
- displacement phase function for \( u_0 \) solution, rad  

\( \varepsilon \)  
- a small parameter less than 1  

\( \omega_n \)  
- natural frequency, rad  

The SDOF differential equation under consideration is
\[
\ddot{u}(t) + \omega_n^2 u(t) + \varepsilon f(u(t), \dot{u}(t)) = 0 \tag{D1}
\]

The symbol \( \varepsilon \) is employed as a "bookkeeping" device as it denotes the relative order of magnitude of terms. For instance, an \( \varepsilon^0 \) term is an order of magnitude greater than an \( \varepsilon^1 \) term, and an \( \varepsilon^1 \) term is an order of magnitude greater than an \( \varepsilon^2 \) term, etc.

The method of multiple scales (see refs. 45 and 71) is applied to obtain a first-order solution to equation (D1).

First, the solution \( u(t) \) is represented as an expansion: a first-order approximation \( u_0 \) plus a second-order correction \( u_1 \) plus higher-order corrections if desired.
\[
\dot{u}(t; \varepsilon) = \varepsilon^0 u_0(t) + \varepsilon^1 u_1(t) + \varepsilon^2 u_2(t) + \ldots
\]

Then, new time scales are introduced to replace the single time scale. Since higher-order effects are usually not evident until much time has passed in a response, the assigning of a fast time scale for first-order effects and increasingly slower time scale for higher-order effects is a logical approach and the underlying basis for the method of multiple scales.
The time scales are defined as follows:

\[ T_0 = \epsilon^0 t \]
\[ T_1 = \epsilon^1 t \]
\[ T_2 = \epsilon^2 t, \text{ etc.} \]

The transformation of the ordinary derivatives on time to partial derivatives is shown below.

\[
\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon^1 \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \ldots
\]

\[
\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + \epsilon^1 \frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2 \left( \frac{\partial^2}{\partial T_1^2} + 2 \frac{\partial^2}{\partial T_0 \partial T_2} \right) + \ldots
\]  

For the present study, first-order \((O(\epsilon^0))\) solutions are desired. In order to calculate a first-order solution, all perturbation expansions must be carried to second-order \((O(\epsilon^1))\).

Thus, expanding to \(O(\epsilon^1)\), the response of equation (D1) is represented by

\[ u(T_0, T_1; \epsilon) = \epsilon^0 u_0(T_0, T_1) + \epsilon^1 u_1(T_0, T_1) \]  

(D3)

where \(u_0(T_0, T_1)\) is the first-order solution to be solved for.

Substituting equation (D3) into equation (D1) and employing the derivative transforms in equation (D2) up to \(O(\epsilon^1)\) results in the following partial differential equation of motion,  

\[
\left[ \frac{\partial^2}{\partial T_0^2} + \omega_n^2 u_0 \right] \epsilon^0 + \left[ \frac{\partial^2}{\partial T_0 \partial T_1} \right] + 2 \frac{\partial^2}{\partial T_0 \partial T_2} + \omega_n^2 u_1 + f(u_0, \frac{\partial u_0}{\partial T_0}) \epsilon^1 = 0
\]

(D4)
Each coefficient of an order of $\varepsilon$ independently satisfies equation (D4).

Setting the coefficient of $\varepsilon^0$ equal to zero results in the following partial differential equation governing the first-order solution $u_0$,

$$\frac{\partial^2 u_0}{\partial T^2} + \omega_n^2 u_0 = 0$$  \hspace{1cm} (D5)

The first-order solution $u_0$ may be then written as

$$u_0(T_0, T_1) = D(T_1)e^{i\omega_n T_0} + D^*(T_1)e^{-i\omega_n T_0}$$  \hspace{1cm} (D6)

where $D(T_1)$ is an unknown complex function and $D^*(T_1)$ is its complex conjugate.

Another form of $u_0$ is

$$u_0(T_0, T_1) = a(T_1) \cos[\omega_n T_0 + \beta(T_1)]$$  \hspace{1cm} (D7)

where $a(T_1)$ is the amplitude of the motion and $\beta(T_1)$ is the phase. Note that

$$D(T_1) = \frac{a(T_1)}{2} e^{i\beta(T_1)}$$  \hspace{1cm} (D8)

and

$$D^*(T_1) = \frac{a(T_1)}{2} e^{-i\beta(T_1)}$$

Returning to equation (D4) and setting the coefficient of $\varepsilon^1$ equal to zero results in the following partial differential equation governing the second-order solution correction $u_1$, 


Substituting where convenient in the above equation for $u_0$

$$\frac{\partial^2 u_1}{\partial T_0^2} + \omega_n^2 u_1 = -2 \frac{\partial^2 u_0}{\partial T_0 \partial T_1} - f(u_0, \frac{\partial u_0}{\partial T_0})$$

results in

$$\frac{\partial^2 u_1}{\partial T_0^2} + \omega_n^2 u_1 = -2i\omega_n \left( \frac{dD}{dT_1} e^{i\omega_n T_0} - \frac{dD^*}{dT_1} e^{-i\omega_n T_0} \right) - f(u_0, \frac{\partial u_0}{\partial T_0}) \quad (D9)$$

The elimination of terms on the right-hand-side of equation (D9) that give rise to "secular terms" in the $u_1$ solution of equation (D9) will result in differential equations which will be used to solve for $a(T_1)$ and $b(T_1)$.

Secular terms are those terms in a solution which grow in size without bound as time increases. Such terms are the result of forcing a system with functions having a forcing frequency equal to the natural frequency of the system. So, terms on the right-hand-side of equation (D9) that have the frequency $\omega_n$ or $-\omega_n$ must be set equal to zero to eliminate secular terms in the solution of equation (D9). Setting equal to zero in equation (D9) terms with the frequency $\omega_n$ results in,

$$-2i\omega_n \frac{dD}{dT_1} e^{i\omega_n T_0} - [\text{the part of } f(u_0, \frac{\partial u_0}{\partial T_0}) \text{ the exhibits the frequency } \omega_n] = 0 \quad (D10)$$

Setting equal to zero terms with the frequency $-\omega_n$ in equation (D9) would result in the complex conjugate of equation (D10) and would in the end give the same results as equation (D10); therefore, such analysis is not shown.
Real part:

\[ a \frac{d^2 \theta}{dT_1^2} - \frac{1}{2\pi} \int_{0}^{2\pi} f(u_0, \frac{\partial u_0}{\partial T_0}) \cos(\omega_n T_0 + \beta) \, dT_0 = 0 \quad (D13) \]

Equations (D12) and (D13) are the set of differential equations which are to be solved for \( a(T_1) \) and \( \beta(T_1) \) given some specific \( f(u_0, \frac{\partial u_0}{\partial T_0}) \).

Once obtained, the solutions for \( a(T_1) \) and \( \beta(T_1) \) are substituted into equation (D7) to give the first-order perturbation solution in terms of two unknown constants, \( a(T_1 = 0) \) and \( \beta(T_1 = 0) \). To get the final solution form, the initial conditions \( u(t = 0) = u(0) \) and \( \frac{du}{dt}(t = 0) = \dot{u}(0) \) are applied to replace the constants \( a(T_1 = 0) \) and \( \beta(T_1 = 0) \) with more meaningful constants.

Shown below are the solution derivations for damping types II and III.

**Type II Damping**

For type II damping,

\[ f(u_0, \frac{\partial u_0}{\partial T_0}) = \frac{\bar{c}_{II}}{m} \left| \frac{\partial u_0}{\partial T_0} \right| \frac{\partial u_0}{\partial T_0} \]

where \( \bar{c}_{II} = \frac{c_{II}}{\epsilon} \).

Using the \( u_0 \) form in equation (C7) \( f(u_0, \frac{\partial u_0}{\partial T_0}) \) is rewritten as
In order to integrate this \( f(u_0, \frac{\partial u_0}{\partial T_0}) \) in equations (D12) and (D13), the limits on the integral must be broken up in such a way as to eliminate the need for the unintegrable absolute value signs. This problem is solved by recognizing the following,

\[
- \frac{c_{II}}{m} a_2 w_2 \sin^2 (\omega_n T_0 + \beta) \quad (\omega_n T_0 + \beta) = 0 \text{ to } \pi \\
- \frac{c_{II}}{m} a_2 w_2 \sin^2 (\omega_n T_0 + \beta) \quad (\omega_n T_0 + \beta) = \pi \text{ to } 2\pi
\]

Substituting equation (D14) into equation (D12) and carrying out the integration yields

\[
\frac{da}{dT_1} = -\omega_n \frac{4}{3} \frac{a^2}{\pi} \frac{c_{II}}{m}
\]

Integrating by separation of variables and applying the condition \( a(T_1 = 0) = a(0) \) to eliminate the constant of integration gives

\[
a(T_1) = \frac{a(0) \, 3\pi m}{4 \, \frac{c_{II}}{m} \omega_n a(0) \, T_1 + 3\pi m}
\]

Substituting equation (D14) into equation (D13) and carrying out the integration results in

\[
\frac{ds}{dT_1} = 0
\]
Thus,

\[ \beta = \text{constant} = \beta(0) \]  

(D16)

Substituting equation (D15) and (D16) into equation (D7) gives

\[ u_0 = \frac{a(0) 3\pi m}{4 \bar{C}_{III} \omega_n a(0) T_1 + 3\pi m} \cos \left[ \omega_n T_0 + \beta(0) \right] \]

The initial conditions \( \frac{\partial u_0}{\partial T_0} (T_0 = 0, T_1 = 0) = \dot{u}(0) \) and \( u_0(T_0 = 0, T_1 = 0) = u(0) \) are applied to replace \( a(0) \) and \( \beta(0) \) with the more meaningful constants. The result is

\[ u_0 = \frac{3\pi m [u(0) \cos(\omega_n T_0) + \frac{\dot{u}(0)}{\omega_n} \sin(\omega_n T_0) - \frac{u(0)}{\omega_n} \sin(\omega_n T_0)]}{4 \bar{C}_{III} \sqrt{\dot{u}(0)^2 + \omega_n^2 u(0)^2} T_1 + 3\pi m} \]  

(D17)

Equation (D17) is the first order \((0(\varepsilon^0))\) approximation to the solution \( u(t) \). Replacing \( u_0 \) with \( u(t) \), \( T_0 \) with \( t \), \( T_1 \) with \( \varepsilon t \), and \( \bar{C}_{III} \) with \( C_{III}/\varepsilon \), gives the following final form for the first-order \( u(t) \) solution,

\[ u(t) = \frac{3\pi m [u(0) \cos(\omega_n t) + \frac{\dot{u}(0)}{\omega_n} \sin(\omega_n t) - \frac{u(0)}{\omega_n} \sin(\omega_n t)]}{4 C_{III} \sqrt{\dot{u}(0)^2 + \omega_n^2 u(0)^2} t + 3\pi m} \]

Type III Damping

For type III damping,

\[ f(u_0, \frac{\partial u_0}{\partial T_0}) = \bar{C}_{III} \frac{\partial u_0}{\partial T_0} \]

where \( \bar{C}_{III} = C_{III}/\varepsilon \).
Using the $u_0$ form of equation (D7), $f(u_0, \frac{\partial u_0}{\partial T_0})$ is rewritten as

$$f(u_0, \frac{\partial u_0}{\partial T_0}) = \frac{C_{\text{III}}}{m} a^2 \cos^2(\omega_n T_0 + \beta) (-a) \omega_n \sin(\omega_n T_0 + \beta)$$  \hspace{1cm} (D18)

Substituting equation (D18) into equation (D12) and integrating yields

$$\frac{da}{dT_1} = \frac{-C_{\text{III}}}{m} \frac{a^3}{8}$$

Integrating by separation of variables and applying the condition $a(T_1 = 0) = a(0)$ to eliminate the constant of integration gives

$$a(T_1) = 2a(0) \sqrt{\frac{m}{C_{\text{III}}} a^2(0) T_1 + 4m}$$  \hspace{1cm} (D19)

Substituting equation (D18) into equation (D13) and integrating results in

$$\frac{d\beta}{dT_1} = 0$$

Thus,

$$\beta = \text{constant} = \beta(0)$$  \hspace{1cm} (D20)

Substituting equations (D19) and (D20) into equation (D7) gives
The initial conditions $u_0(T_0 = 0, T_1 = 0) = u(0)$ and $u_0(T_0 = 0, T_1 = 0) = u(0)$ are applied to replace $a(0)$ and $\beta(0)$ with the more meaningful constants. The result is

$$u_0 = 2 \sqrt{\frac{m}{\bar{C}_{III} a(0)^2 T_1 + 4m}} \cos[\omega_n T_0 + \beta(0)]$$

Equation (D21) is the first-order ($O(\varepsilon^0)$) approximation to the solution $u(t)$. Replacing $u_0$ with $u(t)$, $T_0$ with $t$, $T_1$ with $\varepsilon t$, and $\bar{C}_{III}$ with $C_{III}/\varepsilon$ gives the following final form for the first-order $u(t)$ solution,

$$u(t) = 2 \sqrt{\frac{m}{C_{III} (u(0)^2 + \dot{u}(0)^2/\omega_n^2) t + 4m}} \times \left[ \cos(\omega_n t) + \frac{\dot{u}(0)}{u(0) \omega_n} \sin(\omega_n t) \right]$$

Note that equations (D17) and (D21) have undamped periods which suggests that the damping effect known as "period elongation" is a second-order effect. Both perturbation solutions have amplitude decay.
PSEUDO-FORCE NEWMARK ALGORITHM

INITIAL SPECIFICATIONS:
- Material constants
- Initial displacements and velocities
- Integration time step
- Iteration error tolerance
- Limit on number of iterations

INITIAL CALCULATIONS:
- Mass, stiffness, and viscous damping matrices
- Initial value of nonviscous damping matrix
- Initial accelerations
- Effective stiffness matrix

TIME LOOP

INCREMENT TIME

Calculate first estimate of displacements, velocities, accelerations and nonviscous damping matrix

CALCULATE EFFECTIVE LOAD VECTOR

Solve for new estimate of displacements, velocities, accelerations, and nonviscous damping matrix

New Estimate
Equal Old Estimate
Within Specified Error Tolerance?

Yes

Set latest estimates to be the actual values for this time

No

Repeated

Reached Limit on Number of Iterations?

Yes

End

No

Reached Desired Final Time?