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Formal Power Series Approach to Nonlinear Systems with Static Output Feedback ⋆

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Abstract: The goal of this paper is to compute the generating series of a closed-loop system when the plant is described in terms of a Chen-Fliess series and static output feedback is applied. The first step is to reconsider the so called Wiener-Fliess connection consisting of a Chen-Fliess series followed by a memoryless function. Of particular importance will be the contractive nature of this map, which is needed to show that the closed-loop system has a Chen-Fliess series representation. To explicitly compute the generating series, two Hopf algebras are needed, the existing output feedback Hopf algebra used to describe dynamic output feedback, and the Hopf algebra of the shuffle group. These two combinatorial structures are combined to compute what will be called the Wiener-Fliess feedback product. It will be shown that this product has a natural interpretation as a transformation group acting on the plant and preserves the relative degree of the plant.

Keywords: Nonlinear systems, feedback control, Hopf algebras
AMS subject classification: 93C10, 93B52, 93B25

1. INTRODUCTION

Let F_c and F_d be two nonlinear input-output systems represented by Chen-Fliess functional series (Fliess, 1981). It was shown in Ferfera (1979); Gray & Li (2005) that the feedback interconnection of two such systems always renders a closed-loop system in the same class. Its corresponding generating series, written as the *feedback product* $c@d$, can be efficiently computed in terms of a combinatorial Hopf algebra which is commutative, graded and connected (Duffaut Espinosa, et al., 2016; Foissy, 2015; Gray, et al., 2014). Convergence of the closed-loop system was characterized in detail by Thitsa & Gray (2012). Variations of the feedback product were used to solve system inversion problems (Gray, et al., 2014b) and trajectory generation problems (Duffaut Espinosa & Gray, 2017).

What does not fit so neatly into this existing framework is the important case where the dynamical system F_d in the feedback path is replaced with a memoryless function f_d, the so called static output feedback connection. The central problem here is that the loop contains a cascade connection of a Chen-Fliess series and a memoryless function, an object with an algebraic nature not entirely compatible with the algebras used to analyze the dynamic feedback case. Therefore, the goal this paper is to address this problem by showing how to adapt existing algebraic tools for the analysis of static feedback systems. The first step is to reconsider the so called Wiener-Fliess connection consisting of a Chen-Fliess series followed by a memoryless function (Gray & Thitsa, 2012). Of particular importance will be the contractive nature of this map, which is needed to show that the closed-loop system has a Chen-Fliess series representation. Next the focus turns to actually computing this generating series. What is needed in this regard are *two* Hopf algebras, the output feedback Hopf algebra described above, and the Hopf algebra of the shuffle group. These two combinatorial structures will be combined to compute what will be called the *Wiener-Fliess feedback product*. It will be shown that this product has a natural interpretation as a transformation group acting on the plant and preserves the relative degree of the plant. The challenging problem of showing when the closed-loop system has a *locally convergent* generating series will be deferred to future work.

The paper is organized as follows. The next section provides a summary of the concepts related to Chen-Fliess series and their interconnections. Section 3 characterizes the Wiener-Fliess cascade interconnection. Section 4 describes the Hopf algebra of the shuffle group. Finally, the static feedback system is analyzed in Section 5. The conclusions of the paper and directions for future work are given in the last section.

2. PRELIMINARIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. Its *length* is

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$|\eta| = k$. In particular, $|\eta|_{x_i}$ is the number of times the letter $x_i \in X$ appears in η. The set of all words including the empty word, $\emptyset$, is denoted by X^*, and $X^+ := X^* \backslash \emptyset$. The set X^* forms a monoid under catenation. The set of all words with prefix η is written as ηX^*. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is denoted by (c, η) and called the *coefficient* of η in c. A series c is *proper* when $(c, \emptyset) = 0$. The *support* of c, supp(c), is the set of all words having nonzero coefficients. The *order* of c, ord(c), is the length of the minimal length word in its support. Normally, c is written as a formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. It constitutes an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product uniquely specified by the shuffle product of two words

$$(x_i\eta) ⧢ (x_j\xi) = x_i(\eta ⧢ (x_j\xi)) + x_j((x_i\eta) ⧢ \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta ⧢ \emptyset = \emptyset ⧢ \eta = \eta$ (Fliess, 1981). The subset of all proper formal power series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is denoted by $\mathbb{R}^\ell_p\langle\langle X\rangle\rangle$. The set $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is an ultrametric space with the ultrametric

$$\kappa(c, d) = \sigma^{\text{ord}(c-d)},$$

where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $\sigma \in]0, 1[$. For brevity, $\kappa(c, 0)$ is written as $\kappa(c)$, and $\kappa(c, d) = \kappa(c - d)$. The ultrametric space $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \kappa)$ is known to be Cauchy complete (Berstel & Reutenauer, 1988). The following types of contraction maps will be useful.

Definition 1. Given metric spaces (E, d) and (E', d'), a map $f : E \longrightarrow E'$ is said to be a *strong contraction map* if $\forall s, t \in E$, it satisfies the condition $d'(f(s), f(t)) \le \alpha d(s, t)$ where $\alpha \in [0, 1[$. If $\alpha = 1$, then the map f is said to be a *weak contraction map* or a *non-expansive map.*

In the event that the letters of X commute, the set of all corresponding formal power series is denoted by $\mathbb{R}^\ell[[X]]$. For any series $c \in \mathbb{R}^\ell[[X]]$, the natural number $\varpi(c)$ corresponds to the order of its proper part $c - (c, \emptyset)$.

2.1 Fliess operators

Let $\mathfrak{p} \ge 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_\mathfrak{p} = \max\{\|u_i\|_\mathfrak{p} : 1 \le i \le m\}$, where $\|u_i\|_\mathfrak{p}$ is the usual $L_\mathfrak{p}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_\mathfrak{p}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_\mathfrak{p}$ norm and $B^m_\mathfrak{p}(R)[t_0, t_1] := \{u \in L^m_\mathfrak{p}[t_0, t_1] : \|u\|_\mathfrak{p} \le R\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Given any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the corresponding *Chen-Fliess series* is

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0), \tag{1}$$

where $E_\emptyset[u] = 1$ and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau$$

with $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$ (Fliess, 1981). If there exists constants $K, M > 0$ such that

$$|(c, \eta)| \le KM^{|\eta|}\, |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_\mathfrak{p}(R)[t_0, t_0+T]$ into $B^\ell_\mathfrak{q}(S)[t_0, t_0+T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray & Wang, 2002). This map is referred to as a *Fliess operator*. Here $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ will denote the set of all such *locally convergent* generating series. In the absence of any convergence criterion, (1) only defines an operator in a formal sense.

2.2 Interconnections of Fliess operators

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_cF_d = F_{c ⧢ d}$, respectively (Fliess, 1981). When Fliess operators F_c and F_d with $c \in \mathbb{R}^\ell_{LC}\langle\langle X'\rangle\rangle$ and $d \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, where $|X'| = m + 1$, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X'^*} (c, \eta)\, \psi_d(\eta)(\mathbf{1}) \tag{2}$$

(Ferfera, 1980). Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X'\rangle\rangle$ to the set of vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, End$(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x'_i\eta) = \psi_d(x'_i) \circ \psi_d(\eta)$ with $\psi_d(x'_i)(e) = x_0(d_i ⧢ e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$.

When two Fliess operators F_c and F_d are interconnected to form a feedback system with F_c in the forward path and F_d in the feedback path, the generating series of the closed-loop system is denoted by the *feedback product* $c@d$. It can be computed explicitly using the Hopf algebra of coordinate functions associated with the underlying *output feedback group* (Gray, et al., 2014). Define the set of *unital* Chen-Fliess series $\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}^m\langle\langle X\rangle\rangle\}$, where I denotes the identity map. It is convenient to introduce the symbol δ as the (fictitious) generating series for the identity map. That is, $F_\delta := I$ such that $I + F_c := F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}^m\langle\langle X_\delta\rangle\rangle$. The central idea is that $(\mathscr{F}_\delta, \circ, I)$ forms a group under the composition

$$F_{c_\delta} \circ F_{d_\delta} = (I + F_c) \circ (I + F_d) = F_{c_\delta \circ d_\delta},$$

where $c_\delta \circ d_\delta := \delta + c \circledcirc d$, $c \circledcirc d := d + c\,\tilde{\circ}\, d_\delta$, and $\tilde{\circ}$ denotes the *mixed* composition product (Gray & Li, 2005). The *mixed* composition product in general can be defined as

$$c\,\tilde{\circ}\, d_\delta = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(\mathbf{1}),$$

where $c \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle, d_\delta \in \mathbb{R}^m\langle\langle X_\delta\rangle\rangle$ with $|X'| = m+1$ and ϕ_d is analogous to ψ_d in (2) except here $\phi_d(x_i)(e) = x_ie + x_0(d_i ⧢ e)$ with $d_0 := 0$. Equivalently, $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ forms a group. The following theorem states that the mixed composition can be viewed as a right group action of $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ on $\mathbb{R}^\ell\langle\langle X'\rangle\rangle$.

Theorem 2. (Gray & Duffaut Espinosa, 2013) Given $c \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle$ and $d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then $(c\,\tilde{\circ}\, d_\delta)\,\tilde{\circ}\, e_\delta = c\,\tilde{\circ}\,(d_\delta \circ e_\delta)$.

The next lemma states that the mixed composition product distributes on the left over the shuffle product.

Lemma 3. (Gray & Li, 2005) If $c, d \in \mathbb{R}^\ell\langle\langle X'\rangle\rangle$ with $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$ such that $|X'| = m+1$, then

$$(c \sqcup\!\sqcup\, d)\,\tilde{\circ}\, e_\delta = (c\,\tilde{\circ}\, e_\delta) \sqcup\!\sqcup\, (d\,\tilde{\circ}\, e_\delta).$$

For the group of unital Chen-Fliess series, the coordinate maps for the corresponding Hopf algebra H have the form

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}^\ell : c \mapsto (c, \eta),$$

where $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, $\eta \in X^*$. The commutative product is taken to be

$$\boldsymbol{m} : a_\eta \otimes a_\xi \mapsto a_\eta a_\xi,$$

where the unit $\mathbf{1}_\delta$ is defined to map every c to $\mathbb{1} = [11\cdots 1] \in \mathbb{R}^\ell$. The product of vectors used in the above definition is the Hadamard product. If the *degree* of a_η is defined as $\deg(a_\eta) = 2\,|\eta|_{x_0} + |\eta|_{x_1} + 1$, then H is graded and connected with $H = \bigoplus_{k\geq 0} H_k$, where H_k is the set of all elements of degree k and $H_0 = \mathbb{R}\mathbf{1}_\delta$ (Foissy, 2015). The coproduct Δ is defined so that the formal power series product $c \circledcirc d$ for the group $\mathbb{R}^\ell\langle\langle X_\delta\rangle\rangle$ satisfies

$$\Delta a_\eta(c, d) = a_\eta(c \circledcirc d) = (c \circledcirc d, \eta).$$

Of primary importance is the following lemma which describes how the group inverse $c_\delta^{\circ -1} := \delta + c^{\circ -1}$ is computed.

Lemma 4. (Gray, et al., 2014) The Hopf algebra $(H, \boldsymbol{m}, \Delta)$ has an antipode S satisfying $a_\eta(c^{\circ -1}) = (Sa_\eta)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}\langle\langle X\rangle\rangle$.

With this concept, the generating series for the feedback connection, $c@d$, can be computed explicitly.

Theorem 5. (Gray, et al., 2014) For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X'\rangle\rangle$ where $|X| = m+1$ and $|X'| = \ell + 1$, it follows that $c@d = c\,\tilde{\circ}\,(-d \circ c)_\delta^{\circ -1}$.

2.3 Shuffle group

The following theorem describes the *shuffle group.*

Theorem 6. (Gray, et al., 2014b) The set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ is a group under the shuffle product. In particular, the shuffle inverse of any such series c is

$$c^{\sqcup\!\sqcup -1} = ((c, \emptyset)(\mathbf{1} - c'))^{\sqcup\!\sqcup -1} = (c, \emptyset)^{-1}(c')^{\sqcup\!\sqcup *},$$

where $c' := \mathbf{1} - c/(c, \emptyset)$ is proper and $(c')^{\sqcup\!\sqcup *} := \sum_{k\geq 0} (c')^{\sqcup\!\sqcup k}$.

More generally, if $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, then the shuffle inverse is defined componentwise by $(c^{\sqcup\!\sqcup -1})_i = c_i^{\sqcup\!\sqcup -1}$, where $i = 1, 2, \cdots \ell$. Hence, $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \sqcup\!\sqcup)$ also possesses a group structure.

Example 7. Let $c = \mathbf{1} - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$. Observe that, $c' = x_1$, and hence, $c^{\sqcup\!\sqcup -1} = x_1^{\sqcup\!\sqcup *} = \sum_{k\geq 0} k!\, x_1^k$.

3. WIENER-FLIESS CONNECTIONS

This section describes the cascade connection shown in Figure 1 of a Chen-Fliess series F_c with a proper series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and a formal function $f_d : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$ defined without loss of generality at $z = 0$. Such configurations are called *Wiener-Fliess* connections. The connection is

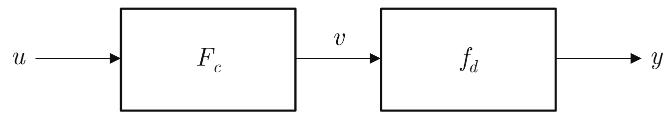

Fig. 1. Wiener-Fliess connection

known to generate another well defined Chen-Fliess series who generating series is given in the following theorem.

Theorem 8. (Gray & Thitsa, 2012) Let $X = \{x_0, x_1, \ldots, x_m\}$ and $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_\ell\}$. Given a Chen-Fliess series F_c with $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ and formal function $f_d : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$ with a generating series $d \in \mathbb{R}^k[[\tilde{X}]]$ at $z = 0$, that is,

$$f_d(z) = \sum_{\tilde{\eta}\in\tilde{X}^*} (d, \tilde{\eta}) z^{\tilde{\eta}},$$

the composition $f_d \circ F_c$ has a generating series in $\mathbb{R}^k\langle\langle X\rangle\rangle$ given by the Wiener-Fliess composition product

$$d\,\hat{\circ}\, c = \sum_{\tilde{\eta}\in\tilde{X}^*} (d, \tilde{\eta}) c^{\sqcup\!\sqcup\, \tilde{\eta}}, \tag{3}$$

where $c^{\sqcup\!\sqcup\, \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_j}} := c_{i_1} \sqcup\!\sqcup\, c_{i_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_{i_j}$.

The following theorem shows that the Wiener-Fliess composition product is left linear.

Theorem 9. If $d, e \in \mathbb{R}^k[[\tilde{X}]]$ and $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$, then $(d+e)\,\hat{\circ}\, c = (d\,\hat{\circ}\, c) + (e\,\hat{\circ}\, c)$.

Proof: Observe

$$\begin{aligned}(d+e)\,\hat{\circ}\, c &= \sum_{\tilde{\eta}\in\tilde{X}^*} (d+e, \tilde{\eta}) c^{\sqcup\!\sqcup\, \tilde{\eta}} \\ &= \sum_{\tilde{\eta}\in\tilde{X}^*} (d, \tilde{\eta}) c^{\sqcup\!\sqcup\, \tilde{\eta}} + \sum_{\tilde{\eta}\in\tilde{X}^*} (e, \tilde{\eta}) c^{\sqcup\!\sqcup\, \tilde{\eta}} \\ &= (d\,\hat{\circ}\, c) + (e\,\hat{\circ}\, c).\end{aligned}$$

■

The next lemma will be used to show that the Wiener-Fliess composition product has certain contractive properties.

Lemma 10. Let $X = \{x_0, x_1, \ldots, x_m\}$ and $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_\ell\}$. If $\eta \in \tilde{X}^+$ and $c, \tilde{c} \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ then

$$\kappa(\eta\,\hat{\circ}\, c, \eta\,\hat{\circ}\,\tilde{c}) \leq \max\{\kappa(c), \kappa(\tilde{c})\}^{(|\eta|-1)}\kappa(c, \tilde{c}).$$

Proof: The proof is by induction on the length of η. If $\eta = \tilde{x}_i \in \tilde{X}$ then

$$\begin{aligned}\kappa(\tilde{x}_i\,\hat{\circ}\, c, \tilde{x}_i\,\hat{\circ}\,\tilde{c}) &= \kappa(c^{\sqcup\!\sqcup\, \tilde{x}_i}, \tilde{c}^{\sqcup\!\sqcup\, \tilde{x}_i}) \\ &= \kappa(c_i, \tilde{c}_i) \\ &\leq \kappa(c, \tilde{c}).\end{aligned}$$

Hence, the base case is proved. Now assume the hypothesis is true for $|\eta| = k \geq 1$. Let $\hat{\eta} = \tilde{x}_j\eta$, where $\tilde{x}_j \in \tilde{X}$ and $\eta \in \tilde{X}^k$. Then

$$\begin{aligned}\kappa(\hat{\eta}\,\hat{\circ}\, c, \hat{\eta}\,\hat{\circ}\,\tilde{c}) &= \kappa(c^{\sqcup\!\sqcup\, \tilde{x}_j\eta}, \tilde{c}^{\sqcup\!\sqcup\, \tilde{x}_j\eta}) \\ &= \kappa(c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta}, \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}) \\ &= \kappa(c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta} - \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}) \\ &= \kappa((c_j \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\, \eta} - c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}) + \\ &\qquad (c_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta} - \tilde{c}_j \sqcup\!\sqcup\, \tilde{c}^{\sqcup\!\sqcup\, \eta}))\end{aligned}$$

$$\begin{aligned}&\leq \max\{\kappa(c_j \sqcup\!\sqcup c^{\sqcup\!\sqcup\,\eta} - c_j \sqcup\!\sqcup \tilde{c}^{\sqcup\!\sqcup\,\eta}),\\ &\quad \kappa(c_j \sqcup\!\sqcup \tilde{c}^{\sqcup\!\sqcup\,\eta} - \tilde{c}_j \sqcup\!\sqcup \tilde{c}^{\sqcup\!\sqcup\,\eta})\}\\ &= \max\{\kappa(c_j \sqcup\!\sqcup (c^{\sqcup\!\sqcup\,\eta} - \tilde{c}^{\sqcup\!\sqcup\,\eta})),\\ &\quad \kappa((c_j - \tilde{c}_j) \sqcup\!\sqcup \tilde{c}^{\sqcup\!\sqcup\,\eta})\}.\end{aligned}$$

By the triangle inequality and the induction hypothesis,

$$\begin{aligned}\kappa(\hat{\eta} \,\hat{\circ}\, c, \hat{\eta} \,\hat{\circ}\, \tilde{c}) &\leq \max\{\kappa(c) \, \max\{\kappa(c), \kappa(\tilde{c})\}^{(|\eta|-1)}\kappa(c,\tilde{c}),\\ &\quad \kappa(\tilde{c})^{|\eta|}\,\kappa(c,\tilde{c})\}\\ &\leq \max\{\kappa(c),\kappa(\tilde{c})\}^{|\eta|}\,\kappa(c,\tilde{c}),\end{aligned}$$

which proves the claim. ■

For a fixed $d \in \mathbb{R}^k[[\tilde{X}]]$ define the map $d\,\hat{\circ} : \mathbb{R}^\ell_p\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^k\langle\langle X\rangle\rangle : c \mapsto d\,\hat{\circ}\,c$. The following theorem describes the contractive properties of $d\,\hat{\circ}$.

Theorem 11. The map $d\,\hat{\circ}$ is a weak contraction map when $\overline{\omega}(d) = 1$ and a strong contraction map when $\overline{\omega}(d) > 1$.

Proof: Let $c, c' \in \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$. Observe,

$$\begin{aligned}\kappa(d\,\hat{\circ}\,(c), d\,\hat{\circ}\,(c')) &= \kappa(d\,\hat{\circ}\,c, d\,\hat{\circ}\,c')\\ &= \kappa\left(\sum_{\eta\in\tilde{X}^*}(d,\eta)(c^{\sqcup\!\sqcup\,\eta} - c'^{\sqcup\!\sqcup\,\eta})\right)\\ &\leq \sup_{\eta\in\tilde{X}^+}\kappa\big((d,\eta)(c^{\sqcup\!\sqcup\,\eta} - c'^{\sqcup\!\sqcup\,\eta})\big)\\ &= \sup_{k\geq\overline{\omega}(d)}\ \sup_{\eta\in\tilde{X}^k}\kappa(c^{\sqcup\!\sqcup\,\eta}, c'^{\sqcup\!\sqcup\,\eta})).\end{aligned}$$

Applying Lemma 10 gives

$$\begin{aligned}\kappa(d\,\hat{\circ}\,(c), d\,\hat{\circ}\,(c')) &\leq \sup_{k\geq\overline{\omega}(d)}\max\{\kappa(c),\kappa(c')\}^{k-1}\kappa(c,c')\\ &\leq \max\{\kappa(c),\kappa(c')\}^{\overline{\omega}(d)-1}\kappa(c,c').\end{aligned}$$

■

The final theorem of this section states a mixed associativity property involving the mixed composition product and the Wiener-Fliess composition product. This identity plays a key role in determining the generating series of the static feedback connection in Section 5.

Theorem 12. If $d \in \mathbb{R}^k[[\tilde{X}]]$, $c \in \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$ and $e \in \mathbb{R}^m\langle\langle X'\rangle\rangle$ such that $|\tilde{X}| = \ell$ and $|X| = m+1$, then $d\,\hat{\circ}\,(c\,\tilde{\circ}\,e_\delta) = (d\,\hat{\circ}\,c)\,\tilde{\circ}\,e_\delta$.

Proof: The proof is obtained directly from the definition of the Wiener-Fliess composition product in Theorem 8 and a simple generalization of the identity in Lemma 3. ■

4. HOPF ALGEBRA OF THE SHUFFLE GROUP

The goal of this section is to describe the Hopf algebra of the shuffle group as defined in Theorem 6. It is utilized subsequently to develop an algorithm to compute the Wiener-Fliess composition product.

Define the set of formal power series

$$M = \{\, \mathbb{1} + d \,:\, d \in \mathbb{R}^n_p\langle\langle X\rangle\rangle\},$$

where $\mathbb{1} = [1\cdots 1\,1]^T\,\emptyset$. In light of Theorem 6, $(M, \sqcup\!\sqcup)$ forms an Abelian group. Let the set of all maps from M to $\mathbb{R}^n$ be denoted as $\mathrm{Hom}_{\mathbf{set}}(M,\mathbb{R}^n)$. The subset $H \subset \mathrm{Hom}_{\mathbf{set}}(M,\mathbb{R}^n)$ of coordinate maps defined on the group M is

$$H = \{a_\eta \,:\, a_\eta(c) = (c,\eta) \,:\, \eta \in X^*\}.$$

H has an $\mathbb{R}$-algebra structure with addition, scalar multiplication and product defined, respectively, as

$$\begin{aligned}(a_\eta + a_\zeta)(c) &= a_\eta(c) + a_\zeta(c)\\ (ka_\eta)(c) &= k(a_\eta(c))\\ \boldsymbol{m}(a_\eta, a_\zeta)(c) &= a_\eta(c)\odot a_\zeta(c),\end{aligned}$$

where $\eta,\zeta \in X^*$, $k\in\mathbb{R}$, and $\odot$ denotes the Hadamard product on $\mathbb{R}^n$. The unit for the product is given by $a_\emptyset$ with $a_\emptyset(c) = \mathbb{1}$, $\forall c \in M$. Define the coproduct $\Delta : H \longrightarrow H \bigotimes H$ as $\Delta a_\eta(c,d) = a_\eta(c \sqcup\!\sqcup d)$, where $c, d \in M$ and $\eta \in X^*$. The counit map ϵ is defined as

$$\epsilon(a_\eta) = \begin{cases}1 \,:\, \eta = \emptyset\\ 0 \,:\, \text{otherwise.}\end{cases}$$

It is simple to check that $(H, \boldsymbol{m}, a_\emptyset, \Delta, \epsilon)$ forms a commutative and cocommutative bialgebra. The bialgebra is graded based on word length, that is, $H = \bigoplus_{k\in\mathbb{N}_0} H_k$ with $a_\eta \in H_k$ if and only if $|\eta| = k$. Since $\mathbb{R} \cong H_0$ in the category of algebras with ϵ acting as the isomorphism, H is a connected and graded bialgebra. The reduced coproduct Δ' is defined as $\Delta'(a_\eta) = \Delta(a_\eta) - a_\eta \otimes a_\emptyset - a_\emptyset \otimes a_\eta$ if $\eta \neq \emptyset$. For the case of the empty word, $\Delta'(a_\emptyset) = 0$. If $c, d \in \mathbb{R}^n_p\langle\langle X\rangle\rangle$, then their corresponding elements in the shuffle group M are $\mathbb{1} + c$ and $\mathbb{1} + d$, respectively. The shuffle product of two proper series is computed by the reduced coproduct of the corresponding elements in the shuffle group M. For all proper series $c, d \in \mathbb{R}^n_p\langle\langle X\rangle\rangle$ and $\eta \in X^*$, it follows that $(c \sqcup\!\sqcup d, \eta) = \Delta'(a_\eta)(\mathbb{1} + c, \mathbb{1} + d)$. The antipode map $S : H \longrightarrow H$ is given by $S(a_\eta)(c) = a_\eta(c^{\sqcup\!\sqcup\,-1})$. Since, the Hopf algebra is graded and connected the antipode can be computed for any $a_\eta \in H^+$ (where $H^+ := \bigoplus_{k\geq 1} H_k$) as

$$S(a_\eta) = -a_\eta - \sum a'_{(1)} \odot S(a'_{(2)})$$

(Figueroa & Gracia-Bondía, 2005), where the summation is taken over all components of the reduced coproduct $\Delta'(a_\eta)$ written in the Sweedler notation (Abe, 2004; Sweedler, 1969). Therefore, the tuple $(H, \boldsymbol{m}, a_\emptyset, \Delta, \epsilon, S)$ forms a commutative, cocommutative, connected and graded Hopf algebra.

Example 13. Reconsider Example 1, where $c = \mathbf{1} - x_1 \in \mathbb{R}\langle\langle X\rangle\rangle$ so that $c^{\sqcup\!\sqcup\,-1} = \sum_{k\geq 0} k!\,x_1^k$. The goal is to determine $\big(c^{\sqcup\!\sqcup\,-1}, x_1^2\big)$ directly without computing the entire shuffle inverse. Observe

$$a_{x_1^2}(c^{\sqcup\!\sqcup\,-1}) = S(a_{x_1^2})(c),$$

and the reduced coproduct of $a_{x_1^2}$ is

$$\Delta'(a_{x_1^2}) = 2(a_{x_1} \otimes a_{x_1}).$$

Since $\Delta'(a_{x_1}) = 0$, it follows that

$$S(a_{x_1}) = -a_{x_1}.$$

Hence,

$$\begin{aligned}S(a_{x_1^2})(c) &= -a_{x_1^2}(c) - 2\big(a_{x_1}(c)(-a_{x_1}(c))\big)\\ &= 0 - 2(-1(1)) = 2.\end{aligned}$$

Therefore, $(c^{\sqcup\!\sqcup\,-1}, x_1^2) = 2$, as expected.

An inductive algorithm is presented next to compute the coproduct Δ on H. A key feature of the algorithm is a recursively defined *partition* map $\mu : X^* \longrightarrow X^* \otimes X^*$, where $x_j\eta \mapsto (x_j \otimes \emptyset + \emptyset \otimes x_j)\mu(\eta)$ with $\eta \in X^*$, $x_j \in X$, and $\mu(\emptyset) := (\emptyset \otimes \emptyset)$. The definition of the map μ is exactly dual to the definition of the *deshuffle* coproduct $\Delta_{\sqcup\!\sqcup}$ described in (Foissy, 2015). The deshuffle coproduct is described on the coordinate maps a_η for all $\eta \in X^*$ and involves the splitting of the coordinate maps. However, from an algorithmic perspective, it is more natural to split the underlying words as described in the following algorithm.

Algorithm 14. For all $\eta \in X^*$ and $c, d \in M$, the coproduct $\Delta a_\eta(c,d)$ can be computed as:

(1) $\mu(\eta) = \sum \eta_{(1)} \otimes \eta_{(2)}$.
(2) $\Delta a_\eta(c,d) = \sum a_{\eta_{(1)}}(c) \odot a_{\eta_{(2)}}(d)$.

This algorithm can be trivially extended to compute the reduced coproduct.

Algorithm 15. For all $\eta \in X^*$ and $c, d \in M$, the reduced coproduct $\Delta' a_\eta(c,d)$ can be computed as:

(1) If $\eta = \emptyset$, then $\Delta' a_\eta(c,d) = 0$.
(2) Else, $\Delta' a_\eta(c,d) = \Delta a_\eta(c,d) - a_\eta(c) \odot \mathbb{1} - \mathbb{1} \odot a_\eta(d)$.

Let Φ_c be an $\mathbb{R}$-linear homomorphism of algebras defined as $\Phi_c : H \longrightarrow \mathbb{R}^n : a_\eta \mapsto a_\eta(c)$, where $\mathbb{R}^n$ is an $\mathbb{R}$-algebra under the Hadamard product. The maps Φ_c are usually called the *characters* of the Hopf algebra H and form a group under the Hopf convolution product $\star$ defined as

$$\begin{aligned}(\Phi_c \star \Phi_d)(a_\eta) &= \boldsymbol{m} \circ (\Phi_c \otimes \Phi_d) \circ \Delta(a_\eta)\\ &= \sum \Phi_c(a_{\eta_{(1)}}) \odot \Phi_d(a_{\eta_{(2)}})\\ &= \sum a_{\eta_{(1)}}(c) a_{\eta_{(2)}}(d)\\ &= \Delta a_\eta(c,d) = (c \sqcup\!\sqcup d, \eta).\end{aligned}$$

Hence, alternatively, the coproduct can be realized as the Hopf convolution product of the characters of the Hopf algebra H. The group inverse for any character Φ_c is defined as $\Phi_c^{\star -1} = \Phi_{c^{\sqcup\!\sqcup -1}} = \Phi_c \circ S$. It is not hard to see that the group of characters of the Hopf algebra H and the shuffle group M are isomorphic.

Example 16. Suppose $X = \{x_1, x_2\}$. Let $c = 1 - x_1$ and $d = 1 + x_1x_2$. Their shuffle product is computed directly as $c \sqcup\!\sqcup d = 1 - x_1 + x_1x_2 - 2x_1^2x_2 - x_1x_2x_1$. The objective is to find only $(c \sqcup\!\sqcup d, x_1x_2x_1) = \Delta a_{x_1x_2x_1}(c,d)$ using Algorithm 14:

(1) Apply the map μ to compute the partition of the word $x_1x_2x_1$:

$$\begin{aligned}\mu(x_1x_2x_1) &= \mu(x_1)\mu(x_2)\mu(x_1)\\ &= (x_1 \otimes \emptyset + \emptyset \otimes x_1)(x_2 \otimes \emptyset + \emptyset \otimes x_2)\\ &\quad (x_1 \otimes \emptyset + \emptyset \otimes x_1)\\ &= x_1x_2x_1 \otimes \emptyset + x_1x_2 \otimes x_1 + x_1^2 \otimes x_2 +\\ &\quad x_1 \otimes x_2x_1 + x_2x_1 \otimes x_1 + x_2 \otimes x_1^2 +\\ &\quad x_1 \otimes x_1x_2 + \emptyset \otimes x_1x_2x_1.\end{aligned}$$

(2) Compute the coproduct:

$$\begin{aligned}\Delta a_{x_1x_2x_1}(c,d) &= (c, x_1x_2x_1)(d, \emptyset) + (c, x_1x_2)(d, x_1) +\\ &\quad (c, x_1^2)(d, x_2) + (c, x_1)(d, x_2x_1) +\\ &\quad (c, x_2x_1)(d, x_1) + (c, x_2)(d, x_1^2) +\\ &\quad (c, x_1)(d, x_1x_2) + (c, \emptyset)(d, x_1x_2x_1)\\ &= (0)(1) + (0)(0) + (0)(0) + (-1)(0) +\\ &\quad (0)(0) + (0)(0) + (-1)(1) + (1)(0)\\ &= -1.\end{aligned}$$

Therefore, $(c \sqcup\!\sqcup d, x_1x_2x_1) = -1$ as computed from the direct shuffle product calculation.

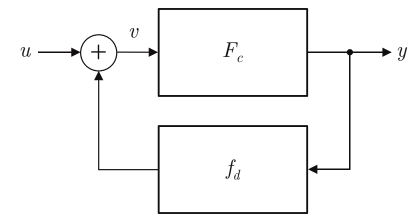

Fig. 2. Fliess operator F_c with static output feedback f_d.

A key observation is that Algorithm 15 can be utilized to compute the Wiener-Fliess composition product (3). Specifically, if $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$, $d \in \mathbb{R}^k[[\tilde{X}]]$, $\tilde{\eta} = \tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_k} \in \tilde{X}^*$, and $\zeta \in X^*$, then $(c^{\sqcup\!\sqcup \tilde{\eta}}, \zeta)$ can be computed as

$$(c^{\sqcup\!\sqcup \tilde{\eta}}, \zeta) = (\Delta'^{\circ(k-1)} a_\zeta)(c'_{i_1}, c'_{i_2}, \ldots, c'_{i_k}),$$

where $c' = \mathbb{1} + c \in M$ and $(\Delta'^{\circ(k-1)} a_\zeta)$ denotes the composition of the reduced coproduct map with itself $k-1$ times and then applied to the coordinate map a_ζ. Computationally, this boils down to splitting the word ζ into k subwords, say $\zeta = \alpha_1\alpha_2\cdots\alpha_k$ where $\alpha_i \in X^*$, and then finding the Hadamard product of the coefficients corresponding to each subword with respect to the series in the argument. That is,

$$(c^{\sqcup\!\sqcup \tilde{\eta}}, \zeta) = (c_{i_1}, \alpha_1) \odot (c_{i_2}, \alpha_2) \odot \cdots \odot (c_{i_k}, \alpha_k).$$

5. CHEN-FLIESS SERIES UNDER STATIC OUTPUT FEEDBACK

Let F_c be a Chen-Fliess series with a proper generating series $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$. Assume it is interconnected with a static formal map f_d with generating series $d \in \mathbb{R}^m[[\tilde{X}]]$ in the additive output feedback configuration shown in Figure 2. (Assume $|X| = m+1$ and $|\tilde{X}| = \ell$.) The first objective of this section is to show that the closed-loop system also has a Chen-Fliess series representation, say $y = F_e[u]$, where $e \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$. If this is the case, then necessarily

$$\begin{aligned}F_e[u] &= y\\ &= F_c[u + f_d(y)]\\ &= F_c[u + f_d \circ F_e[u]]\\ &= F_{c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta}[u]\end{aligned}$$

for any admissible u. Therefore, the series e has to satisfy the fixed point equation

$$e = c\,\tilde{\circ}\,(d\,\hat{\circ}\,e)_\delta. \tag{4}$$

In addition, e must be a proper series for the Wiener-Fliess composition $d\,\hat{\circ}\,e$ to be well defined. It follows directly from the definition of the mixed composition product that if $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ then $c\,\tilde{\circ}\,w_\delta$ is also a proper series for all $w \in \mathbb{R}^m\langle\langle X\rangle\rangle$. The following lemma will be used to show that (4) always has a unique fixed point.

Lemma 17. If $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$, then the map $Q_{c,d} : \mathbb{R}_p^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}_p^\ell\langle\langle X\rangle\rangle : e \mapsto c \,\tilde{\circ}\, (d \,\hat{\circ}\, e)_\delta$ is a strong contraction map in the ultrametric topology on the space $\mathbb{R}_p^\ell\langle\langle X\rangle\rangle$.

Proof: First observe that $\kappa(h_\delta) = \kappa(h)$, $\forall h \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$. Now define two maps, $d_{\,\hat{\circ}\,,\delta} : e \mapsto (d \,\hat{\circ}\, e)_\delta$ and $c_{\,\tilde{\circ}} : f \mapsto c \,\tilde{\circ}\, f_\delta$, where $f \in \mathbb{R}^m\langle\langle X\rangle\rangle$. Note that $Q_{c,d}(e) = (c_{\,\tilde{\circ}} \circ d_{\,\hat{\circ}\,,\delta})(e)$. It is known that $c_{\,\tilde{\circ}}$ is a strong contraction map in the ultrametric topology (Gray & Li, 2005). So, it only needs to be shown that $d_{\,\hat{\circ}\,,\delta}$ is at least a non-expansive map.

Consider first the case where $\overline{\omega}(d) = 1$. By Theorem 11, $\kappa(d_{\,\hat{\circ}\,,\delta}(e)) \leq \kappa(e)$. Therefore, $d_{\,\hat{\circ}\,,\delta}$ is a weak contraction map.

Consider next the case where $\overline{\omega}(d) > 1$. Since $e \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$, $\mathrm{ord}(e) \geq 1$. Therefore, $\kappa(e) \leq \sigma$ with $\sigma \in]0,1[$. By Theorem 11, $\kappa(d_{\,\hat{\circ}\,,\delta}(e)) \leq \sigma\kappa(e)$. Hence, $d_{\,\hat{\circ}\,,\delta}$ is a strong contraction map. ■

The following fixed point theorem establishes the first main result of the section, which follows subsequently.

Theorem 18. The series $c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1} \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ is a unique fixed point of the map $Q_{c,d}$.

Proof: If $e := c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1}$, then

$$\begin{aligned} Q_{c,d}(e) &= c \,\tilde{\circ}\, (d \,\hat{\circ}\, e)_\delta \\ &= c \,\tilde{\circ}\, (d \,\hat{\circ}\, (c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1}))_\delta. \end{aligned}$$

Applying Theorem 12 yields

$$\begin{aligned} Q_{c,d}(e) &= c \,\tilde{\circ}\, ((d \,\hat{\circ}\, c) \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1})_\delta \\ &= c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1} = e. \end{aligned}$$

■

Theorem 19. Given a proper series $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m[[\tilde{X}]]$, the generating series for the closed-loop system in Figure 2 is the *Wiener-Fliess feedback product* $c\hat{@}d := c \,\tilde{\circ}\, (-d \,\hat{\circ}\, c)_\delta^{-1}$.

The computation of $(-d \,\hat{\circ}\, c)$ can be performed via the coproduct of the Hopf algebra of the shuffle group as described in Section 4. The group inverse $(-d \hat{\circ}\, c)_\delta^{-1}$ can be computed via the antipode of the Faà di Bruno type Hopf algebra corresponding to the group $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. (A particularly efficient algorithm appears in Ebrahimi-Fard & Gray (2017).) Hence, the calculation of the generating series for the static feedback case is an interplay between these two very distinct Hopf algebras.

The notion that feedback can described mathematically as a transformation group acting on the plant is well established in control theory. The following theorem describes the situation in the present context.

Theorem 20. The Wiener-Fliess feedback product is a right group action by the additive group $\left(\mathbb{R}^m[[\tilde{X}]], +, 0\right)$ on the set $\mathbb{R}_p^\ell\langle\langle X\rangle\rangle$, where $|X| = m+1$ and $|\tilde{X}| = \ell$.

Proof: Let $d_1, d_2 \in \mathbb{R}^m[[\tilde{X}]]$ and $c \in \mathbb{R}_p^\ell\langle\langle X\rangle\rangle$. It needs to be proven that

$$(c\hat{@}d_1)\hat{@}d_2 = c\hat{@}(d_1 + d_2).$$

From Theorem 19 observe that

$$\begin{aligned} (c\hat{@}d_1)\hat{@}d_2 &= (c\hat{@}d_1) \,\tilde{\circ}\, (-d_2 \,\hat{\circ}\, (c\hat{@}d_1))_\delta^{-1} \\ &= (c \,\tilde{\circ}\, (-d_1 \,\hat{\circ}\, c)_\delta^{-1}) \,\tilde{\circ}\, (-d_2 \,\hat{\circ}\, (c\hat{@}d_1))_\delta^{-1}. \end{aligned}$$

Applying Theorem 2 and then Theorem 19 gives

$$\begin{aligned} &(c\hat{@}d_1)\hat{@}d_2 \\ &\quad= c \,\tilde{\circ}\, \left[(-d_1 \,\hat{\circ}\, c)_\delta^{-1} \circ (-d_2 \,\hat{\circ}\, (c\hat{@}d_1))_\delta^{-1}\right] \\ &\quad= c \,\tilde{\circ}\, \left[(-d_2 \,\hat{\circ}\, (c\hat{@}d_1))_\delta \circ (-d_1 \,\hat{\circ}\, c)_\delta\right]^{-1} \\ &\quad= c \,\tilde{\circ}\, \left[(-d_2 \,\hat{\circ}\, (c \,\tilde{\circ}\, (-d_1 \,\hat{\circ}\, c)_\delta^{-1}))_\delta \circ (-d_1 \,\hat{\circ}\, c)_\delta\right]^{-1}. \end{aligned}$$

In light of Theorem 12,

$$\begin{aligned} &(c\hat{@}d_1)\hat{@}d_2 \\ &\quad= c \,\tilde{\circ}\, \left[\left((-d_2 \,\hat{\circ}\, c) \,\tilde{\circ}\, (-d_1 \,\hat{\circ}\, c)_\delta^{-1}\right)_\delta \circ \left(-d_1 \,\hat{\circ}\, c\right)_\delta\right]^{-1}. \end{aligned}$$

Expanding the group product of $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, it follows that

$$\begin{aligned} &(c\hat{@}d_1)\hat{@}d_2 \\ &\quad= c \,\tilde{\circ}\, \Bigg[\Bigg(\Big(((-d_2 \,\hat{\circ}\, c) \,\tilde{\circ}\, (-d_1 \,\hat{\circ}\, c)_\delta^{-1}) \,\tilde{\circ}\, \left(-d_1 \,\hat{\circ}\, c\right)_\delta\Big) \\ &\qquad + \left(-d_1 \,\hat{\circ}\, c\right)\Bigg)_\delta\Bigg]^{-1}. \end{aligned}$$

Finally, from Theorem 2,

$$(c\hat{@}d_1)\hat{@}d_2 = c \,\tilde{\circ}\, [-d_1 \,\hat{\circ}\, c + (-d_2 \,\hat{\circ}\, c)]_\delta^{-1},$$

so that via the left linearity of Wiener-Fliess composition,

$$(c\hat{@}d_1)\hat{@}d_2 = c \,\tilde{\circ}\, [-(d_1 + d_2) \,\hat{\circ}\, c]_\delta^{-1} = c\hat{@}(d_1 + d_2).$$

■

It is worth noting that for dynamic feedback the transformation group is $(\mathbb{R}^m\langle\langle X\rangle\rangle, +, 0)$, while here it is $(\mathbb{R}^m[[\tilde{X}]], +, 0)$ which plays this role.

The final theorem states that the relative degree of a SISO nonlinear input-output system written in terms of a Chen-Fliess series is invariant under static output feedback. The theorem uses the notation of relative degree of a generating series, namely, that c has relative degree r if and only if there exists some proper $e \in \mathbb{R}\langle\langle X\rangle\rangle$ with $x_1 \notin \mathrm{supp}(e)$ such that

$$c = c_N + Kx_0^{r-1}x_1 + x_0^{r-1}e,$$

where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $K \neq 0$ (Gray, et al., 2014b; Gray & Venkatesh, 2019). This claim is independent of the well known analogous result for systems having a state space realization with relative degree (Isidori, 1995).

Theorem 21. Let $X = \{x_0, x_1\}$ and $c \in \mathbb{R}_p\langle\langle X\rangle\rangle$ have relative degree r. If $d \in \mathbb{R}[[\tilde{x}_1]]$, then $c\hat{@}d$ has relative degree r.

Proof: The proof follows from the formula in Theorem 19 and the relative degree properties summarized in Table 2 of Gray & Venkatesh (2019). ■

Example 22. Consider a normalized forced pendulum equation

$$\ddot{\theta} + \sin\theta = u$$

with input u, angular displacement θ, and output $y = \theta$. Under the feedback law $u = v + \sin\theta$, the system is transformed into a double integrator $\ddot{\theta} = v$. For example, with $\theta(0) = 0$ and $\dot{\theta}(0) = 1$, the closed-loop system is described by

$$y(t) = t + \int_0^t \int_0^{\tau_2} v(\tau_1)\, d\tau_1\, d\tau_2,$$

or equivalently, $y = F_{c\hat{@}d}[v]$ with $c\hat{@}d = x_0 + x_0x_1$. Clearly the series has relative degree two.

The same result can be established via Theorem 19. The following computations were done via Mathematica. It is easily checked that the open-loop system $y = F_c[u]$ has the generating series

$$c = x_0 + x_0x_1 - x_0^3 - x_0^3x_1 + 2x_0^5 + 4x_0^5x_1 + 2x_0^4x_1x_0 + x_0^3x_1x_0^2 + \cdots$$

and has relative degree 2 as expected. The sinusoidal static output feedback map has generating series $d \in \mathbb{R}[[\tilde{x}_1]]$ given by

$$d = \tilde{x}_1 - \frac{1}{3!}\tilde{x}_1^3 + \frac{1}{5!}\tilde{x}_1^5 - \frac{1}{7!}\tilde{x}_1^7 + \cdots$$

Using the computational methods described above and computing the composition antipode for words up to length four, it is found that

$$c\hat{@}d \approx x_0 + x_0x_1 + \mathcal{O}(x_0^6).$$

The terms $\mathcal{O}(x_0^6)$ are the error terms due to the need to truncate all the underlying series at each step of the calculation in the Wiener-Fliess feedback product formula. The order of these error terms can be increased but at a significant computational cost.

6. CONCLUSIONS AND FUTURE WORK

It was shown that the generating series of a closed-loop system consisting of a Chen-Fliess series representation of the plant and formal static output feedback always has Chen-Fliess series representation. To explicitly compute the closed-loop generating series, two Hopf algebras are needed, the existing output feedback Hopf algebra used to describe dynamic output feedback, and the Hopf algebra of the shuffle group. It was then shown that the resulting feedback formula for the closed-loop system has a natural interpretation as a transformation group acting on the plant. It also preserves the relative degree of the plant. Future work will include addressing issues related to local convergence of the Wiener-Fliess feedback product and identifying static feedback invariants using formal power series methods.

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