Analysis of a Non-Equilibrium Vortex Pair as Aircraft Trailing Vortices

Manuel Ayala
Old Dominion University, mayal005@odu.edu

Follow this and additional works at: https://digitalcommons.odu.edu/mae_etds

Part of the Aerodynamics and Fluid Mechanics Commons, and the Mechanical Engineering Commons

Recommended Citation
Ayala, Manuel. "Analysis of a Non-Equilibrium Vortex Pair as Aircraft Trailing Vortices" (2021). Doctor of Philosophy (PhD), Thesis, Mechanical & Aerospace Engineering, Old Dominion University, DOI: 10.25777/b9f8-qx34
https://digitalcommons.odu.edu/mae_etds/335

This Thesis is brought to you for free and open access by the Mechanical & Aerospace Engineering at ODU Digital Commons. It has been accepted for inclusion in Mechanical & Aerospace Engineering Theses & Dissertations by an authorized administrator of ODU Digital Commons. For more information, please contact digitalcommons@odu.edu.
ANALYSIS OF A NON-EQUILIBRIUM VORTEX PAIR AS AIRCRAFT TRAILING VORTICES

by

Manuel Ayala
B.S. August 2019, Universidad de Oriente, Barcelona, Venezuela

A Thesis Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

AEROSPACE ENGINEERING

OLD DOMINION UNIVERSITY
August 2021

Approved by:

Robert L. Ash (Director)
Colin P. Britcher (Member)
Shizhi Qian (Member)
Shortly after the roll-up evolution of the vortex sheet behind the wings of an aircraft, a coherent counter-rotating vortex pair emerges. Presence of this vortex pair in the downstream of an aircraft, creates unsafe conditions for other aircraft, especially near airport runways. Fundamental knowledge of the physics that govern the formation, duration and dissipation of aircraft wake vortices is desirable in order to improve aircraft operational safety. This study uses non-equilibrium pressure theory to develop an accurate model describing the physical behavior of the vortex pair created by an aircraft in the early to mid-field vortex regime. An isolated aircraft vortex is first considered, modeled and compared using several vortex models found in the literature. Additionally, the non-equilibrium model for an isolated vortex is compared with existing wind tunnel data. Eddy viscosity to kinematic viscosity ratio correlation for aircraft trailing vortices has been introduced to satisfy the turbulent energy embedded in the vortex cores. Subsequently, the counter-rotating vortex pair is considered, and detailed derivation of the non-equilibrium vortex pair model is introduced. The model is based on a two-dimensional steady state vortex pair in an unbounded atmosphere. Existence of a vortex pair with non-equilibrium cores embedded in an inviscid fluid medium is discussed. Vortex pairs are characterized by an accompanying isolating “atmosphere”, commonly known as “Kelvin oval”. The non-equilibrium vortex pair model predicts a complete departure from the potential oval size when the pair are near merger.
This thesis is dedicated to my whole family:
My wife for always being a pillar of emotional support and happiness in my life; my parents for
giving me all the tools and advice needed to succeed; my grandparents, uncles, aunts, brothers
and sister for simply being in my life and making it great.
ACKNOWLEDGMENTS

I owe tremendous gratitude towards my advisor, Dr. Robert L. Ash, who provided me with extensive knowledge and encouraged me to take upon a project very dear to him. Thank you for showing me the way towards analytical research. Special thanks go towards Dr. Colin Britcher for serving as committee members and for pointing out some interesting suggestions in this thesis. Also, Dr. Brett Newman for attending the defense and providing his support. I also would like to extend my appreciation towards Dr. Shizi Qian, who served as a committee member.
## NOMENCLATURE

- $x,y$: cartesian coordinates
- $a$: separation between vortex center and midplane
- $B$: separation between vortices
- $k$: vortex strength
- $U$: downward induced vortex pair velocity
- $L$: half-length of oval
- $W$: half-width of oval
- $\bar{L}$: non-dimensional half-length of oval
- $\bar{W}$: non-dimensional half-width of oval
- $\Gamma_0$: root circulation
- $Re_r$: circulation-based Reynolds number
- $r_{core}$: vortex radius
- $\bar{r}$: non-dimensional vortex radius
- $r,\theta,z$: cylindrical coordinates
- $\mathbf{v}$: velocity vector
- $v_\theta$: azimuthal component of velocity
- $v_{\theta,max}$: maximum azimuthal velocity
- $t$: time
- $\rho$: fluid density
- $\nu$: kinematic viscosity
\( \nu_{turb} \)  turbulent kinematic viscosity

\( \mu \)  dynamic viscosity

\( u, v \)  cartesian component velocities

\( P \)  pressure

\( W_a \)  weight of aircraft

\( V \)  aircraft speed

\( b \)  aircraft wingspan

\( U_{\infty} \)  freestream velocity

\( \eta_p \)  pressure relaxation coefficient

\( \eta_v \)  volume viscosity

\( \Omega \)  potential energy

\( \psi \)  velocity potential
# TABLE OF CONTENTS

## Chapter

1. INTRODUCTION ........................................................................................................................... 1
   1.1 Vortex Pair in a Mathematical/Analytical Point of View ....................................................... 2
   1.2 Vortex Pairs as Aircraft Trailing Vortices ............................................................................... 9
   1.3 Vortex Models used for Aircraft Trailing Vortices ............................................................... 12
       1.3.1 Inviscid Vortex Models ................................................................................................. 13
       1.3.2 Viscous Vortex Models ................................................................................................. 13
       1.3.3 Viscous Turbulent Vortex Models ................................................................................ 17

2. NON-EQUILIBRIUM PRESSURE THEORY .................................................................................... 19
   2.1 Non-Equilibrium Laminar Axial Vortex ................................................................................ 20
   2.2 Non-Equilibrium Turbulent Axial Vortex ............................................................................. 24

3. NON-EQUILIBRIUM AIRCRAFT WAKE VORTICES ....................................................................... 27
   3.1 Existence of Non-Equilibrium Vortex Pair in an Unbounded Fluid Region ......................... 27
   3.2 Estimation of local Pressure Relaxation Coefficients employing Aircraft Flight Data ...... 29
   3.4 Comparison of Aircraft Vortex Models .................................................................................... 31
   3.5 Non-Equilibrium Azimuthal Velocity Field of an Aircraft Wake Vortex Pair ....................... 39
   3.6 Non-Equilibrium “Oval” of an Aircraft Wake Vortex Pair ................................................... 47

4. CONCLUSIONS ........................................................................................................................... 57
   Summary ................................................................................................................................... 57
   Future work ............................................................................................................................... 58

REFERENCES .................................................................................................................................. 59

APPENDIX ...................................................................................................................................... 65
   MATLAB Scripts ......................................................................................................................... 65

VITA ............................................................................................................................................... 69
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Estimates of pressure relaxation coefficients in air based on vortex experiments</td>
<td>30</td>
</tr>
<tr>
<td>2. Estimates of inner vortex core radius from different turbulence models</td>
<td>34</td>
</tr>
<tr>
<td>3. Estimates of approximate turbulent eddy viscosity and non-equilibrium vortex core size based on vortex experiments using Iversen [52] data</td>
<td>36</td>
</tr>
<tr>
<td>4. Estimates of approximate turbulent eddy viscosity and non-equilibrium vortex core size based on vortex experiments using Eq. (47)</td>
<td>37</td>
</tr>
<tr>
<td>5. Equations for the size of potential and non-equilibrium oval</td>
<td>51</td>
</tr>
<tr>
<td>6. Non-equilibrium oval size of aircraft vortex pair from Garodz and Clawson [59] flight test data</td>
<td>53</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Schematic representation of the formation of a pair of aircraft vortices</td>
<td>2</td>
</tr>
<tr>
<td>2. Dangerous aerodynamic forces on following aircrafts</td>
<td>3</td>
</tr>
<tr>
<td>3. Schematic representation of the relative lines of motion of a vortex pair moving upward in an unbounded region</td>
<td>5</td>
</tr>
<tr>
<td>4. Schematic representation of the general zones of the vortex life</td>
<td>10</td>
</tr>
<tr>
<td>5. Effective viscosity ratio at different vortex Reynolds number</td>
<td>26</td>
</tr>
<tr>
<td>6. Variation of pressure relaxation coefficient with relative humidity and temperature</td>
<td>29</td>
</tr>
<tr>
<td>7. Normalized tangential velocity distribution of different aircraft vortex models using a non-equilibrium</td>
<td>33</td>
</tr>
<tr>
<td>8. Normalized azimuthal velocity data for the right-hand vortex behind a NACA 0012 at x/c = 10 and x/c = 30 from Devenport, et al. [21]</td>
<td>38</td>
</tr>
<tr>
<td>9. Normalized azimuthal velocity data for the left-hand vortex behind a NACA 0012 at x/c = 10 and x/c = 30 from Devenport, et al. [21]</td>
<td>39</td>
</tr>
<tr>
<td>10. Schematic representation of a counter-rotating pair of aircraft wake vortices</td>
<td>41</td>
</tr>
<tr>
<td>11. Normalized velocity vectors of a non-equilibrium vortex pair assuming B = 1.</td>
<td>46</td>
</tr>
<tr>
<td>12. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 A</td>
<td>53</td>
</tr>
<tr>
<td>13. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 B</td>
<td>53</td>
</tr>
</tbody>
</table>
Figure

14. Dimensions of the non-equilibrium and potential oval for different values of dimensionless inner core size

15. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 A at critical separation (rc = 0.6)
CHAPTER 1
INTRODUCTION

In 1858, Helmholtz \cite{1} reported on his experiments with vortex rings, initiating research studying the behavior and structure of vortices and vortex motions. Vortices appearing as coherent structures are now considered to be basic ingredients in describing transitional and turbulent flows. The study of the Lagrangian behavior of imbedded vortex structures has become an area of intense research since it can lead to understanding fading memory aspects of turbulent flows. Obviously, improved understanding and modeling of vortex structure and vortex motion can contribute to this great unsolved problem of classical physics.

Some of these coherent structures persist in the turbulent atmosphere as strong axial vortices, commonly observed in natural events such as tornadoes, waterspouts, whirlpools, and dust devils. Long-lived axial wingtips or wake vortices are also observed trailing behind cruising aircraft as natural consequences of lift-derived air flow. These large-scale wake vortex pairs form when a range of smaller vortices generated by swirling engine exhaust flows and fluid motion over and around various geometrical appendages merge. The dominant trailing vortices derive from aircraft wing tip flow produced when the low-pressure (higher velocity) driven upper wing surface fluid motion merges with the higher-pressure (lower velocity) under-wing fluid motion, creating a “vorticity sheet.” Subsequently, the left- and right-wing vortex sheet roll-ups evolve into a strong coherent counter-rotating vortex pair (see Fig. 1), which can represent a hazard for following aircraft. This hazard is especially important during landing and takeoff when following aircraft are close to the ground. The physical vortex formation process at any altitude involves local boundary layer separation, roll-up of these vortex sheets, coalescing smaller vortices and formation of the counter-rotating vortex pair. Ultimately, vortex instabilities can lead to merging of the pair, or they can simply decay in the atmosphere.
Figure 1. Schematic representation of the formation of a pair of aircraft vortices

When generated by larger aircraft in the vicinity of busy airports, these trailing vortex pairs can produce extremely dangerous aerodynamic forces on following aircraft (see Fig. 2). Consequently, aircraft trailing vortices have been an area of great interest for more than 50 years. Much of the effort has been toward understanding the instability mechanisms that can influence the vortex pair lifetime. Prior to decay, the maximum swirl velocities, minimum core pressures and associated vortex core sizes of these trailing in-flight vortices appear to vary widely for virtually identical aircraft. This study focuses on the two-dimensional behavior of a long-lived trailing vortex pair with the purpose of demonstrating how atmospheric conditions can result in non-equilibrium pressure-controlled vortex core behavior. A two-dimensional steady-state vortex pair in an unbounded atmosphere has been considered (aircraft wake vortices just beyond the near field axial flow zone in the mid-field region). Although extensive empirical and analytical work has been focused on accurately describing trailing vortices, existing models for describing the velocity and pressure distributions in the vicinity of the cores differ with various aspects of
actual experimental data. This is especially true near the cores because viscous effects and high strain rates dominate.

Figure 2. Dangerous aerodynamic forces on following aircrafts

An extensive literature review focusing on vortex pairs from a mathematical/analytical point of view and vortex pairs as aircraft trailing vortices is presented. Detailed derivation of the non-equilibrium vortex model is introduced and compared to existing models. Experimental aircraft data has been used as the basis to examine the azimuthal velocity behavior predicted using the proposed vortex pair model.

1.1 Mathematical/Analytical Vortex Pair Studies

The experimental work of Helmholtz\(^1\) allowed him to become the first to elucidate key properties of portions of fluid in which vorticity occurs. Ultimately proving that an ideal vortex motion could neither be created from an irrotational flow nor destroyed completely by a natural
potential fluid force, Helmholtz’s research paved the way for the contemporary laws of vortex motion. Thomson (Lord Kelvin) \(^2\), without any knowledge of Helmholtz’s work, studied vortex dynamics, fueled by the idea that atoms could be modeled as vortex configurations in the ether. The behavior and interactions of two atoms was of major concern for Kelvin, resulting in the first study of a vortex pair. In that study, he showed for a steady motion in an unbounded plane, a vortex pair is accompanied by an isolating “atmosphere”, i.e., a fixed closed area of fluid moving forward with the pair. The closed fluid volume is also known as the “Kelvin oval” (see Fig. 3). It is stated in Kelvin’s work, that the motion of the surrounding inviscid fluid must be the same as it would be if the space within the convex surface (“atmosphere”) were occupied by a smooth solid; in reality, the fluid inside is in a rapid motion, circulating around the axes with increasing velocity nearer the centers. Figure 3 represents the convex outline, which refers to the relative streamlines of motion in the interior of this double vortex, consisting of two infinitely long, parallel, straight vortices of equal strength but rotating in opposite directions. The curves are obtained from the following equations:

\[
\frac{y^2}{a} = \frac{2x}{a} \frac{N + 1}{N - 1} - \left(1 + \frac{x^2}{a^2}\right)
\]

where:

\[
\log N = \frac{x + L}{a}.
\]
A direct continuation and extension of Kelvin’s work on vortex pairs was contributed by Hicks \cite{3}. He generalized the Kelvin results by proving for certain steady configurations of point vortices moving through irrotational fluid, three definite regions can be identified: 1) a region of rotational motion (the point vortices), which conserves its identity; (2) a region of irrotational motion surrounding the first, which also retains its identity and volume and travels uniformly through the fluid with an undeformed boundary; and (3) a region of irrotational acyclic motion, outside the second region. The fluid in this region remains at rest at infinity and is never displaced over more than a small distance. By assuming two straight parallel vortices sufficiently far apart, their sections can be considered as circles of radius $c$, with centers at a distance $2a$ and using inviscid theory (stream functions), Hicks managed to obtain accurate dimension estimates of Kelvin’s oval. Then, for a vortex pair in a stream, the stream function was:

$$\psi = -Ux + \frac{k}{2} \log \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}.$$  \hspace{1cm} (3)
Since stagnation points in potential flow occur at $\psi=0$,

$$U_x = \frac{k}{2} \log \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}.$$  \hspace{1cm} (4)

The half-length $L$ of the oval can be obtained by setting $y=0$:

$$UL = \frac{k}{2} \log \frac{(L + a)^2}{(L - a)^2}$$  \hspace{1cm} (5)

$$e^{(2UL)/k} = 1 + \frac{4aL}{(L - a)^2}.$$  \hspace{1cm} (6)

Since the strength of the vortex can be characterized as,

$$\kappa = \frac{\Gamma}{2\pi}$$  \hspace{1cm} (7)

and,

$$U = \frac{\Gamma}{4\pi a},$$  \hspace{1cm} (8)

Eq. (6) becomes,

$$\frac{L}{a} = \frac{1}{4} \left(\frac{L}{a} - 1\right)^2 \left(e^{\frac{L}{a}} - 1\right)$$

$$\bar{L} = \frac{L}{a} = 2.08725$$  \hspace{1cm} (9)
\[ L = 2.08725a . \quad (10) \]

The half-width, \( W \), of the oval can be obtained for \( x \ll 1 \). Recall for \( \varepsilon \ll 1 \), \( \ln(1 + \varepsilon) \approx \varepsilon + O(\varepsilon^2) \), which gives:

\[
\frac{2Ux}{k} \approx \frac{4ax}{W^2 + a^2} \quad \quad (11)
\]

\[
\bar{W} = \frac{W}{a} \approx \sqrt{\frac{2k}{Ua}} - 1 . \quad \quad (12)
\]

Using Eq. (7) and (8) once more:

\[
W \approx a\sqrt{3} = 1.732a \quad \quad (13)
\]

This derivation of the semi-axes of Kelvin’s atmospheric oval will be employed in later sections. Using the Kelvin and Hicks inviscid theory approach to describe the behavior of a vortex pair can be physically sound and, as demonstrated, mathematically proven. However, it is known that near the center of a potential vortex, the velocity and radial gradient (shearing strain rate) increase without bound making it necessary to include viscous forces in order to describe the actual flow. Ting and Tung \[4\] presented an analytical study proving a vortex pair with viscous inner cores can exist in a nonuniform inviscid stream. They modeled the vortex core regions as embedded boundary layer-like domains embedded in inviscid fluid. By introducing two different sets of length and time scales with the larger scale identified with the typical scales of the outer nonuniform flow, they divided the problem into two solutions, an inner-viscous solution, and an outer-inviscid solution. The ratios of the scales are powers of a small parameter, \( \varepsilon \), which is inversely proportional to the square root of the vortex Reynolds number, defined as \( Re_v = \frac{r}{v} \). They showed the time average of the velocity of the center of the vortex, taken over a period on the order of the larger time scale, should agree to the order of \( \varepsilon^2 \). The local space mean of the
velocity of the outer inviscid flow was assumed to be the velocity at the center of the classical inviscid vortex. As a result, the leading terms in the solutions are composed of the classical inviscid vortex pair solutions matched with the solution of a decaying axially symmetric vortex. This analytical study can be applied to the study of the motion of a group of vortices in a nonuniform stream by including effects due to the interference terms. That work can be extended to the motion and decay of vortex rings as well as trailing edge vortex lines.

Norbury [5] focused on finding solutions for a semi-linear elliptic partial differential equation in a bounded domain. Specifically, for a steady-state ideal fluid in \( \mathbb{R}^2 \) which contains bounded regions of vorticity (including a vortex pair), the unknown free vortex pair boundary was removed from consideration. Norbury demonstrated how an associated variational problem possessed a maximiser, which could yield solutions to semi-linear Dirichlet problems in bounded domains. More recently, Cao and Wang [6], found vortex solutions for an ideal incompressible flow in a planar bounded domain by using a variational formulation for the vorticity.

A single, steady-state axial vortex is stable, and its shape does not change. Considering the plane perpendicular to the vortex rotational axis, the two-dimensional streamlines are circles. Pierrehumbert [7] was the first to numerically calculate the steady-state shapes of a pair of compact regions containing constant vorticity of opposite signs and embedded in an irrotational fluid. The main objective of his work was to determine whether exact solutions continue to exist even when the gap between vortices is made arbitrarily small, when holding the outer edges of the vortices fixed. His numerical results supported the existence of steady state solutions even when the gap between vortices was arbitrarily small. Furthermore, as the gap closed, the steady state approached a limiting vortex pair with a cusp on the axis of symmetry. Similarly, Saffman and Szeto [8] employed Newton’s method numerically to show similar types of solution shapes and properties for two equal-strength co-rotating uniform vortices, when rotating steadily about each other. Their main purpose was to demonstrate that a minimum separation distance between vortices must exist in order to sustain steady motion. Their results were also used for discussing the merging of vortices in the wakes of lifting bodies and they addressed the jumbo jet trailing vortex alleviation problem. They concluded that the minimum separation distance
between two vortices required to maintain their stability was \( \frac{r_{\text{core}}}{a} = 0.63 \), where \( a \) is the distance between the vortex centroids, and \( r_{\text{core}} \) represents the inner radius of the vortex.

All the studies mentioned thus far assumed incompressible fluid. Moore and Pullin \(^9\) studied a compressible vortex pair using a finite difference numerical solver. A two-dimensional experimental study showing the formation of vortex couples of opposite signs in a von-Karman wake was developed by Couder and Basdevant \(^{10}\). The evolution of unsteady two-dimensional vorticity structures surrounded by fluid at rest, while allowing the emergence of vortex couples was studied by Duc and Sommeria \(^{11}\). Using a mathematical approach, a steady vortex pair could be considered as a class of nonlinear waves, specifically traveling wave solutions of the incompressible Euler equations. The nonlinear stability of a steady vortex pair in an irrotational flow of an ideal fluid was proven with respect to symmetric perturbations by Burton, et al. \(^{12}\).

**1.2 Vortex Pairs as Aircraft Trailing Vortices**

These earlier analytical and experimental studies provide fundamental knowledge of the behavior of a vortex pair, the mathematical existence of these pairs in a bounded or unbounded region and their stability. However, unanswered fluid dynamic physics-based questions remain. Such is the case for the behavior of aircraft trailing vortices. These vortices originate from the roll-up of a vortex sheet in the downwash of an aircraft and can be considered as two counter-rotating vortices of equal strength. An extensive literature review has been conducted focusing on analytical, experimental, and numerical work specifically on vortex pairs as aircraft trailing vortices.

Commercial interest in finding solutions to aerodynamics problems such as predicting the rolling-up of the primary downwash vortex sheet and the eventual appearance of counter-rotating vortex pairs became an active research area beginning in the late 1960s \(^{13}\). An increasing number of studies focusing on the fluid dynamic behavior have been published subsequently \(^{13}\). Many of these studies were based on actual flight experiments. Chevalier \(^{14}\) used photography to document an in-situ flight test program to describe the characteristics of the vortex behavior and their instabilities, employing smoke grenades activated near the wing tips in flight. One method for classification of the experimental vortex behavior studies assumed
origination from the wing tips and is based on distinctive zones of the vortex life (see Figure 4). In Figure 4, the near field

![Figure 4. Schematic representation of the general zones of the vortex life](image)

extends from wing tip trailing edge approximately ten chord lengths behind the aircraft. Complete roll-up of the vortex sheet has occurred within that zone; all of the lift-based circulation is then attributed to the vortex pair. In this region a series of vortex merging events occur due to other secondary vortices generated by the fuselage, engines, tail and miscellaneous appendages. The final vortex pair appears as an axisymmetric Kelvin vortex pair with a smooth mean velocity profile. In that mid field zone, the vortices decay at a relatively slow rate.

The first study focusing on the near field of a tip vortex was conducted by Grow [15] using a five-hole pressure probe and a vorticity meter. He studied the effect of the wing geometry and the associated boundary layer flows on the trailing vortex. Vortex circulation was increased, and maximum swirl velocity was found to correlate with increasing wing aspect ratio, wing taper ratio and angle of attack. Lee and Schetz [16] investigated vortex formation and behavior from a low aspect ratio lifting surface in the near wake. Measurements were conducted in a wind tunnel employing yaw-head and hot-wire probes at three different Reynolds numbers. In the same study, near field data was employed as initial conditions for a far field computational study employing the parabolized Navier-Stokes equations in terms of vorticity and stream function. Similarly, Shekarriz et al. [17] studied a wing tip vortex generated in the near field in a towing tank using particle displacement image velocimetry (PIV). A more precise study of wing tip vortex formation was performed by Devenport, et al. [18] using a NACA 0012 half wing. Birch, et al. [19] compared the near field vortex behavior of a NACA 0015 airfoil with a high-lift cambered airfoil.
A complete comparison between the azimuthal velocity of a NACA 0012 using a theoretical vortex model with experimental data using PIV techniques was performed by Del Pino, et al. [20]. All of these early studies utilized either a half wing or a lifting surface attached to the wall tunnel, thus generating a single isolated tip vortex. In a later study, Devenport, et al. [21] extended their work employing two separated NACA 0012 half wings. Those results, at 10 chord lengths exhibited laminar vortex cores; the only turbulence that emerged surrounded the cores, and it was formed by the roll-up of and interaction of the wing wakes. At an approximate distance of 30 chord lengths, the cores became turbulent. The turbulent region surrounding the cores doubled in size and apparently was sustained by outward diffusion from the cores. Employing a laser-Doppler velocimeter, the velocity components in the mid field of a lifting hydrofoil vortex wake were reported by Baker, et al. [22].

In the far field, the vortex starts to break down and change shape due to turbulent dissipation and a variety of instability mechanisms. In this region the overall vortex circulation decays rapidly, and dissipation of the vortex structure is observed. Velocity measurements were taken in the far field of a wake in a tow tank by Cliffone, et al. [23] and Sarpkaya [24] who focused on the evolution of the vortex wake in a stratified and unstratified water tunnel. They showed that in a weakly stratified medium, linking of the vortices and/or the cascade of core bursting events are primarily responsible for the breakdown of the vortex pair. Vortex decay measured in terms of velocity and changes in core size was gauged at various distances in the far field by McCormick, et al. [25]. A more complete study focusing on all three zones of vortex life was performed by Allen and Breitsamter [26]. Experimental technique comparisons between a five-hole probe in-situ measurement on an Airbus A321 wind tunnel model and coherent laser radar (lidar) measurements in a full-scale field trial was reported by Harris, et al. [27].

One of the first analytical studies considering the three-dimensional downwash flow from a wing as a simple two-dimensional vortex pair moving through the air was developed by Spriter and Sacks [28]. Their study is considered the standard reference for using the Betz approximation (see section 1.3) to understand the flow physics of the trailing vortex sheet. They found that determination of how vortices are rolled up is directly related to the distance behind the wing.
and upon the lift coefficient, span loading, and aspect ratio of the wing. Scorer and Davenport \cite{29} focused their efforts on an analytical study utilizing three fundamental discussions: (1) The physics of contrails; (2) the motion of streamlines in a vortex pair under the influence of stratification; and (3) the overall stability. An important aspect in the life of the vortex wake of an aircraft is the breakdown due to instabilities. By understanding the governing physics of these instabilities, better prediction of the complete decay of the vortices can allow the aircraft industry to increase air traffic throughput while enhancing aircraft safety.

Several investigations have attempted to understand the underlying physics leading to the destruction of the vortices. The most well-known early analytical study of the underlying long-wave instabilities of the aircraft trailing vortices was developed by Crow \cite{30}. He showed that the mutual induction of a pair of counter-rotating infinite line vortices makes them unstable to coupled sinusoidal disturbances and as the amplitude of the instability grows, the resulting behavior becomes nonlinear. Ultimately, the vortex cores link, and a series of vortex rings is formed. The \textit{Crow instability} is usually cited as the primary mechanism causing the break-up and dissipation of aircraft vortex pairs in cruising flight. Using the seminal Crow instability results, Greene \cite{31} developed an approximate vortex decay model which correlates well with experimental data under some specific atmospheric conditions. A more complete vortex decay model, considering the effects of ground proximity and crosswind shear was presented by Sarpkaya \cite{32}.

Depending on the aircraft geometry, multiple shed vortices can merge to form the final trailing vortex pair. Behind the aircraft, these other vortices merge due to their induced straining field. Bilanin, et al. \cite{33} investigated the dynamic interactions of the aircraft wake vortices focusing on the merging and the later decay using inviscid and viscous models to represent the governing equations for these vortical flows. The merging process occurs more often when both circulations are in the same direction. This condition was extensively studied by Bilanin, et al. \cite{34} and Brandt and Iversen \cite{35}, who developed a merging distance criterion using low-turbulence wind tunnel flow visualization data.
Since vortex velocity and pressure data at aircraft scales is difficult to obtain experimentally, numerical modeling of the behavior of aircraft wake vortices has become a useful tool. Focusing on the instability aspect of aircraft vortices, Holzapfel, et al. [36] performed a large eddy simulation (LES) of a vortex pair superimposed with aircraft induced turbulence and atmospheric turbulence. They observed that short wavelength instability (not to be confused with Crow instability) was triggered by atmospheric and wake turbulence and thus resulted in accelerated decay of the vortex circulation. Recently, Changfoot, et al. [37] developed a parallel three-dimensional hybrid finite volume finite difference code which was implemented to model the trailing vortices shed from the wings of aircraft during transonic flight conditions.

1.3 Vortex Models used for Aircraft Trailing Vortices

Experimental measurements of vorticity, azimuthal and axial velocity distributions generated within actual aircraft trailing edge vortices have provided good insight regarding the behavior and decay of the vortices. However, most experiments rely on wind tunnels as the medium for the acquisition of data, which can be time consuming and come with a high monetary cost [38]. Analytical models have been developed as an alternative approach for predicting velocity distributions in wake vortices with different levels of accuracy. A review of some of the existing vortex models specifically used for aircraft trailing vortices is presented in three categories: Inviscid, Viscous Laminar, and Viscous Turbulent.

1.3.1 Inviscid Vortex Models

One of the first methods describing the circumferential velocity distributions in a lift generated vortex was derived by Betz [39]. He based his theory on the conservation equations for inviscid, two-dimensional vortices. Several assumptions were introduced to obtain a simple model. The vortex was assumed to be completely rolled up, and the rollup process was inviscid. The Betz method does not consider the transition or intermediate stages between the vortex sheet and the final rolled up vortex structure. The method relates the circulation distribution of an isolated wing to the circulation contained in the fully rolled-up vortex. Donaldson [40] extended the work of Betz to take into account circulation distributions which were not monotonic (i.e., wake generated by wings with partial-span flaps).
1.3.2 Viscous Vortex Models

Rankine \[\text{[41]}\] developed a vortex model that takes viscous effects into consideration. He assumed the radial and axial velocity components were equal to zero allowing him to obtain an azimuthal velocity distribution. The Rankine model is a combination of a rigidly rotating core coupled with an outer region represented as a potential vortex. The azimuthal velocity profile of a Rankine vortex is given by:

\[
v_\theta = \begin{cases} 
\frac{\Gamma_o r}{2\pi r_{\text{core}}^2}, & r \leq r_{\text{core}} \\
\frac{\Gamma_o}{2\pi r}, & r > r_{\text{core}} 
\end{cases}
\]

(14)

where \(\Gamma_o\) represents the root circulation and \(r_{\text{core}}\) is the core radius. Although it is regarded as the first vortex model, it is not suitable when assuming unsteady flow because of the shear stress discontinuity at the core interface. The Lamb-Oseen model \[\text{[42]}\] is an exact analytical solution to the two-dimensional Navier-Stokes equation. Lamb \[\text{[42]}\] introduced a potential line vortex with its infinite velocity limit on the centerline as an initial point discontinuity, then subjected it to viscous decay. The model is an exact unsteady laminar solution for the azimuthal velocity profile, given as:

\[
v_\theta = \frac{\Gamma_o}{2\pi r} \left[1 - e^{-\frac{r^2}{4\nu t}}\right].
\]

(15)

A steady-state Lamb-Oseen vortex model also exists provided by Lamb \[\text{[42]}\]

\[
v_\theta = \frac{\Gamma_o}{2\pi r} \left[1 - e^{-1.2526\left(\frac{r}{r_{\text{core}}}\right)^2}\right].
\]

(16)
Through linearization of the continuity and Navier-Stokes equations and assuming the tangential and radial velocities are negligibly small with respect to the freestream, Newman\textsuperscript{[43]} derived the following radial, axial and azimuthal vortex velocity profiles.

\begin{align*}
v_r &= -\frac{A}{2z^2} \left[ e^{-\frac{u_\infty r^2}{4vz}} \right] \quad (17) \\
v_z &= e^{\frac{A}{z} \left[ e^{-\frac{u_\infty r^2}{4vz}} \right]} \quad (18) \\
v_\theta &= \frac{\Gamma_o}{2\pi r} \left[ 1 - e^{-\frac{z r^2}{4vU_\infty}} \right] \quad (19)
\end{align*}

where $A$ is a function of the profile drag of the generating body, $z$ is the axial position and $U_\infty$ is the freestream velocity. Newman’s model is similar to the Lamb-Oseen model. The only difference is that the azimuthal velocity profile depends on the radius and axial position contrary to the Lamb-Oseen model which is time dependent.

An empirically based vortex model that demonstrates great suitability for the experimentally observed actual velocity profiles from aircraft vortices was obtained by Burnham and Hallock\textsuperscript{[44]}. It describes the azimuthal velocity profile as:

\begin{equation}
v_\theta = \frac{\Gamma_o}{2\pi (r + r_{core}^2)} \cdot (20)
\end{equation}

When actual vortex velocity profiles can be measured, this correlation is considered to be a “best fit” for actual wakes. However, the profiles vary widely for nominally identical aircraft, and the sources responsible for this variability are unknown.
By means of Lidar observations of the wake vortex early in their lifespan, Proctor determined the azimuthal velocity profile outside the core radius was:

\[ v_\theta = \frac{\Gamma_o}{2\pi} \left[ 1 - e^{-\theta\left(\frac{r}{b}\right)^{0.75}} \right] r > r_{core} \tag{21} \]

where \( b \) represents the aircraft wingspan. Proctor developed a piecewise formulation of the model to include the core region of the vortex given by:

\[ v_\theta = \begin{cases} 1.4 \frac{\Gamma_o}{2\pi r} \left[ 1 - e^{-1} \left( \frac{r_{core}}{b} \right)^{0.75} \right] \left[ 1 - e^{-1.2527\left(\frac{r}{r_{core}}\right)^2} \right] r \leq r_{core} \\ \frac{\Gamma_o}{2\pi r} \left[ 1 - e^{-10\left(\frac{r}{b}\right)} \right] r > r_{core} \end{cases} \tag{22} \]

Batchelor developed a vortex model referred to as the \textit{q-vortex model} due to the appearance of the swirl strength, which is the ratio of the maximum azimuthal velocity and core axial velocity in the azimuthal velocity profile equation.

\[ v_\theta = \frac{qU_\infty r_o}{r_{core}} \left[ 1 - e^{-\left(\frac{r}{r_{core}}\right)^2} \right] \tag{23} \]

where \( r_{core}(t) = \sqrt{4\nu t + r_0^2} \) is the measured core size as it grows in time due to diffusion. Here \( r_0 \) is the initial vortex core radius at \( t = 0 \). This model has been used extensively for establishing stability theories of the aircraft wake vortices.

Winckelmans, et al. smoothly blended Proctor’s model and adjusted it to a wind tunnel experiment employing a rectangular wing (no flaps, no fuselage) and to two-dimensional vortex roll-up studies, given by:
\[ v_\theta = \frac{\Gamma_o}{2\pi r} \left\{ 1 - \exp \left( - \frac{\beta_i \left( \frac{r}{h} \right)^2}{1 + \left[ \frac{\beta_i}{\beta_o} \left( \frac{r}{h} \right)^{\frac{3}{4}} \right]^p} \right) \right\} \]  

(24)

where \( \beta_o, \beta_i \) and \( p = 10, 500 \) and 3 respectively and are independent parameters.

### 1.3.3 Viscous Turbulent Vortex Models

Several turbulent vortex models sought to adjust viscosity in order to elucidate the effects of turbulence on a vortex structure \[^38]\]. Instead of modeling viscosity, Hoffman and Joubert \[^49]\) proposed that the behavior of a turbulent vortex could be described by a model independent of viscosity. By using a boundary layer analogy, they developed a piecewise model for the inner core region defined by solid body rotation and an adjoining outer region.

\[
\begin{align*}
\Gamma & = 1.83 \left( \frac{r}{r_{core}} \right)^2 \quad r \leq r_{core} \\
\Gamma & = 1 + 2.14 \log_{10} \left( \frac{r}{r_{core}} \right)^2 \quad r > r_{core}
\end{align*}
\]

(25)

where \( r_{core} \) is the circulation at the core radius. Or, in terms of the azimuthal and maximum azimuthal velocity, the unified Hoffman and Joubert \[^49]\) equation is:

\[
v_\theta = v_{\theta,\text{max}} \left[ 1 + \ln \left( \frac{r}{r_{core}} \right) \right].
\]

(26)

Unfortunately, Hoffman and Joubert \[^49]\) found that their model did not correlate well in the transition zone between the two regions, and they proposed additional experimental work to obtain a possible turbulence law in this region. Squire \[^50]\) added eddy viscosity to the Lamb-Oseen vortex model by replacing the viscosity term for \( v_{\text{turb}} = (\nu + \varepsilon) \), where the eddy viscosity was
related to the circulation as $\varepsilon = a_1 \Gamma$. At the time, Squire did not provide for a simple solution for the eddy viscosity or the eddy viscosity proportionality constant $a_1$. Owen $^{[51]}$ using experimental data developed an expression for Squire’s coefficient given as

$$a_1 = 2\pi \Lambda^2 \text{Re}_f^{1/2}$$

(27)

where $\Lambda$ is a function of time, tangential and axial velocity ratios, and the scale of turbulence. Iversen $^{[52]}$ developed a more complex model for the turbulent viscosity using a form of mixing length, which was assumed to be proportional to the core radius and turbulent viscosity. The variable Iversen model and the constant Squire model approach asymptotically the same value of azimuthal velocity at large radii.

$$v_{turb} = \alpha^2 r^2 \left| r \frac{\partial}{\partial r} \left( \frac{\Gamma}{2\pi r^2} \right) \right| + \nu$$

(28)

where, $\alpha$ is the proportionality constant for mixing length.
CHAPTER 2
NON-EQUILIBRIUM PRESSURE THEORY

An attempt to develop a new rigorous framework for theoretical prediction of bulk viscosity in simple fluids was developed by Zuckerwar and Ash\textsuperscript{[53]}. The authors recognized that at high strain rates, a simple fluid, such as air, can depart from its simple equilibrium continuum behavior while remaining incompressible. Using Hamilton’s Principal of Least Action, along with Lagrangian constraints to include molecular departures from equilibrium their model predicted non-equilibrium pressure forces in the Navier-Stokes equation. Their approach enabled a molecular-level departure from vibrational and rotational molecular equilibrium. After some analysis, the authors demonstrated that there are two types of “bulk viscosity” effects – the well-known dissipative effect and a quasi-reversible, constant-pressure non-equilibrium effect which cannot be isolated from bulk viscosity. A second effect was related linearly to the gradient of the material rate of change of pressure\textsuperscript{[53]}. The incompressible modified Navier-Stokes equation can be expressed in vector form as:

\[
\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \left( \frac{\eta_v}{\rho} \frac{\partial P}{\partial t} \right) - \rho \nabla \Omega + \nabla \left[ \left( \eta_v - \frac{2}{3} \mu \right) \nabla \cdot \mathbf{v} \right] + \nabla \times (\mu \nabla \times \mathbf{v}) + 2 \nabla \cdot (\mu \nabla \times \mathbf{v}) \quad (29)
\]

where \( \eta_P \) is the pressure relaxation coefficient, and \( \eta_V \) is the volume or bulk viscosity. Assuming constant thermophysical parameters and neglecting body forces, that conservation of momentum equation can be simplified to:

\[
\rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \eta_p \frac{\partial P}{\partial t} - \left( \eta_v - \frac{4}{3} \mu \right) \nabla (\nabla \cdot \mathbf{v}) - \mu \nabla \times (\nabla \times \mathbf{v}) \quad (30)
\]

Allowing for sound production in otherwise incompressible flow, the modified Navier-Stokes equation can also be expressed employing index notation:
\[
\frac{\rho Dv_i}{Dt} = - \frac{\partial P}{\partial x_i} + \eta_P \frac{D}{Dt} \frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_i^2} + \eta_p \left[ \frac{\partial v_i \cdot \partial P}{\partial x_i \partial x_i} - \frac{(\eta V + \frac{1}{3} \mu)}{\eta_p} \frac{\partial}{\partial x_i} \left( \frac{\partial^2 D}{\partial x_i^2} \right) \right].
\]  

(31)

Ash, Zardadkhan and Zuckerwar \textsuperscript{[54]} logically asserted that the bracketed term in Eq. (31) is a type of acoustic shunt. However, the term should be negligibly small when multiplied by the pressure relaxation coefficient. Thus, the Navier-Stokes equation incorporating pressure relaxation simplifies to:

\[
\frac{\rho Dv_i}{Dt} = - \frac{\partial P}{\partial x_i} + \eta_P \frac{D}{Dt} \frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_i^2}.
\]  

(32)

The vector form can be expressed as:

\[
\frac{\rho Dv}{Dt} = -\nabla P + \eta_P \frac{D}{Dt} \nabla P - \mu \nabla^2 v.
\]  

(33)

This equation possesses a strong resemblance to the conventional Navier-Stokes equations for incompressible flow apart from the extra \( \eta_P \frac{D}{Dt} \nabla P \) term that enables local pressure to depart from its thermodynamics equilibrium state.

### 2.1 Non-Equilibrium Laminar Axial Vortex

Zuckerwar and Ash \textsuperscript{[53]} had predicted analytically that non-equilibrium pressure behavior occurred in a slow viscous Stokes flow past a sphere. Subsequently, Ash, Zardadkhan and Zuckerwar \textsuperscript{[55]} examined steady, incompressible flows with strong streamline gradients (shearing rates of strain) to explore the possible existence of non-equilibrium pressure effects. Naturally recurring vortical flows such as aircraft wake vortices, dust devils and tornadoes exhibit strong streamline curvature since they resemble steady inviscid line vortices. Employing the standard Navier-Stokes equations to model simplified steady-state behavior results in failure because it is not possible to obtain a steady-state solution valid for the entire line vortex domain. As stated in
section 1.3, several models were rendered useless to describe the true axial flow behavior. For instance, Rankine’s model is incompatible with viscous fluid behavior because of its discontinuity in the slope of the velocity distribution. The Lamb-Oseen model cannot predict accurately the flow structure in the vicinity of the vortex rotational axis.

By introducing non-equilibrium pressure terms in the Navier-Stokes equation Ash, Zardadkhan and Zuckerwar \cite{55} derived a steady-state axial vortex velocity distribution valid and continuous throughout the radial flow domain. Using the modified Navier-Stokes equation (Eq. 33) in cylindrical coordinates for steady, axisymmetric incompressible flow and neglecting axial flow, the steady-state radial and azimuthal conservation of momentum equations result in the following.

Radial conservation of momentum:

\[ \rho \frac{v_\theta^2}{r} = \frac{dP}{dr} \]  \hspace{1cm} (34)

Azimuthal conservation of momentum:

\[ 0 = \nu \left( \frac{d^2 v_\theta}{dr^2} + \frac{1}{r} \frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} \right) + \eta_p \left[ \frac{v_\theta dP}{r dr} \right] \]  \hspace{1cm} (35)

The radial component is not affected by the pressure relaxation, and both equations can be combined to yield:

\[ \eta_p \frac{v_\theta^3}{r^2} + \nu \left( \frac{d^2 v_\theta}{dr^2} + \frac{1}{r} \frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} \right) = 0 . \]  \hspace{1cm} (36)

The thusly obtained non-linear ordinary differential equation can be solved for the case of an axial vortex with specified circulation while imposing a physical zero-velocity limit on the vortex
axis. For an axial vortex, the azimuthal velocity on the centerline must be equal to zero and must converge towards a prescribed circulation potential vortex in the far-field. Thus:

\[
\lim_{r \to 0} v_\theta(r) \to 0
\]

\[
\lim_{r \to \infty} v_\theta(r) \to \frac{\Gamma_0}{2\pi r}.
\]

(37)

Dimensionless radius and velocity variables can be formed to rewrite the governing equation in a dimensionless form:

\[
r = \sqrt{\nu \eta} \bar{r}
\]

(38)

\[
v_\theta = \frac{\Gamma_0}{2\pi \sqrt{\nu \eta}} u(\bar{r}).
\]

(39)

Therefore:

\[
\bar{r}^2 \frac{d^2 u}{d\bar{r}^2} + \bar{r} \frac{du}{d\bar{r}} - u + R_\Gamma^2 u^3 = 0
\]

(40)

where \( R_\Gamma \) is the circulation-based Reynolds number defined as: \( R_\Gamma = \frac{\Gamma_0}{2\pi \nu} \). The dimensionless boundary conditions are:

\[
\lim_{r \to 0} u(\bar{r}) \to 0
\]

\[
\lim_{r \to \infty} u(\bar{r}) \to 1.
\]

(41)
The non-linear ODE’s are often solved by introducing a dimensionless independent variable and an associated scaling parameter:

\[ \zeta = \sqrt{\frac{R_f - 2}{2}} \ln \bar{r} \]  

(42)

\[ k = \frac{R_f^2}{\sqrt{2 - R_f^2}}, \quad (R_f \neq 2). \]  

(43)

The transformed equation becomes:

\[ \frac{d^2 u}{d \zeta^2} = -(1 + k^2)u + 2k^2u^3. \]  

(44)

After solving this differential and applying the corresponding boundary conditions, the resulting dimensionless velocity depends only on the circulation-based Reynolds number given as:

\[ u(\bar{r}) = \frac{8\bar{r}}{8\bar{r}^2 + R_f^2}. \]  

(45)

The vortex core radius can be defined as the radial distance where the azimuthal velocity is maximum. By taking the derivative of the dimensionless velocity profile, equating it to zero and transforming to physical variables, the vortex radius and maximum azimuthal velocity are given respectively by:

\[ r_{core} = \frac{\Gamma_o}{4\pi \sqrt{2\nu} \eta_p}. \]  

(46)

\[ v_{\theta,max} = \frac{2\nu}{\eta_p}. \]  

(47)
Reverting to physical variables, the dimensionless velocity can be written as:

\[ v_{\theta} = \frac{\Gamma_o}{2\pi r_{core}} \frac{\left( \frac{r}{r_{core}} \right)}{\left( \frac{r}{r_{core}} \right)^2 + 1}. \]  

(48)

From this final velocity solution, it can be shown that as the pressure relaxation coefficient, \( \eta_p \), approaches zero, the vortex core radius will approach zero, and the maximum azimuthal velocity will become infinite. Therefore, reverting to the inviscid potential vortex solution. As the pressure relaxation asymptotes towards infinity, the inner core of the vortex evolves as a rigid-body rotation.

### 2.2 Non-Equilibrium Turbulent Axial Vortex

As noted, Ash, Zardadkhah and Zuckerwar [55] (Ash, Zuckerwar, and Zardadkhah [55] will be referred to henceforth as AZZ), derived a steady state azimuthal velocity distribution for an incompressible axial vortex employing a modified Navier-Stokes equation that takes into consideration the departure of local fluid pressure from its thermodynamic equilibrium. Since no fluctuations of mean velocity (turbulent flow) were assumed in the derivation, Eq. (46) is for a viscous laminar vortex core radius. Wingtip vortex flows are extremely complex in the near field region as the complete roll-up process involving multiple vortices is largely turbulent, and highly three-dimensional [13].

This study has focused on developing a model for the steady, incompressible aircraft wake vortex pair after the core region becomes turbulent, using the non-equilibrium pressure theory in the mid-field to control the structure. Consequently, an assumption of a turbulent vortex core is justified. Turbulence does not influence the mean velocity profile when the atmospheric turbulence is irrotational, which was proven by Corrsin and Kistler [56], justifying the diminution of turbulence influences on the mean profile away from the core regions. The AZZ axial vortex model assumes that the inner viscous core is controlled by non-equilibrium pressure gradient...
forces in direct response to the coupling of centrifugal forces with unsustainable shearing strain rates near the rotational axis. Moreover, plausible local stress gradients are predicted in the core region. Employing a simple eddy viscosity turbulence model which only influences the steady-state vortex velocity field near the core is therefore warranted. That is:

\[
\sigma_{ij} = (\mu + \mu_{turb}) \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right] = \rho \nu_{turb} \left[ \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right].
\]  

(49)

Ash and Zardadkhan \[57\] employed this simple turbulent eddy viscosity correlation. AZZ used Sinclair’s \[58\] mean turbulent velocity profiles, with local temperature, and pressure surveys for three dust devils to relate turbulent eddy viscosity to local kinematic viscosity. They found

\[
\frac{\mu_{turb}}{\mu} = \frac{\nu_{turb}}{\nu} = 3.2 \pm 1.
\]

(50)

However, dust devils are isolated natural vortices, with circulation levels between 320 m\(^2\)/s and 400 m\(^2\)/s, in that study.

Without direct experimental validation of the theoretical based pressure relaxation coefficient, AZZ \[55\] simply employed the turbulent eddy viscosity correlation with dynamic viscosity to adjust for their steady state axial vortex solution. However, that correlation should be used with extreme caution for coupled, mechanically created vortex pairs or for other circulation ranges. Moreover, a useful eddy viscosity correlation exists for aircraft wake vortex data. Iversen \[52\] used a numerical solution of the decay of a self-similar line vortex with variable eddy viscosity to obtain a correlation function for comparison of scale-model wind tunnel and actual flight data (see Fig. 5). This correlation will be used in later sections to account for the turbulence in the non-equilibrium inner core.
Figure 5. Effective viscosity ratio at different vortex Reynolds number \[^{[52]}\]
CHAPTER 3
NON-EQUILIBRIUM AIRCRAFT WAKE VORTICES

Interestingly, the velocity distribution for an axial vortex derived from non-equilibrium pressure theory has the same functional form as the widely used empirical correlation representing near-surface aircraft trailing line vortices developed by Burnham and Hallock \[44\]. The primary difference is Burnham and Hallock require measured vortex core radii as input whereas the AZZ theory utilizes aircraft circulation and weather data to predict core radius, maximum swirl velocity and pressure drop. Early analytical and empirical vortex models describing laminar and turbulent aircraft wake vortices have been discussed but all differ to some degree with various aspects of experimental data, especially near the core where viscous effects and high strain rates dominate. Consequently, the motivation for this thesis was to develop a model for a steady, incompressible aircraft wake vortex pair using the non-equilibrium pressure theory to describe more accurately the behavior of the long-lived aircraft wake vortex pair after the cores become turbulent (prior to merging or due to other instabilities).

3.1 Existence of Non-Equilibrium Vortex Pair in an Unbounded Fluid Region

Potential flow theory can be used to describe the behavior of various external flows where viscous effects are insignificant. A line vortex or free vortex describes a purely circulating steady motion. By means of inviscid theory, the introduction and superposition of two counter-rotating potential vortices in a uniform stream provides a stream function capable of describing a potential vortex pair. However, near the centers of both potential vortices, the velocity and radial velocity gradients increase without bound making it necessary to include viscous forces in order to describe the actual flow. By incorporating departure of local fluid pressure from its equilibrium state via the conservation of momentum, physically viable velocity gradients in the centers of these axial vortices can be taken into consideration. As shown in section 2.1, a steady state solution for an incompressible axial vortex using non-equilibrium pressure theory can be obtained. The limiting behavior of a non-equilibrium vortex can be described as follows: as the pressure relaxation coefficient approaches zero, the viscous vortex core vanishes, and the
maximum azimuthal velocity becomes infinite—a potential vortex. Thus, we can infer that small non-equilibrium axial vortex core regions can be represented as a potential vortex pair with viscous non-equilibrium pressure cores\cite{4,5,6}. Before using the non-equilibrium axial vortex model to elucidate the behavior of an aircraft wake vortex pair, consider the following question: Can a pair of potential vortices with non-equilibrium centers exist in an unbounded inviscid region of fluid?

We can address this fundamental question by considering several important references. A systematic procedure for studying the decaying motion of a single vortex in a non-uniform flow was provided by Ting and Tung,\cite{4} who modeled vortex core regions as embedded boundary layer like domains embedded in an inviscid fluid. The leading terms in the solutions of the Navier-Stokes equations were composed of the classical inviscid solution along with the solution of a viscous decaying axially symmetric vortex. In that way, they asserted the mathematical existence of a potential vortex with a viscous inner core in a nonuniform stream. Norbury\cite{5} focused on finding solutions for a semi-linear elliptic partial differential equation in a bounded domain. Specifically, for a steady-state ideal fluid in $\mathbb{R}^2$ containing bounded regions of vorticity (including a vortex pair as one case). He proved that steady two-dimensional vortex pairs with vorticity confined to compact regions always existed provided that the vorticity is a Hölder continuous function\footnote{A real or complex function $f$ on a $n$ − dimensional Euclidean space satisfies a Hölder condition when, $|f(x) − f(y)| \leq A \|x − y\|^\mu$, were $A > 0$ and $\mu > 0$} of the stream function. More recently, Cao and Wang\cite{6}, found vortex solutions for an ideal incompressible flow in a planar bounded domain by using a variational formulation for the vorticity. By acknowledging that non-equilibrium viscous effects are confined within background turbulence levels in a real fluid, these earlier references provide justification for invoking a pair of potential vortices with non-equilibrium centers, surrounded by an infinite inviscid domain. Hence, a potential vortex pair with small, embedded sheaths of non-equilibrium fluid in the inner core can exist in an unbounded fluid region.
3.2 Estimation of local Pressure Relaxation Coefficients employing Aircraft Flight Data

From the pressure non-equilibrium version of the Navier-Stokes equation, pressure can depart from thermodynamic equilibrium by means of the pressure relaxation coefficient relating the radial shearing strain rate gradient to the particle-based pressure gradient. Determination of the pressure relaxation coefficient is needed for physical solutions satisfying the modified momentum equation. When air is the medium, the vibrational relaxation times for nitrogen and oxygen are strong functions of humidity. AZZ [55] utilized a mole-fraction weighted averaging approach, incorporating the fundamentally based acoustic influence parameters due to temperature and relative humidity on pressure relaxation coefficient (see Fig. 6). As can be seen in the figure, the theoretically predicted pressure relaxation coefficient can vary from $10^{-8}$ seconds at 50 °C, 100 % Relative Humidity to 30 µseconds for dry air at 0 °C (a factor of 3000). Alternatively, pressure relaxation coefficients can be inferred from experimentally measured axial velocity profiles.

![Figure 6. Variation of pressure relaxation coefficient with relative humidity and temperature](image)
Garodz and Clawson\textsuperscript{[59]} utilized an instrumented tower to measure unsteady trailing vortex velocity profiles generated by Boeing 757 and 767 aircraft, while documenting simultaneously near surface ambient conditions using meteorological instruments. Based on the aircraft geometry, takeoff weight and flyby speed, the initial circulation for each flight was estimated using\textsuperscript{[60]}

$$\Gamma_o = \frac{4W_a}{\pi \rho \omega V b}$$  \hspace{1cm} (51)

where $V$ is the flight speed, $W_a$ is the weight of aircraft, $b$ is the aircraft wingspan and $\rho \omega$ is the ambient density. Local ambient temperature, pressure, and relative humidity were measured and recorded, enabling direct calculation of the pressure relaxation coefficient. For laminar cores, the pressure relaxation coefficient can be estimated using the initial circulation and the measured core radii:

$$\eta_P = 2\nu \left(\frac{4\nu R_{\text{core}}}{\Gamma_o}\right)^2.$$  \hspace{1cm} (52)

As the vortex pair formed and descended, cross winds transported them across the instrumented tower. AZZ\textsuperscript{[55]} noted that actual circulation levels, gradual distortion by ground effect and to some extent small changes in ambient conditions in the time-delayed flow field region precluded direct experimental validation of pressure relaxation coefficients. Furthermore, turbulent effects could not be separated from pressure relaxation effects. In the absence of a prescribed turbulent viscosity, the dynamic viscosity of the fluid was used as a reference. Table 1 compiles the largest maximum azimuthal core velocities measured in the upwind and downwind vortices for the Boeing 757 and a Boeing 767 tower fly-bys along with implied pressure relaxation coefficients. Although the sampled velocity records yield the maximum vortex swirl velocity, the two foot tower anemometer spacing prevented accurate estimation of the inner vortex core radius. Instead, Garodz and Clawson\textsuperscript{[59]} estimated the radius employing an iterative approach based on the Hoffman-Joubert\textsuperscript{[49]} vortex velocity model. This
iterative approach was accomplished by substituting values of \( r_{core} \) into Eq. (26) and solving for the azimuthal velocity at each corresponding level of measured azimuthal velocity. The Hoffman-Joubert model is not considered to be a reliable estimate.

The pressure relaxation coefficient corresponding with the recorded ambient temperature and relative humidity is provided in italics below the vortex-based estimated coefficient. Since the vortex measurements were made on an instrumented tower, the trailing vortices are influenced by ground effect and vortex age. This is especially true for the leeward vortex.

**Table 1. Estimates of pressure relaxation coefficients in air based on vortex experiments**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>( \Gamma_o ) m/s</th>
<th>( V_{\theta,\text{max}} ) (m/s)</th>
<th>( r_{\text{core}} ) (m)</th>
<th>( v ) m/s</th>
<th>( \eta_p ) (( \mu \text{sec} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-757 A 9° C; 74% RH</td>
<td>40.55</td>
<td>99.36</td>
<td>0.061</td>
<td>16.96\times10^{-6}</td>
<td>0.012 (0.512)</td>
</tr>
<tr>
<td>B-757 B 10.5° C; 52% RH</td>
<td>35.41</td>
<td>6.31</td>
<td>0.274</td>
<td>17.07\times10^{-6}</td>
<td>0.323 (0.688)</td>
</tr>
<tr>
<td>B-767 A 7° C; 51% RH</td>
<td>37.36</td>
<td>57.76</td>
<td>0.091</td>
<td>16.71\times10^{-6}</td>
<td>0.031 (0.796)</td>
</tr>
<tr>
<td>B-767 B 21° C; 18% RH</td>
<td>43.53</td>
<td>7.37</td>
<td>0.244</td>
<td>18.16\times10^{-6}</td>
<td>0.180 (1.368)</td>
</tr>
</tbody>
</table>

**3.4 Comparison of Aircraft Vortex Models**

The commercial aircraft tower fly-by measurements of Garodz and Clawson\[^{[59]}\] utilized hot film anemometers spaced at 2-ft. intervals on a 200 ft. tower, limiting resolution of vortex core dimensions. While inviscid ground coupling had begun to spread the pair, crosswinds were the primary mechanism propelling them through the tower. The vortex core regions may have behaved like unsteady laminar flows during initial formation, but the data records show that the core regions were fully turbulent by the time they passed through the tower. Unlike turbulence characterizing an isolated buoyancy-driven atmospheric vortex, the turbulence resulting from the amalgamation of multiple smaller-scale vortices generated physically by an aircraft is
expected to be more intense. Aircraft data parameters and ambient conditions were summarized for the limiting extreme cases in Table 1. For this comparison, the pressure relaxation coefficient was calculated using the temperature and relative humidity of the atmosphere rather than using a laminar vortex-based estimate (the estimated laminar core radius compiled in Table 1 was not actually employed). Although wingtip vortices in the mid to early field are highly turbulent in the inner core region \(^{[13]}\) for simple comparison purposes, if the non-equilibrium inner core vortex was assumed to be laminar and was estimated using Eq. (46), turbulent viscosity is not used. All of the vortex models are scaled with the laminar non-equilibrium vortex radius.

Although the Burnham-Hallock \(^{[44]}\) vortex model is empirical, when the core radius is specified, it is identical with the theoretical non-equilibrium vortex model. Unlike the empirical fit, AZZ utilizes arriving or departing aircraft circulation estimates as input, along with ambient weather conditions, to predict core radius, maximum swirl velocity and centerline pressure deficit—the anticipated hazard conditions. Since both models are expressed in the same mathematical form, by using the same value for the vortex size, the azimuthal velocity output coincides. It will be shown later that the inferred turbulent vortex size is approximately 1/10 the estimated laminar vortex size. From Figure 7, the Rankine \(^{[41]}\) vortex can easily be recognized by the discontinuity between the inner core region which rotates as a solid body with constant vorticity and the outer region behaving as a potential flow with constant vorticity. The Lamb-Oseen \(^{[42]}\) vortex is shown as a continuous profile; however, it decays very rapidly. Proctor’s \(^{[46]}\) model has been adapted from lidar field measurements, and Winkelman’s \(^{[48]}\) vortex model has been smoothly blended from Proctor’s and adjusted to a wind tunnel experiment with a rectangular wing and in two-dimensional vortex roll-up studies. The Proctor and Winkelman vortex models do not correlate well and are the only ones behaving differently. The experimental wind tunnel measurements used to develop these models (Proctor’s \(^{[46]}\) and Winkelman’s \(^{[48]}\)) employed larger inner viscous cores than the ones a laminar non-equilibrium predicts.
The Burnham-Hallock \cite{44} vortex model predicts circulation at the core radius to be one half the far-field circulation, in agreement with non-equilibrium theory. Govindaraju and Saffman \cite{61} examined numerous experimental vortex velocity studies, concluding that the ratio between the circulation at the vortex core radius and the far-field circulation ranged between 0.4 and 0.6. Thus, an inferred inner vortex radius can be estimated using

\[
\rho_{inner \, core} = \frac{\Gamma_o}{4\pi V_{\theta_{Max}}}. \tag{54}
\]

Inner viscous vortex core size has been estimated for the B757-A aircraft flight test using different simple turbulence viscosity models (see Table 2). The turbulent viscosity correlation by Ash and Zardadkhan \cite{57} employing Sinclair’s \cite{58} dust devil data results in an inner vortex size approximately half of the estimated laminar non-equilibrium core. This difference can be attributed to Sinclair’s data, which comes from dust devils with circulation levels between 320
m²/s and 400 m²/s. For comparison, the mechanically generated vortex pair circulation for the B757-A data is approximately a factor of 10 less. Using the Squire-Owen [51] eddy viscosity correlation model, for a value of $a_1 = 0.00005$ the size is 1.13 times bigger. Although this model provides an approximation of the inner core radius, the range of values for $a_1$ provided by Bhagwat and Leishman [62] is too large (0.00005 – 0.0002), creating a difficult selection of an accurate constant for a specific vortex scale. It is important to realize that the maximum azimuthal velocity measurement by Garodz and Clawson [59] is affected by vortex age, vortex meandering, spacing and sensitivity of instruments as well as ground effects. Thus, a difference between any theory-based estimations and inferred measurements is expected.

Table 2. Estimates of inner vortex core radius from different turbulence models

<table>
<thead>
<tr>
<th>Model</th>
<th>$r_{core}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-757 A Inferred Radius Size</td>
<td>0.0325</td>
</tr>
<tr>
<td>Laminar Non-Equilibrium</td>
<td>0.4046</td>
</tr>
<tr>
<td>Sinclair [58]</td>
<td>0.2262</td>
</tr>
<tr>
<td>Squire-Owen [51]</td>
<td>0.0368</td>
</tr>
</tbody>
</table>

From Eq. (46) and Eq. (47), the scaling relationships are $\sqrt{\frac{\eta_p}{2\nu}}$ and its reciprocal $\sqrt{\frac{2\nu}{\eta_p}}$, respectively. If a turbulent eddy viscosity is employed, the ratio of turbulence effects to non-equilibrium pressure relaxation effects can be exploited. Moreover, since the tower fly-by measurements of Garodz and Clawson [59] provided accurate atmospheric data, Eq. (47) can be utilized to estimate the required turbulent eddy viscosity. The B-757 A produced a maximum swirl velocity of 99.4 m/s, 16 seconds after the aircraft passed by the tower, that was 51 m (167 ft.) above the ground where the measured local ambient temperature was approximately 9 °C with a relative humidity of 74%. The pressure relaxation coefficient at the Idaho Falls location (0.829 atm, 9 °C, 74% RH) was 0.512 μsec. Employing the estimated circulation and measured maximum swirl velocity, the ratio of eddy viscosity to kinematic viscosity should be:
\[
\frac{\nu_{turb}}{\eta_p} = \frac{V_{\theta,max}^2}{2}
\]

\[\nu_{turb} = 0.00253 \frac{m^2}{s} .\]

For the B-757 A aircraft flight data, the kinematic viscosity at the local ambient conditions is

\[\nu_{air}(9 \, ^\circ C, 84.0 \, kPa) = 1.696 \times 10^{-5} \frac{m^2}{s} .\]

Employing the estimated flight circulation, \( \Gamma_o = 40.6 \frac{m^2}{s} \), the associated vortex Reynolds number is \( \frac{\Gamma_o}{\nu} = 2.39 \times 10^{-6} .\) From Iversen’s correlation (Figure 3), that ratio can be estimated to be:

\[\frac{\nu_{turb}}{\nu} \approx 149 \] (55)

\[\nu_{turb} = (149)1.696 \times 10^{-5} = 0.002527 \frac{m^2}{s} .\]

On that basis, the Iversen \cite{52} circulation decay correlation is very close to the turbulent eddy viscosity needed to relate the core radius to the pressure relaxation coefficient based on reported ambient conditions and maximum swirl velocity. The size of the inner turbulent vortex core using the obtained ratio, i.e.

\[r_{core} = \frac{\Gamma_o}{5} \frac{\eta_p}{2\pi \nu_{turb}} \] (56)

\[r_{core} = 0.03257 \, m.\]

The tower fly-by data in Table 1 were used to estimate corresponding turbulent eddy viscosity ratios at the measured ambient conditions. For each flight test, the ratio between eddy viscosity and kinematic viscosity was obtained using the Iversen \cite{52} correlation from Figure 3 and shown in Table 3. Additionally, the same ratio was estimated using the pressure relaxation
coefficient, Eq. (47), given in Table 4. Since the instrumented tower fly-by experiments utilized 2-ft. vertical spacing intervals for the reported hot film anemometer locations along the tower, an accurate measurement of the inner core size was not possible. An inferred inner core size was estimated using the circulation of the aircraft and the measured maximum swirl velocity. By using the eddy viscosity ratio from Iversen \cite{52}, (Table 3) the differences between the estimated non-equilibrium core size and the inferred size varied for each test flight. For the cases of largest maximum swirl velocity (B-757 A and B-767 A, downwind), the estimated core radii agreed with the data inferred. However, an order of magnitude disagreement was observed for the B-757 B and B-767 B (leeward) vortex swirl velocity measurements, which represent the smallest measured maximum swirl velocities. Since these low-speed cores were measured in the downwind cores, ground effects and additional decay played a role. A remarkable difference can be observed in the eddy viscosity to viscosity ratio between Table 3 and Table 4. Nevertheless, using the required ratio estimated from the pressure relaxation coefficient and measured maximum swirl velocity, the estimated non-equilibrium core size agrees very closely with the inferred core size.

Table 3. Estimates of approximate turbulent eddy viscosity and non-equilibrium vortex core size based on vortex experiments using Iversen \cite{52} data

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\frac{\Gamma_0}{v}$</th>
<th>$\frac{\nu_{turb}}{v}$</th>
<th>$r_{inferred\ core}$ (m)</th>
<th>$r_{core}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-757 A 9° C; 74% RH</td>
<td>2.392x10^6</td>
<td>149</td>
<td>0.0325</td>
<td>0.0326</td>
</tr>
<tr>
<td>B-757 B 10.5° C; 52% RH</td>
<td>2.074x10^6</td>
<td>100</td>
<td>0.3836</td>
<td>0.0400</td>
</tr>
<tr>
<td>B-767 A 7° C; 51% RH</td>
<td>2.235x10^6</td>
<td>130</td>
<td>0.0514</td>
<td>0.0402</td>
</tr>
<tr>
<td>B-767 B 21° C; 18% RH</td>
<td>2.397x10^6</td>
<td>150</td>
<td>0.4696</td>
<td>0.0549</td>
</tr>
</tbody>
</table>
Table 4. Estimates of approximate turbulent eddy viscosity and non-equilibrium vortex core size based on vortex experiments using Eq. (47)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$\Gamma_0/\nu$</th>
<th>$\nu_{\text{turb}}/\nu_{\text{NE}}$</th>
<th>$r_{\text{inferred core}}$ (m)</th>
<th>$r_{\text{core}}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-757 A 9° C; 74% RH</td>
<td>2.392x10^6</td>
<td>149</td>
<td>0.0325</td>
<td>0.0326</td>
</tr>
<tr>
<td>B-757 B 10.5° C; 52% RH</td>
<td>2.074x10^6</td>
<td>0.8</td>
<td>0.3836</td>
<td>0.4467</td>
</tr>
<tr>
<td>B-767 A 7° C; 51% RH</td>
<td>2.235x10^6</td>
<td>79.5</td>
<td>0.0514</td>
<td>0.0515</td>
</tr>
<tr>
<td>B-767 B 21° C; 18% RH</td>
<td>2.397x10^6</td>
<td>2.1</td>
<td>0.4696</td>
<td>0.4709</td>
</tr>
</tbody>
</table>

Isolated axial vortices have been studied previously using vortex models. Thus, a direct comparison between the non-equilibrium model and measured azimuthal velocity of aircraft wake vortex pair was taken into consideration. Devenport, et al. [21] experimentally studied the wake vortices in the early to mid-field of two rectangular NACA 0012 half wings placed tip to tip separated by a prescribed chord length ratio. Although accurate ambient wind tunnel conditions were not recorded, the measured maximum tangential velocity could be used in Eq. (47) to estimate the pressure relaxation coefficient. Distribution of circulation for each vortex was integrated along circular contours concentric with the vortex centers using measured data. The integrated circulation levels were in the range of $0.60 - 0.67 \text{ m}^2/\text{s}$.

Figure 8 and Figure 9 show the measured averaged azimuthal velocity profile for the right and left vortices at two downstream points. Devenport, et al. [21] stated that at $x/c = 10$, the inner core of the vortex presents laminar behavior and that at $x/c = 30$, the inner core has become completely turbulent. Estimated laminar non-equilibrium velocity distribution shows similar behavior when compared to the wind tunnel vortex. Using an eddy viscosity from the relationship of Eq. (47), the turbulent non-equilibrium velocity distribution was estimated, showing good correlation to the measured vortex. However, for both cases, the size of the inner core predicted by non-equilibrium differs from the measured size.
The difference between the non-equilibrium model and the measured vortices could be attributed to the unknown ambient conditions, specifically the relative humidity which controls the pressure relaxation coefficient. Furthermore, it is important to remember that these wingtip vortex pair experiments were conducted in a stable low-turbulence wind tunnel; thus, an isolated vortex pair in a near infinite medium cannot be replicated. In summary, it has been shown that the non-equilibrium model can be used to estimate aircraft wake vortices azimuthal velocity when accurate ambient conditions are known. Moreover, using an effective eddy viscosity from the correlation of Eq. (47), a turbulent aircraft vortex core can be calculated.

Figure 8. Normalized azimuthal velocity data for the right-hand vortex behind a NACA 0012 at x/c = 10 and x/c = 30 from Devenport, et al. [21] ——— , x/c = 10 ———, x/c 30. Colored lines represent estimated non-equilibrium azimuthal velocity distribution.
Figure 9. Normalized azimuthal velocity data for the left-hand vortex behind a NACA 0012 at $x/c = 10$ and $x/c = 30$ from Devenport, et al.\cite{21}, $x/c = 10$ , $x/c = 30$. Colored lines represent estimated non-equilibrium azimuthal velocity distribution.

3.5 Non-Equilibrium Azimuthal Velocity Field of an Aircraft Wake Vortex Pair

Qualitative and quantitative proof of the non-equilibrium vortex model reliability for laminar and turbulent core regions has been examined in the prior sections. Section 1.3 demonstrated the limitations for various models describing the behavior of axial vortex flows; the majority of which are empirical fits from experimental data. The transient Lamb-Oseen model is an exception, which is an analytical exact solution from the momentum equation. However, this model overpredicts the decay rate of the velocity distribution in the vicinity of the inner core based on measured data. Since high strain rates dominate the inner viscous core of an axial vortex, departure of the local fluid pressure from thermodynamic equilibrium is plausible. Furthermore, literature has been cited justifying boundary layer-like regions such as the separated cores of two counter rotating vortices can be embedded in an otherwise inviscid fluid medium. By means of the non-equilibrium vortex model and knowledge of accurate ambient
conditions, the azimuthal velocity behavior of axial vortex core regions can be estimated, specifically for an axial aircraft wake vortex.

Ash and Zardadkhan\cite{57} have shown recently that the dissipation rate for axial vortices with non-equilibrium cores is slower than for equilibrium Burgers vortices. Furthermore, if the circulation and atmospheric weather conditions for a given aircraft in flight are specified, the Iversen eddy viscosity ratio correlation and weather-based pressure relaxation coefficient can be employed to estimate the maximum swirl velocity and wake vortex core sizes (hazard condition) in the vicinity of actual aircraft. When the embedded cores are small enough to justify the boundary-layer concept of Ting and Tung,\cite{4} it is possible to forecast the size and strength of aircraft wake vortices in the vicinity of airports.

Knowledge of the behavior of aircraft wake vortices has become an important aircraft flight safety consideration \cite{13}. The spatial distance between them affects their velocity distribution and strength, regardless of their rotational orientation \cite{21}\cite{8}. On that basis this study has used the non-equilibrium flow model to elucidate the behavior of a vortex pair produced behind an aircraft. It is first necessary to examine the merits of treating these vortex pairs as potential vortices with embedded turbulent non-equilibrium cores. The equations that model an isolated non-equilibrium axial vortex have been discussed in prior sections. Trigonometric manipulations are required to develop the velocity distributions of a counter-rotating pair of inviscid vortices of equal strength prior to formally considering non-equilibrium influences. If we consider a pair of aircraft wake vortices each centered at $\pm x_o, y_o$ and separated by a distance $2x_o = B$, in two-dimensional Cartesian coordinates the following relations apply (see Fig. 10):

$$\begin{align*}
\tilde{x} &= x \mp x_o \\
\tilde{y} &= y - y_o \\
\tilde{r}^2 &= \tilde{x}^2 + \tilde{y}^2 \\
\theta &= tan^{-1} \frac{\tilde{y}}{\tilde{x}}
\end{align*} \tag{57}$$

From Section 2, the azimuthal velocity distribution for an isolated axial vortex with a non-equilibrium core is:
Figure 10. Schematic representation of a counter-rotating pair of aircraft wake vortices

\[ v_\theta(r) = \frac{\Gamma_o}{2\pi r_{\text{core}}} \frac{r}{(r_{\text{core}})^2 + 1} = \frac{\Gamma_o}{2\pi} r + r_{\text{core}}^{2} + 1. \]

Here, we note that for \( r_{\text{core}} \to 0 \), we have a potential vortex. We will return to this. The velocity profile for a clockwise-rotating vortex (left side), located at \( x_o = -\frac{B}{2}, \ y_o = 0 \), can be written:

\[ v_{\theta, L}(x, y) = \frac{\Gamma_o}{2\pi} \frac{\sqrt{(x + \frac{B}{2})^2 + y^2}}{\sqrt{(x + \frac{B}{2})^2 + y^2 + r_{\text{core}}^2}}. \]  

(58)

Since,

\[ r_L = \sqrt{(x + \frac{B}{2})^2 + y^2} \]  

(59)
\[ \theta_L = \tan^{-1}\left(\frac{y}{x + \frac{B}{2}}\right). \]  

(60)

The vertical and horizontal components of the velocity for the clockwise-rotating vortex can be expressed respectively as:

\[ v_L = -v_{\theta,L} \cos \theta_L \]  

(61)

\[ u_L = v_{\theta,L} \sin \theta_L, \]  

(62)

knowing that:

\[ \sin \theta_L = \frac{y}{\sqrt{(x + \frac{B}{2})^2 + y^2}} \]  

(63)

\[ \cos \theta_L = \frac{x + \frac{B}{2}}{\sqrt{(x + \frac{B}{2})^2 + y^2}}. \]  

(64)

Therefore, the Cartesian velocity components for the clockwise-rotating vortex are:

\[ u_L = \frac{\Gamma_0}{2\pi} \frac{y}{\left(x + \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2} \]  

(65)

\[ v_L = -\frac{\Gamma_0}{2\pi} \frac{x + \frac{B}{2}}{\left(x + \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2}. \]  

(66)

Similarly, we can consider the velocity profile for a counterclockwise-rotating vortex (right side), located at \( x_o = \frac{B}{2}, \ y_o = 0 \), as:
\[ v_{\theta,R}(x,y) = \frac{\Gamma_o}{2\pi} \frac{\sqrt{(x - \frac{B}{2})^2 + y^2}}{(x - \frac{B}{2})^2 + y^2 + r^2_{core}} \] (67)

since:

\[ r_R = \sqrt{(x - \frac{B}{2})^2 + y^2} \] (68)

\[ \theta_R = \tan^{-1}\left(\frac{y}{x - \frac{B}{2}}\right). \] (69)

The vertical and horizontal components of the velocity for the counterclockwise-rotating vortex can be expressed respectively as:

\[ v_R = v_{\theta,R} \cos \theta_R \] (70)

\[ u_R = -v_{\theta,R} \sin \theta_R. \] (71)

Hence,

\[ \sin \theta_R = \frac{y}{\sqrt{(x - \frac{B}{2})^2 + y^2}} \] (72)

\[ \cos \theta_R = \frac{x - \frac{B}{2}}{\sqrt{(x - \frac{B}{2})^2 + y^2}}. \] (73)

Therefore, the Cartesian velocity components for the counterclockwise-rotating vortex are:

\[ u_R = -\frac{\Gamma_o}{2\pi} \frac{y}{(x - \frac{B}{2})^2 + y^2 + r^2_{core}} \] (74)
\[ v_R = \frac{\Gamma_o}{2\pi} \frac{x - \frac{B}{2}}{\left(x - \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2}. \] (75)

Since each velocity component of each vortex was derived from the solution of the modified Navier-Stokes equation for an axial vortex, we can superimpose the vertical and horizontal components of each vortex to obtain the azimuthal velocity field of a quasi-potential vortex pair with non-equilibrium cores

\[ u = -\frac{\Gamma_o}{\pi} \left\{ \frac{xyB}{\left(\left(x + \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right) \left(\left(x - \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right)} \right\} \] (76)

\[ v = \frac{\Gamma_o}{2\pi} \left\{ \frac{\left(x - \frac{B}{2}\right) \left(\left(x + \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right) - \left(x + \frac{B}{2}\right) \left(\left(x - \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right)}{\left(\left(x + \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right) \left(\left(x - \frac{B}{2}\right)^2 + y^2 + r_{\text{core}}^2\right)} \right\}. \] (77)

Now,

\[ a = \frac{B}{2} \] (78)

\[ k = \frac{\Gamma_o}{2\pi}. \] (79)

Therefore, the azimuthal velocity field can be rewritten:

\[ u = -\left\{ \frac{4akxy}{\left((x+a)^2 + y^2 + r_{\text{core}}^2\right) \left((x-a)^2 + y^2 + r_{\text{core}}^2\right)} \right\} \] (80)

\[ v = -\left\{ \frac{2ak\left(a^2 - x^2 + y^2 + r_{\text{core}}^2\right)}{\left((x + a)^2 + y^2 + r_{\text{core}}^2\right) \left((x - a)^2 + y^2 + r_{\text{core}}^2\right)} \right\}. \] (81)
Eqs. (80) and (81) represent the two-dimensional velocity field for a vortex pair with non-equilibrium cores in Cartesian coordinates, with specified strength and spacing between the vortices. However, since we are focusing on a vortex pair produced in the downstream region behind a cruising aircraft as natural consequences of lift-derived air flow, an induced downward descent speed of the vortex pair \( \frac{k}{2a} \) should be included in the velocity field. Therefore, the overall velocity is:

\[
u = -\left\{ \frac{4akxy}{[(x+a)^2 + y^2 + r_{core}^2][(x-a)^2 + y^2 + r_{core}^2]} \right\} \tag{82}
\]

\[
\nu = \frac{k}{2a} - \left\{ \frac{2ak(a^2 - x^2 + y^2 + r_{core}^2)}{[(x+a)^2 + y^2 + r_{core}^2][(x-a)^2 + y^2 + r_{core}^2]} \right\}. \tag{83}
\]

Figure 11 shows a two-dimensional representation of the velocity vector field of a vortex pair with non-equilibrium cores produced by a B-757 at specific ambient conditions (9 C and 74% RH). We must clarify that the vortex spacing used in this representation was selected to be arbitrarily small in comparison with the initial vortex spacing of an aircraft, for visualization purposes. The velocity vectors were normalized with respect to aircraft speed.
Figure 11. Normalized velocity vectors of a non-equilibrium vortex pair assuming $B = 1$. 
3.6 Non-Equilibrium “Oval” of an Aircraft Wake Vortex Pair

The behavior and interaction of a pair of point vortices was firstly studied by Thomson (Lord Kelvin) \[2\], who was deeply interested in modeling atoms and their behavior as vortex configurations in the “ether.” By allowing a pair of vortices to have a steady motion in an unbounded fluid region, he discovered that the pair was characterized by a closed fixed and forward moving area of fluid. This area is known as \textit{Kelvin’s atmosphere} or \textit{Kelvin’s oval} because of the closed contour shape of the convecting streamlines associated with this equal but oppositely rotating vortex pair. The fluid in this region remains at rest at infinity and is never displaced over more than a small distance. The steady motion of a vortex pair in a stream can be described using inviscid theory by Eq. (3) in terms of the Cartesian coordinates, the separation distance between the centroid of a vortex and the mid-plane center, and the vortex strength. Although this oval volume has not been observed directly in the atmosphere, it is considered and used as a key feature in the understanding of the behavior of vortex pair configuration. We can compare the two-dimensional Cartesian-based azimuthal velocity field of a vortex pair with non-equilibrium inner cores with a potential vortex pair by using the definition of the stream function in Eq. (3).

\[ u = \frac{\partial \psi}{\partial y} \]  
\[ v = -\frac{\partial \psi}{\partial x} \]

Thus,

\[ u = -\left\{ \frac{4akxy}{[(x+a)^2 + y^2][(x-a)^2 + y^2]} \right\} \]  
\[ v = \frac{k}{2a} - \left\{ \frac{2ak(a^2 - x^2 + y^2)}{[(x+a)^2 + y^2][(x-a)^2 + y^2]} \right\}. \]
It can be observed that Eqs. (86) and (87) have similar mathematical form when compared to Eqs. (82) and (83). The primary difference is that the non-equilibrium vortex pair velocity field includes the radius of the inner core, \( r_{\text{core}} \), as a variable. From this qualitative similarity, we hypothesize that following a similar procedure to Hicks \(^3\) (Section 1.1) the size and shape of a possible non-equilibrium “oval” can be obtained. Firstly, invoking the definition of a stream function, the mass of the incompressible flow must be conserved. Thus,

\[
\frac{\partial u}{\partial x} = -\frac{4aky(a^4 + 2a^2(r^2_{\text{core}} + x^2 + y^2) + r^4_{\text{core}} - 2r^2_{\text{core}}(x^2 - y^2) - 3x^4 - 2x^2y^2 + y^4)}{(a^2 - 2ax + r^2_{\text{core}} + x^2 + y^2)^2(a^2 + 2ax + r^2_{\text{core}} + x^2 + y^2)^2} \quad (88)
\]

\[
\frac{\partial v}{\partial y} = \frac{4aky(a^4 + 2a^2(r^2_{\text{core}} + x^2 + y^2) + r^4_{\text{core}} - 2r^2_{\text{core}}(x^2 - y^2) - 3x^4 - 2x^2y^2 + y^4)}{(a^2 - 2ax + r^2_{\text{core}} + x^2 + y^2)^2(a^2 + 2ax + r^2_{\text{core}} + x^2 + y^2)^2}. \quad (89)
\]

Therefore, satisfying mass conservation:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \quad (90)
\]

Secondly, the stream function for the non-equilibrium vortex pair, \( \psi_{\text{NE}} \), can be easily obtained by modifying Eq. (85):

\[
\psi_{\text{NE}} = -\int v \, dx . \quad (91)
\]

Thus,

\[
\psi_{\text{NE}} = -\frac{k}{2a} x + \frac{k}{2} \left( \log \frac{(x + a)^2 + y^2 + r^2_{\text{core}}}{(x - a)^2 + y^2 + r^2_{\text{core}}} \right) - \text{constant} . \quad (92)
\]

Following the mathematical procedure of Hicks \(^3\), we can utilize \( \text{constant} = 0 \) and knowing that \( k = 2Ua \), we have
\[
\psi_{NE} = -Ux + \frac{k}{2} \left( \log \left( \frac{(x + a)^2 + y^2 + r_{core}^2}{(x - a)^2 + y^2 + r_{core}^2} \right) \right). \tag{93}
\]

For the oppositely rotating potential pair, mid-plane symmetry exists due to the vortex induced flows impinging on each other. Consequently, stagnation points arise in the velocity field. These stagnation points are characterized by a streamline at \( \psi = 0 \). This streamline enclosing the vortex pair is Kelvin’s oval. Section 1.1 developed the mathematical procedure that yields the size of the oval for a potential vortex pair as a function of the distance between the vortex centroid and the center symmetry plane. By allowing a vortex pair with non-equilibrium centers to exist in an unbounded irrotational fluid and knowing that the mathematical form of the non-equilibrium pair is similar to a potential pair, the size of a possible non-equilibrium oval is desired. Since stagnation points occur at \( \psi = 0 \), from Eq. (93) we obtain:

\[
Ux = \frac{k}{2} \left( \log \left( \frac{(x + a)^2 + r_{core}^2 + y^2}{(x - a)^2 + r_{core}^2 + y^2} \right) \right). \tag{94}
\]

The half-length \( L \) of the oval can be obtained by setting \( y = 0 \):

\[
UL = \frac{k}{2} \log \left( \frac{(L + a)^2 + r_{core}^2}{(L - a)^2 + r_{core}^2} \right) \tag{95}
\]

\[
e^{-\frac{(2UL)}{k}} = 1 + \frac{4al}{(L - a)^2 + r_{core}^2}.
\]

If we use the definition of the vortex strength Eq. (79) and knowing that the induced downward speed of the vortex pair is

\[
U = \frac{\Gamma}{4\pi a}, \tag{96}
\]
Eq. (95) can be rewritten as:

\[
\frac{1}{4} \left( e^{\frac{L}{a}} - 1 \right) \left[ \left( \frac{L}{a} - 1 \right)^2 \right] + \frac{1}{4} \left( e^{\frac{r_{\text{core}}}{a}} - 1 \right) \left( \frac{r_{\text{core}}}{a} \right)^2 = \frac{L}{a} .
\] (97)

Since Eq. (97) is an implicit equation, the introduction of a pair of non-dimensional variables will facilitate its solution, i.e.

\[
\bar{L} = \frac{L}{a}
\] (98)

\[
\bar{r} = \frac{r_{\text{core}}}{a}.
\] (99)

Therefore,

\[
\frac{1}{4} \left( e^{\bar{L}} - 1 \right) \left( (\bar{L} - 1)^2 + \bar{r}^2 \right) = \bar{L} .
\] (100)

In order to obtain the half-length of the non-equilibrium vortex pair, prior knowledge of the inner core size and vortex separation from the center plane must be assessed. The half-width, \( W \), of the non-equilibrium oval can be obtained from Eq. (94) and allowing \( x \ll 1 \):

\[
\frac{(2Ux)}{k} = \log \left( 1 + \frac{4ax}{(x - a)^2 + r_{\text{core}}^2 + W^2} \right).
\] (101)

Recall for \( \varepsilon \ll 1 \), \( \ln(1 + \varepsilon) \approx \varepsilon + O(\varepsilon^2) \), which gives:

\[
\frac{2Ux}{k} \approx \frac{4ax}{W^2 + a^2 + r_{\text{core}}^2}.
\] (102)
Solving for the half-width \( w \), Eq. (102) becomes

\[
W \approx a \sqrt{3 - \left(\frac{r_{\text{core}}}{a}\right)^2}
\]  
(103)

which can be rewritten using the newly introduced non-dimensional variable Eq. (99) as

\[
\bar{W} = \frac{W}{a} \approx \sqrt{3 - \bar{r}^2}.
\]  
(104)

In summary, the approximate size of a non-equilibrium oval given by Eq. (100) and Eq. (104) has been developed using the Hicks\(^3\) procedure to obtain the size of an equivalent Kelvin oval for a potential vortex pair. Table 5 shows a summary of the equations used to estimate the size of the Kelvin oval and the equivalent non-equilibrium oval. On closer examination, a mathematical resemblance between them is observed. Once again, the only difference lies on the non-equilibrium vortex pair which depends not only on the vortex separation but also the ratio of the inner core radius to the separation distance.

<table>
<thead>
<tr>
<th>Vortex Pair</th>
<th>Half-length ( L )</th>
<th>Half-width ( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Potential</td>
<td>( \bar{L} = \frac{1}{4} (\bar{L} - 1)^2 (e^{\bar{L}} - 1) )</td>
<td>( \bar{W} \approx \sqrt{3} )</td>
</tr>
<tr>
<td>Non-Equilibrium</td>
<td>( \bar{L} = \frac{1}{4} [(\bar{L} - 1)^2 + \bar{r}^2](e^{\bar{L}} - 1) )</td>
<td>( \bar{W} \approx \sqrt{3 - \bar{r}^2} )</td>
</tr>
</tbody>
</table>

Using the equations presented in Table 4, the size and shape of the “atmosphere” that surrounds a descending vortex pair in a stream can be estimated. Utilizing the B-757 A and B-757 B flight test data from Table 1, the azimuthal velocity field and streamlines for a non-
equilibrium aircraft wake vortex pair were calculated and are shown in Figures 12 and 13, respectively. The velocity vectors were scaled by a factor of 5 for visualization purposes. In each figure, the non-equilibrium and potential oval are shown. The separation distance between vortices $B$ could not be determined in the Garodz and Clawson\cite{59} flight experiments; therefore, by assuming elliptically loaded wings the separation between vortex centroids is

$$B = \frac{\pi}{4} b$$

where $b$ is the wingspan of the aircraft. From a qualitative perspective, the size and shape of the non-equilibrium oval is the same as the Kelvin oval for the flight test data. This is attributed to the magnitude of the variable $\tilde{r}$, which linearly changes the size of the oval. The dimensionless inner core ratio for each test flight had approximate orders of magnitude between $10^{-3}$ and $10^{-2}$. Therefore, when squared non-equilibrium influence is very small with respect to oval size. Half-length and half-width of the non-equilibrium oval for the other two flight test data cases (B-767 A and B-767 B) was also estimated and is summarized in Table 6. It is observed that for the flight data, the dimensionless inner core sizes are very small and are consistent with the theoretical “boundary layer like” justification\cite{4-6}, enabling the dimensions of the non-equilibrium oval to be virtually the same as the potential oval.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>$S$ (m)</th>
<th>$B$ (m)</th>
<th>$r_{core}$ (m)</th>
<th>$\tilde{r}$</th>
<th>$\bar{L}_{NE}$</th>
<th>$\bar{W}_{NE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-757 A 9° C; 74% RH</td>
<td>38.04</td>
<td>29.87</td>
<td>0.0326</td>
<td>2.18 x10^{-3}</td>
<td>2.0872</td>
<td>1.7320</td>
</tr>
<tr>
<td>B-757 B 10.5° C; 52% RH</td>
<td>38.04</td>
<td>29.87</td>
<td>0.4467</td>
<td>29.90 x10^{-3}</td>
<td>2.0869</td>
<td>1.7318</td>
</tr>
<tr>
<td>B-767 A 7° C; 51% RH</td>
<td>47.55</td>
<td>37.34</td>
<td>0.0515</td>
<td>2.76 x10^{-3}</td>
<td>2.0873</td>
<td>1.7320</td>
</tr>
<tr>
<td>B-767 B 21° C; 18% RH</td>
<td>47.55</td>
<td>37.34</td>
<td>0.4709</td>
<td>25.22 x10^{-3}</td>
<td>2.0870</td>
<td>1.7319</td>
</tr>
</tbody>
</table>
Figure 12. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 A

Figure 13. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 B
Although comprehensive experimental data for aircraft vortex pairs observed in accurately documented ambient conditions are scarce, using the flight test data from Garodz and Clawson \cite{59}, we can infer that the behavior analyzed of the non-equilibrium oval size will be the same independent of the size of the aircraft. Thus, for an aircraft wake vortex pair, the non-equilibrium oval essentially matches the Kelvin oval. Crucially, as shown in Figure 6, the pressure relaxation can vary by two orders of magnitude, depending on ambient temperature and relative humidity. Consequently, under some (warm, humid) ambient conditions, the vortex core size and maximum swirl velocity can produce severe hazard conditions, while being relatively benign during other (cold, dry) ambient conditions.

At some interval of time, in the far-field, the counter-rotating vortex pair start to move laterally, closing the gap distance between them, causing merger and instability \cite{62}. Merging between aircraft wake vortices depends directly on the ratio between the size of inner core and the separation between the centroids $\bar{r}$. Several investigations have taken the objective of finding the critical conditions in which a vortex pair becomes unstable and merges. Overman and Zabusky \cite{63} found a value of $\bar{r}_c = 0.6$ by numerically solving vortex patches which was confirmed experimentally by Griffiths and Hopfinger \cite{64}. Saffman and Szeto \cite{8} found destabilization of two Euler equilibrium numerical solutions for a value of $\bar{r}_c = 0.630$. At these ratios, the dimensionless inner core size will have a direct influence on the size of the non-equilibrium oval. Therefore, a hypothesis arises; as the vortices move towards merger, the “atmospheric” oval size that accompanies the pair will change from the stable potential oval size. Since the non-equilibrium oval size depends directly on the dimensionless inner core size $\bar{r}$, a set of solutions to the dimensionless half-length and dimensionless half-width were obtained numerically and shown in Figure 14.

The dimensionless non-equilibrium half-length and half-width depart from the potential magnitudes approximately at $\bar{r} = 10^{-1}$, and subsequently decays rapidly. For an aircraft vortex pair approaching merger close to the critical condition value of $\bar{r}_c$, the non-equilibrium oval size will be different from the potential oval. A visualization of this difference can be observed in Figure 15. This figure shows the normalized azimuthal velocity field with its respective streamlines for the B-757 A aircraft vortex pair at a critical separation. Knowing that the oval lines
represent the streamlines position of the stagnant flow, it is observed that the streamlines in the domain do not coincide with the size of Kelvin’s oval; however, the streamlines wrap tightly around the non-equilibrium oval line.

![Figure 14. Dimensions of the non-equilibrium and potential oval for different values of dimensionless inner core size](image)

Figure 14. Dimensions of the non-equilibrium and potential oval for different values of dimensionless inner core size.
Figure 15. Normalized velocity vectors and streamlines of the non-equilibrium vortex pair of the test flight data B-757 A at critical separation ($\bar{r}_c = 0.6$).
CHAPTER 4

CONCLUSIONS

4.1 Summary

The present study utilized non-equilibrium pressure theory to develop a theoretical model for the counter-rotating vortex pair produced by aircraft, with the objective of improving our fundamental knowledge of their physical behavior. Zuckerwar and Ash \cite{53} modified the Navier-Stokes equation to incorporate non-equilibrium pressure effects, and by assuming steady state and using cylindrical coordinates, AZZ \cite{55} developed an isolated axial vortex model where the inner viscous core was controlled by non-equilibrium pressure gradient forces in direct response to the coupling of centrifugal forces with unsustainable shearing strain rates near the rotational axis.

An isolated aircraft vortex was modeled using the non-equilibrium model and compared directly with other existing vortex models from the literature review. It was shown that the empirically based Burnham-Hallock \cite{44} vortex model is identical to the theoretical non-equilibrium model when the vortex core radius is known. Unlike Burnham and Hallock, when actual inflight aircraft circulation levels can be estimated, local weather conditions determine the core characteristics and severity of the generated vortex hazard.

Non-equilibrium theory demonstrates that the maximum swirl velocity can vary by a factor of ten depending on weather conditions. Since aircraft trailing vortices have naturally turbulent inner cores, \cite{13} a simple eddy viscosity model was introduced to satisfy the turbulent energy embedded in the non-equilibrium vortex cores. Several eddy viscosity correlations were discussed and compared. By exploiting the relationship between the ratio of turbulence effects to non-equilibrium pressure relaxation effects, the required turbulent eddy viscosity for several aircraft data could be estimated. It was shown that, using this correlation, the turbulent non-equilibrium vortex core size agreed well with the inferred vortex core size.

Non-equilibrium vortex cores can exist in a nominally potential fluid. This was justified by following the works of Ting and Tung \cite{4} who modeled vortex core regions as embedded boundary
layer like domains embedded in an inviscid fluid. Therefore, the non-equilibrium vortex pair is considered a potential vortex pair with small, embedded sheaths of non-equilibrium fluid in the inner core existing in an unbounded fluid region.

A two-dimensional steady state model for a counter-rotating vortex pair in an unbounded domain was derived by superimposing two vortices with non-equilibrium inner cores. The accompanying isolating “atmosphere” enclosing the vortex pair, commonly known as a “Kelvin oval” was investigated for a pair of non-equilibrium vortices (non-equilibrium oval). Using stream functions, it was shown that for a cruising aircraft the non-equilibrium oval size is virtually the same as the Kelvin oval. When increasing dilation of the vortex cores of an aircraft wake occurs or lateral movement of the aircraft vortex pair reduces the spacing between them, the non-equilibrium vortex pair model predicts instability. For a critical magnitude of $\bar{r}$, the non-equilibrium oval size changes drastically from the potential oval size.

4.2 Future work

The current work can be extended. Firstly, the current work is limited to steady state. Since a steady state solution exists for an axial vortex considering non-equilibrium effects, a solution assuming time dependency could be undertaken. Therefore, several physical behaviors of the non-equilibrium vortex pair that occur in the far-field of the downstream of an aircraft can be investigated, such as, merger between vortices and instability. Secondly, although the “atmosphere” enclosing the vortex pair has not been measured directly in the atmosphere, evidence was found that the size of the oval volume changes from the stable potential values when merger is near if non-equilibrium effects are considered. Therefore, an experiment should be undertaken with the sole objective of tracking and measuring the enclosing oval. Future work can include experiments of aircraft wake vortices with no ground effects and where ambient conditions, especially relative humidity, is measured with high accuracy to further evaluate and compare the non-equilibrium vortex model.
REFERENCES


3. W M Hicks, LIX. *The mass carried forward by a vortex*. Philosophical Magazine Series 6, 38:227, 597-612. (1919)


56. **S Corrsin** and **A L Kistler.** *The free-stream boundaries of turbulent flow*. NACA TR 1244 (1955)


APPENDIX

MATLAB SCRIPTS

a. Non-Equilibrium Vortex Pair Visualization

```matlab
%% Input Parameters (B757-200 Aircraft)------> Goradz and Clawson Data
w    = 86638;   % Aircraft weight (kg)
rho  = 1.038;   % Air Density (kg/m^3)
mu   = 1.76e-5; % Dynamic viscosity (kg.s/m)
nu   = 16.96e-6; % Kinematic viscosity (m^2/s)
nuv  = 149*nu;  % Turbulent Eddy viscosity
etap = 0.512e-6 % Pressure relaxation coefficient (microsec)
s    = 38.04;    % Aircraft wingspan (m)
Uinf = 68.9;    % Aircraft speed (m/s)
cir  = (4*w)/(rho*pi*s*Uinf); % Root Circulation
Rcore = cir/(2^(5/2)*pi)*sqrt(etap/nuv); % Vortex Core Radius (m)
B = s*pi/4 ;   % Vortex Separation (m)
a = B/2;       %
k = cir/(2*pi);       % Vortex Separation (m)
U = k/(2*a);

%% Grid Generator
xl = 0.2;
yl = 0.2 ;
dx = 0.0075;
xm = -xl:dx:xl;
ym = -yl:dx:yl;
[x,y] = meshgrid(xm,ym);

%% Superimposed Non-Equilibrium Vortices
u = - (cir.*x.*y.*B)./(pi.*((x+B./2).^2 + y.^2 + Rcore.^2).*((x-B./2).^2 + y.^2 + Rcore.^2));
v = k./2.*a + cir./(2.*pi).*((x-B./2).*((x+B./2)^2 + y.^2 + Rcore.^2)
  - (x+B./2).*((x-B./2)^2 + y.^2 + Rcore.^2))./(((x+B./2).^2 + y.^2 + Rcore.^2).*
  (x-B./2).^2 + y.^2 + Rcore.^2));
psiNE = -U.*x + k./2.*log(((x+a).^2+y.^2 +Rcore.^2)./((x-a).^2+y.^2+Rcore.^2));

%% Potential Flow of Vortex Pair
up2 = -(4.*a.*k.*x.*y)./((a.^2-2.*a.*x+x.^2+y.^2).*a.*x.*y.^2+y.^2));
vp2 = k./(2.*a) - (2.*a.*k.*a.*x.*x+y.^2+y.^2))./((a.^2-2.*a.*x+x.^2+y.^2));
psi = -U.*x + k./2.*log(((x+a).^2+y.^2)./((x-a).^2+y.^2));

%% Kelvin Oval
b_ellipse = 2.08725*a ;  % Half-length of the Kelvin oval
w_ellipse = 1.732*a ;    % Half-width of the Kelvin oval
x_ellipse = b_ellipse*cos(t) ;
y_ellipse = w_ellipse*sin(t) ;
```
%% Possible Non-Equilibrium "Oval" Size
bNE_ellipse = 1.9613*a;  % Half-length of the Non-Equilibrium oval
wNE_ellipse = 1.6248*a;  % Half-width of the Non-Equilibrium oval
xNE_ellipse = bNE_ellipse*cos(t);  
yNE_ellipse = wNE_ellipse*sin(t);

%% Post-processing
figure(1)
quiver(x/B,y/B,u/Uinf,v/Uinf,1,'k'); hold on
plot(xNE_ellipse/B,yNE_ellipse/B,'g-','LineWidth',1);
plot(x_ellipse/B,y_ellipse/B,'r--','LineWidth',1);
startx = xm/B;
starty = -1.847*ones(size(xm/B));
streamline(x/B,y/B,u/Uinf,v/Uinf,startx,starty);
xlabel('x/B','Interpreter','Latex')
ylabel('y/B','Interpreter','Latex')
l=legend('Normalized Velocity Vectors','Non-Equilibrium Oval','Kelvin Oval','Non-Equilibrium Flow Streamlines');
set(l,'Interpreter','Latex');
xlim([-1.8 1.8])
ylim([-1.8 1.8])

b. Comparison of Vortex Models

%% Input Parameters (B757-200 Aircraft)-----> Goradz and Clawson Data
w    = 86638;  % Aircraft weight (kg)
rho  = 1.038;  % Air Density (kg/m^3)
mu   = 1.76e-5;  % Dynamic viscosity (kg.s/m)
nu   = 16.96e-6;  % Kinematic viscosity (m^2/s)
s    = 38.04;  % Aircraft wingspan (m)
Uinf = 68.69;  % Aircraft speed (m/s)
etap = 0.512e-6;  % Pressure relaxation coefficient (microsec)
nut = 150*nu;  % Turbulent Eddy viscosity
cir  = (4*w)/(rho*pi*s*Uinf);  % Circulation
Rcore = cir/(2^(5/2)*pi)*sqrt(etap/nu);  % Vortex Core Radius

%% Values of Radius
dx = 0.0001;
r = linspace(dx,5,1000);
r1 = linspace(dx, Rcore, 500);
r2 = linspace(Rcore + dx, 5,500);
r11 = linspace(dx, 1.4*Rcore,500);
r22 = linspace(1.4*Rcore + dx,5,500);
rr = [r1 r2];
rr2 = [r11 r22];

%% Non-Equilibrium Vortex
Vne = cir/(2*pi).*r./(r.^2 + Rcore^2);

%% Burnham-Hallok Vortex
Vbh = cir./(2.*pi).*r./(r.^2 + Rcore^2);
%% Lamb-Oseen Vortex
Vlb = cir./(2.*pi.*r).*(1-exp(-1.2526.*(r./Rcore).^2));

%% Proctor Vortex
Vp1 = 1.0939.*cir./(2.*pi.*r11).*(1-exp(-10.*(1.4*Rcore./s).^(0.75))).*(1-exp(-1.2527.*(r11./Rcore).^2)); %
First Interval r<=Rcore
Vp2 = cir./(2.*pi.*r22).*(1-exp(-10.*(r22./s).^(0.75))); % Second Interval r>Rcore
Vpt = [Vp1 Vp2];

%% Winckelmans, et al Vortex
bo = 10;
bi = 500;
p = 3;
Vw = cir./(2.*pi.*r).*(1 - exp((-bi.*(r./s).^2)/(1 + ((bi./bo).* (r./s).^(1.2))^p).^(1/p))));

%% Rankine Vortex
Vr1 = (cir.*r1)./(2.*pi.*Rcore.^2); % First Interval r<=Rcore
Vr2 = cir./(2.*pi.*r2); % Second Interval r>Rcore
Vrt = [Vr1 Vr2];

%% Post-processing
figure (1)
plot(r/Rcore,Vne/Uinf,'k-*','LineWidth',1);hold on
plot(r/Rcore,Vbh/Uinf,'ko','LineWidth',1)
plot(r/Rcore,Vlb/Uinf,'k:','LineWidth',1)
plot(rr/Rcore,Vrt/Uinf,'k-.','LineWidth',1)
plot(rr2/Rcore,(Vpt/Uinf),'k-','LineWidth',1)
plot(r/Rcore,Vw/Uinf,'k--','LineWidth',1)
grid on
box on
xlim([0 10])
xlabel('$r / R_{core}$','Interpreter','Latex')
ylabel('$V_{\theta} / U_{\infty}$','Interpreter','Latex')
l = legend('Non-Equilibrium Model','Burnham-Hallock Model','Lamb-Oseen Model','Rankine Model','Proctor Model','Winckelmans, et al Model');
set(l,'Interpreter','latex');

c. Non-Equilibrium Oval Size

syms x
rbar = 0.0001:0.0005:1;

for i = 1:length(rbar)
    rbari = rbar(i);
f(i) = 1/4*(exp(x)-1)*((x-1)^2+rbari^2)-x == 0;
sol(i) = vpasolve(f(i),x,100);
sol2(i) = sqrt(3-rbari^2);
end

fig = figure;
left_color = [0 0 0];
right_color = [0 0 0];
set(fig,'defaultAxesColorOrder',[left_color; right_color]);
grid on
box on
yyaxis left
plot(rbar,sol,'b--','LineWidth',1);
yline(2.08725,'b-','LineWidth',1)
set(gca, 'XScale', 'log')
%set(gca, 'YScale', 'log')
xlabel('$\bar{r}$','Interpreter','Latex')
ylabel('$\bar{L}$','Interpreter','Latex')
ylim([1.7 2.15])

yyaxis right
plot(rbar,sol2,'r--','LineWidth',1);
yline(sqrt(3),'r-','LineWidth',1)
set(gca, 'XScale', 'log')
%set(gca, 'YScale', 'log')
xlabel('$\bar{r}$','Interpreter','Latex')
ylabel('$\bar{W}$','Interpreter','Latex')
set(l, 'Interpreter', 'Latex');
ylim([1.4 1.8])
VITA

MANUEL AYALA
Old Dominion University
Department of Mechanical and Aerospace Engineering
Norfolk, Virginia, 23529

EDUCATION

Master of Science in Mechanical Engineering
Old Dominion University, Norfolk, VA, USA

Expected August 2021

Bachelor of Science in Mechanical Engineering
Universidad de Oriente, Barcelona, VENEZUELA

August 2019

PUBLICATIONS

