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A Formal Power Series Approach to Multiplicative Dynamic and Static Output Feedback

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Abstract: The goal of the paper is two-fold. The first of which is to derive an explicit formula to compute the generating series of a closed-loop system when a plant, given in a Chen-Fliess series description is in multiplicative output feedback connection with another system given in Chen-Fliess series description. In addition, the multiplicative dynamic output feedback connection has a natural interpretation as a transformation group acting on the plant. The second of the two-part goal of this paper is same as the first part albeit when the Chen-Fliess series in the feedback is replaced by a memoryless map, so called multiplicative static feedback connection.

Keywords: Nonlinear systems, Chen-Fliess series, Multiplicative output feedback
AMS subject classification: 93C10, 93B52, 93B25

1. INTRODUCTION

The objective of the document is two fold and works with the Chen-Fliess functional series (Fliess, 1981). There is no need that these input-output systems have a state space realization and thus, the results presented here are independent of any state space embedding when a realization is possible (Fliess, 1983). Firstly, let F_c and F_d be two nonlinear input-output systems represented by Chen-Fliess series. It was shown in Gray & Li (2005) that the *additive feedback* interconnection of two such systems result in a Chen-Fliess series description for the closed-loop system. An efficient computation of the generating series for closed-loop system is facilitated through a combinatorial Hopf algebra (Gray, et al., 2014a; Duffaut Espinosa, et al., 2016). The convergence of the closed-loop system was characterized in Thitsa & Gray (2012). The feedback product formula and its computation were used to solve system inversion problems (Gray, et al., 2014b) and trajectory generation problems (Duffaut Espinosa & Gray, 2017). However, when the nature of interconnection becomes *multiplicative feedback*, the similar set of questions persist in general. It is known that, in single-input single-output setting, the closed-loop system in the affine feedback case (of which multiplicative feedback is a special case) has a Chen-Fliess series description and the computation of feedback formula is facilitated through a combinatorial Hopf algebra (Gray & Ebrahimi-Fard, 2017). The present document, in one part, shows that even in multi-input multi-output setting the closed-loop system under multiplicative feedback has a Chen-Fliess series representation and provides an explicit expression of the closed-loop generating series termed *multiplicative dynamic feedback product*. It will be shown that this feedback product has a natural interpretation as a transformation group acting on the plant. The algorithmic framework for the computation of the multiplicative dynamic feedback product formula for a general multi-input multi-output case and characterization of convergence for the closed-loop system is deferred to future work. Hence, the document is void of a computational example.

Suppose F_d in the feedback path be replaced by a memoryless function f_d which is coined as *additive static feedback* connection (Isidori, 1995), then the closed-loop system for the additive static feedback interconnection is known to have a Chen-Fliess series representation and an explicit expression for the closed-loop generating series, called *Wiener-Fliess feedback product* and an algorithmic framework for computing the feedback product exists in literature (Venkatesh & Gray, 2021). The convergence of the closed-loop system was characterized in Venkatesh (2021). However the questions remain open when the nature of static feedback becomes *multiplicative*. Hence, the second of the two-part goal of this paper is to show that the closed-loop system in multiplicative static feedback connection has a Chen-Fliess series representation and an explicit expression for the closed-loop generating series, termed *multiplicative static feedback product*, is provided. Further, the feedback product is shown as a transformation group acting on the plant. As in the case of multiplicative dynamic feedback product, the algorithmic framework for the computation of the multiplicative static feedback product and characterization of convergence for the closed-loop system is deferred to future work.

The paper is organized as follows. The next section provides a summary of the concepts related to Chen-Fliess series and their interconnections. The section also builds

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the pivotal *multiplicative dynamic output feedback group* and provides a brief discussion on formal static maps and Wiener-Fliess composition. Section 3 is where the multiplicative dynamic feedback connection is analyzed in Section 4 is where the results of the multiplicative static feedback connection is detailed. The conclusions of the paper and directions for future work is given in the last section. Apropos to the page limit, proofs to the results were not furnished and the reader is requested to read the `arXiv`** version fo the document.

2. PRELIMINARIES

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. Its *length* is $|\eta| = k$. The set of all words including the empty word, $\emptyset$, is denoted by X^*, and $X^+ := X^* \backslash \emptyset$. The set X^* forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is denoted by (c, η) and called the *coefficient* of η in c. A series c is *proper* when $(c, \emptyset) = 0$ else it is a *non-proper* series. The *support* of c, $\mathrm{supp}(c)$, is the set of all words having nonzero coefficients. The *order* of c, $\mathrm{ord}(c)$, is the length of the minimal length word in its support. Normally, c is written as a formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The collection of all formal power series over X is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$. The i^{th} component of a series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is denoted by c_i viz $(c_i, \eta) = (c, \eta)_i$. The subset of all proper series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is denoted by $\mathbb{R}^\ell_p\langle\langle X\rangle\rangle$, while the subset of non-proper series is denoted by $\mathbb{R}^\ell_{np}\langle\langle X\rangle\rangle$.

Definition 2.1. A series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is called *purely improper* if c_i is non-proper $\forall i = 1, \ldots, \ell$. The subset of all purely improper series in $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is denoted by $\mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle$.

Observe that $\mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle \subsetneq \mathbb{R}^\ell_{np}\langle\langle X\rangle\rangle$ if $\ell > 1$, otherwise $\mathbb{R}_{pi}\langle\langle X\rangle\rangle = \mathbb{R}_{np}\langle\langle X\rangle\rangle$. For the purpose of the document, the product of two vectors in $\mathbb{R}^n$ is given by the Hadamard product. The *Cauchy product*, $\mathscr{C} : \mathbb{R}^\ell\langle\langle X\rangle\rangle \times \mathbb{R}^\ell\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^\ell\langle\langle X\rangle\rangle$ defined as $(c, d) \mapsto c.d$, where

$$(c.d, \eta) = \sum_{\substack{\zeta,\nu \in X^* \\ \zeta\nu = \eta}} (c, \zeta)\,(d, \nu)$$

Observe that $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ constitutes an associative $\mathbb{R}$-algebra under the Cauchy product. If $d \in \mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle$, then Cauchy inverse of d, denoted by d^{-1} is defined as

$$d_i^{-1} = (d_i, \emptyset)^{-1} \left(\sum_{k \in \mathbb{N}_0} (d_i')^k \right),$$

where $d_i' = 1 - (d_i / (d_i, \emptyset))$. Hence, $\mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle$ forms a group under Cauchy product with $\mathbb{1} = [11 \cdots 1]^t \in \mathbb{R}^\ell$ as the identity element. The shuffle product of two words which is a bilinear product uniquely specified by

$$(x_i\eta) ⧢ (x_j\xi) = x_i(\eta ⧢ (x_j\xi)) + x_j((x_i\eta) ⧢ \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta ⧢ \emptyset = \emptyset ⧢ \eta = \eta$ (Fliess, 1981). The shuffle product of two series, $(c, d) \mapsto c ⧢ d$ is defined as

$$(c ⧢ d, \eta) = \sum_{\substack{\zeta,\nu \in X^* \\ \zeta ⧢ \nu = \eta}} (c, \zeta)\,(d, \nu)$$

Note that $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ forms an associative and commutative $\mathbb{R}$-algebra under the shuffle product. If $d \in \mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle$, then shuffle inverse of d, denoted by $d^{⧢ -1}$ is defined as

$$d_i^{⧢ -1} = (d_i, \emptyset)^{-1} \left(\sum_{k \in \mathbb{N}_0} (d_i')^{⧢ k} \right),$$

where $d_i' = 1 - (d_i / (d_i, \emptyset))$. Hence, $\mathbb{R}^\ell_{pi}\langle\langle X\rangle\rangle$ forms an Abelian group under the shuffle product with $\mathbb{1} = [11 \cdots 1]^t \in \mathbb{R}^\ell$ as the identity element. The set $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ is an ultrametric space with the ultrametric

$$\kappa(c, d) = \sigma^{\mathrm{ord}(c-d)},$$

where $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $\sigma \in]0, 1[$. For brevity, $\kappa(c, 0)$ is written as $\kappa(c)$, and $\kappa(c, d) = \kappa(c - d)$. The ultrametric space $(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \kappa)$ is known to be Cauchy complete (Berstel & Reutenauer, 1988). The following definition of contraction maps will be useful.

Definition 2.2. Given metric spaces (E, d) and (E', d'), a map $f : E \longrightarrow E'$ is said to be a *strong contraction map* if $\forall s, t \in E$, it satisfies the condition $d'(f(s), f(t)) \leq \alpha d(s, t)$ where $\alpha \in [0, 1[$. If $\alpha = 1$, then the map f is said to be a *weak contraction map* or a *non-expansive map*.

In the event that the letters of X commute, the set of all formal power series is denoted by $\mathbb{R}^\ell[[X]]$. The formal series with commuting alphabet is indispensable in definition of the formal static maps in Section 2.4. For any series $c \in \mathbb{R}^\ell[[X]]$, the natural number $\overline{\omega}(c)$ corresponds to the order of its proper part $c - (c, \emptyset)$.

2.1 Chen-Fliess Series

Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R\}$. Given any series $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$, the corresponding *Chen-Fliess series* is

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0), \tag{1}$$

where $E_\emptyset[u] = 1$ and

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau$$

with $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$ (Fliess, 1981). If there exists constants $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|} |\eta|!, \quad \forall \eta \in X^*,$$

then F_c constitutes a well defined mapping from $B^m_{\mathfrak{p}}(R)[t_0, t_0 + T]$ into $B^\ell_{\mathfrak{q}}(S)[t_0, t_0 + T]$ for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray & Wang, 2002). This map is referred to as a *Fliess operator*. Here $\mathbb{R}^\ell_{LC}\langle\langle X\rangle\rangle$ will denote the set of all such *locally convergent* generating series. In the absence of any convergence criterion, (1) only defines an operator in a formal sense.

2.2 Interconnections of Chen-Fliess series

Given Chen-Fliess series F_c and F_d, where $c, d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, the parallel and product connections satisfy $F_c + F_d = F_{c+d}$ and $F_c F_d = F_{c \,\sqcup\!\sqcup\, d}$, respectively (Ree, 1958; Fliess, 1981). The parallel and product connections preserve local convergence and hence the interconnected systems have a Fliess operator representation (Thitsa & Gray, 2012; Venkatesh, 2021). When Chen-Fliess series F_c and F_d with $c \in \mathbb{R}^{k}\langle\langle X'\rangle\rangle$ and $d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion, where $|X'| = \ell + 1$, the composite system $F_c \circ F_d$ has a Chen-Fliess series representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X'^{*}} (c, \eta)\, \psi_d(\eta)(\mathbf{1}) \qquad (2)$$

(Ferfera, 1979). Here $\mathbf{1}$ denotes the monomial $1\emptyset$, and ψ_d is the continuous (in the ultrametric sense) algebra homomorphism from $\mathbb{R}\langle\langle X'\rangle\rangle$ to the set of vector space endomorphisms on $\mathbb{R}\langle\langle X\rangle\rangle$, $\mathrm{End}\left(\mathbb{R}\langle\langle X\rangle\rangle\right)$, uniquely specified by $\psi_d(x'_i\eta) = \psi_d(x'_i) \circ \psi_d(\eta)$ with $\psi_d(x'_i)(e) = x_0(d_i \sqcup\!\sqcup\, e)$, $i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := \mathbf{1}$). By definition, $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. The cascade interconnection preserves local convergence and thus the composite has a Fliess operator representation (Thitsa & Gray, 2012). Given a series $e \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, define a map $\Upsilon_e : \mathbb{R}^{k}\langle\langle X'\rangle\rangle \longrightarrow \mathbb{R}^{k}\langle\langle X\rangle\rangle$ defined as $c \mapsto c \circ e$. The following theorem infers that Υ_e is an $\mathbb{R}$-algebra homomorphism from the shuffle algebra of $\mathbb{R}^{k}\langle\langle X'\rangle\rangle$ to the shuffle algebra of $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$.

Theorem 2.1. (Gray & Li, 2005) Let $c, d \in \mathbb{R}^{k}\langle\langle X'\rangle\rangle$, $e \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $\alpha \in \mathbb{R}$, such that $|X'| = \ell + 1$, then $(\alpha c + d) \circ e = \alpha\,(c \circ e) + (d \circ e)$ and $(c \sqcup\!\sqcup\, d) \circ e = (c \circ e) \sqcup\!\sqcup\, (d \circ e)$.

The composition product is a strong contraction map with respect to its right argument in the ultrametric topology and is stated in the following theorem.

Theorem 2.2. (Gray & Li, 2005) Let $c \in \mathbb{R}^{k}\langle\langle X'\rangle\rangle$ and $d, e \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, such that $|X'| = \ell + 1$, then $\kappa\,(c \circ d, c \circ e) \leq \sigma\kappa\,(d, e)$ where $\sigma \in [0, 1[$.

The *unital shuffle Chen-Fliess series* arise primarily in the multiplicative output dynamic feedback interconnection of Chen-Fliess series as described in Gray & Ebrahimi-Fard (2017) and Section 3 of this document. For $|X| = m + 1$, the set of all unital shuffle Chen-Fliess series, denoted by ${}^{\delta}\mathscr{F}$, is defined as ${}^{\delta}\mathscr{F} = \{I.F_d : d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle\}$, where I denotes the identity operator. It is convenient to introduce a symbol δ as the generating series for the identity map viz. $F_{\delta}[u] = I[u] = u$. Hence, $u.F_d[u] = I.F_d[u] = F_{\delta}.F_d[u] = F_{\delta \,\sqcup\!\sqcup\, d}[u] = F_{\delta_d}[u]$, with ${}^{\delta}d = \delta \sqcup\!\sqcup\, d$. The series $\delta \sqcup\!\sqcup\, d$ is the generating series for the Chen-Fliess series depicting the feedforward product of input with the output of F_d. The set of all generating series for ${}^{\delta}\mathscr{F}$ shall be denoted by $\delta \sqcup\!\sqcup\, \mathbb{R}^{m}\langle\langle X\rangle\rangle$. The cascade interconnection of a Chen-Fliess series F_c and F_d along with the multiplicative feedforward of the input, as shown in Figure 1, is denoted by $F_{c\,\breve{\circ}\,{}^{\delta}d}$ viz. $F_c[u.F_d[u]] = F_c \circ F_{\delta_d}[u] = F_{c\,\breve{\circ}\,{}^{\delta}d}[u]$, where $c\,\breve{\circ}\,{}^{\delta}d$ denotes the *multiplicative mixed composition product* of $c \in \mathbb{R}^{p}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$. The multiplicative

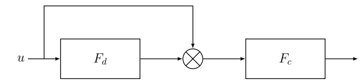

Fig. 1. Cascade connection of Chen-Fliess F_d with F_c along with multiplicative feedforward of input

mixed composition product of c and d , $c\,\breve{\circ}\,{}^{\delta}d$ can be defined as

$$c\,\breve{\circ}\,{}^{\delta}d = \sum_{\eta \in X^{*}} (c, \eta)\, \bar{\phi}_d\,(\eta)\,(\mathbf{1}) = \sum_{\eta \in X^{*}} (c, \eta)\, \eta\,\breve{\circ}\,{}^{\delta}d,$$

where $\bar{\phi}_d : \mathbb{R}\langle\langle X\rangle\rangle \longrightarrow \mathrm{End}\left(\mathbb{R}\langle\langle X\rangle\rangle\right)$ is an $\mathbb{R}$-algebra homomorphism such that $\bar{\phi}_d(x_0)(e) = x_0 e$ and $\bar{\phi}_d(x_i)(e) = x_i(d_i \sqcup\!\sqcup\, e)$. Here $\mathbb{R}\langle\langle X\rangle\rangle$ is taken as an $\mathbb{R}$-algebra under Cauchy product and $\mathrm{End}\left(\mathbb{R}\langle\langle X\rangle\rangle\right)$ is an $\mathbb{R}$-algebra under composition. It is straightforward that multiplicative mixed composition product is linear in its left argument. The following results are already known in the single-input single-output (SISO) setting. However, their multi-input multi-output (MIMO) extensions are straightforward and to avoid reiteration of the proofs, only the statements are provided in this document. The following theorem assets that for any $e \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$, the map $\Gamma_e : \mathbb{R}^{p}\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^{p}\langle\langle X\rangle\rangle$ given by $d \mapsto d\,\breve{\circ}\,{}^{\delta}e$ is an $\mathbb{R}$-algebra endomorphism on the shuffle algebra $\mathbb{R}^{p}\langle\langle X\rangle\rangle$.

Theorem 2.3. (Gray & Ebrahimi-Fard, 2017) Let $c, d \in \mathbb{R}^{p}\langle\langle X\rangle\rangle$ and $e \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$, then $(c \sqcup\!\sqcup\, d)\,\breve{\circ}\,{}^{\delta}e = \left(c\,\breve{\circ}\,{}^{\delta}e\right) \sqcup\!\sqcup \left(d\,\breve{\circ}\,{}^{\delta}e\right)$.

The following theorem states the strong contraction property of the multiplicative mixed composition product which is an essential result in Section 3.

Theorem 2.4. (Gray & Ebrahimi-Fard, 2017) Let $d, e \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ and $c \in \mathbb{R}^{p}\langle\langle X\rangle\rangle$, then $\kappa\left(c\,\breve{\circ}\,{}^{\delta}d, c\,\breve{\circ}\,{}^{\delta}e\right) \leq \sigma^{\mathrm{ord}(c')}\kappa\,(d, e)$, where $c' = c - (c, \emptyset)$, the proper part of c.

Since $\mathrm{ord}\,(c') \geq 1$ and $\sigma \in]0, 1[$, then from Theorem 2.4, the map $\bar{\Gamma}_c : e \mapsto c\,\breve{\circ}\,{}^{\delta}e$ is a strong contraction map in the ultrametric topology. The following theorem states the mixed associativity of the composition and multiplicative mixed composition product. The result in the SISO setting is stated in Gray & Ebrahimi-Fard (2017), and its extension to the MIMO case is purely straightforward.

Theorem 2.5. (Gray & Ebrahimi-Fard, 2017) Let $X' = \{x'_0, \ldots, x'_p\}$ and $c \in \mathbb{R}^{q}\langle\langle X'\rangle\rangle$. Let $d \in \mathbb{R}^{p}\langle\langle X\rangle\rangle$ and $e \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$, then $c \circ \left(d\,\breve{\circ}\,{}^{\delta}e\right) = (c \circ d)\,\breve{\circ}\,{}^{\delta}e$.

2.3 Multiplicative Dynamic Output Feedback Group

The dynamic multiplicative feedback group plays a vital role in computation of the multiplicative dynamic feedback formula, as pictured in SISO setting, in Gray & Ebrahimi-Fard (2017) and in assessing the feedback as a group action in Section 3. Consider the cascade interconnection of two unital shuffle Chen-Fliess series F_{δ_c} and F_{δ_d}, where $c, d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$. The composite system is given by the Chen-Fliess series $F_{\delta_{c \circ {}^{\delta}d}}$, where ${}^{\delta}c \circ {}^{\delta}d$ denotes the *multiplicative composition product* of ${}^{\delta}c$ and ${}^{\delta}d$ and is defined as

$${}^{\delta}c \circ {}^{\delta}d = \delta \sqcup\!\sqcup \left(d \sqcup\!\sqcup\, c\,\breve{\circ}\,{}^{\delta}d\right) = {}^{\delta}\left(d \sqcup\!\sqcup\, c\,\breve{\circ}\,{}^{\delta}d\right). \qquad (3)$$

There is an abuse of notation $\circ$ between (2) and (3), however the meaning of $\circ$ should always be clear from the context. The following theorem states the multiplicative composition product is associative. The result, along with Theorem 2.7 were stated and proven in Lemma 3.6 of Gray & Ebrahimi-Fard (2017) in the SISO setting but the authors' proofs were independent of the SISO assumption. Hence, the statements along with the proofs naturally extend to the MIMO setting.

Theorem 2.6. (Gray & Ebrahimi-Fard, 2017) Let $c, d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then, $\left({}^{\delta}c \circ {}^{\delta}d\right) \circ {}^{\delta}e = {}^{\delta}c \circ \left({}^{\delta}d \circ {}^{\delta}e\right)$.

Observe that (3) and Theorem 2.6 infer that $\delta \sqcup\!\sqcup \mathbb{R}^m\langle\langle X\rangle\rangle$ forms a noncommutative monoid under multiplicative composition product, with the identity element ${}^{\delta}\mathbb{1}$. The following theorem states that the multiplicative mixed composition product is a right action on $\mathbb{R}^q\langle\langle X\rangle\rangle$ by the monoid $(\delta \sqcup\!\sqcup \mathbb{R}^m\langle\langle X\rangle\rangle, \circ)$.

Theorem 2.7. (Gray & Ebrahimi-Fard, 2017) Let $c \in \mathbb{R}^q\langle\langle X\rangle\rangle$ and $d, e \in \mathbb{R}^m\langle\langle X\rangle\rangle$, then $\left(c \,\breve{\circ}\, {}^{\delta}d\right) \breve{\circ}\, {}^{\delta}e = c \,\breve{\circ} \left({}^{\delta}d \circ {}^{\delta}e\right)$.

The prominent question is to find the invertible elements of the monoid $(\delta \sqcup\!\sqcup \mathbb{R}^m\langle\langle X\rangle\rangle, \circ)$. Let $d, e \in \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$ and suppose

$$ {}^{\delta}d \circ {}^{\delta}e = {}^{\delta}\mathbb{1} $$

Applying (3),

$$ e \sqcup\!\sqcup \left(d \,\breve{\circ}\, {}^{\delta}e\right) = \mathbb{1} $$

Observe that $d \in \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$ implies $\mathrm{d} \,\breve{\circ}\, {}^{\delta}e \in \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$ and using Theorem 2.3,

$$ e = \left(d \,\breve{\circ}\, {}^{\delta}e\right)^{\sqcup\!\sqcup\, -1} = d^{\sqcup\!\sqcup\, -1} \,\breve{\circ}\, {}^{\delta}e. $$

Hence, for ${}^{\delta}e$ to be right inverse of ${}^{\delta}d$, the purely improper series e has to satisfy the fixed point equation

$$ e = d^{\sqcup\!\sqcup\, -1} \,\breve{\circ}\, {}^{\delta}e \tag{4} $$

Observe from Theorem 2.4 that the map $e \mapsto d^{\sqcup\!\sqcup\, -1} \,\breve{\circ}\, {}^{\delta}e$ is a strong contraction in the ultrametric space inferring that (4) has a unique fixed point. Suppose ${}^{\delta}e$ is the left inverse of ${}^{\delta}d$ viz ${}^{\delta}e \circ {}^{\delta}d$, then a similar procedure shows that e has to satisfy the equation

$$ d = e^{\sqcup\!\sqcup\, -1} \,\breve{\circ}\, {}^{\delta}d \tag{5} $$

Note that if e is a solution of (4), then e satisfies (5) and also the converse holds true. Hence, e is given the notation $d^{\circ -1}$ and for $d \in \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$, the inverse of ${}^{\delta}d$ exists and is unique, denoted by ${}^{\delta}d^{\circ -1}$ viz. $\left({}^{\delta}d\right)^{\circ -1} = \delta \sqcup\!\sqcup d^{\circ -1} = {}^{\delta}d^{\circ -1}$. Thus, $\delta \sqcup\!\sqcup \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$ forms a group under multiplicative composition product, termed as the *multiplicative dynamic output feedback group* and is formally stated in the following theorem.

Theorem 2.8. $\left(\delta \sqcup\!\sqcup \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle, \circ\right)$ forms a group with ${}^{\delta}\mathbb{1}$ being the identity element.

It is worth noting that Gray & Ebrahimi-Fard (2017) proved Theorem 2.8 for one-dimensional case viz. $m = 1$. In light of Theorem 2.8, Theorem 2.3 and (3) one obtains the following relations for $c \in \mathbb{R}^m_{pi}\langle\langle X\rangle\rangle$:

$$ c^{\circ -1} = c^{\sqcup\!\sqcup\, -1} \,\breve{\circ}\, {}^{\delta}c^{\circ -1} \tag{6} $$

$$ \left(c^{\circ -1}\right)^{\sqcup\!\sqcup\, -1} = c \,\breve{\circ}\, {}^{\delta}c^{\circ -1} $$

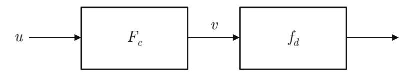

Fig. 2. Wiener-Fliess connection

2.4 Cauchy Algebra of Formal Static Maps

This subsection provides a brief discussion on formal static maps, which are used to describe the memoryless maps in the feedback path of static feedback interconnection, as described in Section 4. Let $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$ and $d \in \mathbb{R}^k[[\tilde{X}]]$. A formal static function $f_d : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ around the point $z = 0$ is defined as

$$ f_d(z) = \sum_{\eta \in \tilde{X}^*} (d, \eta)\, z^{\eta}, $$

where $z \in \mathbb{R}^m$, and $z^{\tilde{x}_i \eta} = z_i z^{\eta}\ \forall \tilde{x}_i \in \tilde{X}, \eta \in \tilde{X}^*$. The base case is taken to be $z^{\emptyset} = 1$. Denote the collection of all formal static maps from $\mathbb{R}^m$ to $\mathbb{R}^k$ as $\mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^m, \mathbb{R}^k\right)$. The series $d \in \mathbb{R}^k[[\tilde{X}]]$ is called the generating series of the static map f_d. A series $d \in \mathbb{R}[[\tilde{X}]]$ is said to be *locally convergent* if there exist constants $K_d, M_d > 0$ such that $|(d, \eta)| \leq K_d M_d^{|\eta|},\ \forall \eta \in \tilde{X}^*$. A series $d \in \mathbb{R}^k[[\tilde{X}]]$ is said to be locally convergent if and only if each component d_i is locally convergent for $i = 1, \ldots, m$.

Theorem 2.9. Let the formal static maps $f_d, f_e : \mathbb{R}^m \longrightarrow \mathbb{R}^k$, with $d, e \in \mathbb{R}^k[[\tilde{X}]]$. The product of the maps $f_d.f_e : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ is a formal static map $f_{d.e}$, where $d.e$ is the Cauchy product of d and e.

Theorem 2.9 asserts that $\mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^m, \mathbb{R}^k\right)$ forms an $\mathbb{R}$-algebra and there is an $\mathbb{R}$-algebra isomorphism from the $\mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^m, \mathbb{R}^k\right)$ to the Cauchy algebra of $\mathbb{R}^k[[\tilde{X}]]$. Let f_d be formal static map, with $d \in \mathbb{R}^k_{pi}[[\tilde{X}]]$. Then from Theorem 2.9, the generating series of the multiplicative inverse of the formal static map f_d, denoted by f_d^{-1}, is given by the Cauchy iverse of d viz. $f_d^{-1} = f_{d^{-1}}$. Hence, the unit group of $\mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^m, \mathbb{R}^k\right)$ is isomorphic to the group $\mathbb{R}^k_{pi}[[\tilde{X}]]$ under Cauchy product.

2.5 Wiener-Fliess Composition of Formal Power Series

This subsection describes the cascade connection shown in Figure 2 of a Chen-Fliess series F_c generated by a proper series $c \in \mathbb{R}^{\ell}_p\langle\langle X\rangle\rangle$ and a formal static map $f_d \in \mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^{\ell}, \mathbb{R}^k\right)$. Such configurations are called *Wiener-Fliess* connections. The connection is known to generate well-defined Chen-Fliess series for the composite system, and its generating series is computed through the *Wiener-Fliess composition product.* The definition of Wiener-Fliess composition product first appeared in Thitsa & Gray (2012), however, the definition was expanded even for $c \in \mathbb{R}^{\ell}_{np}\langle\langle X\rangle\rangle$ in Venkatesh (2021). However, the current document works with the restricted definition.

Theorem 2.10. (Gray & Thitsa, 2012; Venkatesh & Gray, 2021) Let $X = \{x_0, x_1, \ldots, x_m\}$ and $\tilde{X} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\ell}\}$. Given a formal Fliess operator F_c with $c \in \mathbb{R}^{\ell}_p\langle\langle X\rangle\rangle$ and formal function $f_d \in \mathrm{Hom}_{\mathrm{static}}\left(\mathbb{R}^{\ell}, \mathbb{R}^k\right)$, then the composition $f_d \circ F_c$ has a generating series in $\mathbb{R}^k\langle\langle X\rangle\rangle$ given by the Wiener-Fliess composition product

$$d\,\hat{\circ}\,c = \sum_{\tilde{\eta}\in\tilde{X}^*} (d,\tilde{\eta})c^{\sqcup\!\sqcup\,\tilde{\eta}}, \quad (7)$$

where $c^{\sqcup\!\sqcup\,\tilde{x}_i\tilde{\eta}} := c_i \sqcup\!\sqcup\, c^{\sqcup\!\sqcup\,\tilde{\eta}}$ $\forall \tilde{x}_i \in \tilde{X}$, $\forall\tilde{\eta}\in\tilde{X}^*$, and $c^{\sqcup\!\sqcup\,\phi} = 1$.

Observe that if $d \in \mathbb{R}^k_{pi}[[\tilde{X}]]$ and $c \in \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$, then $d\,\hat{\circ}\,c \in \mathbb{R}^k_{pi}\langle\langle X\rangle\rangle$. For a fixed $d\in\mathbb{R}^k[[\tilde{X}]]$ define the map $d_{\,\hat{\circ}} : \mathbb{R}^\ell_p\langle\langle X\rangle\rangle \longrightarrow \mathbb{R}^k\langle\langle X\rangle\rangle : c \mapsto d\,\hat{\circ}\,c$. The Wiener-Fliess connection preserves local convergence and hence, the composite system has a Fliess operator representation (Venkatesh, 2021). The following theorems describe the contractive properties of $d_{\,\hat{\circ}}$ in the ultrametric topology.

Theorem 2.11. (Venkatesh & Gray, 2021) The map $d_{\,\hat{\circ}}$ is a weak contraction map when $\overline{\omega}(d) = 1$ and a strong contraction map when $\overline{\omega}(d) > 1$.

Theorem 2.11 infers that the Wiener-Fliess composition product is at the very least is a weak contraction map with respect to the noncommutative formal series argument. The following theorem is a crucial result used in computing the feedback formula for the multiplicative static feedback connection.

Theorem 2.12. Let $d, d' \in \mathbb{R}^k[[\tilde{X}]]$ and $c\in\mathbb{R}^\ell_p\langle\langle X\rangle\rangle$, then $(d.d')\,\hat{\circ}\,c = (d\,\hat{\circ}\,c) \sqcup\!\sqcup (d'\,\hat{\circ}\,c)$.

Note that $(\mathbb{R}^k[[\tilde{X}]],\cdot)$ is a commutative monoid as the letters of $\tilde{X}$ commute. For a series $c \in \mathbb{R}^\ell_p\langle\langle X\rangle\rangle$, define a map $\Omega_c : \mathbb{R}^k[[\tilde{X}]] \longrightarrow \mathbb{R}^k\langle\langle X\rangle\rangle : d \mapsto d\,\hat{\circ}\,c$. Theorem 2.12 asserts that Ω_c is an $\mathbb{R}$-algebra homomorphism from the Cauchy algebra of $\mathbb{R}^k[[\tilde{X}]]$ to the shuffle algebra of $\mathbb{R}^k\langle\langle X\rangle\rangle$. If c is proper, then via theorem 2.12, it is evident that $d^{-1}\,\hat{\circ}\,c = (d\,\hat{\circ}\,c)^{\sqcup\!\sqcup\,-1}$, provided $d\in\mathbb{R}^k_{pi}[[\tilde{X}]]$. Hence, there is a group homomorphism from the $\mathbb{R}^k_{pi}[[\tilde{X}]]$ group under Cauchy product to the $\mathbb{R}^k_{pi}\langle\langle X\rangle\rangle$ group under shuffle product via the map Ω_c. The following theorem is pivotal in Section 4 and states Wiener-Fliess composition product and multiplicative mixed composition product are mixed associative in light of Theorem 2.3.

Theorem 2.13. Let $d \in \mathbb{R}^p[[\tilde{X}]], c \in \mathbb{R}^q\langle\langle X\rangle\rangle$ and $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$, such that $|\tilde{X}| = q$, then $d\hat{\circ}\left(c\,\breve{\circ}^{\,\delta} e\right) = (d\,\hat{\circ}\,c)\,\breve{\circ}^{\,\delta} e$.

3. CHEN-FLIESS SERIES UNDER MULTIPLICATIVE DYNAMIC OUTPUT FEEDBACK

Let F_c be a Chen-Fliess series with a generating series $c \in \mathbb{R}^q\langle\langle X\rangle\rangle$. Assume it is interconnected with a Chen-Fliess series F_d with a purely improper generating series $d \in \mathbb{R}^m_{pi}\langle\langle X'\rangle\rangle$, as shown in Figure 3. Note that, $|X| = m + 1$ and $|X'| = q + 1$. The primary goal of this section is to show that the closed-loop system has a Chen-Fliess series representation, say $y = F_e[v]$, where $e \in \mathbb{R}^q\langle\langle X\rangle\rangle$. If this is the case, then necessarily

$$\begin{aligned} y &= F_e[v] = F_c[u] = F_c[vF_d[y]] \\ &= F_c[vF_d[F_e[v]]] = F_c[vF_{d\circ e}[v]] \\ &= F_{c\,\breve{\circ}^{\,\delta}(d\circ e)}[v] \end{aligned}$$

for any admissible input v. Therefore, the series e has to satisfy the fixed point equation

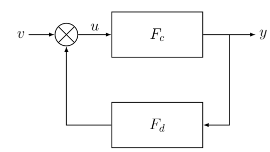

Fig. 3. Chen-Fliess series F_c in multiplicative output feedback with Chen-Flies series F_d

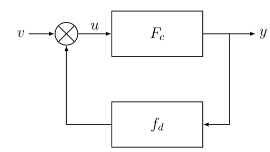

Fig. 4. Chen-Fliess series F_c in multiplicative output feedback with f_d

$$e = c\,\breve{\circ}^{\,\delta}(d\circ e). \quad (8)$$

Observe that, in light of Theorem 2.2 and Theorem 2.4 the map $e \mapsto c\,\breve{\circ}^{\,\delta}(d\circ e)$ is a strong contraction map in the ultrametric space and thus (8) has a unique fixed point. The following thoerem establishes the first main result of this section, which follows immediately.

Theorem 3.1. The series $c\,\breve{\circ}^{\,\delta}\left(d^{\sqcup\!\sqcup\,-1}\circ c\right)^{\circ -1} \in \mathbb{R}^q\langle\langle X\rangle\rangle$ is the unique fixed point of the map $e \mapsto c\,\breve{\circ}^{\,\delta}(d\circ e)$.

Theorem 3.2. Given a series $c \in \mathbb{R}^q\langle\langle X\rangle\rangle$ and a purely improper series $d \in \mathbb{R}^m_{pi}\langle\langle X'\rangle\rangle$ (such that $|X| = m+1$ and $|X'| = q+1$), then the generating series for the closed-loop system in Figure 3 is given by the *multiplicative dynamic feedback product* $c\,\breve{@}\,d := c\,\breve{\circ}^{\,\delta}\left(d^{\sqcup\!\sqcup\,-1}\circ c\right)^{\circ -1}$.

The notion that feedback can described mathematically as a transformation group acting on the plant is well established in control theory (Brockett, 1978). The following theorem describes the situation in the present context.

Theorem 3.3. The multiplicative dynamic feedback product is a right group action by the multiplicative group $\left(\mathbb{R}^m_{pi}\langle\langle X'\rangle\rangle, \sqcup\!\sqcup, \mathbb{1}\right)$ on the set $\mathbb{R}^q\langle\langle X\rangle\rangle$, where $|X| = m+1$ and $|X'| = q+1$.

It is worth noting that for the *additive dynamic feedback product* the transformation group is the additive group $(\mathbb{R}^m\langle\langle X'\rangle\rangle, +, 0)$ while here $(\mathbb{R}^m_{pi}\langle\langle X'\rangle\rangle, \sqcup\!\sqcup, \mathbb{1})$ plays that role.

4. CHEN-FLIESS SERIES UNDER MULTIPLICATIVE STATIC OUTPUT FEEDBACK

Let F_c be a Chen-Fliess series with a proper generating series $c \in \mathbb{R}^q_p\langle\langle X\rangle\rangle$. Assume it is interconnected with a formal static map f_d with a purely improper generating series $d \in \mathbb{R}^m_{pi}[[\tilde{X}]]$, as shown in Figure 4. Note that, $|X| = m+1$ and $|\tilde{X}| = q$. The primary goal of this section is to show that the closed-loop system has a Chen-Fliess series representation, say $y = F_e[v]$, where $e\in\mathbb{R}^q\langle\langle X\rangle\rangle$. If this is the case, then necessarily

$$\begin{aligned} y &= F_e[v] = F_c[u] = F_c[vf_d[y]] \\ &= F_c[vf_d[F_e[v]]] = F_c[vF_{d\,\hat{\circ}\,e}[v]] \\ &= F_{c\,\breve{\circ}^{\,\delta}(d\,\hat{\circ}\,e)}[v] \end{aligned}$$

for any admissible input v. Therefore, the series e has to satisy the fixed point equation

$$e = c \,\breve{\circ}^{\delta}\, (d \,\hat{\circ}\, e). \quad (9)$$

In addition, e must be a proper series for the Wiener-Fliess composition $d \,\hat{\circ}\, e$ to be well defined for arbitrary $d \in \mathbb{R}^m_{pi}[[\tilde{X}]]$. It follows directly from the definition of the multiplicative mixed composition product that if $c \in \mathbb{R}^q_p\langle\langle X\rangle\rangle$ then $c \,\breve{\circ}^{\delta} w$ is also a proper series for all $w \in \mathbb{R}^m\langle\langle X\rangle\rangle$.

Observe that, in light of Theorem 2.11 and Theorem 2.4 the map $e \mapsto c \,\breve{\circ}^{\delta}\, (d \,\hat{\circ}\, e)$ is a strong contraction map in the ultrametric space and thus (9) has a unique fixed point. The following fixed point theorem establishes the first main result of this section, which follows immediately.

Theorem 4.1. The series $c \,\breve{\circ}^{\delta} \left(d^{-1} \,\hat{\circ}\, c\right)^{\circ -1} \in \mathbb{R}^q_p\langle\langle X\rangle\rangle$ is the unique fixed point of the map $e \mapsto c \,\breve{\circ}^{\delta}\, (d \,\hat{\circ}\, e)$.

Theorem 4.2. Given a series $c \in \mathbb{R}^q_p\langle\langle X\rangle\rangle$ and a purely improper series $d \in \mathbb{R}^m_{pi}[[\tilde{X}]]$ (such that $|X| = m+1$ and $|\tilde{X}| = q$), then the generating series for the closed-loop system in Figure 4 is given by the *multiplicative static feedback product* $c \bar{@} d := c \,\breve{\circ}^{\delta} \left(d^{-1} \circ c\right)^{\circ -1}$.

The following theorem describes the transformation group on the plant which characterizes the multiplicative static feedback product.

Theorem 4.3. The multiplicative static feedback product is a right group action by the Abelian multiplicative group $\left(\mathbb{R}^m_{pi}[[\tilde{X}]], \cdot, \mathbb{1}\right)$ on the set $\mathbb{R}^q_p\langle\langle X\rangle\rangle$, where $|X| = m+1$ and $|\tilde{X}| = q$.

It is important to note that for additive static output feedback product, known as *Wiener-Fliess feedback product*, the transformation group is the additive group $(\mathbb{R}^m[[\tilde{X}]], +, 0)$ while for multiplicative static output feedback the Abelian multiplicative group $(\mathbb{R}^m_{pi}[[\tilde{X}]], \cdot, \mathbb{1})$ performs that role.

5. CONCLUSIONS AND FUTURE WORK

It was shown that the closed-loop system of a plant in Chen-Fliess series description in multiplicative output feedback with another system, given by Chen-Fliess series, has a Chen-Fliess series representation. An explicit expression of the closed-loop generating series was derived. The multiplicative dynamic feedback connection has a natural interpretation as a right group action on the plant. The same set of questions were answered when the Chen-Fliess series in the feedback is replaced by a memoryless map. Future work will address the solemn problem regarding the local convergence of the both multiplicative dynamic and static output feedback connections and to identify the feedback invariants.

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