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Nonrecursively Interconnected Fliess Operators Preserve Global Convergence: An Expanded View

Irina M. Winter-Arboleda[†] Luis A. Duffaut Espinosa[‡] W. Steven Gray[†]

***Abstract*— A common representation of an input-output system in nonlinear control theory is the Chen-Fliess functional series or Fliess operator. Such a functional series is said to be *globally convergent* when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. Recent developments have expanded the class of globally convergent Fliess operators. The goal of this paper is to show that the global convergence property is preserved for nonrecursive interconnections (i.e., the parallel, product and cascade connections) involving this largest known class of globally convergent input-output systems. The goal is only partially achieved, however, as some qualification is still needed for the cascade connection.**

***Index Terms*— Nonlinear control systems, Chen-Fliess series, locally convex topological vector spaces, system interconnections.**

***AMS Subject Classifications*—93C10, 47H30, 46A99**

I. Introduction

It is common in nonlinear control theory to represent input-output systems in terms of functional series. The Chen-Fliess series or Fliess operator is one such representation where the terms of the series are indexed by words [6], [7], [11]. Such a functional series is said to be *globally convergent* when there is no a priori upper bound on both the L_1-norm of an admissible input and the length of time over which the corresponding output is well defined. A sufficient condition for global convergence can be described in terms of the asymptotic behavior (in magnitude) of the coefficients of the series via the notion of *Gevrey order*, that is, by a growth rate of the form $KM^n(n!)^s$ for some real $K, M > 0$ and $s \in \mathbb{R}$, where n is word length. In particular, it was shown in [16] that $0 \leq s < 1$ is a sufficient condition for global convergence. However, an interesting example presented by Ferfera in [4], [5] and more recent analysis in [15] in the context of system interconnection suggested that this Gevrey condition is not necessary. The full situation is now described in [17]. It turns out that there is a class of systems whose generating series has Gevrey order $s = 1$ but whose series representation converges globally. Strictly speaking, every globally convergent system is also locally convergent, but this class of systems has a functional series representation with no finite singularities to bound its radius of convergence. Therefore, such systems are called *weakly locally convergent*. They can be viewed as a limiting case in that they reside only in the closure under a semi-norm topology of the set of series whose Gevrey order satisfy $0 \leq s < 1$. So any operator associated with such a generating series is globally convergent. The example of Ferfera is now seen as the earliest known example of such a system.

The example of Ferfera was originally introduced to show that rationality is not preserved under the cascade connection. Rational series have Gevrey order $s = 0$, and, as explained in [17], Ferfera's cascaded system results in a composite system whose Gevrey order is $s = 1$. Therefore, interconnected systems can exhibit a Gevrey order distinct from that of their component systems. However, as observed in [15], all cascades of systems having Gevrey order $s = 0$ have a Fliess operator representation that converges globally. So the property of global convergence *is* preserved in this situation. The goal of this paper is to show that this is true in general for all *nonrecursive* interconnections (i.e., the parallel, product and cascade connections) involving the largest known class of globally convergent systems as described in [17]. The goal is only partially achieved, however, as some qualification is still needed for the cascade connection. The feedback connection is viewed as a recursive connection, and this claim is known to be false as shown in [9, Example 3].

The paper is organized as follows. In Section II, some preliminaries concerning formal power series, Fliess operators and their nonrecursive interconnections are given. In order to make the paper more self-contained, a brief summary of the expanded class of globally convergent Fliess operators in [17] is also provided. The convergence analysis for the parallel and product interconnections is given in Section III. In the subsequent section, the global convergence of the cascade interconnection is addressed. The conclusions are provided in the final section.

II. Preliminaries

A. Fliess Operators and Their Interconnections

A finite nonempty set of noncommuting symbols $X = \{x_0, x_1, \ldots, x_m\}$ is called an *alphabet*. Each element of X is called a *letter*, and any finite sequence of letters from X, $\eta = x_{i_1} \cdots x_{i_k}$, is called a *word* over X. The *length* of η, $|\eta|$, is the number of letters in η. Let $|\eta|_{x_i}$ denote the number of times the letter $x_i \in X$ appears in the word η. The set of all words including the empty word, $\emptyset$, is designated by X^*. It forms a monoid under catenation. Any mapping $c : X^* \to \mathbb{R}^\ell$ is called a *formal power series*. The value of c at $\eta \in X^*$ is written as (c, η) and called the *coefficient* of η in c. Typically, c is represented as the formal sum $c = \sum_{\eta \in X^*}(c, \eta)\eta$. The subset of X^* defined by $\text{supp}(c) = \{\eta : (c, \eta) \neq 0\}$ is called the *support* of c. A series $\hat{c}$ is said to be a *subseries* of c if $\text{supp}(\hat{c}) \subseteq \text{supp}(c)$ and $(\hat{c}, \eta) =$

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(c,η), $\forall\eta \in \operatorname{supp}(\hat{c})$. The collection of all formal power series over X is denoted by $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$. It forms an associative $\mathbb{R}$-algebra under the catenation product and an associative and commutative $\mathbb{R}$-algebra under the *shuffle product*, that is, the bilinear product defined in terms of the shuffle product of two words

$$(x_i\eta) \sqcup\!\sqcup\, (x_j\xi) = x_i(\eta \sqcup\!\sqcup\, (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup\, \xi),$$

where $x_i, x_j \in X$, $\eta, \xi \in X^*$ and with $\eta \sqcup\!\sqcup\, \emptyset = \emptyset \sqcup\!\sqcup\, \eta = \eta$ [6], [14].

One can formally associate with any series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ a causal m-input, ℓ-output operator, F_c, in the following manner. Let $\mathfrak{p} \geq 1$ and $t_0 < t_1$ be given. For a Lebesgue measurable function $u : [t_0, t_1] \to \mathbb{R}^m$, define $\|u\|_{\mathfrak{p}} = \max\{\|u_i\|_{\mathfrak{p}} : \ 1 \leq i \leq m\}$, where $\|u_i\|_{\mathfrak{p}}$ is the usual $L_{\mathfrak{p}}$-norm for a measurable real-valued function, u_i, defined on $[t_0, t_1]$. Let $L^m_{\mathfrak{p}}[t_0, t_1]$ denote the set of all measurable functions defined on $[t_0, t_1]$ having a finite $\|\cdot\|_{\mathfrak{p}}$ norm and $B^m_{\mathfrak{p}}(R_u)[t_0, t_1] := \{u \in L^m_{\mathfrak{p}}[t_0, t_1] : \|u\|_{\mathfrak{p}} \leq R_u\}$. Assume $C[t_0, t_1]$ is the subset of continuous functions in $L^m_1[t_0, t_1]$. Define inductively for each $\eta \in X^*$ the map $E_\eta : L^m_1[t_0, t_1] \to C[t_0, t_1]$ by setting $E_\emptyset[u] = 1$ and letting

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

where $x_i \in X$, $\bar{\eta} \in X^*$, and $u_0 = 1$. The input-output operator corresponding to c is the *Fliess operator*

$$F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0) \quad (1)$$

[6], [7]. The generating series c is said to be of *Gevrey order* $s \in \mathbb{R}$ if there exists constants $K, M > 0$ such that

$$|(c, \eta)| \leq K M^{|\eta|} (|\eta|!)^s, \ \ \forall \eta \in X^*, \quad (2)$$

and s is the smallest number having this property [1], [16]. (Here, $|z| := \max_i |z_i|$ when $z \in \mathbb{R}^{\ell}$.) If $0 \leq s \leq 1$ then F_c constitutes a well defined mapping from $B^m_{\mathfrak{p}}(R_u)[t_0,\, t_0{+}T]$ into $B^{\ell}_{\mathfrak{q}}(S)[t_0,\, t_0 + T]$ for sufficiently small $R_u, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p}{+}1/\mathfrak{q} = 1$ [10]. The set of all such *locally convergent* generating series is denoted by $\mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$. The subset of all generating series with $s = 1$ is denoted by $\mathbb{R}^{\ell}_1\langle\langle X\rangle\rangle$. The least upper bound on $\max\{R_u, T\}$, say $\rho(F_c)$, is called the *radius of convergence* of the operator. It was shown in [3] that $0 < 1/M(m+1) \leq \rho(F_c)$. When $0 \leq s < 1$, the series (1) defines an operator from the extended space $L^m_{\mathfrak{p},e}(t_0)$ into $C[t_0, \infty)$, where

$$\begin{aligned} L^m_{\mathfrak{p},e}(t_0) :=\{&u : [t_0, \infty) \to \mathbb{R}^m : u_{[t_0,t_1]} \in L^m_{\mathfrak{p}}[t_0, t_1], \\ &\forall t_1 \in (t_0, \infty)\}, \end{aligned}$$

and $u_{[t_0,t_1]}$ denotes the restriction of u to $[t_0, t_1]$ [16]. This set of *globally convergent* series is designated by $\mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$. A series c is said to be *globally maximal* with Gevrey order s and growth constants $K, M > 0$ if each component of (c, η) is equal to $KM^{|\eta|}(|\eta|!)^s$, $\forall\eta \in X^*$.

Given Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, the parallel and product connections as shown in Figures 1 and 2 satisfy

$$F_c + F_d = F_{c+d}$$

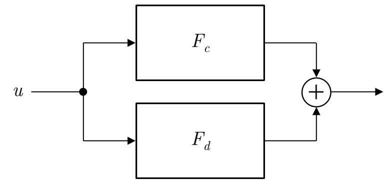

Fig. 1. Parallel connection of two Fliess operators with an adder.

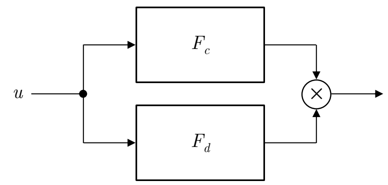

Fig. 2. Product connection of two Fliess operators with a multiplier.

and

$$F_c F_d = F_{c \,\sqcup\!\sqcup\, d},$$

respectively [6]. When Fliess operators F_c and F_d with $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ are interconnected in a cascade fashion as shown in Figure 3, the composite system $F_c \circ F_d$ has the Fliess operator representation $F_{c\circ d}$, where the *composition product* of c and d is given by

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1) \quad (3)$$

[4], [5]. The mapping ψ_d is the continuous (in the ultramet-

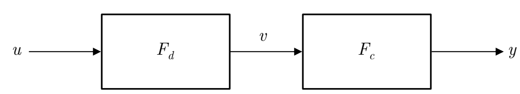

Fig. 3. Cascade connection of two Fliess operators.

ric sense) algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to the set of vector space endomorphism on $\mathbb{R}\langle\langle X\rangle\rangle$, $\operatorname{End}(\mathbb{R}\langle\langle X\rangle\rangle)$, uniquely specified by $\psi_d(x_i\eta) = \psi_d(x_i) \circ \psi_d(\eta)$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup\, e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$). $\psi_d(\emptyset)$ is defined to be the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$. This composition product is associative and $\mathbb{R}$-linear in its left argument.

B. Expanded Set of Globally Convergent Fliess Operators

The set of globally convergent Fliess operators was expanded in [17] beyond the set of systems having generating series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. In this section the situation is briefly summarized as it plays a role in the material that follows. Without lost of generality, it is assumed throughout that $\ell = 1$.

Define for any fixed $R > 0$

$$\|c\|_{\infty,R} = \sup_{\eta \in X^*} \left\{ |(c, \eta)|\, \frac{R^{|\eta|}}{|\eta|!} \right\}$$

and the corresponding normed linear subspace of the $\mathbb{R}$-vector space $\mathbb{R}\langle\langle X\rangle\rangle$ denoted by $S_\infty(R) := \{c \in \mathbb{R}\langle\langle X\rangle\rangle :$

$\|c\|_{\infty,R} < \infty\}$. A sequence $\{c_i\}_{i\in\mathbb{N}}$ in $S_\infty(R)$ converges to $c \in S_\infty(R)$ if and only if $\|c_i - c\|_{\infty,R} \to 0$ as $i \to \infty$. (Hereafter, the shorthand notation $c_i \to c$ is used.) It is easy to show that the spaces $S_\infty(R)$, $R \in \mathbb{R}^+$ are nested. Therefore if $c_i \to c$ in $S_\infty(R)$ and $c_i \to c'$ in $S_\infty(R')$ then $c = c'$. Let $S_{\infty,e} := \cup_{R>0} S_\infty(R)$ and $S_\infty := \cap_{R>0} S_\infty(R)$. The extended space $S_{\infty,e}$ is a locally convex topological vector space equipped with a family of semi-norms $\|\cdot\|_{\infty,R}$, $R \in \mathbb{R}^+$. This semi-norm topology is second countable, and thus first countable. The space is also Hausdorff, in which case, sequentially continuous maps are continuous. A sequence $\{c_i\}_{i\in\mathbb{N}}$ in $S_{\infty,e}$ converges to a (unique) $c \in S_{\infty,e}$ in the semi-norm topology if and only if $\|c_i - c\|_{\infty,R} \to 0$ as $i \to \infty$ for all $R > 0$. Given a series $c \in S_{\infty,e}$, let

$$\bar{R}_c := \sup_{\substack{\|c\|_{\infty,R}<\infty \\ R>0}} R.$$

If $\bar{R}_c = \infty$ then it follows that $c \in S_\infty$. The closure of $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ in the semi-norm topology is denoted by $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. It is shown in [17] that $\mathbb{R}_{LC}\langle\langle X\rangle\rangle = S_{\infty,e}$ and $\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle} = S_\infty$. In addition, the set $\mathbb{R}_1\langle\langle X\rangle\rangle$ can be partitioned as

$$\mathbb{R}_1\langle\langle X\rangle\rangle = \left(\mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}\right) \bigcup \partial\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}.$$

The series in $\mathbb{R}_{LC}\langle\langle X\rangle\rangle \setminus \overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ are referred to as *strongly locally convergent*, whereas the series in $\partial\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$ are said to be *weakly locally convergent*. The relationship among these sets is summarized in Figure 4.

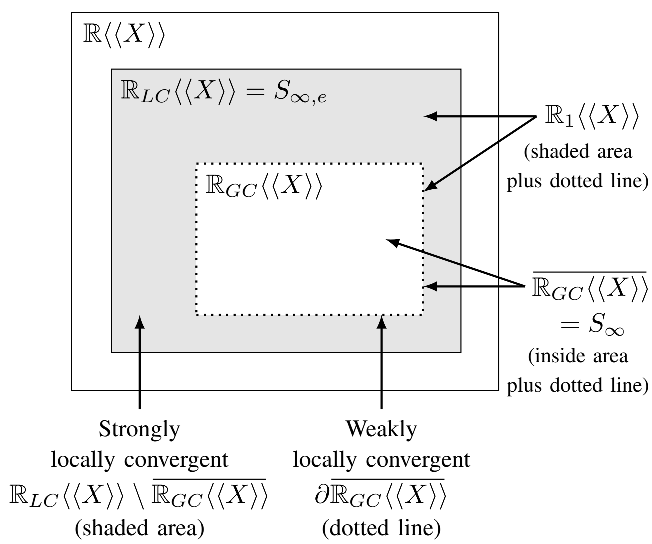

Fig. 4. Relationship between $S_{\infty,e}$, S_∞, Gevrey order and various notions of convergence.

The main results in [17] are given below.

Theorem 1: If c is a strongly locally convergent series, then the radius of convergence of series (1) is finite.

Theorem 2: If c is a weakly locally convergent series, then the radius of convergence of series (1) is infinite.

The next two corollaries provide yet another characterization of strongly and weakly locally convergent series. (The second corollary does not appear in [17].)

Corollary 1: Let c be a strongly locally convergent series. Then there exists a subseries $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ such that each coefficient $(\hat{c}, \eta)$ is growing exactly at the rate $KM^{|\eta|}\,|\eta|!$ for some $K, M > 0$.

Corollary 2: If c is weakly locally convergent then all subseries of c are weakly locally convergent.

Proof: The prove is by contradiction. Assume $\hat{c} \in \mathbb{R}_1\langle\langle X\rangle\rangle$ is a subseries of c which is strongly locally convergent. Then by Theorem 1, $F_{\hat{c}}$ has a finite radius of convergence. This implies that F_c also has a finite radius of convergence (see the proof of Theorem 8 in [17]). Therefore, a contradiction arises since c has infinite radius of convergence in light of Theorem 2. ■

The following example provides a specific example of a weakly globally convergent series, which is the more subtle case.

Example 1: Let $X = \{x_0, x_1\}$ and consider the rational series $x_1^* := \sum_{k\geq 0} x_1^k$. Using the notion of formal power series composition defined in (3), the series considered by Ferfera in [4], [5] is

$$c_F := x_1^* \circ x_1^*. \tag{4}$$

Define two subseries of c_F :

$$c_F^{1/2} = \sum_{k=0}^{\infty} (c_F, x_0^k x_1^k)\, x_0^k x_1^k$$

and

$$c_F^1 = \sum_{k_0,k_1=0}^{\infty} (c_F, x_0^{k_0} x_1^{k_1})\, x_0^{k_0} x_1^{k_1}.$$

Ferfera's central argument in showing that rationality is not preserved under composition was the observation that the coefficients

$$(c_F^{1/2}, x_0^k x_1^k) = k^k, \ \ k \geq 0$$

grow too fast to satisfy (2) when $s = 0$. Therefore, c_F can not be rational. The series $c_F^{1/2}$, on the other hand, was shown in [17] to have Gevrey order $s = 1/2$, and therefore $c \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. The more interesting case, however, is the series c_F^1, which was shown to have Gevrey order 1 with $c_F^1 \in \partial\overline{\mathbb{R}_{GC}\langle\langle X\rangle\rangle}$. Therefore, c_F^1 is weakly locally convergent. The same property can be shown for the full series c_F. □

III. Parallel and Product Interconnections

This section has two objectives. The first is to compute upper bounds on the Gevrey orders of the generating series for the parallel and product interconnections of two Fliess operators with generating series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. The second is to show that these two interconnections preserve the global convergence property. Since global convergence of a Fliess operator is completely characterized by its generating series, that is, F_c is globally convergent if and only if $c \in S_\infty$, it is only necessary to show that S_∞ is closed under addition and the shuffle product. The following technical results will be needed to do this analysis.

Lemma 1: For any $K, M, s > 0$, there exists an integer

$N > 0$ such that

$$KM^n \leq (n!)^s, \tag{5}$$

for all integers $n > N$.

Proof: From Stirling's approximation it follows that $n! \approx \sqrt{2\pi n}(n/e)^n$ for $n \gg 1$. Therefore,

$$\lim_{n\to+\infty} K\frac{M^n}{(n!)^s} = \frac{K}{(2\pi)^{s/2}} \lim_{n\to+\infty} \frac{(e^s M)^n}{\sqrt{n}n^{ns}} = 0,$$

which directly leads to (5). ■

Lemma 2: [16] For any integer $n \geq 0$ and $0 < s \leq 1$ such that $ns \gg 1$ it follows that

$$(ns)! \leq K_s M_s^n (n!)^s,$$

where $(ns)! := \Gamma(ns+1)$, $K_s = ((2\pi)^{1-s}s)^{1/2}$ and $M_s = s^s$.

Lemma 3: [13] (Neoclassical Inequality) For any integer $n \geq 0$, $x, y \in \mathbb{R}^+$, and $p \geq 1$ it follows that

$$\left(\frac{1}{p}\right)^2 \sum_{j=0}^{n} \frac{x^{j/p}}{(j/p)!} \frac{y^{(n-j)/p}}{((n-j)/p)!} \leq \frac{(x+y)^{n/p}}{(n/p)!}.$$

Note that when $p = 1$ above, the result reduces to the binomial theorem.

A. The Parallel Connection

To analyze the parallel connection, it is first shown below how to compute the Gevrey order of the sum of two maximal globally convergent series.

Lemma 4: Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ be maximal globally convergent series with Gevrey order s_c and s_d, respectively. If $b := c + d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ has Gevrey order $s_b = \max\{s_c, s_d\}$.

Proof: First recall that the Gevrey order of a series b is the smallest s satisfying (2). Observe for any $\nu \in X^n$, $n \geq 0$, that

$$\begin{aligned}(b,\nu) = (c,\nu) + (d,\nu) &= K_c M_c^n (n!)^{s_c} + K_d M_d^n (n!)^{s_d} \\ &\leq KM^n (n!)^s, \end{aligned} \tag{6}$$

where $s := \max\{s_c, s_d\}$, $M := \max\{M_c, M_d\}$ and $K := K_c + K_d$. Letting s_b denote the Gevrey order of b, it is clear from (6) that $s_b \leq s < 1$, which implies that $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$. It is shown now that $s_b \not< s$ since considering otherwise would render a contradiction. Suppose $s_b < s$ and there exist constants $K_b, M_b > 0$ such that $(b,\nu) \leq K_b M_b^n (n!)^{s_b}$, $\forall \nu \in X^n$, $n \geq 0$. There is no loss of generality in assuming $s_c \leq s_d$. In which case, $s_b < s = \max\{s_c, s_d\} = s_d$, and therefore,

$$(b,\nu) = K_c M_c^n (n!)^{s_c} + K_d M_d^n (n!)^{s_d} \leq K_b M_b^n (n!)^{s_b}.$$

In particular, this implies that $K_d M_d^n (n!)^{s_d - s_b} \leq K_b M_b^n$. Hence,

$$(n!)^{s_d - s_b} \leq \frac{K_b}{K_d}\left(\frac{M_b}{M_d}\right)^n. \tag{7}$$

Substituting $M' = M_b/M_d$, $K' = K_b/K_d$ and $s' = s_d - s_b$ in (7) gives $K'M'^n \geq (n!)^{s'}$, which contradicts (5) in Lemma 1 since by assumption $s_d - s_b > 0$. Therefore, $s_b = \max\{s_c, s_d\}$. ■

It is now straightforward to compute an upper bound on the Gevrey order of the sum of two arbitrary series in $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$.

Theorem 3: Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with Gevrey order s_c and s_d, respectively. If $b := c + d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with Gevrey order $s_b \leq \max\{s_c, s_d\}$.

Proof: For any $\nu \in X^*$ it follows that

$$|(c+d,\nu)| \leq |(c,\nu)| + |(d,\nu)| \leq (\bar{c},\nu) + (\bar{d},\nu) = (\bar{b},\nu),$$

where $\bar{b}$, $\bar{c}$ and $\bar{d}$, are the maximal globally convergent series corresponding to b, c and d, respectively (that is, each pair, for example b and $\bar{b}$, share the same growth constants). From Lemma 4 it then follows directly that $s_b \leq \max\{s_c, s_d\}$. ■

The fact that the upper bound on the Gevrey order of the sum of two series is the maximum of the Gevrey orders of the component series implies that $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ is closed under addition. The next theorem shows that S_∞ is also closed under addition, and thus, the parallel connection preserves the global convergence of Fliess operators in the broadest known sense.

Theorem 4: The space S_∞ is closed under addition.

Proof: Let $c, d \in S_\infty$. Then clearly

$$\|c+d\|_{\infty,R} \leq \|c\|_{\infty,R} + \|d\|_{\infty,R} < \infty$$

for all $R > 0$. Hence, $c + d \in S_\infty$. ■

B. The Product Connection

The product connection is now addressed. The problem is more difficult since sums are replaced with shuffle products. The first lemma computes the Gevrey order of the shuffle product of two maximal globally convergent series.

Lemma 5: Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ be maximal globally convergent series with Gevrey order s_c and s_d, respectively. If $b := c \sqcup\!\sqcup d$ then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ has Gevrey order $s_b = \max\{s_c, s_d\}$.

Proof: Observe that for any $\nu \in X^n$, $n \geq 0$,

$$\begin{aligned}(b,\nu) = (c \sqcup\!\sqcup d,\nu) &= \sum_{j=0}^{n} \sum_{\substack{\eta\in X^j \\ \xi\in X^{n-j}}} (c,\eta)(d,\xi)(\eta \sqcup\!\sqcup \xi,\nu) \\ &= \sum_{j=0}^{n} K_c M_c^j (j!)^{s_c} K_d M_d^{n-j} ((n-j)!)^{s_d} \binom{n}{j} \\ &= K_c K_d n! \sum_{j=0}^{n} M_c^j M_d^{n-j} \frac{1}{(j!)^{1-s_c}((n-j)!)^{1-s_d}}.\end{aligned}$$

Using Lemma 2 and letting $s := \max\{s_c, s_d\}$, $s' := 1 - s$, $K_s := ((2\pi)^{1-s'}s')^{1/2}$ and $M_s := s'^{s'}$, it follows that

$$\begin{aligned}&(c \sqcup\!\sqcup d,\nu) \\ &\leq K_c K_d n! \sum_{j=0}^{n} M_c^j M_d^{n-j} \frac{(K_s)^2 M_s^n}{(js')!((n-j)s')!}, \\ &= K_c K_d (K_s)^2 M_s^n n! \sum_{j=0}^{n} \frac{\left({M_c}^{1/s'}\right)^{js'} \left({M_d}^{1/s'}\right)^{(n-j)s'}}{(js')!((n-j)s')!}.\end{aligned}$$

Now applying Lemma 3 gives

$$(c \sqcup\!\sqcup d,\nu) \leq \frac{1}{s'} K_c K_d (K_s)^2 M_s^n n! \frac{({M_c}^{1/s'} + {M_d}^{1/s'})^{ns'}}{(ns')!}.$$

In which case, from Lemma 2 is it immediate that

$$(c \sqcup\!\sqcup d, \nu) \leq KM^n (n!)^s, \tag{8}$$

where $M := {M_c}^{1/s'} + {M_d}^{1/s'}$ and $K := K_c K_d K_s/s'$. Since the Gevrey order is the smallest s satisfying (2), if the Gevrey order of $b = c \sqcup\!\sqcup d$ is s_b, then it is clear from (8) that $s_b \leq s < 1$, which automatically implies that $b \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$. It is shown now that $s_b \not< s$ since otherwise a contradiction is obtained. Suppose $s_b < s$ and that there exist constants $K_b, M_b > 0$ such that $(b, \nu) \leq K_b M_b^n (n!)^{s_b}$, $\forall \nu \in X^n$, $n \geq 0$. Without loss of generality assume $s_c \leq s_d$. In which case, $s_b < s = \max\{s_c, s_d\} = s_d$. Thus,

$$(b, \nu) = (c \sqcup\!\sqcup d, \nu) \leq K_b M_b^n (n!)^{s_b},$$

which implies that

$$(c \sqcup\!\sqcup d, \nu) = \sum_{j=0}^{n} \sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (c, \eta)(d, \xi)(\eta \sqcup\!\sqcup \xi, \nu) \leq K_b M_b^n (n!)^{s_b}.$$

In particular, the $j = 0$ term in the summation above must satisfy

$$(c, \emptyset)(d, \nu) = K_c K_d M_d^n (n!)^{s_d} \leq K_b M_b^n (n!)^{s_b},$$

which amounts to the inequality

$$(n!)^{s_d - s_b} \leq \frac{K_b}{K_c K_d} \left(\frac{M_b}{M_d}\right)^n. \tag{9}$$

Letting $M' := M_b/M_d$, $K' := K_b/(K_c K_d)$ and $\bar{s} := s_d - s_b$ in (9) gives $K' M'^n \geq (n!)^{\bar{s}}$, which contradicts (5) in Lemma 1 since by assumption $\bar{s} = s_d - s_b > 0$. Thus, $s_b = \max\{s_c, s_d\}$. ∎

An expression for an upper bound on the Gevrey order of the shuffle product of two arbitrary series in $\mathbb{R}_{GC}\langle\langle X \rangle\rangle$ is computed in the following theorem.

Theorem 5: Let $c, d \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with Gevrey order s_c and s_d, respectively. If $b := c \sqcup\!\sqcup d$ then $b \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ with with Gevrey order $s_b \leq \max\{s_c, s_d\}$.

Proof: First observe that for all $\nu \in X^*$

$$|(c \sqcup\!\sqcup d, \nu)| \leq (\bar{c} \sqcup\!\sqcup \bar{d}, \nu) = (\bar{b}, \nu),$$

where $\bar{b}$, $\bar{c}$ and $\bar{d}$ are maximal globally convergent series corresponding to b, c and d, respectively. From Lemma 5 it then follows directly that $s_b \leq \max\{s_c, s_d\}$. ∎

It is next shown that the product connection preserves global convergence by proving that the shuffle product of two series in S_∞ always produces another series in S_∞. The following lemma is essential.

Lemma 6: For every $c, d \in S_\infty$,

$$\|c \sqcup\!\sqcup d\|_{\infty,R} \leq \|c\|_{\infty,R} \|d\|_{\infty,R}$$

for all $R > 0$.

Proof: Starting with the definition of the norm on $S_\infty(R)$:

$$\|c \sqcup\!\sqcup d\|_{\infty,R} = \sup_{\nu \in X^*} \left\{ |(c \sqcup\!\sqcup d, \nu)| \frac{R^{|\nu|}}{|\nu|!} \right\}$$

$$\leq \sup_{\substack{\nu \in X^n \\ 0 \leq j \leq n \\ n \geq 0}} \left\{ \sum_{\eta \in X^j, \xi \in X^{n-j}} |(c, \eta)| \, |(d, \xi)| \, (\eta \sqcup\!\sqcup \xi, \nu) \frac{R^n}{n!} \right\}$$

$$\leq \sup_{\substack{\nu \in X^n \\ 0 \leq j \leq n \\ n \geq 0}} \left\{ \left(\max_{\eta \in X^j} |(c, \eta)| \, R^j \right) \left(\max_{\xi \in X^{n-j}} |(d, \xi)| \, R^{n-j} \right) \cdot \frac{1}{n!} \sum_{\eta \in X^j, \xi \in X^{n-j}} (\eta \sqcup\!\sqcup \xi, \nu) \right\}.$$

It is easy to show by induction that

$$\sum_{\substack{\eta \in X^j \\ \xi \in X^{n-j}}} (\eta \sqcup\!\sqcup \xi, \nu) = \binom{n}{j}, \ \forall \nu \in X^n.$$

Therefore,

$$\|c \sqcup\!\sqcup d\|_{\infty,R} \leq \sup_{\substack{\nu \in X^n \\ 0 \leq j \leq n \\ n \geq 0}} \left\{ \left(\max_{\eta \in X^j} |(c, \eta)| \frac{R^j}{j!} \right) \left(\max_{\xi \in X^{n-j}} |(d, \xi)| \frac{R^{n-j}}{(n-j)!} \right) \right\}.$$

Since $c, d \in S_\infty$, it is clear that $\|c\|_{\infty,R} < \infty$ and $\|d\|_{\infty,R} < \infty$. This implies that

$$\begin{aligned}
\|c \sqcup\!\sqcup d\|_{\infty,R} &\leq \sup_{\substack{\eta \in X^j \\ 0 \leq j \leq n \\ n \geq 0}} \left\{ |(c, \eta)| \frac{R^j}{j!} \right\} \sup_{\substack{\xi \in X^{n-j} \\ 0 \leq j \leq n \\ n \geq 0}} \left\{ |(d, \xi)| \frac{R^{n-j}}{(n-j)!} \right\} \\
&\leq \sup_{\eta \in X^*} \left\{ |(c, \eta)| \frac{R^{|\eta|}}{|\eta|!} \right\} \sup_{\xi \in X^*} \left\{ |(d, \xi)| \frac{R^{|\xi|}}{|\xi|!} \right\} \\
&= \|c\|_{\infty,R} \|d\|_{\infty,R},
\end{aligned}$$

which completes the proof. ∎

Theorem 6: The space S_∞ is closed under the shuffle product.

Proof: Let $c, d \in S_\infty$. Then from Lemma 6 it follows that

$$\|c \sqcup\!\sqcup d\|_{\infty,R} \leq \|c\|_{\infty,R} \|d\|_{\infty,R} < \infty$$

for all $R > 0$. Hence, $c \sqcup\!\sqcup d \in S_\infty$. ∎

IV. Cascade Interconnection

In this section the global convergence of the cascade connection is addressed. It is instructive to start with a few simple examples.

Example 2: Let $X_0 = \{x_0\}$ and assume $c \in \mathbb{R}_{GC}\langle\langle X_0 \rangle\rangle$ has Gevrey order s_c. Since $c \circ d = c$ for any $d \in \mathbb{R}\langle\langle X \rangle\rangle$, it follows that the Gevrey order s_c is preserved for this particular series composition. □

Example 3: Consider the rational series

$$c = \sum_{n_1, n_2 = 0}^{\infty} K M^{n_1 + n_2} \, x_0^{n_1} x_1 x_0^{n_2} = K(Mx_0)^* x_1 (Mx_0)^*.$$

This series is *input-limited* in the sense that there is a fixed upper bound on $|\eta|_{x_1}$ when $\eta \in \mathrm{supp}(c)$. In this case, the letter x_1, corresponding to the input u in $F_c[u]$, appears exactly once in every word in the support of c. It is known that the composition product preserves rationality when its left argument is input-limited [2], [4], [5]. Therefore, since all rational series have Gevrey order is $s = 0$, the Gevrey order is also preserved for this composition. □

Examples 1-3 provide specific cases in which the Gevrey order of the composition of two series can be determined exactly. The following theorem shows that an explicit upper bound on the Gevrey order of a composition over $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$ can be computed when the left argument of the composition product is input-limited. Unfortunately, at present, no other classes of series are known for which an explicit upper bound on the Gevrey order can be determined.

Theorem 7: Let $c, d \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$ with Gevrey orders s_c and s_d, respectively. If $b := c \circ d$ with c is input-limited, then $b \in \mathbb{R}_{GC}\langle\langle X\rangle\rangle$, and its Gevrey order is $s_b \leq \max\{s_c, s_d\}$.

Proof: Since c is input-limited, there exists some fixed $N \in \mathbb{N}$ such $|\eta|_{x_1} \leq N$, $\forall \eta \in \text{supp}\{c\}$. Therefore, the composition product $b = c \circ d$ can be written in terms of a finite number of sums and shuffle products. It then follows from Theorems 3 and 5 that the Gevrey order of b must satisfy $s_b \leq \max\{s_c, s_d\}$. ■

Now the key conjecture that must be proved in order ensure that global convergence is preserved under cascades in the fullest sense describe in Subsection II-B is given below.

Conjecture 1: The space S_∞ is closed under the composition product.

A possible plan of attack for a proof is given using the following two theorems regarding the completeness of the spaces $S_\infty(R)$, $R > 0$ and S_∞.

Theorem 8: $(S_\infty(R), \|\cdot\|_{\infty,R})$ is a Banach space for any $R > 0$.

Proof: The proof parallels the classical proof for the completeness of l^∞ [12, p. 33]. Fix $R > 0$ and let $\{c_i\}_{i\geq 0}$ be a Cauchy sequence in the normed linear space $S_\infty(R)$. Then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $i, j > N$

$$\|c_i - c_j\|_{\infty,R} = \sup_{\eta\in X^*}\left\{|(c_i - c_j, \eta)|\frac{R^{|\eta|}}{|\eta|!}\right\} < \epsilon.$$

Therefore, given any word $\eta \in X^*$

$$|(c_i - c_j, \eta)|\frac{R^{|\eta|}}{|\eta|!} = \left|(c_i, \eta)\frac{R^{|\eta|}}{|\eta|!} - (c_j, \eta)\frac{R^{|\eta|}}{|\eta|!}\right| < \epsilon, \quad (10)$$

implying that $\{(c_i, \eta)R^{|\eta|}/|\eta|!\}_{i\geq 0}$ is a Cauchy sequence in $\mathbb{R}$. Hence, for each $\eta \in X^*$ define

$$c_\eta = \lim_{i\to\infty}(c_i, \eta)\frac{R^{|\eta|}}{|\eta|!},$$

and let $c := \sum_{\eta\in X^*}(c, \eta)\eta$, where $(c, \eta) := c_\eta\,|\eta|!/R^{|\eta|}$. The claim now is that $c \in S_\infty(R)$. Letting $j \to \infty$ in (10) gives

$$\left|(c_i, \eta)\frac{R^{|\eta|}}{|\eta|!} - c_\eta\right| < \epsilon, \quad i > N. \quad (11)$$

For any fixed i, since $c_i \in S_\infty(R)$, there exists a real number $B_i > 0$ such that $|(c_i, \eta)|\,R^{|\eta|}/\,|\eta|! \leq B_i$ for all $\eta \in X^*$. Therefore, if $i > N$ then for every $\eta \in X^*$

$$|(c, \eta)|\frac{R^{|\eta|}}{|\eta|!} \leq \left|c_\eta - (c_i, \eta)\frac{R^{|\eta|}}{|\eta|!}\right| + |(c_i, \eta)|\frac{R^{|\eta|}}{|\eta|!} \leq \epsilon + B_i.$$

Hence, $c \in S_\infty(R)$. To show completeness, it is only necessary to show that $c_i \to c$ as a sequence in $S_\infty(R)$. From (11) it follows that for any $\eta \in X^*$

$$|(c_i, \eta) - (c, \eta)|\frac{R^{|\eta|}}{|\eta|!} < \epsilon, \quad i > N.$$

Therefore, $\|c_i - c\|_{\infty,R} < \epsilon$ when $i > N$, implying that $c_i \to c$ as desired. ■

Theorem 9: The space S_∞ is complete.

Proof: Given that $S_\infty \subset S_\infty(R)$ is closed, it follows from [8, Proposition 0.24] that S_∞ is a complete metric space since $S_\infty(R)$ is a complete metric space for any fixed $R > 0$. ■

A starting point for proving Conjecture 1 is to use the following lemma from [17].

Lemma 7: Let $c \in S_\infty$ and define $c_N = \sum_{n=0}^{N}\sum_{\eta\in X^n}(c, \eta)\eta$, $N \geq 0$. Then each $c_N \in S_\infty$ and $c_N \to c$ in the semi-norm topology.

The key idea is to select $c, d \in S_\infty$ and define the sequence $\{c_N\}_{N\geq 0}$ as above. Since each $c_N \circ d$ can be written in terms of a finite number of summations and shuffles, then by Theorems 4 and 6 it follows that $c_N \circ d \in S_\infty$. So if it can be shown that $c_N \circ d \to c \circ d$ in the semi-norm topology, then from the completeness of S_∞ it follows that $c \circ d \in S_\infty$, thus proving Conjecture 1. But the exact mechanics for this final step have yet to be realized. The following example illustrates Conjecture 1.

Example 4: Consider the bilinear state space system

$$\begin{aligned}\dot{z_1} &= z_1 z_2, \quad z_1(0) = 1\\ \dot{z_2} &= z_2 u, \quad z_2(0) = 1\\ y &= z_1.\end{aligned}$$

It is easily verified that $y = F_{c_F}[u]$, where c_F is defined in (4). The operator F_{c_F} has an infinite radius of convergence since it was shown in [17] that c_F is weakly locally convergent. The cascade of two such systems has the realization

$$\dot{z} = g_0(z) + g_1(z)u, \quad y = h(z), \quad (12)$$

where

$$g_0(z) = \begin{pmatrix} z_1 z_2\\ z_2 z_3\\ z_3 z_4\\ 0\end{pmatrix}, \quad g_1(z) = \begin{pmatrix}0\\0\\0\\z_4\end{pmatrix},$$

$h(z) = z_1$, and $z_i(0) = 1$ for all i. The corresponding generating series $c_F \circ c_F$ can be computed by iterated Lie derivatives (see [11]) to give

$$\begin{aligned}c_F \circ c_F = &\, 1 + x_0 + 2x_0^2 + 6x_0^3 + 23x_0^4 + x_0^3x_1 + 106x_0^5\\ &+ 9x_0^4x_1 + 3x_0^3x_1x_0 + x_0^3x_1^2 + 568x_0^6 + 68x_0^4x_1\\ &+ 34x_0^3x_1x_0 + 11x_0^4x_1^2 + 11x_0^3x_1x_0^2 + 3x_0^3x_1^2x_0\\ &+ 4x_0^3x_1x_0x_1 + x_0^3x_1^3 + \cdots\end{aligned}$$

Consistent with Conjecture 1, $c_F \circ c_F$ should also be weakly locally convergent and therefore $F_{c_F\circ c_F}$ would have an infinite radius of convergence. In order to test this claim independently, note that the solution of (12) can be written in terms of compositions of functionals as

$$y(t) = F_{c_F\circ c_F}[u](t) = F_c[F_c[F_c[F_c[u]]]](t),$$

where

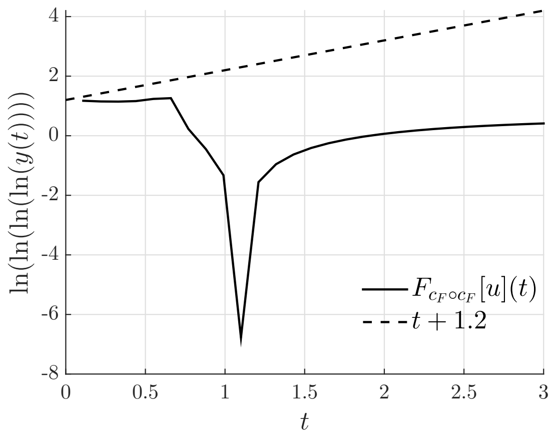

Fig. 5. Response of the operator $F_{c_F \circ c_F}[u]$ when $u = \mathrm{e}^{-t}$ (solid line) on a quadruple logarithmic scale and the bounding function $t + 1.2$ (dotted line).

$$F_c[u](t) = \exp\left(\int_0^t u(\tau)\, d\tau\right).$$

Now given any $u \in L_{\mathfrak{p}}^1[0, T]$ for some arbitrary $T > 0$, $F_c[u]$ is clearly well defined on $[0, T]$. Repeating this argument three more times yields the same conclusion for y. A MatLab simulation of (12) to generate y when $u(t) = \mathrm{e}^{-t}$ is shown in Figure 5. Since the output is four nested exponentials, the response is best viewed by taking four successive logarithms. Note that in the figure the response increases monotonically after approximately $t = 1.1$. The quadruple exponential of $t + 1.2$ (found empirically) bounds the response completely so that there are no finite escape-times no matter how long the simulation is run. This behavior is consistent with that of a globally convergent Fliess operator. $\square$

V. Conclusions

The main result in this paper is that the global convergence property is preserved for parallel and product interconnections using the largest known class of globally convergent Fliess operators. In the process, explicit upper bounds were derived for the Gevrey orders of such interconnections involving generating series from $\mathbb{R}_{GC}\langle\langle X\rangle\rangle$. On the other hand, no general formula of this type is yet known for cascade interconnections. In addition, it has yet to proved that global convergence in the sense described here is preserved under the cascade connection, but a plausible plan of attacked was outlined based on the completeness of the spaces involved. This will be pursued in future work.

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