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Comparative study of $^6$He $\beta$-decay based on different similarity-renormalization-group evolved chiral interactions

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We report on a study of the Gamow-Teller matrix element contributing to $^6$He $\beta$ decay with similarity renormalization group (SRG) versions of momentum- and configuration-space two-nucleon interactions. These interactions are derived from two different formulations of chiral effective field theory ($\chi$EFT)—without and with the explicit inclusion of $\Delta$ isobars. We consider evolution parameters $\Lambda_{\text{SRG}}$ in the range between 1.2 and 2.0 fm$^{-1}$ and, for the $\Delta$-less case, also the unevolved (bare) interaction. The axial current contains one- and two-body terms, consistently derived at tree level (no loops) in the two distinct $\chi$EFT formulations we have adopted here. The $^6$He and $^6$Li ground-state wave functions are obtained from hyperspherical-harmonics (HH) solutions of the nuclear many-body problem. In $A = 6$ systems, the HH method is limited at present to treat only two-body interactions and non-SRG evolved currents. Our results exhibit a significant dependence on $\Lambda_{\text{SRG}}$ of the contributions associated with two-body currents, suggesting that a consistent SRG-evolution of these is needed in order to obtain reliable estimates. We also show that the contributions from one-pion-exchange currents depend strongly on the model (chiral) interactions and on the momentum- or configuration-space cutoffs used to regularize them. These results might prove helpful in clarifying the origin of the sign difference recently found in no-core-shell-model and quantum Monte Carlo calculations of the $^6$He Gamow-Teller matrix element.

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I. INTRODUCTION

Nuclear $\beta$ decays have become, in recent years, a research topic of intense interest. A quantitative understanding of these decays is crucial for a number of experimental endeavors, including the program of experiments planned at the Facility for Rare Isotope Beams (FRIB) to measure weak-interaction rates in nuclei, and neutrinoless double-$\beta$ decay experiments aimed at establishing the Dirac or Majorana nature of the neutrino. In this context, of particular relevance are Gamow-Teller matrix elements (GTMEs). Shell-model calculations have typically failed to reproduce the measured values of these, unless an effective (one-body) Gamow-Teller (GT) operator is used with a nucleon axial coupling constant $g_A$ that is quenched by about 20–30% relative to its free value [1,2]. The shell model also yields rather uncertain estimates [2] for the nuclear matrix elements entering neutrinoless double-$\beta$ decay rates, which are proportional to $g_A^4$. Therefore an understanding of the origin of $g_A$ quenching is important, as is a reliable estimate of the contributions from many-body terms in the weak current.

There have been indications [3–5] that $g_A$ quenching might originate from lack of correlations in shell-model wave functions, and possibly from two-body axial current contributions that tend to decrease the matrix element calculated with the leading one-body GT operator [4]. In this context, it is interesting to note that the Gysbers et al. study [4] consistently finds these two-body contributions to generally have the opposite sign relative to the leading GT contributions in nuclei with mass number $A > 3$. This is in contrast to the results of Refs. [3,5], in which the sign of the one- and two-body contributions is the same, at least in light nuclei with mass number $A \leq 10$, the only ones accessible at this time to Green’s function Monte Carlo (GFMC) methods. The origin of this difference is yet to be clarified. Of course, the comparison of results obtained by different groups is difficult owing to the different models adopted to describe nuclear interactions, and the different methods used to solve the nuclear quantum many-body problem. At this point in time, what can be stated with confidence is that Gysbergs et al. [4] and the authors of Refs. [3,5] only agree on the magnitude of the two-body corrections: they are small in the $A \leq 10$ mass range.

In this work, in an attempt to understand the origin of this discrepancy, we present a calculation of the GTME contributing to the $\beta$-decay of $^6$He, within the hyperspherical-harmonics (HH) method developed by the Pisa group [6,7], and recently extended to deal with $A = 6$ nuclei [8]. The $^6$He and $^6$Li wave functions are obtained from a Hamiltonian including two-nucleon ($2N$) interactions only. Three-nucleon ($3N$) interactions are neglected, since it is not yet possible to incorporate them in HH calculations of $A = 6$ nuclei (although some progress in this direction has been recently made, see Ref. [9], by including the $3N$ contact...
interaction that enters pionless effective field theory at leading order.

We adopt $2N$ interactions obtained in two different formulations of $\chi$EFT: one \cite{10} includes pions and nucleons as degrees of freedom, while the other \cite{11,12} also includes $\Delta$ isobars. To each of these, we apply the similarity renormalization group (SRG) unitary transformation \cite{13} in order to accelerate the convergence rate of the HH expansion. In reference to the nuclear axial currents, we use the chiral models of Refs. \cite{14} and \cite{15} in conjunction with the $\Delta$-less and $\Delta$-full interactions, respectively. These currents are treated without applying the proper SRG transformations. Clearly, the absence of both $3N$ interactions and the proper SRG evolution of interactions and currents does not allow us to obtain a complete and fully consistent description of the process. Nevertheless, having an independent method that can deal with different interactions could prove helpful in clarifying the origin of some of the tensions mentioned above.

The main goal of the present work is to understand the origin of the difference in sign obtained for the two-body contributions to the GTME of $^6$He $\beta$ decay in the no-core shell model (NCSM) and GFMC calculations, reported in Ref. \cite{4} and Refs. \cite{3,5}, respectively. Since use of the next-to-next-to-leading-order (N2LO) interaction of Ref. \cite{10} allows us to achieve a satisfactory convergence in the $A = 6$ HH calculation even without implementing the SRG transformation, we are also in a position to assess the impact of the SRG evolution itself on the GTME, at least as it relates to the $2N$ interaction. However, we should note that the $2N$ interaction adopted here and in the study of Ref. \cite{4} are not the same: specifically, the authors of that work use the next-to-next-to-next-to-leading-order (N4LO500) rather than the N2LO450 model of Ref. \cite{10}. The former is of higher order (N4LO versus N2LO) in the power counting and has a slightly larger cutoff (500 MeV) than the latter (450 MeV).

The other $2N$ interaction we and the authors of Ref. \cite{5} use in the GTME calculations is the NV2-Ia model of Ref. \cite{12}. For this interaction, however, in order to reach convergence in the HH expansion, we are forced to implement the SRG transformation. We consider four different values for the evolution parameter $\lambda_{\text{SRG}}$, namely, $\lambda_{\text{SRG}} = 1.2, 1.5, 1.8,$ and $2.0$ fm$^{-1}$. This allows us to disentangle how two-body axial-current contributions are affected by the input $2N$ interaction model (whether N2LO450 or NV-Ia) and by the corresponding SRG-evolved versions of these models.

The paper is organized as follows. In Secs. II and III we provide a concise review of, respectively, interactions and axial currents, and the HH approach for $A = 6$ nuclei. We report our results in Sec. IV, and close in Sec. V with some concluding remarks. A number of more technical issues having to do with the convergence of the HH method for the $^6$Li and the $^6$He ground states are relegated to Appendices A and B.

II. INTERACTIONS AND AXIAL CURRENTS

In this work we use two different $2N$ chiral interactions. The first one is the next-to-next-to-leading-order (N2LO) model by Entem, Machleidt, and Nosyk \cite{10}. This interaction is derived from a $\chi$EFT including pions and nucleons as degrees of freedom. It is formulated in configuration space and is regularized in this space with two regulators, one ($R_3$) for the short-range components associated with $2N$ contact terms, and the other ($R_L$) for the long-range ones induced by one- and two-pion exchange. Various combinations of $R_3$ and $R_L$ regulators are available, but in this work we have selected the model denoted as NV2-Ia with $(R_3, R_L) = (0.8, 1.2)$ fm.

Below, we will refer to these two interactions as the E and P models by the initial of the first author on the relevant publications, respectively, Refs. \cite{10} and \cite{11}. Both models are evolved using the SRG unitary transformation \cite{13}, in order to improve the convergence of the HH calculation. This SRG evolution leads to momentum-space interactions which are transformed back to coordinate space by standard Fourier transforms. The matrix elements are then computed using the procedure of Ref. \cite{8}.

Since one of our goals is to understand the effect of these SRG-evolved interactions on the GTME, we consider four different values for the evolution parameter $\lambda_{\text{SRG}}$, namely, $\lambda_{\text{SRG}} = 1.2, 1.5, 1.8,$ and $2.0$ fm$^{-1}$. Furthermore, it has been possible with the E interaction to obtain reasonable convergence without implementing any SRG evolution (that is, with the “bare” interaction). This has allowed us to compare directly the bare and SRG calculated GTME, and to assess the role of SRG evolution on this observable (see below). However, we do not account for $3N$ interactions, since SRG evolution for these is not yet available.

Accompanying each of these interactions is a set of $N3LO$ axial currents derived consistently in $\chi$EFT—the formulation that includes pions and nucleons for the E model, and that with, in addition, $\Delta$ isobars for the P model. We provide below their configuration-space expressions in the limit of vanishing momentum transfer of interest here:

(i) The leading-order (LO) term consists of the Gamow-Teller operator

$$ A_{i,a}^{\text{LO}} = -\frac{g_A}{2} t_{i,a} \sigma_i, \quad (2.1) $$

and scales, in a two-body system, as $Q^{-3}$ in the power counting—here, $Q$ denotes generically a low-momentum scale;

(ii) The N2LO terms (scaling as $Q^{-1}$) consist of a relativistic correction to the Gamow-Teller operator

$$ A_{i,a}^{\text{N2LO}}(\text{RC}) = \frac{g_A}{4m^2} t_{i,a} \mathbf{p}_i \times (\sigma_i \times \mathbf{p}_i), \quad (2.2) $$
and of a two-body operator induced by a Δ-isobar intermediate state (this only enters the calculations based on the P interaction)

$$A_{ij,a}^{N3LO}(\Delta) = -(\tau_i \times \tau_j)_a \left[ I_1(r_{ij}; \alpha^3) \sigma_i \times \sigma_j + I_2(r_{ij}; \alpha^3) \sigma_i \times \hat{r}_{ij} \sigma_j \cdot \hat{r}_{ij} \right]$$

$$- \tau_i, a I_1(r_{ij}; \alpha^3) \sigma_j + I_2(r_{ij}; \alpha^3) \hat{r}_{ij} \sigma_j \cdot \hat{r}_{ij} \right] + (i = 2)$$

(2.3)

(iii) The N3LO terms (scaling as $Q^0$) consist of a two-body operator associated with one-pion exchange (OPE):

$$A_{ij,a}^{N3LO}(OPE) = -(\tau_i \times \tau_j)_a \left[ I_1(r_{ij}; \alpha^3) \sigma_i \times \sigma_j + I_2(r_{ij}; \alpha^3) \sigma_i \times \hat{r}_{ij} \sigma_j \cdot \hat{r}_{ij} \right]$$

$$- \tau_i, a I_1(r_{ij}; \alpha^3) \sigma_j + I_2(r_{ij}; \alpha^3) \hat{r}_{ij} \sigma_j \cdot \hat{r}_{ij} \right] + (i = j)$$

and of a two-body contact operator

$$A_{ij,a}^{N3LO}(CT) = I_c(r_{ij}; z_0) (\tau_i \times \tau_j)_a (\sigma_i \times \sigma_j).$$

(2.5)

In Eqs. (2.1)–(2.5), $p_k = -i \nabla_k$, $s_A$, and $r_k$ are the momentum operator, and Pauli spin and isospin operators of nucleon $k$, respectively, $[\ldots, \ldots]$ denotes the anticommutator, and $r_{ij} = r_i - r_j$. Charge-raising (+) or charge-lowering (−) currents follow from $A_{ij,a} = A_{ij,a} \pm i A_{ij,a}$, where the subscript specifies the isospin component. In a many-body system, the one-body operators above are summed over the nucleons $\sum_{i \neq j} A_{ij,a}$, while the two-body ones over the nucleon pairs $\sum_{i \neq j} A_{ij,a}$.

The correlation functions entering the OPE and CT currents and corresponding to the E interaction are regularized by a momentum space cutoff given by $C_A(k) = e^{−k^2/Λ^2}$. They can be written as

$$I_1^E(r; \alpha^E) = -\frac{\alpha^E}{\Lambda r} \int_0^\infty dx x^3 e^{-x^2} j_1(x \Lambda r),$$

(2.6)

$$I_2^E(r; \alpha^E) = \frac{\alpha^E}{\Lambda r} \int_0^\infty dx x^4 e^{-x^2} j_2(x \Lambda r),$$

(2.7)

$$\tilde{I}_E(r; \tilde{\alpha}^E) = -\frac{\tilde{\alpha}^E}{\Lambda} \int_0^\infty dx x^3 e^{-x^2} j_0(x \Lambda r),$$

(2.8)

$$I_c^E(r; \alpha^E_c) = \frac{\alpha^E}{2\pi^2} \int_0^\infty dx x^2 e^{-x^2} j_0(x \Lambda r),$$

(2.9)

where the $j_n(z)$ are spherical Bessel functions, the $\alpha^E$ and $\tilde{\alpha}^E$ denote the combinations of coefficients defined as

$$\alpha^E_1 = \frac{\Lambda^3}{4\pi^2} \frac{g_A}{f_\pi} \left( c_4 + \frac{1}{4m} \right), \quad \alpha^E_2 = \frac{\Lambda^3}{2\pi^2} \frac{g_A c_3}{f_\pi},$$

$$\tilde{\alpha}^E = \frac{\Lambda^2}{8\pi^2} \frac{g_A}{m f_\pi^2} c_4,$$

(2.10)

and $\alpha^E_c$ is the low-energy constant (LEC) that characterizes the contact axial current, its determination is discussed below; note that the $\alpha^E_i$ are adimensional. Here, $g_A$ is the nucleon axial coupling constant $(g_A = 1.2723)$, $f_\pi$ is the pion-decay constant $(f_\pi = 92.4 \text{ MeV})$, and $m_\pi$ and $m$ are the pion and nucleon masses, respectively. The values of the LECs $c_3$ and $c_4$ depend on the interaction model (either E or P) and are listed in Table I.

The (regularized) correlation functions entering the Δ, OPE, and CT currents and corresponding to the P interaction are

$$I_1^P(r; \alpha^P) = -\alpha^P \left( 1 + \mu \right) \frac{e^{-\mu}}{\mu^3} C_R(r),$$

(2.11)

$$I_2^P(r; \alpha^P) = \alpha^P \left( 3 + 3 + \mu^2 \right) \frac{e^{-\mu}}{\mu^2} C_R(r),$$

(2.12)

$$\tilde{I}^P(r; \tilde{\alpha}^P) = -\tilde{\alpha}^P \left( 1 + \mu \right) \frac{e^{-\mu}}{\mu^3} C_R(r),$$

(2.13)

$$I_c^P(r; \alpha^P_c) = \frac{\alpha^P}{2\pi^2} \int_0^\infty dx x^2 e^{-x^2} j_0(x \Lambda r),$$

(2.14)

where $\mu = m_\pi r$, and

$$C_R(r) = 1 - \frac{1}{(r/R_S)^s} e^{(r/R_S)/a_L + 1},$$

(2.15)

Here, $a_L = R_L/2$, and the exponent $s$ is taken as $s = 6$. The $R_S$ and $R_L$ values are $(R_S, R_L) = (0.8, 1.2) \text{ fm}$, consistently with the P model for the nuclear interaction. The correlation functions entering the Δ current of Eq. (2.3) are the same as Eqs. (2.11) and (2.12) but with $\alpha^P \to \alpha^E$. The $\alpha^P_i$ and $\alpha^E_i$ combinations are defined as

$$\alpha^E_1 = \frac{g_A}{8\pi} \frac{m_\pi^3}{f_\pi^2} c_4, \quad \alpha^E_2 = \frac{g_A}{4\pi} \frac{m_\pi^3}{f_\pi^2} c_3,$$

$$\tilde{\alpha}^E = \frac{g_A}{16\pi} \frac{m_\pi^3}{m f_\pi^2} c_4,$$

(2.16)

$TABLE I.$ Values of the LECs $c_3$ and $c_4$ associated with the E [10] and P [11] chiral interactions and used in the accompanying axial currents; they are in units of GeV$^{-1}$. These values are obtained from fits to πN data without (E model) and with (P model) the inclusion of Δ isobars.

<table>
<thead>
<tr>
<th></th>
<th>E model</th>
<th>P model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_3$</td>
<td>−3.61</td>
<td>−0.79</td>
</tr>
<tr>
<td>$c_4$</td>
<td>2.64</td>
<td>1.33</td>
</tr>
</tbody>
</table>
TABLE II. Results for the GTME of the tritium $\beta$ decay for the different components of the currents. Columns labeled LO, and N3LO(CT) refer to the contributions given by the axial currents of Eqs. (2.1), and (2.5), respectively; the column labeled N2LO+N3LO(OPE) refers to the cumulative contribution of the axial currents of Eqs. (2.2), (2.4), and (2.3) in the P-model case. The N3LO(CT) contribution is determined by fitting the LEC $z_0$, used in Eqs. (2.9) and (2.14), to reproduce the observed $\beta$-decay GTME of tritium, $(GT)^{exp/\sqrt{3}} = 0.9511 \pm 0.0013$ [14]. The calculated values of the LEC $z_0$ in fm$^3$ are reported as well in the last column of the table.

<table>
<thead>
<tr>
<th>Model</th>
<th>LO</th>
<th>N2LO+N3LO(OPE)</th>
<th>N3LO(CT) z0 [fm$^3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-SRG1.2</td>
<td>0.9722</td>
<td>-0.0121</td>
<td>-0.0090</td>
</tr>
<tr>
<td>E-SRG1.8</td>
<td>0.9666</td>
<td>-0.0095</td>
<td>-0.0060</td>
</tr>
<tr>
<td>E-SRG2.0</td>
<td>0.9572</td>
<td>-0.0021</td>
<td>-0.0041</td>
</tr>
<tr>
<td>E-bare</td>
<td>0.9446</td>
<td>0.0086</td>
<td>-0.0021</td>
</tr>
<tr>
<td>P-SRG1.2</td>
<td>0.9728</td>
<td>0.0118</td>
<td>-0.0335</td>
</tr>
<tr>
<td>P-SRG1.5</td>
<td>0.9679</td>
<td>0.0182</td>
<td>-0.0348</td>
</tr>
<tr>
<td>P-SRG1.8</td>
<td>0.9620</td>
<td>0.0253</td>
<td>-0.0363</td>
</tr>
<tr>
<td>P-SRG2.0</td>
<td>0.9584</td>
<td>0.0294</td>
<td>-0.0368</td>
</tr>
</tbody>
</table>

with the LECs $c^3_A$ and $c_4^A$ given by

$$c_3^A = -\frac{\hbar^2}{9 m_{\Delta N}}, \quad c_4^A = \frac{\hbar^2}{18 m_{\Delta N}},$$

(2.18)

where $\hbar$ is the nucleon-to-$\Delta$ axial coupling constant ($\hbar_A = 2.74$) and $m_{\Delta N}$ is the $\Delta$-nucleon mass difference ($m_{\Delta N} = 293.1$ MeV). Finally, as per the determination of $z_0$, we note that this LEC is related to the LEC $c_D$ that appears in the 3N contact interaction [15]. Since 3N interactions are altogether ignored in the present work, we fix directly $z_0$ so as to reproduce the experimental value of the GTME in tritium $\beta$ decay, $(GT)^{exp/\sqrt{3}} = 0.9511 \pm 0.0013$ [14], without concerning ourselves with the connection between $z_0$ and $c_D$. We do so for each of the SRG-evolved interactions corresponding to the E and P models. The numerical results for the GTME in tritium $\beta$ decay and the fitted values of $z_0$ are reported in Table II.

III. THE HYPERSONSHERICAL HARMONIC METHOD

The $^6$Li and $^6$He wave functions have been expanded using the HH basis. As reference set of Jacobi vectors for six equal-mass particles we use

$$\xi_{1p} = \sqrt{\frac{5}{3}} \left( r_n - \frac{r_m + r_i + r_k + r_j + r_l}{5} \right),$$
$$\xi_{2p} = \sqrt{\frac{8}{5}} \left( r_n - \frac{r_i + r_k + r_j + r_l}{4} \right),$$
$$\xi_{3p} = \sqrt{\frac{3}{2}} \left( r_i - \frac{r_k + r_j + r_l}{3} \right),$$
$$\xi_{4p} = \sqrt{\frac{4}{3}} \left( r_k - \frac{r_j + r_l}{2} \right),$$
$$\xi_{5p} = r_j - r_l,$$

(3.1)

where $(i, j, k, l, m, n)$ indicates a generic permutation $p$ of the particles. By convention, $p = 1$ is chosen to correspond to $(1,2,3,4,5,6)$. For a given choice of the Jacobi vectors, the hyperspherical coordinates are given by the hyper-radius $\rho$, which is independent on the permutation $p$ of the particles and is defined as

$$\rho = \sqrt{\sum_{j=1}^{5,6} \xi_{jp}^2},$$

(3.2)

and by a set of variables, which in the Zernike and Brinkman representation [16,17], are the polar angles $\xi_{jp}$ of each Jacobi vector and the four additional “hyperspherical” angles $\varphi_{jp}$, with $j = 2, \ldots, 5$, defined as

$$\cos \varphi_{jp} = \frac{\xi_{jp}}{\xi_{1p}^2 + \cdots + \xi_{5p}^2}.$$

(3.3)

Here, $\xi_{jp}$ is the magnitude of the Jacobi vector $\xi_{jp}$. The set of variables $\xi_{1p}, \ldots, \xi_{5p}, \varphi_{2p}, \ldots, \varphi_{5p}$ is denoted hereafter as $\Omega_p$. The expression of the generic $A = 6$ HH function is

$$\mathcal{Y}^{\kappa\lambda\mu}_{\nu} (\Omega_p) = \left\{(\langle Y_{\kappa}(\xi_{1p}) Y_{\lambda}(\xi_{2p}) \rangle)_{\nu} Y_{\lambda} \times \mathcal{P}_{\nu}^{\xi_{3p}, \xi_{4p}, \xi_{5p}} (\varphi_{2p}, \varphi_{3p}, \varphi_{4p}, \varphi_{5p}) \right\}_{LM},$$

(3.4)

where

$$\mathcal{P}_{\nu}^{\xi_{3p}, \xi_{4p}, \xi_{5p}} (\varphi_{2p}, \varphi_{3p}, \varphi_{4p}, \varphi_{5p}) = \mathcal{N}^{\xi_{3p}, \nu_{15}} \mathcal{N}^{\xi_{4p}, \nu_{15}} \mathcal{N}^{\xi_{5p}, \nu_{15}} \left( \sin \varphi_{2p} \mathcal{P}_{\nu}^{(1/2,1/2)} (\cos 2 \varphi_{2p}) \right) \times \mathcal{N}^{\xi_{3p}, \nu_1} \mathcal{N}^{\xi_{4p}, \nu_1} \mathcal{N}^{\xi_{5p}, \nu_1} \left( \cos 2 \varphi_{2p} \mathcal{P}_{\nu}^{(1/2,1/2)} (\cos 2 \varphi_{2p}) \right) \times \mathcal{N}^{\xi_{3p}, \nu_4} \mathcal{N}^{\xi_{4p}, \nu_4} \mathcal{N}^{\xi_{5p}, \nu_4} \left( \cos 2 \varphi_{2p} \mathcal{P}_{\nu}^{(1/2,1/2)} (\cos 2 \varphi_{2p}) \right),$$

(3.5)

and $\mathcal{P}_{\nu}^{a,b}$ are Jacobi polynomials. The coefficients $\mathcal{N}_{\nu_{15}}^{\xi_{3p}, \nu_{15}}$ are normalization factors given explicitly by

$$\mathcal{N}_{\nu_{15}}^{\xi_{3p}, \nu_{15}} = \left[ \frac{2 \nu_{15} \Gamma(\nu_{15} - \frac{1}{2}) \Gamma(\nu_{15} + \frac{3}{2})}{\Gamma(\nu_{15} - \nu_{15} - \ell_1 - \frac{1}{2} \nu_{15} + \ell_1 + \frac{3}{2})} \right]^{1/2},$$

(3.6)

and we have defined

$$\kappa_j = \ell_j + 2n_j + K_{j-1}, \quad \nu_j = K_j + \frac{3}{2} j - 1$$

(3.7)

with $K_1 = \ell_1$ and $K_5 = K$. The integer index $\mu$ labels the set of hyperangular quantum numbers, namely

$$\mu \equiv \{ \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, L_2, L_3, L_4, n_2, n_3, n_4, n_5 \}.$$  

(3.8)

The wave function is constructed to have a well-defined total angular momentum $J$ and third component $L_z$ parity $\pi$ and isospin $T$ (in the following, we ignore the small admixtures between isospin states induced by isospin-symmetry-breaking interactions). Therefore, a complete basis of antisymmetrical hyperangular-spin-isospin states is constructed as follows:

$$\Psi_{\kappa LST} = \sum_{\mu=1}^{360} \phi_{\kappa LST}^{\mu} (i, j, k, l, m, n),$$

(3.9)
The functions \( \gamma_\mu^{KLM}(\Omega_\rho) \) are the HH functions defined in Eq. (3.4), and \( s_i(t_i) \) denotes the spin (isospin) state of nucleon \( i \). Note that the coupling scheme of these spin and isospin states does not follow that of the hyperangular part. This particular choice simplifies the calculation of the interaction matrix elements. The index \( \alpha \) labels the possible sets of hyperangular, spin and isospin quantum numbers compatible with the given values of \( K, L, S, T, J, J, \) and \( \pi \), namely

\[
\alpha \equiv \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, L_2, L_3, L_4, n_2, n_3, n_4, n_5, S_2, S_3, S_4, S_5, T_2, T_3, T_4, T_5\}.
\]

(3.11)

The parity of the state is defined by \( \pi = (-1)^{\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\gamma} \); of course, we include in the basis only those states having the parity of the nuclear state under consideration. By exploiting the sum over the permutation, the antisymmetry on the wave function is imposed by the condition

\[
\ell_5 + S_2 + T_2 = \text{odd}.
\]

(3.12)

This method generates linearly dependent HH states. However, in the basis we only include independent states, obtained by calculating the norm matrix elements and by implementing the Gram-Schmidt orthogonalization procedure (the technique is described in Ref. [8]). This drastically reduces the number of states used in the expansion.

The final form of the six-nucleon bound state wave function can be written as

\[
\psi_6^{TJ} = \sum_{l} \sum_{K} c_{l,\alpha}^{KLSTJ} f_j(\rho) \phi_{\alpha}^{KLSTJ\pi},
\]

(3.13)

where the sum is over the linearly independent antisymmetric states \( \alpha \), and \( c_{l,\alpha}^{KLSTJ} \) are variational coefficients to be determined. The hyper-radial functions \( f_j(\rho) \) are chosen to be

\[
f_j(\rho) = y^{15/2} L_l^{(14)}(y\rho) e^{-y\rho/2},
\]

(4.1)

where \( L_l^{(14)}(y\rho) \) are Laguerre polynomials [18], and \( y \) is a nonlinear variational parameter that is introduced so as to improve the convergence on \( l \). A typical range for \( y \) is 3.5–5.5 fm\(^{-1}\) while the sum over \( l \) is typically carried up to \( l = 20 \).

The expansion coefficients \( c_{l,\alpha}^{KLSTJ} \) are determined by using the Rayleigh-Ritz variational principle. The resulting eigenvalue problem is solved with the procedure of Ref. [19].

Even though the number of states is much reduced, a brute force approach, in which the complete basis of independent states up to a maximum \( K \) is included, is not yet possible. For this reason, we select subsets of basis states, separating them in classes of convergence. Within each class, we analyze the convergence pattern in order to obtain a reliable extrapolation for the binding energy. A fairly detailed discussion of these classes for \( ^6\text{Li} \) is given in Ref. [8]. It is summarized here in Appendix A along with a discussion of the classes of convergence for \( ^6\text{He} \). In the Appendix, we also discuss the extrapolation procedure, and provide tables exhibiting the convergence pattern, within each class and for each nucleus, corresponding to the different interaction models.

### IV. RESULTS

The extrapolated binding energies for the \( ^6\text{Li} \) and \( ^6\text{He} \) ground states corresponding to the E and P models are listed in Table III. We stress again that 3N interactions as well as many-body interactions induced by the SRG transformation are not accounted for. Nevertheless, the results obtained with the SRG-evolved versions of the E and P models happen to be quite close to the experimental values.

We define the reduced GTME as

\[
\text{RME}(K_L, K_H) = \sqrt{2J_f + 1} \langle \psi_{J_f,M(K_L)} | A_z^{(\text{GTME})} \psi_{J_f,M(K_H)} \rangle (J_f, M, 10 | J_f, M),
\]

(4.1)

where \( A_z^{(\text{GTME})} \) is the z component (at vanishing momentum transfer) of the total charge-raising axial current given in Sec. II, and \( (J_f, M, 10 | J_f, M) \) is a Clebsch-Gordan coefficient; note that the \( ^6\text{He} \) and \( ^6\text{Li} \) ground states have \( J^z_f = 0^+ \) and \( J^z_f = 1^+ \), respectively. This matrix element depends explicitly on the maximum value of \( K \) used in the HH expansion of the \( ^6\text{He} \) (\( K_f \)) and \( ^6\text{Li} \) (\( K_L \)) wave functions. Its evaluation is carried out by Monte Carlo integration with \( \sim 30000 \) configurations, which yields a statistical error of the order of \( \sim 1\% \) on the individual components beyond LO of the axial current, except for the \( \text{A_z^{(\text{OPE})}} \) component because of accidental cancellations (see below).

We study separately the convergence of the RME with respect to \( K_H \) and \( K_L \), since the states included in the HH expansions of the \( ^6\text{He} \) and \( ^6\text{Li} \) wave functions are different. We proceed as follows. We fix \( K_L (K_H) \) to the maximum value used in the present work—namely, \( K_L = 12 (K_H = 12) \)—and

<table>
<thead>
<tr>
<th>( ^6\text{Li} )</th>
<th>( ^6\text{He} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>E model</td>
<td>P model</td>
</tr>
<tr>
<td>SRG1.2</td>
<td>32.19(1)</td>
</tr>
<tr>
<td>SRG1.5</td>
<td>33.47(2)</td>
</tr>
<tr>
<td>SRG1.8</td>
<td>33.33(5)</td>
</tr>
<tr>
<td>SRG2.0</td>
<td>32.94(7)</td>
</tr>
<tr>
<td>Exp.</td>
<td>31.99</td>
</tr>
</tbody>
</table>
then compute the matrix element by increasing the value of $K_H$ ($K_L$). The LO RME exhibits an exponential behavior with respect to both $K_H$ and $K_L$, as shown in the left panel of Fig. 1. We fit our results with a function of the form $RME(K) = RME(\infty) + A \exp(-bK)$ for $K_L, K_H \geq 4$, where the parameter $RME(\infty)$ is the extrapolated value corresponding to $K_L \to \infty$ and $K_H \to \infty$. The fits are indicated by the solid and dashed lines. The two extrapolated values are then mediated with the weighted average in order to obtain the final result. The same exponential behavior of the RME is observed when all axial-current contributions up to N3LO are included, see solid lines in the right panel of Fig. 1. It is worthwhile noting, though, that this behavior is essentially driven by the LO term, since higher-order terms only provide small corrections to the RME, see below. Considering separately the contributions beyond LO, we observe that they do not present any particular convergence pattern. However, the calculations for $8 \leq K_L, K_H \leq 12$ are compatible within twice the statistical error bars of the Monte Carlo integration. Therefore, we consider as our best estimate the weighted average between the

![Graph](image-url)

**FIG. 1.** RME values computed as function of the maximum $K$ used in the expansion of the $^6$Li ($K_L$, circles) and $^6$He ($K_H$, squares) wave functions for the SRG2.0 version of the $E$ interaction. The left (right) panel corresponds to results obtained with the LO (up to N3LO) axial current. The solid red (dashed blue) line is a fit to the calculated RMEs as function of $K_L$ ($K_H$); see text for further explanations. All remaining interactions exhibit a similar pattern of convergence.

Table IV. Extrapolated RMEs in $^6$He $\beta$ decay. The results are obtained by using SRG-evolved versions of the $E$ and $P$ models with $\Lambda_{SRG} = 1.2, 1.5, 1.8, 2.0$ fm$^{-1}$; 3W interactions are not included. Columns labeled LO, N2LO(RC), N3LO(OPE), and N3LO(CT) refer to the contributions given by the axial currents of Eqs. (2.1), (2.2), (2.4), and (2.5), respectively; the column labeled N2LO(RC+$\Delta$) refers to the cumulative contribution of the axial currents of Eqs. (2.2) and (2.3). Note that the N3LO(CT) results have been divided out by the $z_0$ values listed in Table II. The errors, when shown, are associated with the extrapolation; when they are not explicitly indicated, they are below the precision reported in the table. For a qualitative comparison, we also list the results of Ref. [5] obtained with the $P$ model (including 3N interactions), and the experimental value from Ref. [21].

<table>
<thead>
<tr>
<th>E model</th>
<th>LO</th>
<th>N2LO(RC)</th>
<th>N3LO(OPE)</th>
<th>N3LO(CT)</th>
<th>N3LO(CT)/$z_0^2$ [fm$^{-3}$]</th>
<th>Full</th>
</tr>
</thead>
<tbody>
<tr>
<td>SRG1.2</td>
<td>2.345(2)</td>
<td>-0.019</td>
<td>-0.038(1)</td>
<td>-0.018</td>
<td>-0.162(1)</td>
<td>2.271(3)</td>
</tr>
<tr>
<td>SRG1.5</td>
<td>2.342(3)</td>
<td>-0.021</td>
<td>-0.029(1)</td>
<td>-0.011</td>
<td>-0.185(1)</td>
<td>2.281(2)</td>
</tr>
<tr>
<td>SRG1.8</td>
<td>2.327(3)</td>
<td>-0.022</td>
<td>-0.019(1)</td>
<td>-0.008</td>
<td>-0.198(1)</td>
<td>2.281(4)</td>
</tr>
<tr>
<td>SRG2.0</td>
<td>2.336(3)</td>
<td>-0.022</td>
<td>-0.013(1)</td>
<td>-0.008</td>
<td>-0.202(1)</td>
<td>2.297(2)</td>
</tr>
<tr>
<td>bare</td>
<td>2.321(9)</td>
<td>-0.023</td>
<td>0.002(1)</td>
<td>-0.004</td>
<td>-0.211(1)</td>
<td>2.303(11)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>P model</th>
<th>LO</th>
<th>N2LO(RC+$\Delta$)</th>
<th>N3LO(OPE)</th>
<th>N3LO(CT)</th>
<th>N3LO(CT)/$z_0^2$ [fm$^{-3}$]</th>
<th>Full</th>
</tr>
</thead>
<tbody>
<tr>
<td>SRG1.2</td>
<td>2.354(1)</td>
<td>-0.033(1)</td>
<td>0.011</td>
<td>-0.066</td>
<td>-0.143(1)</td>
<td>2.265(2)</td>
</tr>
<tr>
<td>SRG1.5</td>
<td>2.331(4)</td>
<td>-0.030(1)</td>
<td>0.016</td>
<td>-0.066</td>
<td>-0.166(1)</td>
<td>2.251(3)</td>
</tr>
<tr>
<td>SRG1.8</td>
<td>2.329(5)</td>
<td>-0.023(1)</td>
<td>0.020</td>
<td>-0.068</td>
<td>-0.177(1)</td>
<td>2.257(4)</td>
</tr>
<tr>
<td>SRG2.0</td>
<td>2.322(6)</td>
<td>-0.019(1)</td>
<td>0.022</td>
<td>-0.070</td>
<td>-0.185(1)</td>
<td>2.260(11)</td>
</tr>
<tr>
<td>NV2-Ia + 3b(VMC) [5]</td>
<td>2.200</td>
<td>0.022</td>
<td>0.039</td>
<td>-0.005</td>
<td>-0.009</td>
<td>2.256</td>
</tr>
<tr>
<td>NV2-Ia + 3b(GFMC) [5]</td>
<td>2.130</td>
<td>0.022</td>
<td>2.201</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exp. [21]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.1609(40)</td>
</tr>
</tbody>
</table>

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values obtained in the range $8 \leq K_r, K_H \leq 12$. We note that the convergence pattern of these contributions is independent of the interaction model (either E or P) and the value of $\Lambda_{\text{SRG}}$. The extrapolated values of the RME for each individual component of the current as well as for the full current are reported in Table IV. We find that two-body currents give a overall correction of opposite sign to the LO contribution, of the order of $\sim 3/4\%$, in line with the results of Refs. [4,22]. However, a closer inspection of the table suggests a more complex situation.

From the first column of Table IV, the LO contribution seems to have a weak dependence on $\Lambda_{\text{SRG}}$: the larger is $\Lambda_{\text{SRG}}$, the smaller is the resulting LO contribution. This same sensitivity is also shown in Fig. 8 of Ref. [4] and, as demonstrated by the authors of that paper, it is removed by including the SRG-induced two-body operators corresponding to the LO current. By comparing the results for the bare E model with its SRG evolved versions, the difference is of the order of $\sim 0.5\%$ and we would have expected a similar difference also in the case of the P model, had we been able to use the bare interaction. However, the results reported in Ref. [5] show that at least a correction of the order of $5\%$ is needed. This extra quenching of the LO contribution comes from $3N$ interaction effects [23].

The N2LO(CT) contribution, which only consists of relativistic corrections to the LO Gamow-Teller operator, appears to be independent of the SRG-evolution parameter for both the E and P models. By contrast, the N2LO($\Delta$) contribution strongly depends on $\Lambda_{\text{SRG}}$, and is responsible for generating the pattern shown in Table IV. It starts off negative for $\Lambda_{\text{SRG}} = 1.2 \text{ fm}^{-1}$, increases monotonically, and becomes positive for $\Lambda_{\text{SRG}} = 2.0 \text{ fm}^{-1}$. In Ref. [5] (with the bare P interaction) this contribution is found to be positive and larger than the negative N2LO(CT) contribution, resulting in an overall positive value for the sum. Here, the situation is reversed, and even at $\Lambda_{\text{SRG}} = 2.0 \text{ fm}^{-1}$ the sum of the N2LO(CT) and N2LO($\Delta$) contributions remains negative. Such a difference is clearly due to SRG-evolution effects.

The N3LO(OPE) contribution also depends strongly on $\Lambda_{\text{SRG}}$, see Table IV. For the E model, it starts off negative at low $\Lambda_{\text{SRG}}$, and increases monotonically as $\Lambda_{\text{SRG}}$ increases, becoming positive in the limit $\Lambda_{\text{SRG}} \to \infty$, corresponding to the bare interaction. This is a clear indication that a proper SRG evolution of the N3LO(OPE) current—as well as the N2LO($\Delta$) current, discussed above—is needed to obtain reliable estimates. Such a program has been partially carried out in Ref. [4]. However, to best of our knowledge, three body induced axial currents generated by the SRG evolution have not been included. The results obtained with the P model show the same behavior as function of $\Lambda_{\text{SRG}}$. However, in this case the N3LO(OPE) contribution is positive for all $\Lambda_{\text{SRG}}$ used, and the calculations seem to go in the direction of Ref. [5] when $\Lambda_{\text{SRG}}$ increases. However, we should point out that, because of cancellations between the terms proportional to $c_3$ and $c_4$, the overall N3LO(OPE) contribution is rather sensitive to the actual values of these LECs, in particular their ratio $c_3/c_4$. Lastly, for this contribution we do not expect significant effects from $3N$ interactions, since the latter do not affect appreciably the short-range behavior of two-nucleon densities. These densities, and the resulting change of sign between the E and P N3LO(OPE) contributions, are studied in the next section.

The N3LO(CT) contributions are found to be negative for both interaction models. When divided out by the LEC $z_0$—column labeled N3LO(CT)/$z_0$—they are almost identical between the E and P models. The results reported in Table IV exhibit a significant dependence on the SRG-evolution parameter. It is interesting to note, however, how these results, when they are multiplied by the fitted values of $z_0$ from Table II, become essentially independent of $\Lambda_{\text{SRG}}$ for the P model. By contrast, in the case of the E model the results remain $\Lambda_{\text{SRG}}$-dependent, albeit the trend is inverted (rather than de-
increasing, they increase as $\Lambda_{\text{SRG}}$ increases. It seems that $z_0$ can absorb, at least partially, the effect of the SRG evolution of the currents. By comparing our results for the P model with those of Ref. [5], there is almost one order of magnitude of difference.

Two-body transition densities

In order to understand the differences between the results obtained with the two different chiral interactions, we compute the two-body transition density, which we define as [5,24]

$$\text{RME}(2b) = 4\pi \int_0^\infty dr r^2 \rho_2^b(r), \quad (4.2)$$

where $r$ is the distance between two nucleons and $2b$ stands for N2LO($\Delta$) (only for the P model), N3LO(OPE), and N3LO(CT). In Fig. 2 we report the two-body densities computed using $\Lambda_{\text{SRG}} = 2.0$ fm$^{-1}$ for the E and P models. Their shape is independent on the $\Lambda_{\text{SRG}}$ value, except for the N2LO($\Delta$) contribution for the P model where for $\Lambda_{\text{SRG}} = 1.2, 1.5, 1.8$ fm$^{-1}$ the two-body transition densities result of opposite sign.

Inspection of the two panels in Fig. 2 indicates that the N3LO(OPE) densities corresponding to the E and P models are rather different. As a matter of fact, the shape of these densities is determined by the cancellation between the two components of the current proportional to the LECs $c_3$ and $c_4$ through $\alpha_1$ and $\alpha_2$ in Eq. (2.4). In Fig. 3 we plot the separated contributions for the two interactions. In the E model there is a double lobe structure for both the $c_3$ and $c_4$ components. This, and the fact that the maxima of the second lobes do not coincide, generate a three-lobe structure with two of the lobes negative and one positive. In the P model, the $c_3$ and $c_4$ components have both just one lobe, which generates a single lobe in the total contribution (see Fig. 3). This is qualitatively consistent with the results reported in Ref. [5].

The difference in the N3LO(OPE) densities of the E and P models originates from that in the corresponding correlation functions entering the current, see Eqs. (2.6)–(2.7) and Eqs. (2.11)–(2.12). We plot those proportional to $c_3$ (with $c_3 = 1$ in units of GeV$^{-1}$) in Fig. 4. In the region $r \lesssim 3$ fm, their shapes are affected by the choice of regulator. This also produces the sign inversion between the E and P model $c_3$ and $c_4$ contributions, shown in Fig. 3.

For the N3LO(CT) contribution, the main difference between the two interactions is the presence of a second tiny lobe at $r \approx 2$ fm in the E model, and the fact that the maximum is shifted towards larger $r$ values (around 1 fm) compared to that in the P model. Also in this case, the origin of the differences among the two-body densities comes from the different behavior of the correlation functions given in Eqs. (2.9) and (2.14). The results obtained with the P model with $\Lambda_{\text{SRG}} = 2.0$ for the N2LO($\Delta$), N3LO(OPE), and N3LO(CT) densities are in qualitative agreement with those of Ref. [5].
V. CONCLUSIONS

In this work, we have reported on a study of the $^6$He GTME, using different chiral two-nucleon interactions, the N2LO450 [10] and NV2-Ia [12] models. Both models have been evolved via SRG unitary transformations corresponding to parameters $\Lambda_{SRG}$ between 1.2 and 2.0 fm$^{-1}$. We have neglected 3N and SRG-induced many-nucleon interaction effects, as well as SRG-induced many-body terms in the nuclear axial current.

The results are summarized in Table IV. We find for both models that all axial-current terms beyond LO yield a cumulative contribution, which (in magnitude) amounts to a 3% correction of the LO Gamow-Teller contribution. We also find this cumulative contribution to have the opposite sign of the LO one, in agreement with the results of Refs. [4,22]. The contributions of two-body currents, in particular of N3LO(OPE), while small, depend strongly on the parameter $\Lambda_{SRG}$, suggesting that a consistent evolution of these currents (together with the one-body current) may be necessary in order to obtain reliable predictions. The same conclusion can also be drawn by considering the results for the tritium $\beta$ decay in Table II.

We have been unable to reproduce the sign of the beyond-LO contributions obtained in Ref. [5] with the bare NV2-Ia interaction. This can be traced back to differences in the contributions associated with the N2LO(RC) and N3LO(CT) currents. The origin of these differences is unclear. We conjecture they might be due to the absence, in the present HH calculation, of the multinucleon terms induced by the SRG transformation in the interactions and currents. By contrast, there is qualitative agreement in the shape of the two-body transition densities calculated here and in Ref. [5].

We have shown that the N3LO(OPE) contribution is opposite in sign for the SRG-evolved N2LO450 and NV2-Ia interactions. The corresponding transition densities in Fig. 2 have different shapes, reflecting the different behavior of the correlation functions entering the N3LO(OPE) current, see Fig. 4. This behavior follows in turn from the different choice of short-range regulators we have adopted in the N2LO450 and NV2-Ia calculations. We note in closing that the sign difference in the N3LO(OPE) contribution obtained in Refs. [4] and [5] may have a similar origin.

ACKNOWLEDGMENTS

We thank G.B. King, S. Pastore, and R.B. Wiringa for a useful email exchange on the effect of 3N interactions on $\beta$-decay matrix elements. We also thank M. Piarulli for her help during the implementation of the SRG version of the Norfolk potential. This research is supported by the U.S. Department of Energy, Office of Nuclear Science, under Contracts No. DE-AC05-06OR23177 (A.G. and R.S.). The calculations were made possible by grants of computing time from the National Energy Research Supercomputer Center (NERSC).

APPENDIX A: CLASSES OF CONVERGENCE

In this Appendix we define the classes of convergence in which we separate the HH states. This definition is based on a couple of criteria. The first one is that, as the value $\ell_{\text{sum}} = \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5$ increases, so does the centrifugal barrier, which keeps nucleons apart from each other, thus reducing the effect of correlations induced by the nuclear interactions. The second criterion accounts for the fact that the $2N$ interaction favors two-body correlations and so the HH states with nonzero quantum numbers for the couple $(i, j)$ are privileged. These states can be easily selected by imposing $\ell_i = 0$ with $i = 1, 2, 3, 4$. Furthermore, the HH states can also be classified on the basis of their $\text{LST}$ quantum numbers (or partial waves). Indeed, in the $^6$Li and $^6$He nuclei the most important partial waves are the $S$ and $D$ waves, while all the others give small contribution to the binding energy. In Tables V and VI we report the properties of the HH states used to define a given class for, respectively, $^6$Li and $^6$He. For each class $i$ we also give the maximum value of $K$ we have adopted ($K_{\text{max}}$). A more detailed discussion of the class definition for $^6$Li can be found in Ref. [8]. Here, we only note that in the case of $^6$He we divide the HH states in six different classes. Classes $C_1^5$ and $C_2^5$ are the main components of the $^6$He ground state, since they correspond to two-body correlated states having $L = 0$ and 2, respectively. For both of them we reach $K$ values up to $K_{\text{max}} = K_{\text{max}} = 12$. Classes $C_3^5$ and $C_4^5$ contain HH states that generate many-body correlations for the $S$ and $D$ wave, respectively. For this reason, their contribution to the binding energy is smaller and we stop at $K_{\text{max}} = K_{\text{max}} = 10$. Class $C_5^5$ contains HH states with $L = 1$, which are less important in the construction of the wave function. We therefore keep $K$ values up to $K_{\text{max}} = 8$.

TABLE V. Definition of the classes of hyperangular-spin-isospin states $\psi^\text{LST}_{n,i,j}^\text{CH}$ [see Eq. (3.9)] used for the $^6$Li bound state as given in Ref. [8]. The classes are defined by selecting particular values of the total orbital angular momentum $L$ and total spin $S$ (indicated by the spectroscopic notation $2S+1L_J$), and the value of $\ell_{\text{sum}} = \ell_1 + \cdots + \ell_5$, given in the second and third column, respectively. In the last column, the maximum $K$ value adopted in the expansion is reported for each class.

<table>
<thead>
<tr>
<th>Class</th>
<th>Partial waves</th>
<th>$\ell_{\text{sum}}$</th>
<th>$K_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1^5$</td>
<td>$^3S_1$</td>
<td>$\ell_{\text{sum}} = 0$</td>
<td>14</td>
</tr>
<tr>
<td>$C_2^5$</td>
<td>$^3D_1$, $^3D_1$, $^7D_1$</td>
<td>$\ell_{\text{sum}} = 2$, $\sum_{i=1,4} \ell_i = 0$</td>
<td>12</td>
</tr>
<tr>
<td>$C_3^5$</td>
<td>$^3S_1$</td>
<td>$\ell_{\text{sum}} = 2$</td>
<td>10</td>
</tr>
<tr>
<td>$C_4^5$</td>
<td>$^3D_1$, $^3D_1$, $^7D_1$</td>
<td>$\ell_{\text{sum}} = 2$, not included in $C_2^5$</td>
<td>10</td>
</tr>
<tr>
<td>$C_5^5$</td>
<td>$^1P_1$, $^3P_1$, $^3P_1$</td>
<td>$\ell_{\text{sum}} = 2$</td>
<td>8</td>
</tr>
<tr>
<td>$C_6^5$</td>
<td>$^5F_1$, $^7F_1$, $^7G_1$</td>
<td>$\ell_{\text{sum}} = 4$</td>
<td>8</td>
</tr>
</tbody>
</table>

TABLE VI. Same as Table V but for $^6$He.

<table>
<thead>
<tr>
<th>Class</th>
<th>Partial waves</th>
<th>$\ell_{\text{sum}}$</th>
<th>$K_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1^6$</td>
<td>$^3S_0$</td>
<td>$\ell_{\text{sum}} = 0$</td>
<td>12</td>
</tr>
<tr>
<td>$C_2^6$</td>
<td>$^3D_0$</td>
<td>$\ell_{\text{sum}} = 2$, $\sum_{i=1,4} \ell_i = 0$</td>
<td>12</td>
</tr>
<tr>
<td>$C_3^6$</td>
<td>$^3S_0$</td>
<td>$\ell_{\text{sum}} = 2$</td>
<td>10</td>
</tr>
<tr>
<td>$C_4^6$</td>
<td>$^3D_0$</td>
<td>$\ell_{\text{sum}} = 2$, not included in $C_2^6$</td>
<td>10</td>
</tr>
<tr>
<td>$C_5^6$</td>
<td>$^3P_0$</td>
<td>$\ell_{\text{sum}} = 2$</td>
<td>8</td>
</tr>
<tr>
<td>$C_6^6$</td>
<td>$^5F_0$</td>
<td>$\ell_{\text{sum}} = 4$</td>
<td>8</td>
</tr>
</tbody>
</table>
for these. Finally, class $C_b^H$ consists of HH states with $L = 3$. Their contribution to the binding energy is tiny and so we select $K_{BM} = 8$.

**APPENDIX B: CONVERGENCE OF THE HH EXPANSION**

In this Appendix we study the convergence of the $^6$Li and $^6$He binding energies and discuss the extrapolation method. The convergence is studied class by class. When studying the convergence of a generic class $C_i$, we include in the expansion all the HH states with $K_i \leq K_i$ and then vary $K_i$ between a minimum value and $K_{BM}$. For the other classes $C_j$ with $j \neq i$, we include all HH states up to $K_{BM}$. Note that for classes $C_i^L$ and $C_j^L$ ($C_i^H$ and $C_j^H$) in $^6$Li ($^6$He), because of the procedure used for the selection of the linearly independent HH states, we cannot include, respectively, classes $C_i^L$ and $C_j^L$ ($C_i^H$ and $C_j^H$). The $^6$Li and $^6$He binding energies are listed in Tables VII and VIII.

We assume that for each class of convergence the behavior of the binding energy as function of $K$ is exponential, namely

$$B_i(K) = B_i(\infty) + a_i e^{-b_i K},$$  

(B1)

where $B_i(\infty)$ is the asymptotic binding energy of class $C_i$ as $K \to \infty$. The parameters $a_i$ and $b_i$ depend on the interaction model and on the specific class of HH states we are studying. The values of $b_i(K)$ are those reported in Tables VII and VIII. By defining the function

$$\Delta_i(K) = B_i(K) - B_i(K - 2),$$  

(B2)

it is possible to compute, for each class, the “missing” binding energy due to the truncation of the expansion to a finite $K_{BM}$ as illustrated in Ref. [25], namely,

$$\langle \Delta B \rangle_i = \sum_{K=K_{BM}+2,K_{BM}+4,...} \Delta_i(K).$$  

(B3)

By using Eq. (B1), we obtain

$$\langle \Delta B \rangle_i = \Delta_i(K_{BM}) \frac{1}{e^{b_i} - 1}.$$  

(B4)

The “total missing” binding energy is then computed as

$$\langle \Delta B \rangle_T = \sum_{i=1,2} \Delta_i(K_{BM}) \frac{1}{e^{b_i} - 1}.$$  

(B5)

In order to determine the coefficients $b_i$ for each class, we proceed as follows. For classes $C_i^L$, $C_j^L$, and $C_i^H$, we estimate the $b_i$ by performing a fit to the binding energy values of Tables VII and VIII, using Eq. (B1). We propagate the error on
we have considered for $^6$Li. In parentheses, we report the errors on the extrapolation; note that (0) indicates that the error does not affect the last digit reported in the result. We use the central value of the interval as the best estimate,

$$b_i = \frac{\Delta_i(K_{IM} - 2)}{\Delta_i(K_{IM})} = e^{2b_i}. \quad (B8)$$

In such cases, we use these $b_i$ to obtain the missing energy, and estimate the error as half of this missing energy. Finally, for classes $C_i^L$ and $C_i^H$ it is not possible to obtain reliable values for the $b_i$. Therefore, we estimate the missing binding energy as $\Delta_i(K_{IM})$ and the error as half of it. In Tables IX and X we report the missing binding energy with the associated error for each of the six classes we have considered.

### Table IX. Missing binding energies corresponding to the E and P interaction models, obtained for each of the six classes of convergence we have considered for $^6$Li. In parentheses, we report the errors on the extrapolation; note that (0) indicates that the error does not affect the last digit reported in the result.

<table>
<thead>
<tr>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$K_3$</th>
<th>$K_4$</th>
<th>$K_5$</th>
<th>$K_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>E model</td>
<td>SRG1.2</td>
<td>SRG1.5</td>
<td>SRG1.8</td>
<td>SRG2.0</td>
<td>Bare</td>
</tr>
<tr>
<td>P model</td>
<td>SRG1.2</td>
<td>SRG1.5</td>
<td>SRG1.8</td>
<td>SRG2.0</td>
<td></td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.007(0)$</td>
<td>$0.016(0)$</td>
<td>$0.022(0)$</td>
<td>$0.025(0)$</td>
<td>$0.175(0)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.003(0)$</td>
<td>$0.018(2)$</td>
<td>$0.066(5)$</td>
<td>$0.118(7)$</td>
<td>$0.557(19)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.015(8)$</td>
<td>$0.030(15)$</td>
<td>$0.046(23)$</td>
<td>$0.049(24)$</td>
<td>$0.105(53)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.004(2)$</td>
<td>$0.013(6)$</td>
<td>$0.035(18)$</td>
<td>$0.054(27)$</td>
<td>$0.231(116)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.004(0)$</td>
<td>$0.020(1)$</td>
<td>$0.061(1)$</td>
<td>$0.102(3)$</td>
<td>$0.319(25)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.008(4)$</td>
<td>$0.032(16)$</td>
<td>$0.080(40)$</td>
<td>$0.118(59)$</td>
<td>$0.302(151)$</td>
</tr>
<tr>
<td>$C_{i^L}$</td>
<td>$0.033(9)$</td>
<td>$0.113(23)$</td>
<td>$0.288(50)$</td>
<td>$0.442(70)$</td>
<td>$1.515(200)$</td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.008(0)$</td>
<td>$0.016(0)$</td>
<td>$0.023(1)$</td>
<td>$0.026(1)$</td>
<td></td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.002(0)$</td>
<td>$0.016(2)$</td>
<td>$0.071(4)$</td>
<td>$0.173(10)$</td>
<td></td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.016(8)$</td>
<td>$0.030(15)$</td>
<td>$0.041(20)$</td>
<td>$0.051(25)$</td>
<td></td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.003(1)$</td>
<td>$0.012(6)$</td>
<td>$0.032(16)$</td>
<td>$0.060(30)$</td>
<td></td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.005(1)$</td>
<td>$0.019(1)$</td>
<td>$0.074(60)$</td>
<td>$0.123(24)$</td>
<td></td>
</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.008(4)$</td>
<td>$0.030(15)$</td>
<td>$0.074(37)$</td>
<td>$0.120(60)$</td>
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</tr>
<tr>
<td>$C_{i^H}$</td>
<td>$0.034(9)$</td>
<td>$0.107(22)$</td>
<td>$0.292(75)$</td>
<td>$0.527(76)$</td>
<td></td>
</tr>
</tbody>
</table>
TABLE X. Same as Table IX but for $^6$He.

<table>
<thead>
<tr>
<th></th>
<th>E model</th>
<th>P model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SRG1.2</td>
<td>SRG1.5</td>
</tr>
<tr>
<td>$C_{1}^{H}$</td>
<td>0.051(1)</td>
<td>0.088(3)</td>
</tr>
<tr>
<td>$C_{2}^{H}$</td>
<td>0.006(1)</td>
<td>0.030(5)</td>
</tr>
<tr>
<td>$C_{3}^{H}$</td>
<td>0.009(4)</td>
<td>0.018(9)</td>
</tr>
<tr>
<td>$C_{4}^{H}$</td>
<td>0.009(4)</td>
<td>0.020(6)</td>
</tr>
<tr>
<td>$C_{5}^{H}$</td>
<td>0.0002(1)</td>
<td>0.006(3)</td>
</tr>
<tr>
<td>Tot.</td>
<td>0.078(7)</td>
<td>0.162(13)</td>
</tr>
</tbody>
</table>