Fall 2013

Identifiability of Additive, Time-Varying Actuator and Sensor Faults by State Augmentation

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IDENTIFIABILITY OF ADDITIVE, TIME-VARYING ACTUATOR AND SENSOR FAULTS BY STATE AUGMENTATION

by

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A Thesis Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

ELECTRICAL AND COMPUTER ENGINEERING

OLD DOMINION UNIVERSITY
December 2013

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ABSTRACT

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Faults in dynamical systems can have serious safety and reliability implications. For example, actuator and sensor faults have been factors in past incidents and mishaps in many aerospace systems. A large body of research is devoted to developing methods to detect and identify actuator and sensor faults in such systems.

One fault detection and identification method employs state augmentation, whereby a set of time-varying faults of interest are modeled as outputs of exogenous linear, time-invariant systems and augmented to the state of the nominal system model. The resulting model represents the system dynamics due to a particular actuator-sensor fault configuration. Typically, a filter is associated with each model, and a test matches the model most closely associated with the present system state estimates and measurements. A significant portion of the model-based fault detection and identification literature is concerned with the design of such filters. A basic requirement of these techniques is that the modeled fault configuration of interest be identifiable.

Recent research has led to a set of necessary and sufficient conditions for identifiability of additive step faults. Such faults manifest themselves as, for example, a stuck control surface or a constant sensor bias. This thesis extends these results by presenting necessary and sufficient conditions for identifiability of additive, time-varying faults affecting arbitrary combinations of actuators and sensors, either alone or simultaneously.

The application of the main theorems is illustrated with two case studies, which provide some insight into how the conditions may be used to check the identifiability of fault configurations of interest for a given system. It is shown that while state augmentation can be used to identify certain fault configurations, other fault configurations cannot be identified. Furthermore, one limitation of model-based methods is that innumerable fault configurations are possible. However, identifiability of known, credible fault configurations can be tested using the theorems presented in this thesis.

Para Sara y Henry: Ustedes son mi vida.

ACKNOWLEDGEMENTS

I am deeply grateful to my thesis advisor, Dr. Oscar R. González, for his remarkable commitment to understanding and for his patience and frequent guidance over the course of the research for this thesis and its production. He is an excellent example of the kind of researcher I endeavor to be.

Additionally, I am indebted to Dr. Suresh M. Joshi, for very generously allowing me to solve the problem he conceived of and for helping me at critical times to consider alternative and meaningful ways to present the results of this research.

I am very appreciative of the time and attention that Dr. W. Steven Gray gave in offering ways to improve the quality of the results in this thesis, and for his suggestions of future research paths for this work.

Furthermore, I would like to thank Dr. Dimitrie C. Popescu, for his time and excellent feedback about improving the clarity of fundamental aspects of this research.

NOMENCLATURE

$\mathbb{C}$	Set of complex numbers
$\mathbb{R}$	Set of real numbers
$\emptyset$	Empty set
s	Complex variable
$\mathfrak{Re}(\cdot)$	Real part of a complex number
$\mathbb{R}(s)$	Field of rational functions in s
$\{A, B, C, D\}$	Realization of a linear, time-invariant system
(C, A, B)	Triple characterizing a system realization $\{A, B, C, 0\}$
$\Sigma(s)$	Rosenbrock System Matrix (RSM)
$D_i(s)$	Monic greatest common divisor of all the nonzero ith-order minors of $\Sigma(s)$
$\epsilon_i(s)$	ith invariant polynomial
$\Lambda(\cdot)$, $(\Lambda_u(\cdot)$	Set of (unstable) eigenvalues
$\mathcal{G}(s)$	Set of input-zero direction vectors of $\Sigma(s)$, possibly a subspace
$\overline{\mathcal{G}}(s)$	Subspace spanned by the non-input-zero direction vectors of $\Sigma(s)$
$\Lambda(s)$, $(\Lambda_e(s))$	Diagonal matrix of (extended) invariant polynomials of the system
λ	Eigenvalue or invariant zero, depending on context
$S(s)$, $S_e(s)$	Smith form and extended Smith form of the Rosenbrock System Matrix, respectively
I_n	$n \times n$ identity matrix
$u_s(t - t_0)$	Unit step function starting at time $t = t_0$

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CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

A fault in a dynamical system is a state which may result in a malfunction or failure of the system [1]. Faults represent unpermitted deviations of properties or parameters of a system from an acceptable condition, and malfunctions and failures are, respectively, intermittent and permanent interruptions of a system's ability to fulfill a desired function [2]. In dynamical systems employing control effectors and measurements to control the system's behavior, actuator and sensor faults may lead to failures characterized by, for example, instability and loss of control.

In aerospace applications, actuator and sensor faults can have serious implications for system safety and reliability. For example, actuator faults such as rudder runaway have been implicated in multiple aviation incidents (for example, see [3] and [4]). Other actuator faults such as undesired control surface oscillations (that is, the *oscillatory failure case*) can increase the structural loads on an aircraft, possibly compromising its structural integrity in flight [5, 6]. Finally, sensor faults such as constant bias have contributed to the failure of missions such as NASA's Demonstration of Autonomous Rendezvous Technology (DART) [7]. In the last example, the constant bias represents an *error*, or a deviation between the measured value and the true value [2]. Many more examples of aviation incidents and accidents where actuator or sensor faults were contributing factors can be found in [8].

At present, safety and reliability concerns related to aircraft actuators and sensors are primarily addressed through hardware redundancy-based techniques [9, 10]. The counterpart to hardware redundancy is generally referred to as *analytical redundancy*, a broad class of techniques which make use of mathematical models of a system to detect and identify actuator and sensor faults. Such model-based fault detection and identification (FDI) methods have received significant attention in the literature over the last several decades. For surveys on a variety of model-based FDI techniques, see [11–14].

One particular technique uses multiple models, where each model corresponds to the nominal system state augmented by a different fault configuration of interest. Typically, a bank of detection filters is used to estimate the present state of the aircraft, and multiple-hypothesis testing determines if a fault has occurred [15–17]. To date, several authors have proposed FDI techniques based on multiple models, where the models are developed by state augmentation (for example, see [16–21]). A key requirement in these techniques is that each of the faulty-system states represented by the models be identifiable [22]. This requirement motivates the study of when such faults may or may not be identifiable, particularly for the base case, that is, when state augmentation alone is used for FDI.

An important class of faults is the additive step fault, such as a stuck system input or constant bias in a measurement. Identification of constant measurement biases was initially treated in [23]. The preliminary conditions for identifiability of bias-type actuator and sensor faults were presented in [17], and a subsequent detailed analysis was given in [24]. A complete characterization of a set of necessary and sufficient conditions for identifiability of all combinations of additive, constant actuator and sensor faults, including numerical examples, can be found in [22]; a fundamental contribution of these conditions is that they provide steps to reveal precisely which combinations of additive, constant actuator-sensor faults can and cannot be identified using state augmentation alone. This thesis is concerned with the development of a similar set of necessary and sufficient conditions for identifiability of additive, actuator and sensor faults for the time-varying case, using state augmentation alone.

1.2 PROBLEM STATEMENT

The results presented in [22] and [24] fully address identifiability of additive, constant faults by multiple-model state augmentation. This thesis presents a set of necessary and sufficient conditions for identifiability of additive, time-varying faults affecting arbitrary combinations of: (1) actuators only, (2) sensors only, and (3) actuators and sensors, simultaneously, for the case when state augmentation is the sole FDI method employed. That is, this thesis extends the results in [22] and [24] to the case of time-varying faults.

1.3 SCOPE

Although the results of this thesis can be readily extended to the discrete-time case, the treatment of identifiability in this thesis is entirely in the continuous-time domain. Furthermore, the systems under study are assumed to be linear, time-invariant systems. Finally, the necessary and sufficient conditions for fault identifiability presented herein address the case where state augmentation alone is used.

1.4 ORGANIZATION

The remaining chapters of this thesis are organized as follows: Chapter 2 provides the definitions and concepts to be used in subsequent chapters; Chapter 3 develops a state-space representation for time-varying actuator and sensor faults with examples of several important classes of faults; Chapter 4 presents the main results, that is, a set of necessary and sufficient conditions for identifiability of additive, time-varying faults affecting arbitrary combinations of: (a) actuators only, (b) sensors only, and (c) actuators and sensors, simultaneously; Chapter 5 provides two case studies of practical systems to illustrate how the results presented in Chapter 4 may be applied; finally, Chapter 6 presents the conclusions of the research.

CHAPTER 2

DEFINITIONS AND FUNDAMENTAL CONCEPTS

This chapter provides a review of the following definitions and concepts relevant to the development of conditions for fault identifiability presented in this thesis:

1. observability and detectability,
2. the Rosenbrock System Matrix,
3. the Smith form,
4. normal rank,
5. degeneracy,
6. multi-input, multi-output zeros:
    (a) system zeros,
    (b) (extended) invariant zeros,
    (c) transmission zeros,
7. zero direction vectors,
8. input-zero direction vectors.

In some instances, more than one working definition for a particular term exists in the literature, depending on the date of publication and any prevailing assumptions made therein. The definition of the *invariant zeros* of a multi-input, multi-output (MIMO) system represents one such example (see [25] for a summary of several interpretations). The given definitions are assumed throughout this thesis.

2.1 OBSERVABILITY AND DETECTABILITY

Consider a system with state-space representation given by

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; x(0) = x_0, \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{l\times n}$, and $D \in \mathbb{R}^{l\times m}$. Furthermore, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the state, input, and output vectors, respectively. In many practical systems, not all of the states may be available for measurement, so state estimators, or observers, may be used to estimate the state by making use of the output vector $y(t)$ and the input vector $u(t)$. For such methods to work, a system of the form given by Equations (1) and (2) must generally be *observable*, or at least *detectable*. These terms and their associated tests are given subsequently.

Definition 1. The system given by Equations (1) and (2) is said to be *observable* if there exists a time $t_1 > 0$ such that any initial state x_0 can be uniquely determined from $y(t), \;\; t \in [0, t_1]$ [26].

Two tests for observability given in [27] are the *PBH rank test* and the *PBH eigenvector test*:

Test 1. PBH Rank Test. The system given by Equations (1) and (2) is *observable* if and only if for every eigenvalue λ_i of A, that is for every $\lambda_i \in \Lambda(A)$, where $\Lambda(A)$ denotes the set of eigenvalues of A,

$$\operatorname{rank} \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = n$$

for $i = 1, 2, \ldots, n$.

A test equivalent to Test 1 follows.

Test 2. PBH Eigenvector Test. The system given by Equations (1) and (2) is *observable* if and only if there does not exist a nonzero $\gamma \in \mathbb{C}^n$ such that

$$\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} \gamma = 0$$

for $i = 1, 2, \ldots, n$.

If either Test 1 or 2 fails for any value λ_i, $i = 1, \ldots, n$, then λ_i is considered to be an *unobservable eigenvalue* of A.

Definition 2. An eigenvalue λ is *asymptotically stable* if and only if $\mathfrak{Re}(\lambda) < 0$. Let $\Lambda_u(A)$ denote the unstable eigenvalues of A.

Definition 3. The system given by Equations (1) and (2) is said to be *detectable* if and only if all of the unobservable eigenvalues are asymptotically stable [26].

Remark 1. A system is *detectable* if and only if either Test 1 or 2 is satisfied for $\{\lambda_j : \mathfrak{Re}(\lambda_j) \geq 0\}$, $j = 1, 2, \ldots, k$, where $k \leq n$ [26].

Remark 2. Since observability requires that Test 1 be satisfied for all eigenvalues of A, observability implies detectability. However, because detectability requires only those eigenvalues with non-negative real parts to be observable, an unobservable system may still be detectable. Finally, a system may be detectable, but if it has unobservable eigenvalues with negative real parts then the system is still not observable. Thus, observability implies detectability, but detectability does not imply observability.

Definition 4. A system having state-space representation of the form given by Equations (1) and (2) is *identifiable* if the pair (C, A) is detectable or observable.

2.2 THE ROSENBROCK SYSTEM MATRIX

The Rosenbrock System Matrix and some of its important properties are used in the proofs of conditions for time-varying actuator and sensor fault identifiability (see Chapter 4). The derivation of the RSM follows.

Consider the system given by Equations (1) and (2). This system can be represented in the frequency domain by its one-sided Laplace transformation as

$$s\hat{x}(s) - x_0 = A\hat{x}(s) + B\hat{u}(s), \tag{3}$$

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s), \tag{4}$$

where $\hat{x}(s), \hat{u}(s)$, and $\hat{y}(s)$ are the Laplace transformations of the state, input, and output vectors, respectively. Furthermore, x_0 is the initial condition at time $t = 0$, that is, $x(0)$. Here, the one-sided Laplace transform $\hat{\Upsilon}(s)$ of a function $\Upsilon(t)$ is as commonly defined, that is,

$$\hat{\Upsilon}(s) = \int_{0^-}^{\infty} \Upsilon(t) e^{-st}\, dt. \tag{5}$$

Now, the representation given by Equations (3) and (4) can be expressed as

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(s) \\ \hat{u}(s) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{y}(s) \end{bmatrix}. \tag{6}$$

The coefficient matrix in (6) is referred to as the *Rosenbrock System Matrix (RSM)* of the system having realization $\{A, B, C, D\}$. Furthermore, if $D = 0$ then the system is said to be *strictly proper*. Let

$$\Sigma_{(C,A,B)}(s) = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{bmatrix} sI - A & -B \\ C & 0 \end{bmatrix} & \begin{array}{c} n \\ l \end{array} \end{array} \tag{7}$$

denote the RSM of a system having such a realization characterized by the triple (C, A, B), where the dimensions of the block rows and columns are as indicated.

Definition 5. The *normal rank* of the RSM associated with the triple (C, A, B) is equal to the number of linearly independent rows or columns of $\Sigma_{(C,A,B)}(s)$ over the field of rational functions in s, or $\mathbb{R}(s)$ [28].

Let $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = r$ be the normal rank of $\Sigma_{(C,A,B)}(s)$, with $0 < r \leq n + \min\{m, l\}$.

Definition 6. The *Smith form* of $\Sigma_{(C,A,B)}(s)$ is defined as

$$S(s) = \begin{bmatrix} \Lambda(s) & 0_{r\times(n+m-r)} \\ 0_{(n+l-r)\times n} & 0_{(n+l-r)\times(n+m-r)} \end{bmatrix}.$$

where $\Lambda(s) \triangleq \text{diag}\begin{bmatrix} \epsilon_1(s), & \ldots & , \epsilon_r(s) \end{bmatrix}$, and $\text{diag}(\cdot)$ denotes a diagonal matrix whose diagonal entries are determined by

$$\epsilon_i(s) = D_i(s)/D_{i-1}(s), \; i = 1, \ldots, r,$$

where $D_i(s)$ is defined as the monic greatest common divisor of all the nonzero *ith*-order minors of $\Sigma_{(C,A,B)}(s)$, and $D_0(s) \triangleq 1$ [28].

Definition 7. Each $\epsilon_i(s)$, for $i = 1, \ldots, r$, in Definition 6 represents an *invariant polynomial*.

Definition 8. The product of the invariant polynomials in Definition 7 is the *invariant zero polynomial* of the system having realization $\{A, B, C, 0\}$.

Remark 3. Since $\Lambda(s) = \text{diag}\begin{bmatrix} \epsilon_1(s), & \dots & ,\epsilon_r(s) \end{bmatrix}$, the Smith form of $\Sigma_{(C,A,B)}(s)$ is equivalently expressed as

$$S(s) = \left[\begin{array}{cccc|c} \epsilon_1(s) & & & & \\ & \epsilon_2(s) & & & \\ & & \ddots & & 0 \\ & & & \epsilon_r(s) & \\ \hline & & 0 & & 0 \end{array}\right]. \tag{8}$$

The case of extending $\epsilon_i(s)$ to include $i = r+1, \dots, \kappa$, where $\kappa = n + \min\{m, l\}$, was considered in [29]. Let $\Lambda_e(s) \triangleq \text{diag}\begin{bmatrix} \epsilon_1(s), & \dots & ,\epsilon_r(s), & \epsilon_{r+1}(s), & \dots & ,\epsilon_\kappa(s) \end{bmatrix}$, where $\epsilon_{r+1}(s), \dots, \epsilon_\kappa(s)$ are identically zero.

Definition 9. The terms $\epsilon_1(s), \dots, \epsilon_r(s), \epsilon_{r+1}(s), \dots, \epsilon_\kappa(s)$ are the *extended invariant polynomials* of $\Sigma_{(C,A,B)}(s)$.

Remark 4. Observe that all of the ith-order minors $D_i(s)$ for $i = r+1, \dots, \kappa$ are zero. Thus, either by this observation or by inspection of the form of matrix in Equation (8), all of the extended invariant polynomials $\epsilon_{r+1}(s), \dots, \epsilon_\kappa(s)$ are identically zero. This fact will be used in the subsequent development of multi-input, multi-output zeros of $\Sigma_{(C,A,B)}(s)$.

Definition 10. The *extended Smith form*, considered in [29], can be expressed as

$$S_e(s) = \begin{cases} \begin{bmatrix} \Lambda_e(s) & 0_{\kappa\times(m-l)} \end{bmatrix}, & m > l, \\ \Lambda_e(s), & m = l, \\ \begin{bmatrix} \Lambda_e(s) \\ 0_{(l-m)\times\kappa} \end{bmatrix}, & m < l. \end{cases} \tag{9}$$

Let $\kappa = n + \min\{m, l\}$ and $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = r \leq \kappa$.

Definition 11. The *extended invariant zero polynomial* of the system is the product $\epsilon_1(s) \cdot \ldots \cdot \epsilon_r(s) \cdot \epsilon_{r+1} \cdot \ldots \cdot \epsilon_\kappa(s)$.

Remark 5. Since the Smith form (extended Smith form) of $\Sigma_{(C,A,B)}(s)$ is found by pre- and post-multiplying $\Sigma_{(C,A,B)}(s)$ by unimodular matrices (that is, nonsingular polynomial matrices with constant determinants) it follows that $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = \text{rank}\{S_e(s)\}$, and for every $\lambda \in \mathbb{C}$, $\text{rank}\{\Sigma_{(C,A,B)}(\lambda)\} = \text{rank}\{S_e(\lambda)\}$. For $r < \kappa$, $\epsilon_{r+1}(s), \dots, \epsilon_\kappa(s) = 0$. For $r = \kappa$, $\Lambda_e(s)$ has full rank. Therefore, any of the possible forms of $S_e(s)$ given in Equation (9) has full rank, and $\epsilon_1(s), \ldots, \epsilon_\kappa(s) \neq 0$. As in Remark 4, this fact will be used in the subsequent development of multi-input, multi-output zeros.

2.3 MULTI-INPUT, MULTI-OUTPUT ZEROS

Consider a system $\Sigma_{(C,A,B)}(s)$, where $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = r \leq \kappa$.

Definition 12. The *zero polynomial* of $\Sigma_{(C,A,B)}(s)$ is the monic greatest common divisor of all the rth-order nonzero minors determined by complementing $[\lambda I - A]$ with the appropriate $r - n$ rows of $\begin{bmatrix} C & 0 \end{bmatrix}$ and columns of $\begin{bmatrix} B^T & 0^T \end{bmatrix}^T$ [28].

Definition 13. The *system zeros* of $\Sigma_{(C,A,B)}(s)$ are the roots of the zero polynomial [28].

Example 1. Consider a system with RSM given by

$$\Sigma_{(C,A,B)}(s) = \left[\begin{array}{cccc|cc} s & 0 & 0 & 0 & 1 & 0 \\ 0 & s+2 & 0 & 0 & 0 & 1 \\ 0 & 0 & s+3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s+4 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right],$$

where the partitioned areas identify the block entries of the RSM given by Equation (7). For this example, there are four states, two inputs, and three outputs. Furthermore, it can be verified that $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = r = \kappa = 6$. Now, composing a square matrix from $[sI - A]$ and $[-B]$ by taking two rows of C and D at a time, it can also be verified that the only monic nonzero 6th-order minor satisfying the condition that all of $[sI - A]$ is included is $(s+3)(s+4)$. Thus, this minor is also the monic greatest common divisor, and the system zeros of $\Sigma_{(C,A,B)}(s)$ are the values of s which satisfy $(s+3)(s+4) = 0$, that is, the system zeros are $\{-3, -4\}$.

Definition 14. The *invariant zeros* are the roots, including algebraic multiplicity, of the invariant polynomials $\epsilon_1(s), \ldots, \epsilon_r(s)$ given in Definition 7 [28].

Example 2. In calculating the invariant zeros for the system given in Example 1, it can be verified that the monic greatest common divisors $D_1(s), D_2(s), \ldots, D_5(s)$ are all 1, and the monic greatest common divisor $D_6(s)$ is $(s+4)$, (the other nonzero 6th-order minor was given in Example 1 as $(s+3)(s+4)$, where $(s+4)$ is the monic greatest common divisor of these two minors). Thus, the invariant zero of $\Sigma_{(C,A,B)}(s)$ is -4. Furthermore, the Smith form of $\Sigma_{(C,A,B)}(s)$ is given by

$$S(s) = \begin{bmatrix} I_5 & 0 \\ 0 & s+4 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Remark 6. Since by Remark 5 $\text{rank}\{\Sigma_{(C,A,B)}(z)\} = \text{rank}\{S(z)\}$ for any $z \in \mathbb{C}$ then if z is an invariant zero there is at least one $i \in \{1, \ldots, r\}$ such that $\epsilon_i(z) = 0$. Thus, the set of invariant zeros is

$$\{z \in \mathbb{C} |\, \text{rank}\{\Sigma_{(C,A,B)}(z)\} < \Sigma_{(C,A,B)}(s)\}.$$

This is the definition of invariant zeros given in [25, p. 1418].

Definition 15. The set of *extended invariant zeros* is

$$\{z \in \mathbb{C} |\, \text{rank}\{\Sigma_{(C,A,B)}(z)\} < \kappa,$$

or equivalently, the roots of the extended invariant polynomials $\epsilon_1(s), \ldots, \epsilon_\kappa(s)$ given in Definition 9.

Remark 7. Observe that when $r = \kappa$ then the sets of invariant and extended invariant zeros coincide. In Equation (10), $\kappa = r = 6$, therefore the invariant zeros and extended invariant zeros coincide. When $r < \kappa$, every complex number is an extended invariant zero.

Definition 16. A system $\Sigma_{(C,A,B)}(s)$ is said to be *degenerate* if

$$\text{rank}\{\Sigma_{(C,A,B)}(\lambda_i)\} < \kappa$$

for any $n+1$ distinct scalars $\lambda_i \in \mathbb{C}$, where $1 \leq i \leq n+1$ [25].

Since this definition implies that $\text{rank}\{\Sigma_{(C,A,B)}(\lambda_i)\} < \kappa$ for all $\lambda_i \in \mathbb{C}$, it is also implied that

$$\text{rank}\{\Sigma_{(C,A,B)}(s)\} < \kappa,$$

for all $s \in \mathbb{C}$, that is, $r < \kappa$. Therefore, a degenerate system is one which equivalently has (a) normal rank less than full rank, and (b) the set of extended invariant zeros equal to $\mathbb{C}$.

Remark 8. The invariant zeros are a subset of the system zeros, but the extended invariant zeros are not necessarily a subset of the system zeros.

Definition 17. An *input-decoupling zero* of a system is any value $\lambda_0 \in \Lambda(A)$ which satisfies

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A & B \end{bmatrix} < n. \tag{11}$$

Definition 18. An *output-decoupling zero* of a system is any value $\lambda_0 \in \Lambda(A)$ which satisfies

$$\operatorname{rank}\begin{bmatrix} \lambda_0 I - A \\ C \end{bmatrix} < n. \tag{12}$$

Remark 9. The input decoupling (output decoupling) zeros correspond to uncontrollable (unobservable) eigenvalues of $\Sigma_{(C,A,B)}(s)$.

Definition 19. The transmission zeros of $\Sigma_{(C,A,B)}(s)$ are those invariant zeros which are not input-decoupling and/or output-decoupling zeros [28].

Example 3. Consider the set of system zeros determined in Example 1, that is, $\{-3, -4\}$. A check of these values in Equations (11) and (12) reveals that -3 is an input-decoupling zero, and -4 is an input-output-decoupling zero. Thus, there are no transmission zeros for the given system.

Definition 20. An *invariant zero direction vector* is a nonzero vector ℓ which satisfies

$$\Sigma(\lambda_0)\ell = 0, \tag{13}$$

where λ_0 is an invariant zero or an extended invariant zero.

Let $\ell = \begin{bmatrix} \ell_n^T & \ell_m^T \end{bmatrix}^T$.

Definition 21. An *input-zero direction vector* is a vector ℓ_m, perhaps the zero vector, such that Equation (13) is satisfied when λ_0 is an invariant zero.

Example 4. Consider the RSM given in Example 1. It can be verified that when $s = -4$, $\operatorname{rank}\{\Sigma_{(C,A,B)}(-4)\} = 5$. Furthermore, the set of invariant zero directions may be given by $\left\{\begin{bmatrix} 0 & 0 & 0 & \alpha & 0 & 0 \end{bmatrix}^T\right\}$, where $\alpha \in \mathbb{C}$ is nonzero. Observe that $0 \in \mathbb{C}^{n+m}$ always satisfies Equation (13), however, the zero vector is not an invariant zero direction. Since the invariant zero is also an output-decoupling zero, the set of input-zero direction vectors is a subspace consisting only of the zero vector, given by $\left\{\begin{bmatrix} 0 & 0 \end{bmatrix}^T\right\}$.

Remark 10. Consider the following statement from [28], where the notation has been made to be consistent with this thesis:

> [W]hen $\min\{m, l\} = l < m$, there are nonzero vectors satisfying $\Sigma_{(C,A,B)}(\lambda_0)\ell = 0$ with λ_0 not necessarily being an invariant zero of the system. This situation becomes accentuated, that is, Equation (13) is satisfied for values of λ_0 that are not necessarily zeros of the system, when $\text{rank}\{\Sigma_{(C,A,B)}(s)\} < \kappa$.

Under the definitions given in this thesis, a system satisfying $\text{rank}\{\Sigma_{(C,A,B)}(s)\} < \kappa$ has been defined as (a) a degenerate system, and (b) a system with extended invariant zeros equal to $\mathbb{C}$ (see Definition 16 and Remark 7, respectively). By Definition 20 when $\text{rank}\{\Sigma_{(C,A,B)}(s)\} < \kappa$, (λ_0, ℓ) satisfying Equation (13) is an (extended invariant zero, invariant zero direction vector) pair of the system $\Sigma_{(C,A,B)}(s)$. Example 5 illustrates this case, and Example 6 illustrates the case when $l < m$, and $\Sigma_{(C,A,B)}(\lambda_0)\ell = 0$ for nonzero ℓ and λ_0 not an invariant zero.

Example 5. Consider the system given in RSM form as

$$\Sigma_{(C,A,B)}(s) = \left[\begin{array}{cccc|cc} s & 0 & 0 & 0 & 1 & 0 \\ 0 & s+2 & 0 & 0 & 0 & 1 \\ 0 & 0 & s+3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s+4 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right],$$

where $l < m$ (that is, $l = 1$ and $m = 2$). It can be verified that $\text{rank}\{\Sigma_{(C,A,B)}(s)\} = r = 4 < \kappa = 5$. Furthermore, it can be verified that the system zeros are the roots of $\det(sI - A)$. Proceeding as in Example 1, the system zeros are $\{0, -2, -3, -4\}$, and the Smith form is

$$S(s) = \left[\begin{array}{cc|cc} I_3 & 0 & 0 & 0 \\ 0 & s+4 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

By Definition 14, the invariant zeros are the roots of the invariant zero polyomial, that is, $1 \cdot 1 \cdot 1 \cdot (s+4)$. Thus, the invariant zero is $\{-4\}$. It can be verified that for $s = -4$, the set of invariant zero direction vectors can be given by

$$\left\{\alpha_1 \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \alpha_2 \begin{bmatrix} 1 & 0 & 0 & 0 & 4 & 0 \end{bmatrix}^T, \alpha_1 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}^T\right\},$$

where $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3$, and $\alpha_i \neq 0$, and the set of input-zero direction vectors is a subspace which spans $\mathbb{C}^2$.

For $s \neq -4$, the set of invariant zero direction vectors is

$$\left\{\beta_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \beta_2 \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}^T\right\}, \; s = 0,$$
$$\left\{\beta_1 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \beta_2 \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}^T\right\}, \; s = -2,$$
$$\left\{\beta_1 \begin{bmatrix} \frac{-1}{s} & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \beta_2 \begin{bmatrix} 0 & \frac{-1}{s+2} & 0 & 0 & 0 & 1 \end{bmatrix}^T\right\}, \; s \neq 0 \text{ and } s \neq -2,$$

where $\beta_i \in \mathbb{C}$, $i = 1, 2$ and $\beta_i \neq 0$. Observe that for all $s \neq 0$ and $s \neq -2$ that the set of input-zero direction vectors is $\mathbb{C}^2$. When $s = 0$ the subspace of input-zero direction vectors is $\left\{\beta \begin{bmatrix} 0 & 1 \end{bmatrix}^T\right\}$ for $\beta \in \mathbb{C}$. When $s = -2$ the subspace of input-zero direction vectors is $\left\{\beta \begin{bmatrix} 1 & 0 \end{bmatrix}^T\right\}$ for $\beta \in \mathbb{C}$.

Example 6. Consider the RSM given by

$$\Sigma_{(C,A,B)}(s) = \left[\begin{array}{ccc|cc} s & 1 & -1 & 1 & 0 \\ -1 & s+2 & -1 & 1 & 1 \\ 0 & -1 & s+1 & 1 & 2 \\ \hline 0 & -1 & 0 & 0 & 0 \end{array}\right].$$

It can be verified that for this example, the system zeros of $\Sigma_{(C,A,B)}(s)$ are $s = \{-1, -2\}$, and that $s = -2$ is the only invariant zero. Furthermore, there is one output-decoupling zero at $s = -1$. The set of invariant zero direction vectors associated are

$$\left\{\alpha_1 \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T, \alpha_2 \begin{bmatrix} -1 & 0 & 2 & 0 & 1 \end{bmatrix}^T\right\},$$

where $\alpha_i \in \mathbb{C}$, $i = 1, 2$, such that $\alpha_i \neq 0$. Thus, the input-zero direction vectors are $\mathbb{C}^2 \backslash \{0\}$.

Now, when $s \neq -2$, the nullspace of $\Sigma_{(C,A,B)}(s)$ is

$$\left\{\beta \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{-1}{s+1} & -1 & 1 \end{bmatrix}^T\right\}, \; s \neq -1,$$
$$\left\{\beta \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \end{bmatrix}^T\right\}, \; s = -1,$$

where $\beta \in \mathbb{C}$. Observe that for $s = -1$, ℓ has the form

$$\left\{\beta \begin{bmatrix} \star & \star & \star & 0 & 0 \end{bmatrix}^T\right\},$$

where the last m components of ℓ are identically zero.

CHAPTER 3

TIME-VARYING FAULT MODELING

This chapter presents a representation for time-varying faults by treating them as outputs of a linear time-invariant system (LTI) driven only by initial conditions. It is assumed in the subsequent development that the fault of interest has a one-sided Laplace transform as defined by Equation (5). The subsequent development also assumes the notation in [22] with some modification, to represent the time-varying fault case.

3.1 A REPRESENTATION FOR TIME-VARYING FAULTS

Let $f(t)$ be a vector of faults. Such faults may be modeled as the output of an LTI system having state-space representation given by

$$\dot{x}_f(t) = A_f x_f(t), \ x_f(0) = x_{f_0}, \tag{14}$$

$$f(t) = C_f x_f(t). \tag{15}$$

where $A_f \in \mathbb{R}^{n_f \times n_f}, C_f \in \mathbb{R}^{\mu \times n_f}, x_f(t) \in \mathbb{R}^{n_f}$, and $f(t) \in \mathbb{R}^{\mu}$. It is assumed that C_f has full row rank, that is, $\mathrm{rank}(C_f) = \mu$. Taking the Laplace transform of Equations (14) and (15), and solving for $\hat{x}(s)$ in Equation (14) gives

$$\hat{x}(s) = (sI - A_f)^{-1} x_{f_0}, \tag{16}$$

$$\hat{f}(s) = C_f \hat{x}(s). \tag{17}$$

Now, substituting Equation (16) into Equation (17) gives

$$\hat{f}(s) = C_f (sI - A_f)^{-1} x_{f_0}.$$

Thus, the frequency domain representation of the fault vector is the zero-input response of the system given by $C_f(sI - A_f)^{-1} x_{f_0}$.

3.1.1 EXAMPLES OF REPRESENTATIONS FOR TIME-VARYING FAULTS

In this section, the derivation of a state-space representation for step faults, ramp faults, and sinusoidal faults is presented. The examples are derived using the methods in [30].

Example 7. The Step Fault. Consider a single step fault affecting the ith actuator (sensor). Such a fault may be modeled as

$$f_i(t) = \alpha_i u_s(t - t_0),$$

where $\alpha_i \in \mathbb{R}$, and $u_s(t - t_0) \in \mathbb{R}$ is a unit step function associated with the ith actuator (sensor), starting time $t = t_0$.

First, observe that $\dot{f}_i(t) = 0$ for $t > t_0$. Furthermore, since $f_i(t)$ is to be modeled by an LTI system, it can be assumed that the fault is initiated at $t_0 = 0^+$. Thus, the initial conditions can be expressed as $f_i(0^+) = \alpha_i$. Now, for $t > 0$, the following relations are true

$$\dot{f}_i(t) = 0, \tag{18}$$

$$f_i(t) = \alpha_i. \tag{19}$$

By comparing Equations (18) and (19) with Equations (14) and (15), it can be verified that for a single step fault, $f_i(t) = 1 \cdot x_{f_i}(t) = 1 \cdot \alpha_i$, and $\dot{x}_{f_i}(t) = 0 \cdot \alpha_i$. Therefore, $A_{f_i} = 0$, $C_{f_i} = 1$ and $x_{f_i}(0) = \alpha_i$. Thus, the single step-fault state-space representation is given by

$$\begin{aligned} \dot{x}_{f_i}(t) &= 0 x_{f_i}(t), \ x_{f_i}(0) = \alpha_i, \\ f_i(t) &= x_{f_i}(t). \end{aligned}$$

Now, suppose μ of the m actuators (μ of the l sensors) are affected by step faults, so that $f(t) = \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_\mu(t) \end{bmatrix}^T$. Assuming that each fault affects only one actuator (sensor), and actuator (sensor) fault i is decoupled from actuator (sensor) fault j, for $i \neq j$, then such multiple actuator (sensor) step faults can be represented as

$$\begin{aligned} \dot{x}_f(t) &= 0_{\mu\times\mu} x_f(t), \ x_f(0) = \alpha, \\ f(t) &= I_\mu x_f(t), \end{aligned}$$

where $\alpha \in \mathbb{R}^\mu$, and $\mu = n_f$.

Example 8. The Ramp Fault. Consider a single ramp fault affecting the ith actuator (sensor). Such a fault may be modeled as

$$f_i(t) = \alpha_i \cdot (t - t_0) u_s(t - t_0),$$

where α_i and $u_s(t - t_0)$ are as defined in Example 7. Additionally, consider the fault after t_0, where the first two derivatives are

$$\begin{aligned} \dot{f}_i(t) &= \alpha_i, \\ \ddot{f}_i(t) &= 0, \end{aligned}$$

which can be expressed as

$$\begin{bmatrix} \dot{f}_i(t) \\ \ddot{f}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_i(t) \\ \dot{f}_i(t) \end{bmatrix}.$$

Letting $x_{f_i}(t) = \begin{bmatrix} f_i(t) & \dot{f}_i(t) \end{bmatrix}^T$, the ramp fault can be expressed with the state-space representation given by

$$\begin{aligned} \dot{x}_{f_i}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{f_i}(t), \; x_{f_i}(0) = x_{f_{i0}}, \\ f_i(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_{f_i}(t). \end{aligned}$$

By inspection, $A_{f_i} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C_{f_i} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Furthermore, the initial conditions are found to be $x_{f_{i0}} = \begin{bmatrix} 0 & \alpha_i \end{bmatrix}^T$.

Now, suppose that μ of the m actuators (μ of the l sensors) are affected by ramp faults. Then under the assumption that the faults are decoupled, the multiple-fault configuration can be expressed in the state-space representation given by

$$\begin{aligned} \dot{x}_f(t) &= \operatorname{diag} \begin{bmatrix} A_{f_1} & A_{f_2} & \cdots & A_{f_\mu} \end{bmatrix} x_f(t), \; x_f(0) = x_{f_0}, \\ f(t) &= \operatorname{diag} \begin{bmatrix} C_{f_1} & C_{f_2} & \cdots & C_{f_\mu} \end{bmatrix} x_f(t). \end{aligned}$$

Observe that for this example, $n_f = 2\mu$.

Example 9. The Sinusoidal Fault. Consider a single sinusoidal fault affecting the ith actuator (sensor). Such a fault may be modeled as

$$f_i(t) = \alpha_i \sin(\omega_i t) + \beta_i \cos(\omega_i t), \; t > 0.$$

where α_i, β_i, and $\omega_i \in \mathbb{R}$. Proceeding as in Example 8, the first two derivatives of $f_i(t)$ are

$$\begin{aligned}\dot{f}_i(t) &= \alpha_i \omega_i \cos(\omega_i t) - \beta_i \omega_i \sin(\omega_i t),\\ \ddot{f}_i(t) &= -\alpha_i \omega_i^2 \sin(\omega_i t) - \beta_i \omega_i^2 \cos(\omega_i t),\end{aligned}$$

which can be put into the state equation

$$\begin{bmatrix} \dot{f}_i(t) \\ \ddot{f}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix} \begin{bmatrix} f_i(t) \\ \dot{f}_i(t) \end{bmatrix}.$$

Again, letting $x_{f_i}(t) = \begin{bmatrix} f_i(t) & \dot{f}_i(t) \end{bmatrix}^T$, the fault model state-space representation is

$$\begin{aligned}\dot{x}_{f_i}(t) &= \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix} x_{f_i}(t), \ x_{f_i}(0) = x_{f_{i_0}},\\ f_i(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_{f_i}(t),\end{aligned}$$

where the initial conditions can be found to be $x_{f_{i_0}} = \begin{bmatrix} \beta_i & \alpha_i \omega_i \end{bmatrix}^T$. Now, suppose again that μ of the m actuators (μ of the l sensors) are affected by sinusoidal faults. Then under the assumption that the faults are decoupled, the multiple-fault configuration can be expressed in the state-space representation given by

$$\begin{aligned}\dot{x}_f(t) &= \operatorname{diag}\begin{bmatrix} A_{f_1} & A_{f_2} & \cdots & A_{f_\mu} \end{bmatrix} x_f(t), \ x_f(0) = x_{f_0},\\ f(t) &= \operatorname{diag}\begin{bmatrix} C_{f_1} & C_{f_2} & \cdots & C_{f_\mu} \end{bmatrix} x_f(t).\end{aligned}$$

Again, observe that $n_f = 2\mu$.

Remark 11. The actuator (sensor) faults affecting each actuator (sensor) need not be identical, so that the general the state-space representation for arbitrary faults may given by

$$\begin{aligned}\dot{x}_f(t) &= \operatorname{diag}\begin{bmatrix} A_{f_1} & A_{f_2} & \cdots & A_{f_\mu} \end{bmatrix} x_f(t), \ x_f(0) = x_{f_0},\\ f(t) &= \operatorname{diag}\begin{bmatrix} C_{f_1} & C_{f_2} & \cdots & C_{f_\mu} \end{bmatrix} x_f(t).\end{aligned}$$

Example 10. Consider the case when three actuators (sensors) are subject to a step fault with amplitude α_1, a ramp fault with slope α_2, and sinusoidal fault with frequency ω_1 and amplitude α_3, respectively. In this case, the state-space representation for the fault dynamics may be given by

$$\dot{x}_f(t) = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_1^2 & 0 \end{array}\right] x_f(t),\ x_f(0) = \left[\begin{array}{c|cc|cc} \alpha_1 & 0 & \alpha_2 & 0 & \omega_1\alpha_3 \end{array}\right]^T,$$

$$f(t) = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\right],$$

where the entries on the block diagonal of A_f represent each A_{f_i}, $i = 1, 2, 3$, and the entries on the block diagonal of C_f represent each C_{f_i}, $i = 1, 2, 3$.

Remark 12. The fault dynamics modeled in Equations (14) and (15) neglect *process noise* and *measurement noise*, that is, the faults appear as perfect models of, for example, steps, ramps, and sinusoids. In practical applications, however, faults may not manifest themselves as such perfect representations. Therefore, in subsequent discussion Equations (14) and (15) will be replaced by

$$\dot{x}_f(t) = A_f x_f(t) + \omega_{f,p}(t),\ x_f(0) = x_{f_0} \tag{20}$$

$$f(t) = C_f x_f(t) + \omega_{f,m}(t), \tag{21}$$

respectively, where $\omega_{f,p}(t) \in \mathbb{R}^{n_f}$ represents fictitious process noise and $\omega_{f,m}(t) \in \mathbb{R}^{\mu}$ represents fictitious measurement noise.

3.2 ACTUATOR FAULT MODELING

This section presents a method for modeling a system affected by additive, time-varying actuator faults. It is assumed that time-varying actuator faults may affect none, some, or all of the actuators. For such cases, the notation presented in [22] is adopted with some modifications for the time-varying case. The constant actuator fault case considered in [22] is also derived as a special case of the time-varying case.

Consider a system given by Equations (1) and (2). If m_k of the m actuators are affected by time-varying faults, the system dynamics can be represented as

$$\dot{x}(t) = Ax(t) + \sum_{j\in\mathcal{F}_{ak}} b_j\overline{u}_j(t) + \sum_{j\notin\mathcal{F}_{ak}} b_j u_j(t) + \omega_p(t), \tag{22}$$

$$= Ax(t) + \overline{B}^k\overline{u}^k(t) + B^k u^k(t) + \omega_p(t), \tag{23}$$

$$y(t) = Cx(t) + \omega_s(t), \tag{24}$$

where $A \in \mathbb{R}^{n\times n}$, $\overline{B}^k = \sum_{j\in\mathcal{F}_{ak}} b_j \in \mathbb{R}^{n\times m_k}$, $B^k = \sum_{j\notin\mathcal{F}_{ak}} b_j \in \mathbb{R}^{n\times(m-m_k)}$, $C \in \mathbb{R}^{l\times n}$, $x(t) \in \mathbb{R}^n$, $\overline{u}^k(t) \in \mathbb{R}^{m_k}$, $u^k(t) \in \mathbb{R}^{m-m_k}$, $y(t) \in \mathbb{R}^l$, $\omega_p(t) \in \mathbb{R}^n$, and $\omega_s(t) \in \mathbb{R}^l$. Furthermore, $\mathcal{F}_{ak}$ in Equation (22) denotes the set of indices corresponding to the failed actuators, $\overline{u}_j(t) \in \mathbb{R}$ denotes a faulty input associated with a faulty actuator at time t, $u_j(t) \in \mathbb{R}$ denotes a non-faulty input associated with a non-faulty actuator at time t, and $b_j \in \mathbb{R}^n$ denotes the particular column of B (see Equation (1)) affected by the appropriate faulty or non-faulty actuator. Thus, Equation (23) represents the system subject to a particular actuator fault configuration.

Now consider the state-space representation for time-varying faults given by Equations (20) and (21). Observe that such a representation may be adapted to account for the actuator faults in Equation (23) as

$$\dot{x}_a(t) = A_a x_a(t) + \omega_{p_a}(t), \;\; x_a(0) = x_{a_0}, \tag{25}$$

$$\overline{u}^k(t) = C_a x_a(t) + \omega_{s_a}(t), \tag{26}$$

where $A_a \in \mathbb{R}^{n_a\times n_a}$, $C_a \in \mathbb{R}^{m_k\times n_a}$, $x_a(t) \in \mathbb{R}^{n_a}$, and $\omega_{p_a}(t) \in \mathbb{R}^{n_a}$ and $\omega_{s_a}(t) \in \mathbb{R}^{m_k}$ are fictitious actuator fault process and measurement noise, respectively. Also observe that Equations (25) and (26) model the actuator faults present in Equation (23). Thus, Equations (23) and (24) can be viewed as an LTI system with m_k faulty inputs given as the output of the LTI system represented by Equations (25) and (26). That is, after substituting Equation (26) into Equation (23), the interconnected system can be represented in augmented state-space form as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & \overline{B}^k C_a \\ 0 & A_a \end{bmatrix}}_{A_\xi^k} \underbrace{\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix}}_{\xi^k(t)} + \underbrace{\begin{bmatrix} B^k \\ 0 \end{bmatrix}}_{B_\xi^k} u^k(t) + \underbrace{\begin{bmatrix} \overline{B}^k \omega_{s_a}(t) + \omega_p(t) \\ \omega_{p_a}(t) \end{bmatrix}}_{\omega_{\xi_p}^k(t)}, \tag{27}$$

$$y(t) = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_\xi^k} \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} + \omega_s(t). \tag{28}$$

The system above can be expressed compactly by making the appropriate substitutions indicated by the braces as

$$\dot{\xi}^k(t) = A_\xi^k \xi^k(t) + B_\xi^k u^k(t) + \omega_{\xi_p}^k(t), \tag{29}$$

$$y(t) = C_\xi^k \xi^k(t) + \omega_s(t). \tag{30}$$

Equations (29) and (30) model the general case of additive, time-varying actuator faults when such faults can be modeled as the outputs of an LTI system.

3.2.1 A SPECIAL CASE FOR TIME-VARYING ACTUATOR FAULTS: THE ACTUATOR STEP FAULT

In the case of step faults, $A_a = 0_{m_k \times m_k}$, and $C_a = I_{m_k}$, as shown in Example 7. Thus, Equation (27) reduces to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A & \overline{B}^k \\ 0 & I_{m_k} \end{bmatrix} \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B^k \\ 0 \end{bmatrix} u^k(t) + \begin{bmatrix} \overline{B}^k \omega_{s_a}(t) + \omega_p(t) \\ \omega_{p_a}(t) \end{bmatrix}. \tag{31}$$

Remark 13. Equation (31) is identical to Equation (4) in [22], where the case of actuator step faults was considered.

3.3 SENSOR FAULT MODELING

This section presents a method for modeling a system affected by additive, time-varying sensor faults, where such faults may affect none, some, or all of the sensors. It is also shown that the constant sensor bias considered in [22] is a special case of time-varying sensor bias.

Consider again a system given by Equations (1) and (2). If q of the l sensors for the given system are affected by time-varying faults, the system dynamics can be represented as

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega_p(t), \tag{32}$$

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{33}$$

$$= Cx(t) + \begin{bmatrix} 0 \\ y_s(t) \end{bmatrix} + \omega_s(t), \tag{34}$$

where $A, B, C, x(t), u(t), y(t), \omega_p(t)$, and $\omega_s(t)$ are as previously defined, and $y_1(t) \in \mathbb{R}^{l-q}$ and $y_2(t) \in \mathbb{R}^q$ represent the vectors containing the fault-free sensor measurements and the faulty sensor measurements, respectively. Finally, $y_s(t) \in \mathbb{R}^q$ is the vector containing the additive time-varying sensor faults affecting the q faulty sensors.

Furthermore, consider the fault dynamics represented by Equations (14) and (15). Such a representation can be modified to address the specific case of sensor faults as

$$\dot{x}_s(t) = A_s x_s(t) + \omega_{p_s}(t), x_s(0) = x_{s_0}, \tag{35}$$

$$y_s(t) = C_s x_s(t) + \omega_{s_s}(t), \tag{36}$$

where $A_s \in \mathbb{R}^{n_s \times n_s}$, $C_s \in \mathbb{R}^{q \times n_s}$, $x_s(t) \in \mathbb{R}^{n_s}$, $y_s(t) \in \mathbb{R}^q$, and $\omega_{p_s}(t) \in \mathbb{R}^{n_s}$ and $\omega_{s_s}(t) \in \mathbb{R}^q$ are fictitious sensor fault process and measurement noise, respectively. Observe that Equations (35) and (36) model the sensor faults present in Equation (34). Thus, Equations (32) and (34) can be viewed as an LTI system with q of its outputs affected by the time-varying bias given by the output of the LTI system represented by Equations (35) and (36). The interconnected system can be represented in augmented state-space form after substituting Equation (36) into Equation (34) as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_s(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0_{n \times n_s} \\ 0_{n_s \times n} & A_s \end{bmatrix}}_{A_\eta} \underbrace{\begin{bmatrix} x(t) \\ x_s(t) \end{bmatrix}}_{\eta(t)} + \underbrace{\begin{bmatrix} B \\ 0_{n_s \times m} \end{bmatrix}}_{B_\eta} u(t) + \underbrace{\begin{bmatrix} \omega_p(t) \\ \omega_{p_s}(t) \end{bmatrix}}_{\omega_{\eta_p}(t)}, \tag{37}$$

$$y(t) = \underbrace{\begin{bmatrix} C_1 & 0_{(l-q) \times n} \\ C_2 & C_s \end{bmatrix}}_{C_\eta} \begin{bmatrix} x(t) \\ x_s(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \omega_{s_1}(t) \\ \omega_{s_2}(t) + \omega_{s_s}(t) \end{bmatrix}}_{\omega_{\eta_s}(t)}. \tag{38}$$

The system above can be expressed compactly by making the appropriate substitutions indicated by the braces as

$$\dot{\eta}(t) = A_\eta \eta(t) + B_\eta u(t) + \omega_{\eta_p}(t), \tag{39}$$

$$y(t) = C_\eta \eta(t) + \omega_{\eta_s}(t). \tag{40}$$

Equations (39) and (40) model the general case of additive, time-varying sensor faults when such faults can be modeled as the outputs of an LTI system.

3.3.1 A SPECIAL CASE FOR TIME-VARYING SENSOR FAULTS: THE SENSOR STEP FAULT

It was suggested in the beginning of this section that the additive, constant sensor fault case (that is, the constant bias case) represents a special case of the additive, time-varying sensor fault case. In the case of sensor step faults, $A_s = 0_{q\times q}$, and $C_s = I_q$, (see Example 7). Thus, when all of the sensor faults are step faults, Equations (37) and (38) reduce to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \omega_p(t) \\ \omega_{p_s}(t) \end{bmatrix}, \tag{41}$$

$$y(t) = \begin{bmatrix} C_1 & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} x(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} \omega_{s_1}(t) \\ \omega_{s_2}(t) + \omega_{s_s}(t) \end{bmatrix}. \tag{42}$$

Remark 14. Equations (41) and (42) are identical to Equations (17) and (18) in [22], where the case of sensor step faults was considered.

3.4 SIMULTANEOUS ACTUATOR AND SENSOR FAULT MODELING

This section presents the modeling of simultaneous, time-varying actuator and sensor faults. The derivation is a relatively straightforward combination of the results from Sections 3.2 and 3.3. Furthermore, the special case of simultaneous step faults in the actuators and sensors is shown to be identical to the results presented in [22].

Consider the case of time-varying actuator faults represented by Equations (27) and (28) and the case of time-varying sensor faults represented by Equations (37) and (38). Now, in order to represent simultaneous actuator-sensor faults, it is sufficient to augment the states in the form $\begin{bmatrix} x(t)^T & x_a(t)^T & x_s(t)^T \end{bmatrix}^T$. Thus, the augmented system can be expressed as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \\ \dot{x}_s(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & \overline{B}^k C_a & 0_{n\times n_s} \\ 0_{n_a\times n} & A_a & 0_{n_a\times n_s} \\ 0_{n_s\times n} & 0_{n_s\times n_a} & A_s \end{bmatrix}}_{A_\varphi} \underbrace{\begin{bmatrix} x(t) \\ x_a(t) \\ x_s(t) \end{bmatrix}}_{\varphi(t)} + \underbrace{\begin{bmatrix} B^k \\ 0 \\ 0 \end{bmatrix}}_{B_\varphi} u^k(t)$$

$$+\underbrace{\begin{bmatrix} \overline{B}^k\omega_{s_a}(t)+\omega_p(t) \\ \omega_{p_a}(t) \\ \omega_{p_s}(t) \end{bmatrix}}_{\omega_{\varphi p}(t)}. \tag{43}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 & 0_{(l-q)\times n_a} & 0_{(l-q)\times n_s} \\ C_2 & 0_{q\times n_a} & C_s \end{bmatrix}}_{C_\varphi} \begin{bmatrix} x(t) \\ x_a(t) \\ x_s(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \omega_{s,1}(t) \\ \omega_{s,2}(t)+\omega_{s_s}(t) \end{bmatrix}}_{\omega_{\varphi_s}(t)}. \tag{44}$$

The system above can be expressed compactly, by making the appropriate substitutions indicated by the braces, as

$$\dot{\varphi}(t) = A_\varphi\varphi(t) + B_\varphi u^k(t) + \omega_{\varphi_p}(t), \tag{45}$$

$$y(t) = C_\varphi\varphi(t) + \omega_{\varphi_s}(t). \tag{46}$$

Equations (45) and (46) model the general case of simultaneous, additive, time-varying actuator-sensor faults when such faults can be modeled as the outputs of an LTI system.

3.4.1 A SPECIAL CASE FOR SIMULTANEOUS, TIME-VARYING ACTUATOR-SENSOR FAULTS: THE ACTUATOR-SENSOR STEP FAULT

For the case when all simultaneous actuator-sensor faults are step faults, $A_a = 0_{m_k\times m_k}, A_s = 0_{q\times q}, C_a = I_{m_k}$, and $C_s = I_q$, (see Example 7), and Equations (43) and (44) reduce to

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A & \overline{B}^k & 0_{n\times q} \\ 0_{m_k\times n} & 0_{m_k\times m_k} & 0_{m_k\times q} \\ 0_{q\times n} & 0_{q\times m_k} & 0_{q\times q} \end{bmatrix} \begin{bmatrix} x(t) \\ x_a(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} B^k \\ 0 \\ 0 \end{bmatrix} u^k(t)$$

$$+\begin{bmatrix} \overline{B}^k\omega_{s_a}(t)+\omega_p(t) \\ \omega_{p_a}(t) \\ \omega_{p_s}(t) \end{bmatrix}, \tag{47}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 & 0_{(l-q)\times m_k} & 0_{(l-q)\times q} \\ C_2 & 0_{q\times m_k} & I_q \end{bmatrix} \begin{bmatrix} x(t) \\ x_a(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} \omega_{s,1}(t) \\ \omega_{s,2}(t)+\omega_{s_s}(t) \end{bmatrix}. \tag{48}$$

Remark 15. Equations (47) and (48) are identical to Equations (31) and (33) presented in [22], where the case of simultaneous actuator-sensor step faults was considered.

CHAPTER 4

IDENTIFIABILITY OF TIME-VARYING ACTUATOR AND SENSOR FAULTS

This chapter presents the main results of this thesis, that is, a set of necessary and sufficient conditions for the identifiability of additive, time-varying faults affecting combinations of

(1) actuators only,

(2) sensors only, and

(3) actuators and sensors, simultaneously.

The primary contribution of this thesis is in the provision of Theorems 1-3 in Sections 4.2-4.4, respectively. The conditions are presented as three separate theorems accompanied by a proof for each of the indicated fault configurations. It is further shown through corollaries that, when all of the faults are step faults, a set of necessary and sufficient conditions for items (1), (2), and (3) above reduce to those presented in [22]. The proofs of the additive, time-varying case and the special, step-fault case rely upon several rank assumptions, which are presented subsequently.

4.1 ASSUMPTIONS

It is assumed in the proofs presented in this chapter that the following rank assumptions hold:

$(A1)$ $\text{rank}(\overline{B}^k) = m_k$, where $\overline{B}^k$ corresponds to the columns of B associated with the m_k faulty actuators,

$(A2)$ $\text{rank}(C) = l$,

$(A3)$ $\text{rank}(C_a) = m_k$, where C_a represents the output matrix associated with an exogenous LTI system which generates the m_k time-varying actuator fault signals,

($A4$) $\text{rank}(C_s) = q$, where C_s represents the output matrix associated with an exogenous LTI system which generates the q time-varying sensor fault signals,

($A5$) $\text{rank}(C_1) = l - q$, where C_1 corresponds to the rows of C associated with the $l - q$ non-faulty sensors, and

($A6$) $\text{rank}(C_2) = q$, where C_2 corresponds to the rows of C associated with the q faulty sensors.

Assumptions ($A1$) and ($A2$) follow directly from [22], that is: (1) it is assumed that inputs associated with linearly dependent columns of $\overline{B}^k$ have been aggregated and that $\overline{B}^k$ is full column rank, and (2) it is assumed that outputs associated with linearly dependent rows of C have been aggregated and that C is full row rank. If ($A1$) or ($A2$) do not hold, faults associated with linearly dependent columns of $\overline{B}^k$, or linearly dependent rows of C will not be uniquely identifiable. Similarly, Assumptions ($A3$) and ($A4$) follow from the representation of time-varying actuator and sensor faults shown in Sections 3.2 and 3.3, respectively. Finally, Assumptions ($A5$) and ($A6$) follow from Assumption ($A2$).

4.2 TIME-VARYING ACTUATOR FAULT IDENTIFIABILITY

This section presents a necessary and sufficient condition for time-varying actuator fault identifiability using state augmentation. As discussed in Section 2.1, the ability to estimate the state of such a system depends upon the properties of observability and detectability. In particular, identifiability of the time-varying actuator faults as presented in Equations (27) and (28) require detectability (respectively, observability) of the pair (C_ξ^k, A_ξ^k). Mathematically, a fault is identifiable if the augmented system is detectable. However, in some practical applications, observability is preferred [22]. (The property of observability allows arbitrary placement of the closed-loop eigenvalues in the detection filter). Thus, this thesis presents conditions for both detectability and observability. Consider the augmented system given by Equations (27) and (28), the model of an LTI system subject to additive, time-varying actuator faults. Before the conditions for identifiability and the associated proof for them are presented, preliminary notation development and lemmas to be used in the proof are given.

First, let $\Sigma_{(C,A,\overline{B}^k)}(s)$ denote the RSM given by

$$\begin{bmatrix} sI - A & -\overline{B}^k \\ C & 0 \end{bmatrix},$$

and let $\Gamma^{\xi}(s) \in \mathbb{C}^{n+m_k}$ denote the nullspace of $\Sigma_{(C,A,\overline{B}^k)}(s)$, where $s \in \mathbb{C}$. Furthermore, let $\Gamma^{\xi}_{m_k}(s) \in \mathbb{C}^{m_k}$ denote the subspace spanned by the last m_k components of a basis for $\Gamma^{\xi}(s)$. The elements of $\Gamma^{\xi}_{m_k}(s)$ are characterized next.

1. When s is an (extended) invariant zero but not an output-decoupling zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, then $\Gamma^{\xi}_{m_k}(s) = \mathcal{G}^{\xi}_{m_k}(s) \cup \{0\}$, where $\mathcal{G}^{\xi}_{m_k}(s)$ is the set of all input-zero directions of $\Sigma_{(C,A,\overline{B}^k)}(s)$,

2. When s is an (extended) invariant zero and an output-decoupling zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, then $\Gamma^{\xi}_{m_k}(s) = \mathcal{G}^{\xi}_{m_k}(s) = \{0\}$ is the only input-zero direction of $\Sigma_{(C,A,\overline{B}^k)}(s)$, and

3. When s is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, then $\Gamma^{\xi}_{m_k}(s) = \overline{\mathcal{G}}^{\xi}_{m_k}(s)$ is the subspace spanned by all of non-input-zero directions of $\Sigma_{(C,A,\overline{B}^k)}(s)$.

Observe that the introduced notation, that is, $\mathcal{G}^{\xi}_{m_k}(s)$ and $\overline{\mathcal{G}}^{\xi}_{m_k}(s)$, correspond to the situations in Examples 5 and 6, respectively.

Lemma 1. *The pair (C_a, A_a) is detectable (observable) if and only if the pair $(\overline{B}^k C_a, A_a)$ is detectable (observable).*

Proof. Observe that the pair $(\overline{B}^k C_a, A_a)$ is detectable (observable) if and only if

$$\operatorname{rank} \begin{bmatrix} sI - A_a \\ \overline{B}^k C_a \end{bmatrix} = n_a \text{ for } s \in \Lambda_u(A_a) \ (s \in \Lambda(A_a)),$$

$$\operatorname{rank} \begin{bmatrix} I_n & 0 \\ 0 & \overline{B}^k \end{bmatrix} \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix} = n_a \text{ for } s \in \Lambda_u(A_a) \ (s \in \Lambda(A_a)).$$

By Sylvester's inequality (see [31]) and Assumptions $(A1)$ and $(A3)$,

$$\operatorname{rank} \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} I_n & 0 \\ 0 & \overline{B}^k C_a \end{bmatrix} \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix}.$$

Therefore, $\operatorname{rank} \begin{bmatrix} sI - A_a \\ \overline{B}^k C_a \end{bmatrix} = \operatorname{rank} \begin{bmatrix} sI - A_a \\ C_a \end{bmatrix}$ for $s \in \Lambda_u(A_a)$ $(s \in \Lambda(A_a))$, so that the pair (C_a, A_a) is detectable (observable) if and only if the pair $(\overline{B}^k C_a, A_a)$ is detectable (observable). $\square$

Lemma 2. *The pair (C_ξ^k, A_ξ^k) is detectable (observable) if and only if all of the following conditions are satisfied:*

(i) the pair (C, A) is detectable (observable),

(ii) (C_a, A_a) is detectable (observable), and

(iii) for (λ_a, v) an (eigenvalue, eigenvector) pair of A_a with $\lambda_a \in \Lambda_u(A_a)$ ($\lambda_a \in \Lambda(A_a)$), $C_a v \notin \Gamma_{m_k}^\xi(\lambda_a)$.

Proof. Applying the PBH eigenvector test, the pair (C_ξ, A_ξ) is detectable (observable) if and only if

$$\begin{bmatrix} sI - A & -\overline{B}^k C_a \\ 0_{n_a \times n} & sI - A_a \\ C & 0_{l \times n_a} \end{bmatrix} \begin{bmatrix} \zeta \\ v \end{bmatrix} = 0$$

is satisfied only by the trivial solution (that is, $\begin{bmatrix} \zeta^T & v^T \end{bmatrix}^T = 0$) for $s \in \Lambda_u(A) \cup \Lambda_u(A_a)$ (for $s \in \Lambda(A) \cup \Lambda(A_a)$). The first n columns of the PBH test matrix are independent for $s \in \Lambda_u(A)$ (for $s \in \Lambda(A)$) if and only if the pair (C, A) is detectable (observable). The last n_a columns are independent for $s \in \Lambda_u(A_a)$ (for $s \in \Lambda(A_a)$) if and only if the pair (C_a, A_a) is detectable (observable) (see Lemma 1). For $s \notin \Lambda(A_a)$ the last n_a columns are independent of the first n columns. For $s = \lambda_a \in \Lambda_u(A_a)$ ($s = \lambda_a \in \Lambda(A_a)$) the last n_a columns are independent of the first n columns if and only if

$$\underbrace{\begin{bmatrix} \lambda_a I - A & -\overline{B}^k \\ C & 0 \end{bmatrix}}_{\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)} \begin{bmatrix} \zeta \\ C_a v \end{bmatrix} = 0 \tag{49}$$

does not have a nontrivial solution for (λ_a, v) an (eigenvalue, eigenvector) pair of A_a, that is, if and only if $C_a v \notin \Gamma_{m_k}^\xi(\lambda_a)$. □

The following theorem provides a necessary and sufficient condition for the identifiability of additive, time-varying actuator-only faults.

Theorem 1. *The pair (C_ξ^k, A_ξ^k) is detectable (observable) if and only if all of the following conditions are satisfied:*

(i) for (λ_a, v) an (eigenvalue, eigenvector) pair of A_a with $\lambda_a \in \Lambda_u(A_a)$ ($\lambda_a \in \Lambda(A_a)$), when $l < m_k$ and when λ_a is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$, then $C_a v \notin \overline{\mathcal{G}}_{m_k}^\xi(\lambda_a)$,

(ii) the pair (C, A) is detectable (observable),

(iii) the pair (C_a, A_a) is detectable (observable), and

(iv) for (λ_a, v) an (eigenvalue, eigenvector) pair of A_a with $\lambda_a \in \Lambda_u(A_a)$ ($\lambda_a \in \Lambda(A_a)$), when λ_a is an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$, then $C_a v \notin \mathcal{G}^{\xi}_{m_k}(\lambda_a)$.

Proof. The pair (C^k_ξ, A^k_ξ) is detectable (observable) if and only if Conditions $(i)-(iii)$ of Lemma 2 are satisfied. Thus, Conditions (ii) and (iii) of Theorem 1 are established. It remains only to be shown that Conditions (i) and (iv) of Theorem 1 now hold if and only if Condition (ii) of Lemma 2 holds. Now, suppose that Conditions (i) and (iv) of Theorem 1 hold and that Condition (iii) of Lemma 2 does not. Then Equation (49) has a nontrivial solution. The following two possible cases will be considered and shown to result in a contradiction:

1. $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)\} < \text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(s)\}$, and
2. $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)\} = \text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(s)\}$ with
   (a) $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(s)\} = n + \min\{m_k, l\}$, or
   (b) $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(s)\} < n + \min\{m_k, l\}$.

Case 1 leads to a nontrivial solution if and only if $(\lambda_a, C_a v)$ is an (invariant zero, input-zero direction) pair of $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$, a contradiction of Condition (iv) of Theorem 1, that is, $C_a v \in \mathcal{G}^{\xi}_{m_k}(\lambda_a)$. Case 2(a) leads to a nontrivial solution if and only if $l < m_k$ and $C_a v \in \overline{\mathcal{G}}^{\xi}_{m_k}(\lambda_a)$, a contradiction of Condition (i) of Theorem 1. Case 2(b) implies that $\Sigma_{(C,A,\overline{B}^k)}(s)$ is degenerate, that is, every $\lambda_a \in \mathbb{C}$ is an invariant zero $\Sigma_{(C,A,\overline{B}^k)}(s)$. Thus, Case 2(b) leads to a nontrivial solution if and only if $C_a v \in \mathcal{G}^{\xi}_{m_k}(\lambda_a)$, a contradiction of Condition (iv) of Theorem 1. Thus, Conditions $(i)-(iv)$ of Theorem 1 are necessary and sufficient for identifiability of the pair (C^{ξ}, A^{ξ}). □

Remark 16. For Case 2(a), if $l \geq m_k$ then $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$ has full column rank, and $\begin{bmatrix} \zeta^T & (C_a v)^T \end{bmatrix}^T = 0$ is the only solution to Equation (49).

The following corollary considers the case when the geometric multiplicity of λ_a is equal to n_a. One such instance is when $A_a = 0_{n_a \times n_a}$, (a step fault in $n_a = m_k$ actuators).

Corollary 1. *Let* $\lambda_a \in \Lambda_u(A_a)$ $(\lambda_a \in \Lambda(A_a))$ *have geometric multiplicity* γ_a. *If* $\gamma_a = n_a$ *then Condition (iii) in Lemma 2 becomes*

(*i*) $l \geq m_k$, *and*

(*ii*) $\Sigma_{(C,A,\overline{B}^k)}(s)$ *has no invariant zeros at* $s = \lambda_a$.

Proof. Let $g \in \Gamma^{\xi}_{m_k}(\lambda_a)$, and consider whether there exists $v \in \mathbb{R}^{n_a}$ such that $C_a v = g$, where v is an eigenvector of A_a in the associated eigenspace W, with $\dim(W) = \gamma_a$. Next, observe that $C_a : \mathbb{R}^{n_a} \to \mathbb{R}^{m_k}$ is a surjective linear transformation since C_a has full row rank, that is, $\text{rank}(C_a) = m_k$ (see Assumption A3). Therefore, for any $g \in \mathbb{R}^{m_k}$, there always exists a vector $\tau \in \mathbb{C}^{n_a}$ such that $C_a \tau = g$. Now, when $\gamma_a = n_a$, $W = \mathbb{C}^{n_a}$, and any nonzero vector in $\mathbb{C}^{n_a}$ is an eigenvector. Thus, for any nonzero solution τ, let $v = \tau$, and it follows that $C_a v = g$. Therefore, when λ_a has geometric multiplicity $\gamma_a = n_a$, $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$ must have full column rank so that $\Gamma^{\xi}_{m_k}(\lambda_a) = \{0\}$, implying that $l \geq m_k$ and that λ_a is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$ □

Corollary 2. *For the special case when all of the faults are constant biases, that is, a step fault in each of the* m_k *faulty actuators, Conditions (i) – (iv) of Theorem 1 reduce to:*

(*i*) $l \geq m_k$,

(*ii*) *the pair* (C, A) *is detectable (observable), and*

(*iii*) $\Sigma_{(C,A,\overline{B}^k)}(s)$ *has no invariant zeros at* $s = 0$.

Proof. First, observe that for step faults, $n_a = m_k$, $A_a = 0_{n_a \times n_a}$ and $C_a = I_{n_a \times n_a}$ (see Example 7). Thus, A_a has one distinct eigenvalue at zero with geometric multiplicity n_a, that is, any nonzero vector $v \in \mathbb{R}^{n_a}$ is an eigenvector of A_a, and by Corollary 1, Conditions (*i*) and (*iii*) of Corollary 2 are established. Furthermore, since (C_a, A_a) is observable, Condition (*iii*) of Theorem 1 is not needed. The detectability (observability) requirement for the pair (C, A) is maintained. □

Remark 17. The conditions in Corollary 2 are identical to the conditions for constant, bias-type actuator fault identifiability presented in Theorem 1 in [22].

Example 11. Consider a system with two actuator faults and one output given by

$$\Sigma_{(C,A,\overline{B}^k)}(s) = \begin{bmatrix} s+1 & 0 & -1 & 0 \\ 0 & s+2 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

and suppose that the fault dynamics are given by

$$A_a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}; \; C_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be verified that both pairs (C, A) and (C_a, A_a) are observable, thus, Conditions (ii) and (iii) of Theorem 1 are satisfied. Furthermore, checking Condition (iv) of Theorem 1 by first determining any invariant zeros of $\Sigma_{(C,A,\overline{B}^k)}(s)$, it can be verified that $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(s)\} = 3$, and for $\Lambda(A_a) = \{-1, 1\}$, $\text{rank}\{\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)\} = 3$. Therefore, $\Sigma_{(C,A,\overline{B}^k)}(s)$ has no invariant zeros, and Condition (iv) of Theorem 1 is satisfied. Now, all that remains is a check of Condition (i) of Theorem 1.

Observe that $\overline{\mathcal{G}}^{\xi}_{m_k}(s)$ has the form

$$\begin{bmatrix} \alpha & -(s+1)/(s+2) & 1 \end{bmatrix}^T, \; s \neq -2,$$
$$\begin{bmatrix} \alpha & 1 & 0 \end{bmatrix}^T, \; s = -2,$$

where $\alpha \in \mathbb{C}$ such that $\alpha \neq 0$. Now, consider the two eigenvalues of A_a. When $\lambda_a = -1$, a basis vector for $\overline{\mathcal{G}}^{\xi}_{m_k}(-1)$ is $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $C_a v = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \notin \overline{\mathcal{G}}^{\xi}_{m_k}(-1)$. When $\lambda_a = 1$, a basis vector for $\overline{\mathcal{G}}^{\xi}_{m_k}(1)$ is $\begin{bmatrix} -2 & 3 \end{bmatrix}^T$, and $C_a v = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \notin \overline{\mathcal{G}}^{\xi}_{m_k}(-1)$, satisfying Condition (i) of Theorem 1. Thus, Conditions $(i) - (iv)$ of Theorem 1 are satisfied, and the fault is identifiable.

Example 12. Consider a different system with two actuator faults and one output given by

$$\Sigma_{(C,A,\overline{B}^k)}(s) = \begin{bmatrix} s-1 & 0 & -1 & 0 \\ 0 & s+2 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

where the fault dynamics are exactly those given in Example 11. It can be verified that Conditions $(ii) - (iv)$ of Theorem 1 are again satisfied. Only a check of Condition (i) of Theorem 1 remains. Now, $\overline{\mathcal{G}}^{\xi}_{m_k}(s)$ has the form $\begin{bmatrix} -(s-1)/(s+2) & 1 \end{bmatrix}^T$. When $\lambda_a = -1$ a basis for $\overline{\mathcal{G}}^{\xi}_{m_k}(-1)$ is $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$, and $C_a v = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \notin \overline{\mathcal{G}}^{\xi}_{m_k}(-1)$. When $\lambda_a = 1$, a basis for $\overline{\mathcal{G}}^{\xi}_{m_k}(1)$ is $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $C_a v = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \in \overline{\mathcal{G}}^{\xi}_{m_k}(1)$. Thus, Condition (i) of Theorem 1 is not satisfied and the fault is not identifiable. In particular, $\lambda_a = 1$ is not an observable eigenvalue of the pair (C^{ξ}, A^{ξ}), and since $\mathfrak{Re}\{1\} \geq 0$ the pair is also not detectable.

4.3 TIME-VARYING SENSOR FAULT IDENTIFIABILITY

This section presents conditions for time-varying sensor fault identifiability. Consider the augmented system given by Equations (37) and (38), the model of an arbitrary LTI system subject to time-varying sensor faults. The identifiability of such a fault requires that the pair (C_η, A_η) be detectable (observable). The following theorem gives a necessary and sufficient condition for identifiability of time-varying sensor faults.

Theorem 2. *The pair (C_η, A_η) is detectable (observable) if and only if all of the following conditions are satisfied:*

(i) the pair (C, A) is detectable (observable),

(ii) the pair (C_s, A_s) is detectable (observable), and

(iii) when (λ_s, ζ) and (λ_s, ψ) are eigenvalue, eigenvector pairs of A and A_s, respectively, and λ_s is not a detectable (observable) eigenvalue of the pair (C_1, A), then $C_2\zeta \neq \alpha C_s\psi$, where $\alpha \in \mathbb{C}$.

Proof. The pair (C_η, A_η) is detectable (observable) if and only if

$$\operatorname{rank}\left(\begin{bmatrix} sI - A & 0_{n\times n_s} \\ 0_{n_s\times n} & sI - A_s \\ C_1 & 0_{(l-q)\times n_s} \\ C_2 & C_s \end{bmatrix}\right) = n + n_s$$

for $s \in \Lambda_u(A) \cup \Lambda_u(A_s)$ (for $s \in \Lambda(A) \cup \Lambda(A_s)$). The first n columns of the PBH test matrix are linearly independent for all $s \in \Lambda_u(A)$ (for all $s \in \Lambda(A)$) if and only if (C, A) is detectable (observable). The last n_s columns are linearly independent for $s \in \Lambda_u(A_s)$ (for $s \in \Lambda(A_s)$) if and only if (C_s, A_s) is detectable (observable). Furthermore, whenever $s \notin \Lambda(A)\cup\Lambda(A_s)$ the last n_s columns are linearly independent of the first n columns. Now, let $s = \lambda_s \in \Lambda_u(A_s) \cup \Lambda_u(A)$ ($s = \lambda_s \in \Lambda(A_s) \cup \Lambda(A)$). The last n_s columns are linearly independent from the first n columns if and only if

$$\begin{bmatrix} \lambda_s I - A & 0_{n\times n_s} \\ 0_{n_s\times n} & \lambda_s I - A_s \\ C_1 & 0_{(l-q)\times n_s} \\ C_2 & C_s \end{bmatrix} \begin{bmatrix} \zeta \\ \psi \end{bmatrix} = 0 \tag{50}$$

only for $\begin{bmatrix} \zeta^T & \psi^T \end{bmatrix}^T = 0$. Now, suppose there exists $\begin{bmatrix} \zeta^T & \psi^T \end{bmatrix}^T \neq 0$ such that (50) is satisfied. Then

$$(\lambda_s I - A)\zeta = 0 \tag{51}$$

$$(\lambda_s I - A_s)\psi = 0 \tag{52}$$

$$C_1\zeta = 0 \tag{53}$$

$$C_2\zeta + C_s\psi = 0, \tag{54}$$

must hold. Since the pair (C, A) is detectable (observable) $\psi \neq 0$, and since the pair (C_s, A_s) is detectable (observable) $\zeta \neq 0$. The observation that ψ in Equation (52) is nonzero requires that ψ be an eigenvector of A_s associated with λ_s, thus satisfying Equation (52). The remaining equations may be expressed as

$$\begin{bmatrix} \lambda_s I - A \\ C_1 \end{bmatrix} \zeta = 0, \tag{55}$$

$$C_2\zeta + C_s\psi = 0. \tag{56}$$

If λ_s is not a detectable (observable) eigenvalue of (C_1, A) then Equation (55) is satisfied. Now, Equation (56) is satisfied if and only if $C_2\zeta \neq \alpha C_s\psi$, where $\alpha \in \mathbb{C}$. Thus, Conditions (i)–(iii) of Theorem 2 are necessary and sufficient for identifiability of the pair (C_η, A_η). □

The following corollary considers the special case when λ_s has geometric multiplicity equal to n_s.

Corollary 3. *Let $\lambda_s \in \Lambda_u(A_s) \cap \Lambda_u(A)$ $(s \in \Lambda(A_s) \cap \Lambda(A))$ have geometric multiplicity equal to γ_s, and let ψ be an eigenvector of A_s from the associated eigenspace V corresponding to the eigenvalue λ_s. Observe that $\dim(V) = \gamma_s$. If $\gamma_s = n_s$ then Condition (iii) in Theorem 2 becomes*

(iii) λ_s is a detectable (observable) eigenvalue of the pair (C_1, A).

Proof. Observe that $C_s : \mathbb{R}^{n_s} \to \mathbb{R}^q$ is a surjective linear transformation since $\text{rank}(C_s) = q$ (see Assumption (A4)). Therefore, there always exists a vector, $p \in \mathbb{C}^{n_s}$, such that $C_s p = -C_2\zeta$. Now, when $\gamma_s = n_s$, $V = \mathbb{C}^{n_s}$, and any vector in $\mathbb{C}^{n_s}$ is an eigenvector. Then, for any solution p let $\psi = p$, and it follows that $C_s\psi = -C_2\zeta$. That is, whenever $\gamma_s = n_s$, Equation (56) is satisfied. Thus, it is required that λ_s be a detectable (observable) eigenvalue of the pair (C_1, A). □

Corollary 4. *For the special case when all of the faults are constant biases, that is, step faults, Conditions* $(i) - (iii)$ *in Theorem 2 reduce as follows:*

(i) *the pair* (C, A) *is detectable (observable), and*

(ii) $s = 0$ *is a detectable (observable) eigenvalue of the pair* (C_1, A).

Proof. First, observe that Condition (i) of Theorem 2 is identical to Condition (i) of Corollary 4. Now, from Example 7, $A_s = 0$ and $C_s = I$. Thus, Condition (ii) of Theorem 2 is no longer necessary, since the pair (C_s, A_s) is observable. Furthermore, note that the only eigenvalue of A_a is $s = 0$, having algebraic and geometric multiplicity $q = n_s$, and by Corollary 3, Condition (ii) of Corollary 4 is established, that is, if (C_1, A) is not observable when $s = 0$ then the pair is not detectable. □

Remark 18. Conditions (i) and (ii) of Corollary 4 are exactly those presented in Theorem 2 of [22] for the sensor step fault case.

Example 13. Consider the system given by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \; C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $C_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}$ (the fault-free output) and $C_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}$ (the fault-affected output). Furthermore, let $A_s = -1$ and $C_s = -1$. Observe that the pairs (C, A) and (C_s, A_s) are observable. Therefore, Conditions (i) and (ii) of Theorem 2 are satisfied. Now, the augmented system is given by

$$\begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+1 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Note that (C_η, A_η) is not observable, because the pair (C_1, A) is not observable. However, since the unobservable eigenvalue at $s = -1$ is asymptotically stable, (C_1, A) is detectable, and the pair (C_η, A_η) is detectable, that is, $C_2\zeta \neq \alpha C_s \psi$, where $\alpha \in \mathbb{C}$.

4.4 SIMULTANEOUS TIME-VARYING ACTUATOR AND SENSOR FAULT IDENTIFIABILITY

This section presents a necessary and sufficient condition for identifiability of simultaneous, time-varying actuator-sensor faults. Consider the augmented system given by Equations (43) and (44), the model of an arbitrary LTI system subject to simultaneous, time-varying actuator-sensor faults. Prior to presenting the proof of a necessary and sufficient condition for identifiability of the pair (C_φ, A_φ) given by Equations (43) and (44), some preliminary notation is given.

Let $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ denote the Rosenbrock System Matrix given by

$$\begin{array}{cc} & \begin{array}{cc} n & m_k \end{array} \\ \begin{array}{c} n \\ l-q \end{array} & \begin{bmatrix} sI - A & -\overline{B}^k \\ C_1 & 0 \end{bmatrix} \end{array} .$$

with associated nullspace $\Gamma^\varphi(s)$, where $s \in \mathbb{C}$, and let $\Gamma^\varphi_{m_k}(s)$ be the subspace spanned by the last m_k components of a basis for $\Gamma^\varphi(s)$. As in Section 4.2, depending on the value of s, $\Gamma^\varphi_{m_k}(s)$ either includes the set of all input-zero direction vectors of $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ or it is a subspace of non-input-zero direction vectors. The former is denoted by $\mathcal{G}^\varphi_{m_k}(s)$ and the latter by $\overline{\mathcal{G}}^\varphi_{m_k}(s)$.

Theorem 3. *The pair (C_φ, A_φ) is detectable (observable) if and only if all of the following conditions are satisfied:*

(i) for $\lambda_{a,s} \in \Lambda_u(A_a) \cap \Lambda_u(A_s)$ $(\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s))$ not an invariant zero of $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ with v and ψ eigenvectors of A_a and A_s associated with $\lambda_{a,s}$, respectively, when $l < m_k + q$ then either $C_a v \notin \overline{\mathcal{G}}^\varphi_{m_k}(\lambda_{a,s})$ or $C_2\zeta \neq \alpha C_s \psi$, for $\alpha \in \mathbb{C}$,

(ii) for $\lambda_a \in \Lambda_u(A_a)$ $(\lambda_a \in \Lambda(A_a))$ not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(\lambda_a)$, where v is an eigenvector of A_a and $\psi = 0$, when $l < m_k$ then $C_a v \notin \overline{\mathcal{G}}^\xi_{m_k}(\lambda_a)$,

(iii) the pair (C, A) is detectable (observable),

(iv) the pair (C_a, A_a) is detectable (observable),

(v) the pair (C_s, A_s) is detectable (observable),

(vi) for $\lambda_s \in \Lambda_u(A_s)$ $(\lambda_s \in \Lambda(A_s))$ not a detectable (observable) eigenvalue of the pair (C_1, A), where ψ is an eigenvector of A_s associated with λ_s and $v = 0$, $C_2\zeta \neq \alpha C_s \psi$, for $\alpha \in \mathbb{C}$,

(vii) *for* $\lambda_a \in \Lambda_u(A_a)$ $(\lambda_s \in \Lambda(A_a))$ *an invariant zero of* $\Sigma_{(C,A,\overline{B}^k)}(s)$, *where* v *is an eigenvector of* A_a *and* $\psi = 0$, $C_a v \notin \mathcal{G}^{\xi}_{m_k}(\lambda_a)$, *and*

(viii) *for* $\lambda_{a,s} \in \Lambda_u(A_a) \cap \Lambda_u(A_s)$ $(\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s))$ *an invariant zero of* $\Sigma_{(C_1,A,\overline{B}^k)}(s)$, *where* v *and* ψ *are eigenvectors of* A_a *and* A_s *associated with* $\lambda_{a,s}$, *respectively, either* $C_a v \notin \mathcal{G}^{\varphi}_{m_k}(\lambda_{a,s})$ *or* $C_2\zeta \neq \alpha C_s \psi$, *for* $\alpha \in \mathbb{C}$.

Proof. The pair (C_φ, A_φ) is detectable (observable) if and only if

$$\operatorname{rank}\left(\begin{bmatrix} sI - A & -\overline{B}^k C_a & 0_{n \times n_s} \\ 0_{n_a \times n} & sI - A_a & 0_{n_a \times n_s} \\ 0_{n_s \times n} & 0_{n_s \times n_a} & sI - A_s \\ C_1 & 0_{(l-q)\times n_a} & 0_{(l-q)\times n_s} \\ C_2 & 0_{q \times n_a} & C_s \end{bmatrix}\right) = n + n_a + n_s$$

for $s \in \Lambda_u(A) \cup \Lambda_u(A_a) \cup \Lambda_u(A_s)$ (for $s \in \Lambda(A) \cup \Lambda(A_a) \cup \Lambda(A_s)$. The first n columns are linearly independent for all $s \in \Lambda_u(A)$ (for all $s \in \Lambda(A)$) if and only if the pair (C, A) is detectable (observable), establishing Condition *(iii)* of Theorem 3. The next n_a columns are linearly independent for all $s \in \Lambda_u(A_a)$ (for all $s \in \Lambda(A_a)$) if and only if the pair (C_a, A_a) is detectable (observable), establishing Condition *(iv)* of Theorem 3. The last n_s columns are linearly independent for all $s \in \Lambda_u(A_s)$ (for all $s \in \Lambda(A_s)$) if and only if the pair (C_s, A_s) are detectable (observable), establishing Condition (v) of Theorem 3. When $s \notin \Lambda(A_a) \cup \Lambda(A_s)$ then all of the columns are mutually independent. Now, when $s = \lambda_{a,s} \in \Lambda_u(A_a) \cup \Lambda_u(A_s)$ (when $s = \lambda_{a,s} \in \Lambda(A_a) \cup \Lambda(A_s)$), all of the columns are mutually independent if and only if

$$\begin{bmatrix} \lambda_{a,s} I - A & -\overline{B}^k C_a & 0_{n \times n_s} \\ 0_{n_a \times n} & \lambda_{a,s} I - A_a & 0_{n_a \times n_s} \\ 0_{n_s \times n} & 0_{n_s \times n_a} & \lambda_{a,s} I - A_s \\ C_1 & 0_{(l-q)\times n_a} & 0_{(l-q)\times n_s} \\ C_2 & 0_{q \times n_a} & C_s \end{bmatrix} \begin{bmatrix} \zeta \\ v \\ \psi \end{bmatrix} = 0 \tag{57}$$

only for $\begin{bmatrix} \zeta^T & v^T & \psi^T \end{bmatrix}^T = 0$. Now, suppose there exists $\begin{bmatrix} \zeta^T & v^T & \psi^T \end{bmatrix}^T \neq 0$ such that Equation (57) is satisfied. Then

$$(\lambda_{a,s}I - A)\zeta - \overline{B}^k C_a v = 0 \tag{58}$$

$$(\lambda_{a,s}I - A_a)v = 0 \tag{59}$$

$$(\lambda_{a,s}I - A_s)\psi = 0 \tag{60}$$

$$C_1\zeta = 0 \tag{61}$$

$$C_2\zeta + C_s\psi = 0, \tag{62}$$

must hold. The possible cases to be considered are listed in Table 1. Observe that Case (a) is always a solution to Equation (57). Case (b) gives a nontrivial solution if and only if the pair (C_s, A_s) is not detectable (observable), thus establishing Condition (v) of Theorem 3. Case (c) gives a nontrivial solution if and only if the pair (C_a, A_a) is not detectable (observable), thus establishing Condition (iv) of Theorem 3. Case (d) gives a nontrivial solution if and only if Conditions (iv) and (v) do not hold. Case (e) gives a nontrivial solution if and only if the pair (C, A) is not detectable (observable), thus establishing Condition (iii) of Theorem 3. When

Case	$\zeta = 0$	$v = 0$	$\psi = 0$
(a)	T	T	T
(b)	T	T	F
(c)	T	F	T
(d)	T	F	F
(e)	F	T	T
(f)	F	T	F
(g)	F	F	T
(h)	F	F	F

TABLE 1: General form of possible solutions to Equation (57).

$v = 0$ as in Case (f), Equations (58)-(62) reduce to Equations (55) and (56). Thus, by Theorem 2, Case (f) gives a nontrivial solution to Equations (58)-(62) if and only if for $\lambda_s \in \Lambda_u(A_s)$ ($\lambda_s \in \Lambda(A_s)$) not a detectable (observable) eigenvalue of the pair (C_1, A), $C_2\zeta \neq \alpha C_s\psi$, where ψ is an eigenvector of A_s associated with λ_s and $\alpha \in \mathbb{C}$, thus establishing Condition (vi) of Theorem 3.

When $\psi = 0$ as in Case (g), Equations (58)-(62) reduce to Equation (49). Thus, by Theorem 1, Case (g) gives a nontrivial solution to Equations Equations (58)-(62) if and only if either:

(g.1) for $\lambda_a \in \Lambda_u(A_a)$ ($\lambda_a \in \Lambda(A_a)$) not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, when $l < m_k$ then $C_a v \in \overline{\mathcal{G}}^{\xi}_{m_k}(\lambda_a)$, where v is an eigenvector of A_a, or

(g.2) for $\lambda_a \in \Lambda_u(A_a)$ ($\lambda_a \in \Lambda(A_a)$) an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, $C_a v \in \mathcal{G}^{\xi}_{m_k}(\lambda_a)$, where v is an eigenvector of A_a.

Thus, Conditions (ii) and (vii) of Theorem 3 are established.

Finally, Case (h) gives a nontrivial solution to Equations (58)-(62) only for $\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s)$. With Equations (59) and (60) satisfied, the remaining equations can be expressed as

$$\begin{bmatrix} \lambda_{a,s} I - A & -\overline{B}^k \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} \zeta \\ C_a v \end{bmatrix} = 0. \tag{63}$$

$$C_2 \zeta + C_s \psi = 0. \tag{64}$$

Equations (63) and (64) together have nontrivial solutions if and only if either:

(h.1) for $\lambda_{a,s} \in \Lambda_u(A_a) \cap \Lambda_u(A_s)$ ($\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s)$) not an invariant zero of $\Sigma_{(C_1,A,\overline{B}^k)}(s)$, when $l < m_k + q$ then both $C_a v \in \overline{\mathcal{G}}^{\varphi}_{m_k}(\lambda_{a,s})$ and $C_2 \zeta = \alpha C_s \psi$, where v and ψ are eigenvectors of A_a and A_s, respectively, associated with $\lambda_{a,s}$, $\zeta \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, or

(h.2) for $\lambda_{a,s} \in \Lambda_u(A_a) \cap \Lambda_u(A_s)$ ($\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s)$) an invariant zero of $\Sigma_{(C_1,A,\overline{B}^k)}(s)$, both $C_a v \in \mathcal{G}^{\varphi}_{m_k}(\lambda_{a,s})$ and $C_2 \zeta = \alpha C_s \psi$, where v and ψ are eigenvectors of A_a and A_s, respectively, associated with $\lambda_{a,s}$, $\zeta \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$.

Thus, Conditions (i) and ($viii$) of Theorem 3 are established, and Conditions (i) – ($viii$) of Theorem 3 are together a necessary and sufficient condition for detectability (observability) of the pair $(C^{\varphi}, A^{\varphi})$. □

Corollary 5. *Let λ_a be an eigenvalue of A_a with geometric multiplicity γ_a and λ_s be an eigenvalue of A_s with geometric multiplicity γ_s, where $\lambda_a = \lambda_s = \lambda_{a,s}$. If $\gamma_a = n_a$ and $\gamma_s = n_s$ then Conditions (i) and ($viii$) of Theorem 3 become*

(i) $l \geq m_k + q$,

(iii) $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ *has no invariant zeros at* $\lambda_{a,s}$,

respectively.

Proof. The proof follows directly from Corollaries 1 and 3. If $\gamma_s = n_s$ then $C_2\zeta = \alpha C_s\psi$, where $\alpha \in \mathbb{C}$, and if $\gamma_a = n_a$ then $C_a v$ is always part of a solution to Equation (63), if such a solution exists. Therefore, (63) must be satisfied only by the trivial solution, implying that full column rank is necessary. Thus, when $\gamma_a = n_a$ and $\gamma_s = n_s$, then $l \geq m_k + q$ and $\lambda_{a,s}$ must not be an invariant zero of $\Sigma_{(C_1, A\overline{B}^k)}(s)$. □

Corollary 6. *For the special case when all of the simultaneous faults are constant biases, that is, step faults, and $v, \psi \neq 0$, Conditions $(i) - (viii)$ of Theorem 3 reduce as follows:*

- (i) $l \geq m_k + q$,
- (ii) *the pair (C, A) is detectable (observable), and*
- (iii) $\Sigma_{(C_1, A, \overline{B}^k)}(s)$ *has no invariant zeros at $\lambda_{a,s}$.*

Proof. Observe that when all the actuator and sensor faults are step faults that $\gamma_a = n_a$ and $\gamma_s = n_s$. Thus, Conditions (i) and (iii) of Corollary 6 follow directly from Corollary 5. Furthermore, since $A_a = 0_{n_a \times n_a}$, $C_a = I_{n_a}$, $A_s = 0_{n_s \times n_s}$, and $C_s = I_{n_s}$, both pairs (C_a, A_a) and (C_s, A_s) are observable, and Conditions (iv) and (v) of Theorem 3 are no longer needed. Finally, since $v, \psi \neq 0$, Conditions (ii), (vi), and (vii) of Theorem 3 are no longer needed. □

Remark 19. Conditions $(i) - (iii)$ of Corollary 6 are equivalent to those presented in Theorem 3 of [22] for the simultaneous actuator-sensor step fault case.

Example 14. Consider the system given by

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix};\ B = \begin{bmatrix} -1 \\ 1 \end{bmatrix};\ C = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$

where identical faults affect the first sensor and actuator, that is,

$$A_a = A_s = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix};\ C_a = C_s = \begin{bmatrix} 1 & 0 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} -1 & 1 \end{bmatrix};\ C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

where C_1 corresponds to the fault-free outputs, and C_2 corresponds to the faulty outputs. It can be verified that Conditions $(i) - (vii)$ of Theorem 3 are satisfied, and that $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ has one invariant zero at $s = -2$, an eigenvalue common to both A_a and A_s. Now, a basis for the nullspace of $\Sigma_{(C_1,A,\overline{B}^k)}(-2)$ is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. Thus, $\mathcal{G}^{\varphi}_{m_k}(-2) = \mathbb{R}\backslash\{0\}$, and $C_a v = 1 \in \mathcal{G}^{\varphi}_{m_k}(-2)$. Furthermore, $C_2\zeta = \alpha C_s \psi$, . Therefore, Condition $(viii)$ of Theorem 3 is violated when $\lambda_{a,s} = -2$, so the pair $(C^{\varphi}, A^{\varphi})$ is not observable, but the pair is detectable (since $\lambda_{a,s} = -2$ is stable).

Example 15. Consider the system and faults given in Example 14, where all quantities are as given, except that now

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}.$$

It can be verified that Conditions $(i) - (vii)$ of Theorem 3 are again satisfied. Now, $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ has one invariant zero at $s = 2$, an eigenvalue common to both A_a and A_s. Furthermore, $C_2\zeta = \alpha C_s \psi$, where $\alpha \in \mathbb{C}$. Therefore, Condition $(viii)$ of Theorem 3 is violated when $\lambda_{a,s} = 2$, and the pair $(C^{\varphi}, A^{\varphi})$ is neither observable nor detectable (since the eigenvalue at 2 is unstable).

CHAPTER 5

CASE STUDIES

This chapter presents two case studies in order to provide some illustration of the relevance and application of the theorems presented in Chapter 4. Each case study is taken from a practical example in the literature. The linearized model of each dynamical system being analyzed is given, along with the particular fault models and augmented system being tested for identifiability. The approach for constructing augmented systems to represent a particular fault configuration is presented first.

5.1 CASE STUDY APPROACH

To illustrate how the augmented system models were constructed, consider the case where m_k of the m actuators and q of the l sensors are affected by time-varying faults, where it is not necessarily the case that any of the faults are identical, that is, in general,

$$\begin{aligned} A_{a_i} &\neq A_{a_j}, i \neq j \\ A_{s_i} &\neq A_{s_j}, i \neq j \\ C_{a_i} &\neq C_{a_j}, i \neq j \\ C_{s_i} &\neq C_{s_j}, i \neq j. \end{aligned}$$

For a set of arbitrary, simultaneous actuator-sensor faults,

$$\begin{aligned} A_a &= \text{diag}\begin{bmatrix} A_{a_1} & \dots & A_{a_{m_k}} \end{bmatrix}, \\ A_s &= \text{diag}\begin{bmatrix} A_{s_1} & \dots & A_{s_q} \end{bmatrix}, \\ C_a &= \text{diag}\begin{bmatrix} C_{a_1} & \dots & C_{a_{m_k}} \end{bmatrix}, \\ C_s &= \text{diag}\begin{bmatrix} C_{s_1} & \dots & C_{s_q} \end{bmatrix}. \end{aligned}$$

Thus, for such an arbitrary fault configuration, the PBH test matrix can be constructed as

$$\left[\begin{array}{c|cccc|cccc}
sI-A & -\bar{b}_1C_{a_1} & -\bar{b}_2C_{a_2} & \dots & -\bar{b}_{m_k}C_{a_{m_k}} & 0 & 0 & \dots & 0 \\
\hline
0 & sI-A_{a_1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & sI-A_{a_2} & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & sI-A_{a_{m_k}} & 0 & 0 & \dots & 0 \\
\hline
0 & 0 & 0 & \dots & 0 & sI-A_{s_1} & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & sI-A_{s_2} & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & sI-A_{s_q} \\
\hline
C_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\hline
C_{2_1} & 0 & 0 & \dots & 0 & C_{s_1} & 0 & \dots & 0 \\
C_{2_2} & 0 & 0 & \dots & 0 & 0 & C_{s_2} & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
C_{2_q} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & C_{s_q}
\end{array}\right],$$

where if $m_k = 0$ then the second block column and the second block row are removed, and if $q = 0$ then the last block column and the last block row are removed, leaving only the original system in a no-fault configuration.

By inspection of the PBH test matrix, it can be verified that there are 2^{m_k} possible time-varying actuator fault combinations, that is, fault or no fault for each actuator, including the case when there are no actuator faults. When each actuator is subject to more than one particular fault model, the number of possible fault cases grows rapidly. For example, if each actuator is subject to the same n distinct faults then the number of cases to consider is n^{m_k}. Furthermore, there are 2^q possible time-varying sensor fault combinations, including the no-fault condition. If each sensor is subject to the same m possible configurations then the number of cases to consider grows to m^q. Thus, in total there are $n^{m_k}m^q$ actuator-sensor fault combinations to consider, including the no-fault configuration. The unique configuration when there are no faulty actuators or sensors will be treated as a simultaneous actuator-sensor fault configuration in each case study.

5.2 CASE STUDY 1: A VTOL AIRCRAFT

Consider the 4th-order, linearized vertical-plane dynamics of a vertical takeoff and landing (VTOL) aircraft, flying in the airspeed range of 60-170 knots, given in [21] and [32] given by

$$A = \begin{bmatrix} -0.0336 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.5200 & 4.9900 \\ 0 & 0 \end{bmatrix};$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The state vector is comprised of the horizontal velocity, vertical velocity, pitch rate, and pitch angle, that is, $x(t) = \begin{bmatrix} v & w & q & \theta \end{bmatrix}^T$. The control vector is comprised of the collective pitch angle and longitudinal cyclic pitch angle, that is, $u(t) = \begin{bmatrix} A_0 & B_1 \end{bmatrix}^T$. The collective pitch angle input controls the vertical motion, and the longitudinal cyclic pitch angle input controls the horizontal velocity [21]. The output vector is comprised of the horizontal velocity and vertical velocity, that is, $y(t) = \begin{bmatrix} v & w \end{bmatrix}^T$. It can be verified that $\Lambda(A) = \{2.8174 \cdot 10^{-1} \pm i9.7701 \cdot 10^{-2}, -3.3318 \cdot 10^{-1}, -1.9809\}$, and that $\{A, B, C, 0\}$ is a minimal realization, that is, both controllable and observable.

The case of oscillatory faults in either or both actuators was considered in [21], wherein a primary research goal was the design of a detection filter to identify such faults. This case study seeks to validate the primary contribution of this thesis, that is, a set of conditions for additive, time-varying fault identifiability using state augmentation alone. The faults considered in this case study are

1. actuator oscillatory faults and sensor step faults in the following configurations:
   (a) actuator-only oscillatory faults,
   (b) sensor-only step faults,
   (c) actuator oscillatory faults with simultaneous sensor step faults;
2. actuator ramp faults and sensor step faults in the following configurations:
   (a) actuator-only ramp faults,

(b) sensor-only step faults,

(c) actuator ramp faults with simultaneous sensor step faults.

These fault configurations were chosen to illustrate how identifiability for some particular fault configuration may not hold under another fault configuration. The theorems of Chapter 4 are able to completely characterize the cases of non-identifiability.

5.2.1 ACTUATOR OSCILLATORY FAULTS AND SENSOR STEP FAULTS

The actuator faults are modeled as sinusoids, as shown in Example 9, where $\omega = 2\pi$ rad·s^{-1} as given in [21], and the sensor faults are modeled as step faults. The construction of the augmented system for this example and all subsequent examples follows from the general PBH matrix given in Section 5.1. This approach is illustrated in the following construction of all possible forms of A_a, depending upon which actuator is subject to faults. The approach is also applied for subsequent cases where A_s is constructed for the possible sensor faults.

$$A_{a_k} = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix}; \; C_{a_k} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

where when

$$k = \begin{cases} 0, & A_a = [\,], \\ 1, & A_a = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix} & [\,] \\ [\,] & [\,] \end{bmatrix}, \\ 2, & A_a = \begin{bmatrix} [\,] & [\,] \\ [\,] & \begin{bmatrix} 0 & 1 \\ -4\pi^2 & 0 \end{bmatrix} \end{bmatrix}, \\ 3, & A_a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4\pi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4\pi^2 & 0 \end{bmatrix}. \end{cases}$$

The notation "[ ]" indicates an empty matrix, so that these entries contribute null rows and columns to A_a and C_a. The construction of C_a is done in an identical manner for this example, and for subsequent examples, where the contents of each A_{a_k} and C_{a_k} reflect the particular fault being modeled.

Actuator-Only Faults

For the three actuator-only fault configurations, all fault cases are identifiable. In particular,

1. $l \geq m_k$ for all fault configurations, thus Condition (i) in Theorem 1 is automatically satisfied,

2. the pairs (C, A) and (C_a, A_a) are observable for all three configurations, thus Conditions (ii) and (iii) of Theorem 1 are satisfied,

3. Condition (iv) of Theorem 1 is satisfied automatically for the case when only one actuator fails at a time, since $(C, A, \bar{b}_i)$ has no invariant zeros for $i = 1, 2$. For the case when both actuators fail, $(C, A, \overline{B}^k)$ has two invariant zeros given by $\{-0.29 \pm i2.2\}$, however $\Lambda(A_a) = \{\pm 6.28318530717959\}$. Thus, Condition (iv) of Theorem 1 is also satisfied in the case of both actuators failing, since $\lambda_a \in \Lambda(A_a)$ is not an invariant zero of $(C, A, \overline{B}^k)$.

The preceding results show that all actuator-only oscillatory faults at $\omega = 2\pi$ rad$\cdot$s^{-1} are identifiable, as was the case in the fault-detection filter design in [21]. It can be verified that if $\omega = 0$ rad$\cdot$s^{-1} (that is, a step fault) then all actuator-only faults are still identifiable. The Conditions of Theorem 1 provide a useful means to test particular frequencies of interest, such as those known to be associated with the oscillatory failure case discussed in Section 1.1.

Sensor-Only Faults

For the given sensor suite, there are three possible fault configurations. That is,

1. either the horizontal velocity or vertical velocity measurements may be biased, or

2. both may be biased.

As discussed in Section 5.1, more cases may be possible if, for example, multiple step-fault amplitudes are considered. The results presented in this section show the theoretical identifiability of sensor step faults of an arbitrary amplitude, where each sensor may also subject to step faults of different amplitudes. A test of Conditions $(i)-(iii)$ of Theorem 2 shows that

1. the pairs (C, A) and (C_s, A_s) are observable, thus Conditions (i) and (ii) of Theorem 2 are satisfied,

2. for the case when either the horizontal velocity measurement or the vertical velocity measurement are biased, the pair (C_1, A) is observable and

3. for the case when both measurements are biased, the pair (C_1, A) is observable with respect to $\lambda_s \in \{0, 0\}$, that is, A has no zero eigenvalues, thus Condition (iii) of Theorem 2 is also satisfied.

Remark 20. Observe that for the step-fault case, the conditions in Corollary 4 may be used to test identifiability. However, the general, time-varying conditions of Theorem 2 are used here, to illustrate their application.

Simultaneous Actuator-Sensor Faults

For the case of simultaneous actuator-sensor faults, there are 10 possible configuration, including the no-fault case. Observe that $\Lambda(A_a) \cup \Lambda(A_s)\{\pm 2\pi, 0\}$. The possible configurations were tested against Conditions $(i)-(viii)$ of Theorem 3, and the results are presented subsequently:

1. there are five cases when $l < m_k + q$, however, Condition (i) of Theorem 3 is satisfied,

2. $l \geq m_k$, therefore Condition (ii) of Theorem 3 is satisfied,

3. the pairs $(C, A), (C_a, A_a)$, and (C_s, A_s) are all observable, therefore Conditions $(iii)-(v)$ of Theorem 3 are satisfied,

4. $\lambda_s = 0$ is not an eigenvalue of the pair (C_1, A) for any C_1, that is, $\operatorname{rank}\left\{\begin{bmatrix} -A^T & C_1^T \end{bmatrix}^T\right\} = 4$, therefore Condition (vi) of Theorem 3 is satisfied,

5. $\lambda_a = 0$ is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$ for any $\overline{B}^k$, therefore Condition (vii) of Theorem 3 is satisfied,

6. there are five cases when $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ has invariant zeros:

   (a) when the horizontal velocity measurement is biased and

      i. the collective pitch angle input has oscillations at 2π rad·s^{-1},

      ii. the longitudinal cyclic pitch angle input has oscillations at 2π rad·s^{-1},

   (b) when the vertical velocity measurement is biased and

      i. either or both inputs have oscillations at 2π rad·s^{-1}.

   (c) For all five cases, Condition $(viii)$ of Theorem 3 is satisfied since $\lambda_{a,s} = 0$, is not one of the invariant zeros of $\Sigma_{(C_1,A,\overline{B}^k)}(s)$.

Thus, when any combination of sinusoidal actuator faults (at $\omega = 2\pi$ rad·s^{-1}) in conjunction with sensor step faults (where there are 9 such faults and one no-fault case) all configurations are identifiable by the conditions of Theorem 3.

5.2.2 ACTUATOR RAMP FAULTS AND SENSOR STEP FAULTS

For the case when actuators are subject to ramp faults, and the sensors are subject to step faults, the following results can be verified using the conditions of (1) Theorem 1 for actuator-only faults, (2) Theorem 2 for sensor-only faults, and (3) Theorem 3 for simultaneous actuator-sensor faults.

Actuator-Only Faults

For any of the three combinations of actuator ramp faults in the absence of any sensor faults. all of the conditions of Theorem 1 are satisfied, and the faults are identifiable. In particular,

1. all cases of actuator-only ramp faults satisfy $l \geq m_k$, therefore Condition (i) of Theorem 1 is satisfied,

2. the pairs(C, A) and (C_a, A_a) are observable, therefore Conditions (ii) and (iii) of Theorem 1 are satisfied, respectively, and

3. only the case when both inputs are faulty generates invariant zeros in $\Sigma_{(C,A,\overline{B}^k)}(s)$, that is, the invariant zeros given by $\{-0.29 \pm i2.2\}$ do not intersect with the zero eigenvalue of A_a. Therefore, Condition (iv) of Theorem 1 is satisfied.

Simultaneous Actuator-Sensor Faults

For the 10 possible fault configurations, where one configuration is the no-fault case, a check of the conditions in Theorem 3 show that

1. for $\lambda_{a,s} \in \Lambda(A_a) \cap \Lambda(A_s) = \{0\}$ (where $\lambda_{a,s}$ is not an invariant zero of $(C_1, A, \overline{B}^k)$), there are five cases when $l < m_k + q$, and in all five cases, $C_a v \in \overline{\mathcal{G}}^{\varphi}_{m_k}(0)$. For each of these cases, since the geometric multiplicity of the zero eigenvalue of A_s is $n_s = q$, it is always the case that $C_2\zeta = \alpha C_s\psi$, where $\alpha \in \mathbb{C}$. Thus, Condition (i) of Theorem 3 is not satisfied for these five cases, and the faults are not identifiable,

2. $l \geq m$, therefore Condition (ii) of Theorem 3 is satisfied,

3. the pairs $(C, A), (C_a, A_a)$, and (C_s, A_s) are observable, therefore Conditions $(iii) - (v)$ of Theorem 3 are satisfied,

4. $\lambda_s = 0$ is not an eigenvalue of the pair (C_1, A), therefore Condition (vi) of Theorem 3 is satisfied,

5. λ_a is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$ for any $\overline{B}^k$, therefore Condition (vii) of Theorem 3 is satisfied, and

6. for the four cases when $\Sigma_{(C_1,A,\overline{B}^k)}(s)$ has invariant zeros, that is when only one or the other measurement is biased in conjunction with only one or the other input being faulty, none of the invariant zeros are at the origin. Thus for all four such fault cases, Condition $(viii)$ of Theorem 3 is satisfied.

5.3 CASE STUDY 2: A RESEARCH UAV

Consider the 6th-order linearized longitudinal dynamics for the Cranfield A3 Observer, a fixed-wing research UAV presented in [33]. As noted in [22], the UAV is in cruise condition, and the airframe is in a gust-insensitive configuration. The dynamics are given by

$$A = \begin{bmatrix} -0.146 & -0.016 & 0.557 & -9.809 & 0 & 0.001 \\ -0.63 & -4.487 & 34.57 & 0.161 & 0 & 0 \\ 0.001 & 0.039 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -0.016 & -1 & 0 & 35.2 & 0 & 0 \\ 665.7 & -6.89 & 0 & 0 & 0 & -8.57 \end{bmatrix}; \; B = \begin{bmatrix} 0 & -1.368 \\ 0 & -19.96 \\ 0 & -15.96 \\ 0 & 0 \\ 0 & 0 \\ 45910 & 0 \end{bmatrix},$$

and the output is specified as

$$C = \begin{bmatrix} 1 & -0.014 & 0.019 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.984 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The state vector consists of the forward speed, vertical speed, pitch rate, pitch angle, altitude, and engine rpm, that is, $x(t) = \begin{bmatrix} v & w & q & \theta & h & N_E \end{bmatrix}^T$. The control vector consists of engine thrust and elevator deflection, that is, $u(t) = \begin{bmatrix} u_T & u_e \end{bmatrix}^T$. The output vector consists of the measured speed error, pitch rate, pitch angle, perturbed altitude, and engine rpm, that is, $y(t) = \begin{bmatrix} v_e & q & \theta & h_e & N_E \end{bmatrix}^T$. It can be verified that $\Lambda(A) = \{0, -4.8345, 1.9641\cdot10^{-1}, -4.0534\cdot10^{-1} \pm i2.0428\cdot10^{-1}, -8.6482\}$ and that the system realization $\{A, B, C, 0\}$ is minimal, that is, the system is both controllable and observable.

5.3.1 ACUATOR OSCILLATORY FAULTS AND SENSOR STEP FAULTS

The faults considered in this section are oscillatory faults in the inputs, and step faults in the measurements. That is, each actuator and sensor are respecively subject to faults of the form

$$A_{a_k} = \begin{bmatrix} 0 & -1 \\ -\omega^2 & 0 \end{bmatrix}; \; C_{a_k} = \begin{bmatrix} 1 & 0 \end{bmatrix};$$

$$A_{s_k} = 0; \; C_{s_k} = 1,$$

where the composition of A_a and A_s from A_{a_k} and A_{s_k} follow from the approach presented in Section 5.2.1. Furthermore, for this case study, let $\omega = 4.835$ rad·s^{-1}, so that A_a and A have an eigenvalue in common.

Actuator-Only Faults

For any of the three combinations of actuator oscillatory faults (when $\omega = 4.835$ rad·s^{-1}), all of the conditions of Theorem 1 are satisfied, that is, in the case when either or both actuators are faulty in the absence of sensor bias,

1. for $\lambda_a = \{\pm 4.835\}$ is not an invariant zero of $\Sigma_{(C,A,\overline{B}^k)}(s)$, $l \geq m_k$, thus, Condition (i) of Theorem 1 is satisfied,

2. both pairs (C, A) and $(C_a. A_a)$ are observable, therefore Conditions (ii) and (iii) of Theorem 1 are satisfied, respectively, and

3. for the possible actuator fault configurations with ω fixed, $\Sigma_{(C,A,\overline{B}^k)}(s)$ has no invariant zeros, and Condition (iv) of Theorem 1 is satisfied.

Thus, all oscillatory, actuator-only faults for $\omega = 4.835$ rad·s^{-1} are identifiable.

Sensor-Only Faults

For the case of sensor-only faults in the form of step faults, there are 31 fault cases $(2^5 - 1)$. Of these, there are 16 cases of non-identifiability. In particular, a test against the conditions of Theorem 2 show that

1. in all 31 fault cases, the pairs (C, A) and (C_s, A_s) are observable, thus Conditions (i) and (ii) of Theorem 2 are satisfied. and

2. in the 16 cases of non-identifiability, $\lambda_s = 0$ with geometric multiplicity equal to $n_s = q$, but 0 is not an observable eigenvalue of the pair (C_1, A). Therefore, Condition (iii) of Theorem 2 is not satisfied, and the faults are not identifiable.

Simultaneous Actuator-Sensor Faults

For the case of simultaneous actuator-sensor faults, there are 93 possible configurations for the given faults (including the no fault configuration). Of these cases, it can be verified by checking the conditions of Theorem 3 that there are 48 cases of non-identifiability. The 48 non-identifiable fault configurations are all due to violation of Condition (vi) of Theorem 3. In particular,

1. $\Lambda(A_a) \cap \Lambda(A_s) = \emptyset$, thus Condition (i) and $(viii)$ of Theorem 3 are satisfied,

2. $l \geq m_k$, therefore Condition (ii) of Theorem 3 is satisfied,

3. the pairs (C, A), (C_a, A_a), and (C_s, A_s) are all observable, therefore, Conditions $(iii) - (v)$ of Theorem 3 are all satisfied,

4. the only eigenvalue of A_s is 0, which is not an observable eigenvalue of the pair (C_1, A). In each of these cases, $v = 0$ and $C_2\zeta = \alpha C_s\psi$, where $\alpha \in \mathbb{C}$, yet an actuator fault is present, thus these five cases represent non-identifiable fault configurations,

5. $\Sigma_{(C,A,\overline{B}^k)}(s)$ has no invariant zeros for any $\overline{B}^k$, therefore Condition (vii) of Theorem 3 is satisfied, and

6. $\Lambda(A_a) \cap \Lambda(A_s) = \emptyset$, therefore Condition $(viii)$ of Theorem 3 is satisfied.

CHAPTER 6

CONCLUSIONS AND FUTURE RESEARCH

This chapter summarizes the key results of the research. In addition, some directions for future research in the area of model-based fault detection and identification are considered.

6.1 FAULT IDENTIFIABILITY

The fundamental problem addressed in this thesis was the determination of a set of necessary and sufficient conditions for identifiability of additive, time-varying actuator and sensor fault configurations occurring as (1) actuator-only faults, (2) sensor-only faults, and (3) simultaneous actuator-sensor faults.

6.1.1 ACTUATOR-ONLY FAULTS

A necessary and sufficient condition for the identifiability of additive, time-varying actuator-only faults was given in Theorem 1. With the conditions of Theorem 1, a designer may determine whether or not particular actuator-only fault configurations will be identifiable under model-based FDI using state augmentation. One result of interest for the actuator-only fault configuration is in the case when there are more faulty inputs than there are sensors. In such cases, faults may still be identifiable, provided all of the conditions of Theorem 1 are satisfied. The situation is different when all of the faults are step faults, where it is required that there be more outputs than faulty actuators. This is the case discussed in Theorem 1 in [22], where the conditions were also derived in Section 2 in Chapter 4.

6.1.2 SENSOR-ONLY FAULTS

A necessary and sufficient condition for the identifiability of additive, time-varying sensor-only faults was given in Theorem 2. With the conditions of Theorem 2, a designer may determine whether or not particular sensor-only fault configurations will be identifiable using model-based state augmentation FDI. As discussed in Section 6.1.1, if all of the steps are step faults, the conditions for identifiability reduce to the conditions presented in Theorem 2 in [22], which were also derived in Section 4 in Chapter 4.

6.1.3 SIMULTANEOUS ACTUATOR-SENSOR FAULTS

A necessary and sufficient condition for the identifiability of additive, time-varying simultaneous actuator-sensor faults was given in Theorem 3. With the conditions of Theorem 3, a designer may determine whether or not particular simultaneous actuator-sensor fault configurations will be identifiable using model-based state augmentation FDI. It was observed that the special case when all simultaneous faults are step faults and $v, \psi = 0$, the conditions for fault identifiability reduce to those conditions given in Theorem 3 in [22]. These conditions were also derived in Section 6 of Chapter 4.

6.2 FUTURE RESEARCH

Immediate problems which need to be addressed include:

1. developing the conditions for identifiability for the multiplicative-only fault case (for example, loss-of-effectiveness faults),
2. extending the theorems of this thesis to include the possibility of simultaneous configurations of multiplicative-additive
    (a) actuator-only faults,
    (b) sensor-only faults, and
    (c) simultaneous actuator-sensor faults.

Additional research problems of interest include:

1. exploring the development of necessary and sufficient conditions for identifiability of actuators and sensors in certain nonlinear systems,
2. characterizing any frequency dependence of identifiability for period faults.

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Typeset using LaTeX.