Undetermined Coefficients: A Fully Generalized Approach

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Reducing Time-Complexity

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20 March 2021

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Motivation

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In the fields of physics, engineering, chemistry, and mathematics, we often encounter linear differential equations with constant coefficients which model the behavior of physical systems we wish to examine.

In particular, such differential equations often are non-homogeneous, requiring students and researchers alike to employ common solution techniques to accurately model the system for a given configuration.

For non-homogeneous equations of this form whose non-homogeneous function is exponential, sinusoidal, or polynomial in form, or a product of any of those three, the method of undetermined coefficients is a technique often taught as a solution method "by hand."

Motivation

In this presentation, I outline the development of a fully-generalized application of this method which does not require the reader to work with annihilator operators[1]or additional related ODEs[2], and only requires an understanding of summation notation and calculus. Additionally, this method provides a straightforward way to develop a program to implement the technique, and potentially reduces the time-complexity for solutions with comparisons to other methods.

In future work, I intend to create a program dedicated to this task and compare solution times with existing algorithms.

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[1]Dennis Zill. *A First Course in Differential Equations with Modeling Applications* (2017), 149-156

[2]Oswaldo Branco De Oliveira. "A formula substituting the undetermined coefficients and the annihilator methods" (2012)

Past Works

In prior articles[3] [4], authors have previously derived this method using differential operators and discussed the potential implications for reducing time-complexity of such solutions.

The present derivation was conducted without knowledge of the previous two articles. In the present case, I have emphasized the practical implications of varying each parameter (the order of the differential equation, the order of the non-homogeneous polynomial, and the multiplicity of the characteristic roots with the particular solution) on the **B** matrix, which allows for straightforward implementation for any interested student.

[3] Herman Gollwitzer. "Matrix Patterns and Undetermined Coefficients" (1994)

[4] John Zobitz. "Pascal Matrices and Particular Solutions to Differential Equations" (2003)

Introductory Concepts

Form of Differential Equation

We wish to examine solutions to a M-th order, non-homogeneous, linear differential equation with constant coefficients of the form,

$$p_M y^{(M)}(t) + p_{M-1} y^{(M-1)}(t) + p_{M-2} y^{M-2}(t) + \cdots \\ + p_3 y'''(t) + p_2 y''(t) + p_1 y'(t) + p_0 y(t) = g(t)$$

Where p_m are real-valued constants.

This can be expressed more compactly as,

$$\sum_{m=0}^{M} p_m y^{(m)}(t) = g(t)$$

For differential equations of this form, solutions can be comprised of the combination of a homogeneous and a particular solution[5],

$$y(t) = y_c(t) + Y_P(t)$$

Where $y_c(t)$ is the solution to the homogeneous equation,

$$\sum_{m=0}^{M} p_m y_c^{(m)}(t) = 0$$

And $Y_P(t)$ is the solution to the non-homogeneous component of the equation.

[5] Boyce and DiPrima. *Elementary Differential Equations and Boundary Value Problems* (2008), 175-185

Method of Undetermined Coefficients

One implementation of this solution technique is known as the *Method of Undetermined Coefficients* and is commonly taught to students in introductory Ordinary Differential Equations courses.

The Method of Undetermined Coefficients is suited to equations of this sort with a non-homogeneous component, $g(t)$, which is comprised of terms which either have a finite number of derivatives (e.g. polynomials) or whose successive derivatives are cyclic (e.g. exponentials and sinusoids).

In order to generalize this technique, we will need to develop the most generalized product of these three classes of functions and show that their successive derivatives also have a finite number of terms.

For any two exponential terms, $g_1(t) = c_1 e^{a_1 t}$ and $g_2(t) = c_2 e^{a_2}$, the product of these terms can be combined into a single exponential as,

$$g_1(t)g_2(t) = c_1 c_2 e^{a_1 t} e^{a_2 t} = c_1 c_2 e^{(a_1 + a_2)t} = c e^{at}$$

Where in the final step we combine $c_1 c_2 = c$ and $a_1 + a_2 = a$. Thus the product of any two general exponential terms can be treated as if they are a single exponential term.

Form of $g(t)$: Sinusoids

Using Euler's Formula[6],

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We can rewrite any sinusoidal term as either the real or imaginary component of a complex exponential,

$$\cos\theta = \mathrm{Re}\left(e^{i\theta}\right) \qquad \text{OR} \qquad \sin\theta = \mathrm{Im}\left(e^{i\theta}\right)$$

[6] Arfken, Weber, and Harris. *Mathematical Methods for Physicists* (2012), 41

Thus the product of any real exponential term and any sinusoid can be expressed as,

$$\begin{aligned} e^{at}\cos bt &= e^{at}\,\mathrm{Re}\left(e^{ibt}\right) \\ &= \mathrm{Re}\left(e^{at}e^{ibt}\right) \\ &= \mathrm{Re}\left(e^{(a+ib)t}\right) \\ &= \mathrm{Re}\left(e^{\alpha t}\right) \end{aligned} \qquad \begin{aligned} e^{at}\sin bt &= e^{at}\,\mathrm{Im}\left(e^{ibt}\right) \\ &= \mathrm{Im}\left(e^{at}e^{ibt}\right) \\ &= \mathrm{Im}\left(e^{(a+ib)t}\right) \\ &= \mathrm{Im}\left(e^{\alpha t}\right) \end{aligned}$$

Where we have defined the complex quantity: $\alpha = a + ib$.

Form of $g(t)$: All Terms

Taking advantage of the Leibniz Formula[7] for the generalized m-th order product rule,

$$(fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)} g^{(k)}$$

Where,

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$$

[7] Arfken, Weber, and Harris. *Mathematical Methods for Physicists* (2012), 113

We can demonstrate the m-th order derivative of the product of any complex exponential term and any n-th order polynomial is a finite sequence such that,

$$\frac{d^m}{dt^m}\left[e^{\alpha t}\left(\sum_{n=0}^{N} g_n t^n\right)\right] = \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!}\left(\frac{d^{m-k}}{dt^{m-k}}e^{\alpha t}\right)\left(\frac{d^k}{dt^k}\sum_{n=0}^{N} g_n t^n\right)$$

Form of $g(t)$: All Terms

Thus,

$$
\begin{aligned}
\frac{d^m}{dt^m}\left[e^{\alpha t}\left(\sum_{n=0}^{N} g_n t^n\right)\right] &= \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!}\left(\alpha^{m-k}e^{\alpha t}\right)\left(\sum_{n=k}^{N} g_n \frac{n!}{(n-k)!} t^{n-k}\right) \\
&= \sum_{k=0}^{m}\sum_{n=k}^{N} g_n \frac{m!\,n!}{k!\,(m-k)!\,(n-k)!}\alpha^{m-k}e^{\alpha t}t^{n-k}
\end{aligned}
$$

Which shows the m-th derivative of the product of any exponential, sinusoidal, and polynomial function is a finite sequence of terms.

Since this product is a finite sequence, we can use the method of undetermined coefficients to solve any non-homogeneous differential equation with a $g(t)$ of the above form.

Multiplicity of Solutions

In the study of the Method of Undetermined Coefficients, one requirement is that the assumed solutions for the particular component of the equation must be linearly independent from the terms in the homogeneous component.

If a term in the assumed solution for the particular component is not linearly independent, such a term cannot contribute to the particular solution.

The correction for the issue of non-linearly independent solutions is to multiply the offending term by a factor of t raised to the number of times that root appears in the characteristic equation, which is known as the *multiplicity*.

Multiplicity of Solutions

In order to account for the multiplicity, we first must solve for the roots of the corresponding characteristic equation and check whether our assumed solution includes terms which are linearly dependent on the corresponding homogeneous solution.

We then multiply our assumed solution for $Y_P(t)$ by a factor of t^ϵ, where ϵ is the multiplicity. Our assumed solution therefore takes the form,

$$Y_P = e^{\alpha t} \sum_{n=0}^{N} A_n t^{n+\epsilon}$$

The Solution

We begin with a differential equation of the form,

$$\sum_{m=0}^{M} p_m y^{(m)}(t) = e^{\alpha t} \sum_{n=0}^{N} g_n t^n$$

After solving the characteristic equation for its roots and checking $g(t)$ for any linearly dependent terms, we assume a particular solution of the form,

$$Y_P = e^{\alpha t} \sum_{n=0}^{N} A_n t^{n+\epsilon}$$

M-th Order: $Q(\alpha)$

We now introduce a function, $Q(\alpha)$, based on the characteristic equation,

$$Q(\alpha) = \sum_{m=0}^{M} p_m \alpha^m$$

We can express the k-th derivative,

$$\frac{d^k Q}{d\alpha^k} = Q_k = \sum_{m=k}^{M} p_m \frac{m!}{(m-k)!} \alpha^{m-k}$$

We can then group terms on the left-hand side by powers of t,

$$
\begin{array}{c|l}
\boldsymbol{t^{\epsilon}e^{\alpha t}} & A_0Q_0 + A_1(\epsilon+1)Q_1 + A_2\frac{(\epsilon+2)(\epsilon+1)}{2!}Q_2 + A_3\frac{(\epsilon+3)!}{3!((\epsilon+3)-3)!}Q_3 + \cdots \\
\boldsymbol{t^{\epsilon+1}e^{\alpha t}} & A_1Q_0 + A_2(\epsilon+2)Q_1 + A_3\frac{(\epsilon+3)(\epsilon+2)}{2!}Q_2 + A_4\frac{(\epsilon+4)!}{3!((\epsilon+4)-3)!}Q_3 + \cdots \\
\boldsymbol{t^{\epsilon+2}e^{\alpha t}} & A_2Q_0 + A_3(\epsilon+3)Q_1 + A_4\frac{(\epsilon+4)(\epsilon+3)}{2!}Q_2 + A_5\frac{(\epsilon+5)!}{3!((\epsilon+5)-3)!}Q_3 + \cdots \\
\vdots & \qquad\qquad\qquad\qquad\vdots
\end{array}
$$

M-th Order: Multiplicity and $Q(\alpha)$

Returning to our definitions, non-zero multiplicity implies that α is a root to the characteristic polynomial. If α is a root with multiplicity ϵ, then we expect the characteristic polynomial can be factored such that,

$$\sum_{m=0}^{M} p_m r^m = (r-\alpha)^\epsilon P_{M-\epsilon}(r)$$

Where $P_z(r)$ is a z-th order polynomial of variable r. We then expect for any k-th order derivative, all such derivatives evaluated at $r=\alpha$ will be zero while $k<\epsilon$. This further implies that all such Q_k will also be zero.

Comparing with the right hand side of the equation by each term, we can develop a system of equations depending on the value of the multiplicity.

M-th Order: $\epsilon = 0$

We begin by comparing terms on both sides of our equation by powers of t,

$$\begin{aligned}
g_0 &= A_0Q_0 + A_1Q_1 + A_2Q_2 + A_3Q_3 + \cdots \\
g_1 &= A_1Q_0 + 2A_2Q_1 + 3A_3Q_2 + 4A_4Q_3 + \cdots \\
g_2 &= A_2Q_0 + 3A_3Q_1 + 6A_4Q_2 + 10A_5Q_3 + \cdots \\
\vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}$$

Which can be represented by the matrix equation,

$$\mathbf{B_0}\vec{A} = \vec{g}$$

Where,

$$\vec{A} = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix} \qquad \text{and} \qquad \vec{g} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}$$

And $\mathbf{B_0}$ is,

$$\begin{bmatrix}
Q_0 & Q_1 & Q_2 & Q_3 & \cdots & Q_{N-3} & Q_{N-2} & Q_{N-1} & Q_N \\
0 & Q_0 & 2Q_1 & 3Q_2 & \cdots & (N-3)Q_{N-4} & (N-2)Q_{N-3} & (N-1)Q_{N-2} & NQ_{N-1} \\
0 & 0 & Q_0 & 3Q_1 & \cdots & \frac{(N-3)(N-4)}{2!}Q_{N-5} & \frac{(N-2)(N-3)}{2!}Q_{N-4} & \frac{(N-1)(N-2)}{2!}Q_{N-3} & \frac{N(N-1)}{2!}Q_{N-2} \\
0 & 0 & 0 & Q_0 & \cdots & \frac{(N-3)!}{3!(N-6)!}Q_{N-6} & \frac{(N-2)!}{3!(N-5)!}Q_{N-5} & \frac{(N-1)!}{3!(N-4)!}Q_{N-4} & \frac{N!}{3!(N-3)!}Q_{N-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & Q_0 & (N-2)Q_1 & \frac{(N-1)(N-2)}{2!}Q_2 & \frac{N!}{3!(N-3)!}Q_3 \\
0 & 0 & 0 & 0 & \cdots & 0 & Q_0 & (N-1)Q_1 & \frac{N(N-1)}{2!}Q_2 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & Q_0 & NQ_1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_0
\end{bmatrix}$$

Where the upper-right values assume large N and $M > N$.

M-th Order: $\epsilon = 0$

Which can be equivalently expressed in terms of the binomial coefficient as,

$$\begin{bmatrix} \binom{0}{0}Q_0 & \binom{1}{1}Q_1 & \binom{2}{2}Q_2 & \cdots & \binom{N-2}{N-2}Q_{N-2} & \binom{N-1}{N-1}Q_{N-1} & \binom{N}{N}Q_N \\ 0 & \binom{1}{0}Q_0 & \binom{2}{1}Q_1 & \cdots & \binom{N-2}{N-3}Q_{N-3} & \binom{N-1}{N-2}Q_{N-2} & \binom{N}{N-1}Q_{N-1} \\ 0 & 0 & \binom{2}{0}Q_0 & \cdots & \binom{N-2}{N-4}Q_{N-4} & \binom{N-1}{N-3}Q_{N-3} & \binom{N}{N-2}Q_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N-2}{0}Q_0 & \binom{N-1}{1}Q_1 & \binom{N}{2}Q_2 \\ 0 & 0 & 0 & \cdots & 0 & \binom{N-1}{0}Q_0 & \binom{N}{1}Q_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \binom{N}{0}Q_0 \end{bmatrix}$$

Where, again, the upper-right values assume large N and $M > N$.

In general, the Q_k terms truncate when $k > N$ or $k > M + \epsilon$, whichever comes first.

Using this formulation, we find the coefficient matrix, $\mathbf{B_0}$, takes the form of a upper-triangular Pascal matrix, whose diagonal terms are multiplied by successive values of Q_k until Q_k goes to zero.

As multiplicity increases, the columns of $\mathbf{B}_\epsilon$ shift leftward, and any terms below the main diagonal go to zero due to the multiplicity.

M-th Order: Conclusions

Taken together, we can generate a matrix for any given multiplicity as,

$$\begin{bmatrix} \binom{\epsilon}{\epsilon}Q_\epsilon & \binom{\epsilon+1}{\epsilon+1}Q_{\epsilon+1} & \binom{\epsilon+2}{\epsilon+2}Q_{\epsilon+2} & \cdots & \binom{\epsilon+N-2}{\epsilon+N-2}Q_{\epsilon+N-2} & \binom{\epsilon+N-1}{\epsilon+N-1}Q_{\epsilon+N-1} & \binom{\epsilon+N}{\epsilon+N}Q_{\epsilon+N} \\ 0 & \binom{\epsilon+1}{\epsilon}Q_\epsilon & \binom{\epsilon+2}{\epsilon+1}Q_{\epsilon+1} & \cdots & \binom{\epsilon+N-2}{\epsilon+N-3}Q_{\epsilon+N-3} & \binom{\epsilon+N-1}{\epsilon+N-2}Q_{\epsilon+N-2} & \binom{\epsilon+N}{\epsilon+N-1}Q_{\epsilon+N-1} \\ 0 & 0 & \binom{\epsilon+2}{\epsilon}Q_\epsilon & \cdots & \binom{\epsilon+N-2}{\epsilon+N-4}Q_{\epsilon+N-4} & \binom{\epsilon+N-1}{\epsilon+N-3}Q_{\epsilon+N-3} & \binom{\epsilon+N}{\epsilon+N-2}Q_{\epsilon+N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{\epsilon+N-2}{\epsilon}Q_\epsilon & \binom{\epsilon+N-1}{\epsilon+1}Q_{\epsilon+1} & \binom{\epsilon+N}{\epsilon+2}Q_{\epsilon+2} \\ 0 & 0 & 0 & \cdots & 0 & \binom{\epsilon+N-1}{\epsilon}Q_\epsilon & \binom{\epsilon+N}{\epsilon+1}Q_{\epsilon+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \binom{\epsilon+N}{\epsilon}Q_\epsilon \end{bmatrix}$$

Where, again, the upper-right values assume large N and $M > N$.

We can then populate the matrix (assuming indices start at 0) with the proper coefficients using the simplified notation,

$$A_{ij} = \begin{cases} \binom{\epsilon + j}{\epsilon + j - i} Q_{\epsilon+j-i} & \text{if } j \geq i \text{, and } \epsilon + j - i \leq M \\ 0 & \text{otherwise} \end{cases}$$

This notation allows for simple programming to implement this solution technique.

Example

Given the differential equation,

$$y''' - y'' - 8y' + 12y = e^{2t}(t^2 + 1)$$

We first examine the characteristic equation, which can be factored as,

$$(r - 2)^2(r + 3) = 0$$

Therefore giving the roots $r = \{2, -3\}$, of which $r = 2$ is a double root.

Since the non-homogeneous component has an exponential term with $\alpha = 2$, our particular solution will have a multiplicity of 2 with respect to the homogeneous solution.

We therefore propose a particular solution of the form,

$$Y_P = e^{2t} \sum_{n=0}^{3} A_n t^{n+2}$$

Again using the characteristic equation, we can find our Q_k values as,

$$\begin{aligned} Q_0 &= (2)^3 - (2)^2 - 8(2) + 12 = 0 \\ Q_1 &= 3(2)^2 - 2(2) - 8 = 0 \\ Q_2 &= 6(2) - 2 = 10 \\ Q_3 &= 6 \end{aligned}$$

Remembering to generate the 4×4 upper triangular Pascal matrix and shift it left two columns, we compose the matrix equation,

$$\begin{bmatrix} Q_2 & Q_3 & 0 & 0 \\ 0 & 3Q_2 & 4Q_3 & 0 \\ 0 & 0 & 6Q_2 & 10Q_3 \\ 0 & 0 & 0 & 10Q_2 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Which yields the solution,

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 27/250 \\ -1/75 \\ 1/60 \\ 0 \end{bmatrix}$$

Example

Therefore, we can provide the full general solution to this differential equation as,

$$y = C_1 e^{-3t} + C_2 e^{2t} + C_3 t e^{2t} + e^{2t}\left(\frac{27}{250}t^2 - \frac{1}{75}t^3 + \frac{1}{60}t^4\right)$$

Where the constants C_1, C_2 and C_3 are fixed by initial conditions.

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