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Mellin representation of the graviton bulk-to-bulk propagator in AdS space

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The correlation functions of the conformal $\mathcal{N} = 4$ super Yang-Mills theory at large coupling constant are reduced via AdS/CFT correspondence [1–3] to Witten diagrams in AdS space. A powerful method to calculate Witten diagrams in AdS space is to represent bulk-to-bulk propagators, calculate the tree-level “star” integrals with vertices over the AdS space, and then convert the remaining integrals over the flat space into Mellin transforms of the conformal ratios using Symanzik’s star formula [4] (see the discussion in Ref. [5]). For the scalar propagator with mass $m^2 = (\Delta - d)\Delta$ the Mellin representation of a bulk-to-bulk propagator has the form [5,6]

$$\Pi^d_{\Lambda}(x,y) = -\frac{i\Gamma(d/2)}{4\pi^{(d+1)/2}} \int_{-\infty}^\infty \frac{d\lambda}{\lambda} \left(\Gamma\left(\frac{d}{2}, -\lambda\right) - \Gamma\left(\frac{d}{2}, -\lambda - \frac{r}{1-r}\right)\right) \times \int d^d z \frac{\Gamma\left(\frac{d}{2} + \lambda\right)\Gamma\left(\frac{d}{2} - \lambda\right)}{\Gamma\left(\frac{d}{2}, \lambda\right)\Gamma\left(\frac{d}{2}, -\lambda\right)} f_{\Lambda}(u),$$

(1)

Here we used Poincaré coordinates $x = (x^0, x^i)$, where $\tilde{x}$ is a $d$-dimensional Euclidean vector (our metric is $dx^2 = \frac{1}{\Delta^d}[(dx^0)^2 + dx^2]$), with the size of AdS space $R = 1$.

The above equation looks like the integral of the product of two bulk-to-boundary propagators with unphysical complex masses $m = \pm i\sqrt{\Delta - \lambda^2}$ over the usual flat space and over $\lambda$. The easiest way to prove this formula is to calculate explicitly the integral over $z$ in the right-hand side (r.h.s.) of Eq. (1). One obtains (cf. Ref. [7])

$$\Pi^d_{\Lambda}(x,y) = \frac{1}{\Delta^{(d+1)/2}} \left(\Gamma\left(\frac{d}{2} + \lambda\right)\Gamma\left(\frac{d}{2} - \lambda\right)\Gamma\left(\frac{d}{2}, \lambda\right)\Gamma\left(\frac{d}{2}, -\lambda\right)\right)$$

(2)

where $|x - z|^2 = (x^0)^2 + (\tilde{x} - \tilde{z})^2$,

$$u(x,y) = \frac{(x^0 - y^0)^2 + (\tilde{x} - \tilde{y})^2}{2\lambda^d y^0}$$

is the chordal distance between points $x$ and $y$, and

$$f_{\Lambda}(u) = \frac{\Gamma\left(\frac{d}{2} + \lambda\right)}{\Gamma\left(\frac{d}{2}, \lambda\right)} (1 - r)^{-d/2} \frac{\Gamma\left(\frac{d}{2} + \lambda\right)}{\Gamma(\lambda)} \times F\left(\frac{d}{2}, 1 - d/2, 1 + \lambda, -\frac{r}{1-r}\right).$$

(3)

Here $F$ is the hypergeometric function $_2F_1$ and the variable $r(u)$ is defined as

$$r(u) = \frac{1 + u - \sqrt{u(2 + u)}}{1 + u + \sqrt{u(2 + u)}}.$$

(4)

Substituting the integral (2) into Eq. (1) we get

$$\Pi^d_{\Lambda}(u) = -\frac{i}{2\pi^{(d+1)/2}} \int_{-\infty}^\infty \frac{d\lambda}{\lambda} \left(\Gamma\left(\frac{d}{2}, -\lambda\right)\right)$$

(5)

Since $r < 1$ and the function

$$F\left(\frac{d}{2}, 1 - d/2, 1 + \lambda, -\frac{r}{1-r}\right) = \Gamma(1 + \lambda)\Gamma^{-1}(\lambda)$$

(6)

is regular in the right half-plane and behaves like $\lambda^{(d/2)-1}$ as $\text{Re}\lambda \to \infty$, one can close the contour over $\lambda$ in Eq. (5) in the right semiplane and get the result as a residue at $\lambda = \Delta - \frac{d}{2}$,

$$\Pi^d_{\Lambda}(u) = \frac{f_{\Lambda}(u)}{\pi^{d/2}(2 \Delta - d)} = \frac{x^{-(d/2)}\Gamma(\Delta)}{2\Gamma(\Delta - d/2 + 1)} F\left(\frac{d}{2}, 1 - d/2, \Delta - d/2, 1, -\frac{r}{1-r}\right).$$

(7)

It is easy to see that the r.h.s. of Eq. (7) is equal to the bulk-to-bulk scalar propagator [8].

As we mentioned above, the formula (1) is extremely convenient for the calculation of Witten diagrams in the Mellin representation, so it would be advantageous to get a similar expression for the bulk-to-bulk graviton propagator. This propagator can be represented as [9]
\[ G^{\alpha\beta;\mu\nu}(x, y) = (\partial^\alpha \partial^\mu u \partial^\beta \partial^\nu u + \alpha \leftrightarrow \beta) \Pi_\delta^d(u) + g_{\alpha\beta} g^{\mu\nu} H(u) + \{(D^\alpha[\partial^\beta \partial^\mu u \partial^\nu u X(u)] + D^\alpha[\partial^\beta \partial^\mu u \partial^\nu u Y(u)] + \alpha \leftrightarrow \beta) + D^\alpha[\partial^\beta Z(u)] g^{\mu\nu} + (\alpha \leftrightarrow \mu, \beta \leftrightarrow \nu)\}, \]

where \( D_\mu \) is a covariant derivative and

\[ H(u) = -\frac{2}{d-1} \left[ (1 + u)^2 \Pi_\delta^d(u) - (d - 2)(1 + u) \int_u^\infty du' \Pi_\delta^d(u') \right] \]

The remaining three functions \( X(u), Y(u), \) and \( Z(u) \) are gauge artifacts. Hereafter the Greek indices from the first half of the alphabet refer to the point \( x \) and from the second to \( y \).

The Mellin representation of the graviton propagator has the form

\[ G^{\alpha\beta;\mu\nu}(x, y) = \frac{i \Gamma(d/2)}{2(d-1) \pi^{d/2}} \int_{-\infty}^{\infty} d\lambda \frac{\Gamma(d/2 + \lambda + 2) \Gamma(d/2)}{\pi^{d/2} (|x - z|^2)^{d/2 + \lambda}} \times \frac{(y^0)^{d/2 - \lambda + 2} \Gamma(d/2)}{(|y - z|^2)^{(d/2) - \lambda}} \]

\[ J^{ai}(x - y) J^{bj}(x - y) E_{ij,kl} \delta_{\mu\nu} \]

where

\[ J^{\mu i}(x - z) = \delta^{\mu i} - 2 \frac{(x - z)^{\mu i}(x - z)^i}{|x - z|^2} \]

(11)

(12)

The Mellin integrand in the formula (10) to the bulk-to-boundary propagator of the graviton. The general solution of the Dirichlet problem with the boundary data \( \hat{h}_{\ab} \) has the form (10)

\[ \hat{h}_{\beta}^a(x) = \frac{(d + 1) \Gamma(d)}{(d - 1) \Gamma(d/2)} \int \frac{d^dz}{\pi^{d/2}} \frac{(y^0)^d}{(|x - z|^2)^{d/2} + \lambda} J^{ai}(x - y) \]

\[ J^{bj}(x - z) E_{ij,kl} \delta_{\mu\nu} \hat{h}_{ab}. \]

We see that, similarly to the scalar case, the Mellin representation (10) looks like an integral of the product of two bulk-to-boundary propagators with unphysical complex graviton masses \( m = \pm i \sqrt{\frac{d^2}{4} - \lambda^2} \) over the usual flat space and over \( \lambda \).

Now let us prove Eq. (10). The central point of the proof is the calculation of the following integral:

\[ I^{\alpha\beta;\mu\nu}(x, y, \lambda) = \frac{1}{d^2} \left[ \Gamma(d/2) \right] \frac{\Gamma(d/2 + \lambda - 2) \Gamma(d/2)}{\Gamma(d/2 - \lambda) \Gamma(\lambda - 1) \Gamma(\lambda - 1)} \frac{1}{|x - z|^2} \frac{1}{|y - z|^2} \frac{1}{|y - z|^2 - \lambda^2} \]

\[ J^{ai}(x - y) J^{bj}(x - y) E_{ij,kl} \delta_{\mu\nu} \]

(15)

It can be decomposed in the same set of structures as the propagator (8),

\[ I^{\alpha\beta;\mu\nu}(x, y, \lambda) = (\partial^\alpha \partial^\mu u \partial^\beta \partial^\nu u + \alpha \leftrightarrow \beta) G_\lambda(u) + g_{\alpha\beta} g^{\mu\nu} H_\lambda(u) + \{(D^\alpha[\partial^\beta \partial^\mu u \partial^\nu u X_\lambda(u)] + D^\alpha[\partial^\beta \partial^\mu u \partial^\nu u Y_\lambda(u)] + \alpha \leftrightarrow \beta) + D^\alpha[\partial^\beta Z_\lambda(u)] g^{\mu\nu} + (\alpha \leftrightarrow \mu, \beta \leftrightarrow \mu)\} + D^\alpha[\partial^\beta Z_\lambda(u)] g^{\mu\nu} + D^\alpha[\partial^\beta Z_{\lambda - \lambda}(u)] g^{\nu\beta}. \]

A straightforward but somewhat lengthy calculation yields (cf. Ref. [7])

\[ G_\lambda(u) = \left[ \left( \frac{d}{2} - 1 \right)^2 - \lambda^2 \right] f_\lambda(u) + (\lambda \leftrightarrow -\lambda), \]

\[ H_\lambda(u) = 2(1 + u)^2 f_\lambda(u) - \frac{2}{d} \left[ \frac{d^2}{4} - \lambda^2 \right] f_\lambda(u) + 2(d - 2)(1 + u) F_\lambda(u) + (\lambda \leftrightarrow -\lambda). \]

(17)

for the two physical structures and
\[ \frac{d^2}{4} - \lambda^2 \] \[ X_{\lambda}(u) = \left[ (1 + u)^2 - \frac{1}{d} \right] f''_\lambda(u) + \left[ \left( \frac{d}{2} + 1 \right)^2 - \lambda^2 \right] (1 + u) f'_\lambda(u) + d \left( \frac{d^2}{4} - \lambda^2 \right) f_\lambda(u) + (\lambda \leftrightarrow -\lambda), \]
\[ \frac{d^2}{4} - \lambda^2 \] \[ Y_{\lambda}(u) = \left[ (1 + u)^2 - \frac{1}{d} \right] f'''_\lambda(u) + (d + 1)(1 + u) f''_\lambda(u) + \frac{d(d + 1)}{2} f'_\lambda(u) + (\lambda \leftrightarrow -\lambda), \]
\[ \frac{d^2}{4} - \lambda^2 \] \[ Z_{\lambda}(u) = \left[ (1 + u)^3 - \frac{1}{d} \right] f'''_\lambda(u) + f'''_\lambda(u) + \left[ (1 + d - 2\lambda)(1 + u) + \left( \frac{d}{2} + \frac{2}{d} \lambda^2 - \frac{1}{d} \right) \right] f'_\lambda(u) + f''_\lambda(u) \]
\[ + 2(d - 1)(1 + u) f'_\lambda(u) + f''_\lambda(u) + \left[ (2(d - 1)\lambda + (2 - d - 2\lambda) \left( \frac{d^2}{4} - \lambda^2 \right) \right] [F_{\lambda}(u) + F_{-\lambda}(u)] \quad (18) \]

for three gauge-dependent ones. Here
\[ F_\lambda(u) = - \int^\infty_u f_\lambda(u) dv = - \frac{\Gamma\left( \frac{d}{2} + \lambda \right)}{\Gamma(d/2)} \frac{\Gamma\left( (d - 2)/4 \right)(1 - r)^{1 - (d/2)}}{(d - 2 + 2\lambda)} F\left( \frac{d}{2} - 1, 2 - \frac{d}{2}, 1 + \lambda, -\frac{r}{1 - r} \right). \quad (19) \]

One can easily see that the function \( F\left( \frac{d}{2} - 1, 2 - \frac{d}{2}, 1 + \lambda, -\frac{r}{1 - r} \right) \) is also regular at the right half-plane and behaves like \( \lambda^{(d/2) - 1} \) as \( \text{Re}\lambda \to \infty \); cf. Eq. (6).

Let us now return to the proof of Eq. (10), which can be rewritten as
\[ G^{\alpha \beta; \mu \nu}(x, y, \lambda) = \frac{i(d - 1)^{-1} - 1}{4\pi^{(d/2) + 1}} \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} I^{\alpha \beta; \mu \nu}(x, y, \lambda). \quad (20) \]

Let us discuss the two gauge-invariant structures \( G(u) \) and \( H(u) \). The corresponding terms in the r.h.s of Eq. (20) are
\[ (\partial^\alpha \partial^\beta \partial^\mu \partial^\nu u + \alpha \leftrightarrow \beta) \frac{i(d - 1)^{-1} - 1}{4\pi^{(d/2) + 1}} \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} G(u) + g^{\alpha \beta} g^{\mu \nu} \frac{i(d - 1)^{-1} - 1}{2\pi^{(d/2) + 1}} \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} H(u) \]
\[ = (\partial^\alpha \partial^\beta \partial^\mu \partial^\nu u + \alpha \leftrightarrow \beta) \frac{i(d - 1)^{-1} - 1}{2\pi^{(d/2) + 1}} \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} \left[ \frac{d}{2} - 1 \right]^2 - \lambda^2 \] \[ f'_\lambda(u) + f''_\lambda(u) + \left( \frac{d}{2} + \frac{2}{d} \lambda^2 - \frac{1}{d} \right) \right] f'_\lambda(u) + f''_\lambda(u) \]
\[ \times \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} \left[ 2(1 + u)^2 f'_\lambda(u) - \frac{2}{d} \left( \frac{d^2}{4} - \lambda^2 \right) f'_\lambda(u) + 2(d - 2)(1 + u) F_{\lambda}(u) \right] \quad (21) \]

As we discussed above [see Eqs. (3), (6), and (19)], the functions \( f'_\lambda(u) \) and \( f''_\lambda(u) \) are regular in the right half-plane and decrease as \( \lambda^{(d/2) - 1} \) as \( \text{Re}\lambda \to \infty \), so one can close the contour over \( \lambda \) and take the residue at \( \lambda = \frac{d}{2} \). One obtains
\[ G^{\alpha \beta; \mu \nu}(x, y) = \frac{f_{\lambda}(d/2)}{d\pi^{(d/2) + 1}} (\partial^\alpha \partial^\mu \partial^\beta \partial^\nu u + \alpha \leftrightarrow \beta) - \frac{2}{(d - 1)\pi^{(d/2) + 1}} \left[ (1 + u)^2 f_{\lambda}(d/2)(u) + (d - 2)(1 + u) F_{\lambda}(d/2)(u) \right] g^{\alpha \beta} g^{\mu \nu} \]
\[ + \text{gauge-dependent structures} \quad (22) \]

which coincides with Eqs. (8) and (9) since \( \Pi_{\lambda}(d/2) = \frac{1}{d\pi^{(d/2) + 1}} f_{\lambda}(d/2)(u) \). Thus, we proved that the integral (10) can serve as a graviton bulk-to-bulk propagator in the gauge \( D_{\alpha} G^{\alpha \beta; \mu \nu} = 0 \). It should be mentioned that a similar but somewhat more complicated representation of the graviton propagator was obtained in Ref. [6]. It has a function \( \frac{1}{d\pi^{(d/2) + 1}} \) in place of \( \frac{1}{d\pi^{(d/2) + 1}} \) in Eq. (10), as well as additional terms proportional to the tensor structure obtained from that of Eq. (10) by the replacement \( E_{ijkl} \to \delta_{ij} \delta_{kl} \) and terms proportional to the \( g^{\mu \nu} g^{\alpha \beta} \) structure.

For completeness, let us briefly discuss the gauge boson propagator [11]
\[ G_{\alpha \mu}(x, y) = g^{\alpha \mu} \partial^\nu S(u), \quad (23) \]

where the second structure depends on the choice of gauge.

The Mellin representation of this propagator has the form [6]
\[ G^{\alpha \mu}(x, y) = \frac{i\Gamma(d/2)}{4\pi^{(d/2) + 1}} \int_{-\infty}^{\infty} \frac{d\lambda}{(d/2)^2 - \lambda^2} \]
\[ \times \int_{-\infty}^{\infty} \frac{d\lambda}{\pi^{(d/2) + 1}} \frac{(d/2)^2 - \lambda^2}{\Gamma(d/2 + 2\lambda) + 2\Gamma(d/2 + \lambda)} \]
\[ \times \int_{-\infty}^{\infty} \frac{d\lambda}{\pi^{(d/2) + 1}} \frac{(d/2)^2 - \lambda^2}{\Gamma(-\lambda)(1 + \lambda) + 2\Gamma(d/2 + \lambda)} \]
\[ \times \frac{\Gamma(-\lambda)(1 + \lambda) + 2\Gamma(d/2 + \lambda)}{\Gamma(-\lambda)(1 + \lambda) + 2\Gamma(d/2 + \lambda)} \]
\[ \times J^{\alpha \mu}(x - z) \delta_{jk} J^{k\mu}(z - y). \quad (24) \]
the solution of the Einstein equations. Again, the gauge condition for the propagator (24) is \( D_{\alpha}G^{\alpha\beta}(x, y) = 0 \).

We have represented the graviton bulk-to-bulk propagator in the form of the Mellin integral of the product of bulk-to-boundary propagators (with nonphysical masses). This formula permits us to apply the Mellin-transformation method of Ref. [5] to Witten diagrams with graviton (and gauge boson) propagators.

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