Photon Impact Factor and $k_T$ Factorization for DIS in the Next-to-Leading Order

Ian Balitsky
Old Dominion University, ibalitsk@odu.edu

Giovanni A. Chirilli

Follow this and additional works at: https://digitalcommons.odu.edu/physics_fac_pubs

Part of the Elementary Particles and Fields and String Theory Commons, and the Quantum Physics Commons

Original Publication Citation

This Article is brought to you for free and open access by the Physics at ODU Digital Commons. It has been accepted for inclusion in Physics Faculty Publications by an authorized administrator of ODU Digital Commons. For more information, please contact digitalcommons@odu.edu.
Photon impact factor and $k_T$ factorization for DIS in the next-to-leading order

Ian Balitsky*
Physics Department, Old Dominion University, Norfolk, Virginia 23529, and Theory Group, Jlab, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

Giovanni A. Chirilli
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA

(Received 19 July 2012; published 15 January 2013)

The photon impact factor for the Balitsky-Fadin-Kuraev-Lipatov pomeron is calculated in the next-to-leading order approximation using the operator expansion in Wilson lines. The result is represented as a next-to-leading order $k_T$-factorization formula for the structure functions of small-$x$ deep inelastic scattering.

DOI: 10.1103/PhysRevD.87.014013 PACS numbers: 12.38.Bx, 12.38.Cy

I. INTRODUCTION

It is well known that the small-$x$ behavior of structure functions of deep inelastic scattering (DIS) is determined by the hard pomeron contribution. In the leading order, the pomeron intercept is determined by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [1] and the pomeron residue (the $\gamma^*\gamma^*$-pomeron vertex) is given by the so-called impact factor. To find the small-$x$ structure functions in the next-to-leading order (NLO), one needs to know both the pomeron intercept and the impact factor. The NLO pomeron intercept was found many years ago [2] but the analytic expression for the NLO impact factor is obtained for the first time in the present paper.

We calculate the NLO impact factor using the high-energy operator expansion of the $T$ product of two vector currents in Wilson lines (see, e.g., the reviews [3,4]). Let us recall the general logic of an operator expansion. In order to find a certain asymptotical behavior of an amplitude by operator product expansion (OPE) one should

(i) identify the relevant operators and factorize an amplitude into a product of coefficient functions and matrix elements of these operators,
(ii) find the evolution equations of the operators with respect to the factorization scale,
(iii) solve these evolution equations,
(iv) convolute the solution with the initial conditions for the evolution and get the amplitude.

Since we are interested in the small-$x$ asymptotics of DIS it is natural to factorize in rapidity: we introduce the rapidity divide $\eta$ which separates the “fast” gluons from the “slow” ones.

As a first step, we integrate over gluons with rapidities $Y > \eta$ and leave the integration over $Y < \eta$ for the later time; see Fig. 1.

It is convenient to use the background field formalism: we integrate over gluons with $\alpha > \sigma = e^{\eta}$ and leave gluons with $\alpha < \sigma$ as a background field, to be integrated over later. Since the rapidities of the background gluons are very different from the rapidities of gluons in our Feynman diagrams, the background field can be taken in the form of a shock wave due to the Lorentz contraction. To derive the expression of a quark (or gluon) propagator in this shock-wave background, we represent the propagator as a path integral over various trajectories, each of them weighted with the gauge factor $P \exp(ig \int dx_{\mu} A^\mu)$ ordered along the propagation path. Now, since the shock wave is very thin, quarks (or gluons) do not have time to deviate in a transverse direction so their trajectory inside the shock wave can be approximated by a segment of the straight line. Moreover, since there is no external field outside the shock

---

*balitsky@jlab.org

FIG. 1 (color online). Rapidity factorization. The impact factors with $Y > \eta$ are given by diagrams in the shock-wave background. Wilson-line operators with $Y < \eta$ are denoted by dotted lines.
As usual, we label operators by hats and \( p/C22 \) (sometimes called “color dipole”). The LO impact factor line gauge factor

\[
\begin{align*}
\langle T \{ \hat{\psi}(x) \hat{\psi}(y) \} \rangle_A & = - \int d^d z \delta(z_s) \frac{(\hat{f} - \hat{f})}{2\pi^2(x - z)^2} \hat{p}_2 \\
& \quad \times U_z \frac{(\hat{f} - \hat{f})}{2\pi^2(x - z)^2}.
\end{align*}
\]

As usual, we label operators by hats and \( \langle \hat{O} \rangle_A \) means the vacuum average of the operator \( \hat{O} \) in the presence of an external field \( A \). Hereafter, we use the notations \( x_* = \frac{\rho^2}{2} - \frac{x^*}{2} \) and \( \frac{x_1}{x_2} = \frac{x}{x} \) [and our metric is \((1, -1, -1, -1)\)]. Note that the Regge limit in the coordinate space can be achieved by rescaling

\[
\begin{align*}
x & \rightarrow \rho x \frac{2}{s} p_1 + x \frac{2}{s\rho} p_2 + x_\perp, \\
y & \rightarrow \rho y \frac{2}{s} p_1 + y \frac{2}{s\rho} p_2 + y_\perp.
\end{align*}
\]

with \( \rho \rightarrow \infty \); see the discussion in Refs. [6,7].

The result of the integration over gluons with rapidities \( Y > \eta \) gives the impact factor—the amplitude of the transition of virtual photon in two-Wilson-lines operators (sometimes called “color dipole”). The LO impact factor is a product of two propagators (2) (see Fig. 3)

\[
\begin{align*}
\langle T \{ \hat{\psi}(x) \gamma^\mu \hat{\psi}(x) \gamma^\nu \hat{\psi}(y) \gamma^\nu \hat{\psi}(y) \} \rangle_A & = \frac{s^2}{2^n \pi^{n+1} x y z} \int d^d z_{1\perp} d^d z_{2\perp} \frac{\text{tr}(U_z \hat{U}^\dagger)}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)^3} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \\
& \times [2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) - \kappa^2(\zeta_1 \cdot \zeta_2)] + O(\alpha_s).
\end{align*}
\]

where the Sudakov variable \( \alpha_x \) is defined as usual, \( k = \alpha_x p_1 + \beta p_2 + k_\perp \). We define the lightlike vectors \( p_1 \) and \( p_2 \) such that \( q = p_1 - x_B p_2 \) and \( p = p_2 + \frac{\omega}{m} p_1 \), where \( q \) is the virtual photon momentum, \( p \) is the momentum of the target particle, and \( x_B = Q^2/s \ll 1 \) is the Bjorken variable (at large energies \( s \approx 2 p \cdot q \)). The structure of the propagator in a shock-wave background looks as follows (see Fig. 2):

\[ \text{Free propagation from initial point } x \text{ to the point of intersection with the shock wave } z \]
\[ \times \text{Interaction with the shock wave described by the Wilson-line operator } U_z \]
\[ \times \text{Free propagation from point of interaction } z \text{ to the final point } y. \]

Here we introduced the conformal vectors [8,9]:

\[
\begin{align*}
\kappa & = \kappa_x - \kappa_y, \\
\kappa_x & = \frac{\sqrt{x}}{2x} \left( \frac{p_1}{x - x^2 p_2 + x_\perp} \right), \\
\zeta_i & = \left( \frac{p_1}{s} + z_i p_2 + z_\perp \right),
\end{align*}
\]

and the notation \( R = \frac{\kappa^2(\zeta_1 \cdot \zeta_2)}{2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \). The above equation is explicitly Möbius invariant. In addition, it is easy to check that \( \frac{\partial}{\partial z} (\text{r.h.s}) = 0 \).

Our goal is the NLO contribution to the right-hand side of Eq. (4), but first let us briefly discuss the three remaining steps of the high-energy OPE. The evolution equation for color dipoles has the form [5,10]

\[
\frac{d}{d\eta} \text{tr}(U_z \hat{U}^\dagger) = \frac{\alpha_x}{2\pi^2} \int d^d z_{1\perp} \frac{z_{2\perp}^2}{z_{12}^2 z_{23}^2} \\
\times \left[ \text{tr}(U_z \hat{U}^\dagger) \text{tr}(U_z \hat{U}^\dagger) \right] + \text{NLO contribution}.
\]

(To save space, hereafter \( z_i \) stands for \( z_i \perp \) so \( \frac{z_{2\perp}}{z_{12}} \equiv \frac{z_{2\perp}}{z_{12\perp}} \).) The explicit form of the NLO contributions can be found in Refs. [4,11,12] while the argument of the coupling constant in the above equation (following from the NLO calculations) is discussed in Refs. [13,14].

FIG. 2 (color online). Propagator in a shock-wave background.
It is worth noting that we performed the OPE program outlined above for the scattering of scalar “particles” in $\mathcal{N} = 4$ super Yang-Mills and obtained the explicit expression for the four-point correlator of scalar operators at high energies in the next-to-leading order [7]. In QCD, the analytic solution of the evolution equation for color dipoles with a running-coupling constant is not known at present. This prevents us from getting the explicit NLO amplitude as in the $\mathcal{N} = 4$ case. We can, however, perform the first two steps in our OPE program discussed in the Introduction: calculate the coefficient function (impact factor) and find the evolution equation for color dipoles. The next two steps, solution of the evolution equation (6) with appropriate initial conditions and the eventual comparison with experimental DIS data are discussed in many papers (see, e.g., [15]). It is worth noting that, contrary to the comparison with experimental DIS data are discussed in many papers (see, e.g., [15]), the argument of the coupling constant at the next-to-leading order level. Thus, the argument of the coupling constant at the NLO level is determined solely by the evolution equation for color dipoles. For numerical estimates involving the impact factor, one can take $\alpha_s (|x - y|)$ as the first approximation since the characteristic transverse distances in the impact factor are $\sim |x - y|$.

The paper is organized as follows: in Secs. II and III we calculate the NLO impact factor in the coordinate representation (the results of these sections were published previously in a Brief Report [16]). The Mellin representation of the impact factor is presented in Sec. IV and V contains the impact factor in the momentum representation for the forward case corresponding to deep inelastic scattering. Finally, we present the NLO BFKL kernel and discuss the $k_T$ factorization for DIS in Sec. VI.

II. CALCULATION OF THE NLO IMPACT FACTOR

Now we would like to repeat the steps of operator expansion discussed above to the NLO accuracy. A general form of the expansion of the $T$ product of the electromagnetic currents in color dipoles looks as follows:

$$
(x - y)^4 T \{ \tilde{\mathcal{J}}(x) \gamma^\mu \tilde{\mathcal{J}}(y) \}
= \int d^2 z_1 d^2 z_2 \left\{ \frac{1}{x^2} + \frac{\alpha_s}{\pi} \right\} \text{tr}[\hat{U}_{z_1}^\eta \hat{U}_{z_2}^\eta]
+ \int d^2 z_3 \mathcal{I}_{\mu LO}^N (z_1, z_2, z_3, \eta)
\times \left[ \text{tr}[\hat{U}_{z_1}^\eta \hat{U}_{z_2}^\eta] \text{tr}[\hat{U}_{z_3}^\eta \hat{U}_{z_2}^\eta] - N_c \text{tr}[\hat{U}_{z_2}^\eta \hat{U}_{z_2}^\eta] \right].
$$

Unfortunately, in terms of the Wilson-line approach, there is no direct way to get the NLO impact factor for the BFKL pomeron. One needs first to find the coefficient in front of the four-Wilson-line operator (which we will also call the NLO impact factor) and then linearize it.

The structure of the NLO contribution is clear from the topology of diagrams in the shock-wave background; see Fig. 4 below. Also, the term $\sim 1 + \frac{\alpha_s}{\pi}$ can be restored from the requirement that at $U = 1$ (no shock wave) one should get the perturbative series for the polarization operator $1 + \frac{\alpha_s}{\pi} + O(\alpha_s^3)$.

In our notations

$$
\mathcal{I}_{\mu LO}^N (z_1, z_2) = \frac{2 \mathcal{R}^2}{\pi^2 (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[ (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)
- \frac{1}{2} \kappa^2 (\zeta_1 \cdot \zeta_2) \right],
$$

which corresponds to the well-known expression for the LO impact factor in the momentum space.

The NLO impact factor is given by the diagrams shown in Fig. 4. The calculation of these diagrams is similar to the calculation of the NLO impact factor for scalar currents in the $\mathcal{N} = 4$ SYM carried out in our previous paper [12]. The gluon propagator in the shock-wave background at $x_0 > 0 > y_0$ in the lightlike gauge $p^{\mu}_A A_\mu = 0$ is given by [17,18]

$$
\langle T \{ \hat{A}_\mu^a(x) \hat{A}_\nu^b(y) \}\rangle
= - \frac{i}{2} \int d^4 z \delta(z_0) \frac{x_0 \delta_{\mu \ell} - p_{2\mu}(x - z)_{\ell}^\perp}{\pi^2 ((x - z)^2 + i \epsilon)^2}
\times U_{z_1}^{ab} \frac{1}{\delta_\nu^\perp} \frac{y_0 \delta_{\nu \ell}^\perp - p_{2\nu}(y - x)_{\ell}^\perp}{\pi^2 ((z - y)^2 + i \epsilon)^2},
$$

where $\frac{1}{\delta_\nu^\perp}$ can be either $\frac{1}{\delta_\nu + i \epsilon}$ or $\frac{1}{\delta_\nu - i \epsilon}$ which leads to the same result. [This is obvious for the leading order and correct in NLO after subtraction of the leading-order contribution, see Eq. (16) below.]

The diagrams in Fig. 4 can be calculated using the conformal integral
To find the NLO impact factor, we consider the operator
\[ \langle T\{ \bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(y)\gamma^\nu \psi(y)\}\rangle_A \]
\[ \int d^2z_1 d^2z_2 \frac{\alpha_3}{2\pi^3} \alpha_3 P_{ij} \left( \frac{z_1^2}{z_3^2} \right) \int d^2z_3 \left[ \frac{\sigma}{\alpha} \int d^2z_3 \frac{z_1^2}{z_3^2} \right] \]
\[ \times \left[ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \text{tr}\{ U_{z_1} U_{z_3} \} - N_c \text{tr}\{ U_{z_1} U_{z_3} \} \right]. \] (14)

The NLO matrix element \( \langle T\{ \bar{\psi}(x)\gamma^\mu \psi(x)\bar{\psi}(y)\gamma^\nu \psi(y)\}\rangle_A \) is given by Eq. (11) while the subtracted term is
\[ \int d^2z_1 d^2z_2 \frac{\alpha_3}{2\pi^3} \alpha_3 P_{ij} \left( \frac{z_1^2}{z_3^2} \right) \int d^2z_3 \left[ \frac{\sigma}{\alpha} \int d^2z_3 \frac{z_1^2}{z_3^2} \right] \]
\[ \times \left[ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \text{tr}\{ U_{z_1} U_{z_3} \} - N_c \text{tr}\{ U_{z_1} U_{z_3} \} \right]. \] (15)

as follows from Eq. (6). The \( \alpha \) integration is cut from above by \( \sigma = e^\eta \) in accordance with the definition of operators \( \bar{U}^\eta \); see Eq. (1). Subtracting (15) from (11) we get
\[ \langle I^{(2)}_{\mu \nu}(z_1, z_2, z_3; \eta) \rangle = \langle I^{(2)}_{\mu \nu}(z_1, z_2, z_3, \eta) \rangle \]
\[ - \langle I^{(2)}_{\mu \nu}(z_1, z_2, z_3; \eta) \rangle \]
\[ \frac{\sigma}{\alpha} \int d^2z_3 \frac{z_1^2}{z_3^2} \right] \]
\[ \times \left[ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \text{tr}\{ U_{z_1} U_{z_3} \} - N_c \text{tr}\{ U_{z_1} U_{z_3} \} \right]. \] (16)

where \( C \) is the Euler constant. Note that one should expect the NLO impact factor to be conformally invariant since it is determined by tree diagrams in Fig. 4. However, as discussed in Refs. [4,7,11], formally, the lightlike Wilson lines are conformally (M"obius) invariant but the longitudinal cutoff \( \alpha < \sigma \) in Eq. (1) violates this property so the term \( -\ln Z_2 \) in the right-hand side of Eq. (16) is not invariant. As was demonstrated in these papers, one can define a composite operator in the form
\[ \left\{ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \right\}_a = \text{tr}\{ \bar{U}_{z_1}^{(a)} \bar{U}_{z_3}^{(a)} \} + \frac{\alpha_3}{2\pi} \int d^2z_3 \frac{z_1^2}{z_3^2} \]
\[ \times \left[ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3}^{(a)} \} \text{tr}\{ U_{z_1} U_{z_3}^{(a)} \} \right] \]
\[ - N_c \text{tr}\{ \bar{U}_{z_1}^{(a)} \bar{U}_{z_3}^{(a)} \} \right\}_a \]
\[ \frac{4\alpha_3^2}{\sigma^2 z_1^2 z_3^2} + O(\alpha_3^2). \] (17)

where \( a \) is an arbitrary constant. It is convenient to choose the rapidity-dependent constant \( a \rightarrow ae^{-2\eta} \) so that the \( \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \right\}_a \) does not depend on \( \eta = \ln \sigma \) and all the rapidity dependence is encoded into \( a \) dependence. Indeed, it is easy to see that \( \frac{d}{da} \left\{ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \right\}_a = 0 \) and \( \frac{d}{da} \times \left[ \text{tr}\{ \bar{U}_{z_1} \bar{U}_{z_3} \} \right\}_a \) is determined by the NLO Balitsky-Kovchegov kernel which is a sum of the conformal part and the running-coupling part with our \( O(\alpha_3^2) \) accuracy [4,12].

Rewritten in terms of composite dipoles (17), the operator expansion (7) takes the form

\[ T(\bar{\psi}(x)\gamma^{\mu}\bar{\psi}(x)\gamma^{\nu}\bar{\psi}(y)) = \int \frac{dz_1dz_2}{z_{12}^4} \left\{ I_{LO}(z_1, z_2) \left[ 1 + \frac{\alpha_s}{\pi} \right] \text{tr}[\hat{U}_{z_1}\hat{U}_{z_2}^{\dagger}] \right\}_{a_0} + \frac{z_{12}^2}{4\pi^2} \int dz_3 z_3^{12} \alpha_s \left[ \frac{\kappa^2(\xi_1^\alpha\cdot\xi_3^\beta)(\xi_1^\alpha\cdot\xi_3^\beta) - 2C}{2(\kappa\cdot\xi_1^\alpha)(\kappa\cdot\xi_3^\beta)} \right] I_{NLO}^{\mu\nu}(x_1, x_2, z_{12}) \right\}_{a_0} - N_c \text{tr}[\hat{U}_{z_1}\hat{U}_{z_2}^{\dagger}] \right\}_{a_0}. \]  

(18)

We need to choose the “new rapidity cutoff” \( a \) in such a way that all the energy dependence is included in the matrix elements of Wilson-line operators so the impact factor should not depend on energy. A suitable choice of \( a \) is given by \( a_0 = -\kappa^2 + i\epsilon = -\frac{4\pi\alpha_s}{\alpha_s - \alpha_s} + i\epsilon \) so we obtain

\[ (x-y)^4 T(\bar{\psi}(x)\gamma^{\mu}\bar{\psi}(x)\gamma^{\nu}\bar{\psi}(y)) = \int \frac{dz_1dz_2}{z_{12}^4} \left\{ I_{LO}(z_1, z_2) \left[ 1 + \frac{\alpha_s}{\pi} \right] \text{tr}[\hat{U}_{z_1}\hat{U}_{z_2}^{\dagger}] \right\}_{a_0} + \frac{z_{12}^2}{4\pi^2} \int dz_3 z_3^{12} \alpha_s \left[ \frac{\kappa^2(\xi_1^\alpha\cdot\xi_3^\beta)(\xi_1^\alpha\cdot\xi_3^\beta) - 2C}{2(\kappa\cdot\xi_1^\alpha)(\kappa\cdot\xi_3^\beta)} \right] I_{NLO}^{\mu\nu}(x_1, x_2, z_{12}) \right\}_{a_0} - N_c \text{tr}[\hat{U}_{z_1}\hat{U}_{z_2}^{\dagger}] \right\}_{a_0}. \]  

(19)

Here the composite dipole \( \text{tr}[\hat{U}_{z_1}\hat{U}_{z_2}^{\dagger}] \) is given by Eq. (17) with \( a_0 = -\frac{4\pi\alpha_s}{\alpha_s - \alpha_s} + i\epsilon \) while \( I_{LO}^{\mu\nu}(x_1, x_2, z_{12}) \) and \( I_{NLO}^{\mu\nu}(x_1, x_2, z_{12}) \) are given by Eqs. (8) and (13), respectively.

\[ I_{NLO}^{\alpha\beta}(x, y; z_1, z_2) = \frac{\alpha_s N_c}{4\pi^2} R^2 \left[ \frac{\kappa^2}{(\kappa\cdot\xi_1^\alpha)(\kappa\cdot\xi_3^\beta)} \left( \begin{array}{c} \xi_1^\alpha \xi_3^\beta + \xi_1^\beta \xi_3^\alpha \\ 4 \ln \frac{1}{R} + 1 - 2 \left( \ln \frac{1}{R} + 2 \right) - 2 - 2 \left( \ln \frac{1}{R} + 2 \right) \end{array} \right) \right] \left( \begin{array}{c} \xi_1^\alpha \xi_3^\beta + \xi_1^\beta \xi_3^\alpha \\ 4 \ln \frac{1}{R} + 1 - 2 \left( \ln \frac{1}{R} + 2 \right) \end{array} \right) \]  

(20)

\[ - \frac{2}{\kappa^2} \left[ \ln \frac{1 - R}{R} + \ln \frac{1}{R} - 3 \left( \ln \frac{1}{R} + 2 \right) + 6 \ln \frac{1}{R} + 2 + 3 \frac{2}{R^2} \right] \left( \begin{array}{c} \xi_1^\alpha \xi_3^\beta + \xi_1^\beta \xi_3^\alpha \\ 4 \ln \frac{1}{R} + 1 - 2 \left( \ln \frac{1}{R} + 2 \right) \end{array} \right). \]  

(21)

where \( \text{Li}_2(z) \) is the dilogarithm. Here one easily recognizes five conformal tensor structures discussed in Ref. [19].

While it is easy to see that

\[ \frac{d}{dx_{\mu}} \left( \frac{1}{x-y} \right)^4 T(\bar{\psi}(x)\gamma^{\mu}\bar{\psi}(y)) = 0, \]  

(22)

one should be careful when checking the electromagnetic gauge invariance in the next-to-leading order. The reason is that the composite dipole \( \hat{U}_{a_0}(z_1, z_2) \) depends on \( x \) via the rapidity cutoff \( a_0 = -\frac{4\pi\alpha_s}{\alpha_s - \alpha_s} + i\epsilon \), so from Eq. (21) we get
Using the leading-order BFKL equation in the dipole form [linearization of Eq. (6)]

\[ a \frac{d}{da} \hat{U}_a(z_1, z_2) = \frac{\alpha_s N_c}{4\pi^2} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ \hat{U}_a(z_1, z_3) + \hat{U}_a(z_2, z_3) - \hat{U}_a(z_1, z_2) \right], \]

we obtain the following consequence of gauge invariance:

\[
\frac{\partial}{\partial x_\mu} \frac{1}{(x-y)^4} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y^\beta} \hat{U}^{NLO}_{a\beta}(x, y; z_i) = \frac{\alpha_s}{\pi} \frac{y_+}{x_-(x-y)^6} R \left[ \left( \frac{1}{2R} - 3 - \ln R \right) \frac{\partial}{\partial y^\beta} \ln k^2 + \left( \frac{\ln R}{R} + \frac{5}{2R} - \frac{1}{2R^2} \right) \frac{\partial}{\partial y^\beta} [\ln(k \cdot \zeta_1) + \ln(k \cdot \zeta_2)] \right].
\]

We have verified that the expression (23) satisfies the above equation.

### IV. PHOTON IMPACT FACTOR IN THE MELLIN REPRESENTATION

In preparation for the Fourier transformation, we calculated the Mellin transform of the photon impact factor (23). We project the impact factor on the conformal eigenfunctions of the BFKL equation [20]

\[ E_{\nu, n}(z_{10}, z_{20}) = \left[ \frac{z_{12}}{z_{10} z_{20}} \right]^{1+i\nu} \left[ \frac{z_{12}}{z_{10} z_{20}} \right]^{1+i\nu - \frac{R}{2}} \]

(here \( \bar{z} = z_x + iz_y, \tilde{z} = z_x - iz_y, z_{10} \equiv z_1 - z_0 \) etc.). Since electromagnetic currents are vectors, the only nonvanishing contribution comes from a projection on the eigenfunctions with spin 0 and spin 2. The spin-0 projection has the form (throughout the paper we reserve the notation \( \gamma \) for \( \frac{1}{2} + i\nu \))

\[
\left( 1 + \frac{\alpha_s}{\pi} \right) J_{a\beta}^{LO}(x, y; z_0, \nu) + J_{a\beta}^{NLO}(x, y; z_0, \nu) = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \left[ \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y^\beta} \left( 1 + \frac{\alpha_s}{\pi} \right) J_{a\beta}^{LO}(x, y; z_1, z_2) + J_{a\beta}^{NLO}(x, y; z_1, z_2) \right] \left( \frac{z_{12}}{z_{10} z_{20}} \right)^\gamma
\]

\[
= \frac{1}{4\pi^2} B(\bar{\gamma}, \tilde{\gamma}) \Gamma(2 - \gamma) \left( \frac{k^2}{(2\kappa \cdot \bar{z}_0)^2} \right)^\gamma - \frac{\gamma \bar{\gamma}}{3} (2S_1 + S_2)_{\mu\nu} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_1(\gamma) \right] \\
- 2S_{2\mu\nu} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_2(\gamma) \right] + 2\gamma (S_2 - S_3)_{\mu\nu} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_3(\gamma) \right] \\
+ \frac{1}{2} \frac{\gamma \bar{\gamma}}{3 - 2\gamma} \left[ \frac{1}{3} S_1 - \frac{2}{3} S_2 + S_3 - 2S_4 \right]_{\mu\nu} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_5(\gamma) \right] \\
+ (S_1 + S_2)_{\mu\nu} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_4(\gamma) \right].
\]

where \( \gamma = \frac{1}{2} + i\nu, \bar{\gamma} = 1 - \gamma = \frac{1}{2} - i\nu \) and

\[ F_1(\gamma) = F(\gamma) + \frac{X_\gamma}{\gamma \bar{\gamma}}, \quad F_2(\gamma) = F(\gamma) - 1 + \frac{1}{2\gamma \bar{\gamma}} + X_\gamma, \quad F_3(\gamma) = F(\gamma) + \frac{1}{2} X_\gamma, \]

\[ F_4(\gamma) = F(\gamma) - \frac{6}{\gamma \bar{\gamma}} + \frac{3}{\gamma \bar{\gamma}^2} - 2X_\gamma, \quad F_5(\gamma) = F(\gamma) + \frac{3\gamma \bar{\gamma} X_\gamma + 1 - 2\gamma \bar{\gamma}}{\gamma \bar{\gamma}(2 + \gamma \bar{\gamma})}, \]

\[ F(\gamma) = \frac{2\gamma^2}{3} + 1 - \frac{2\gamma^2}{\sin^2 \pi \gamma} - 2C \gamma + \frac{X_\gamma - 2}{\gamma \bar{\gamma}}, \]

and
\[ S_{1}^{\mu\nu} = \frac{\partial \ln \kappa^2}{\partial x_{\mu}} \frac{\partial \ln \kappa^2}{\partial y_{\nu}}, \quad S_{2}^{\mu\nu} = \frac{\partial \ln \kappa^2}{\partial x_{\mu}} \frac{\partial \ln \kappa^2}{\partial y_{\nu}}, \]
\[ S_{3}^{\mu\nu} = \frac{\partial \ln \kappa^2}{\partial x_{\mu}} \frac{\partial \ln \kappa \cdot \zeta_0}{\partial y_{\nu}}, \quad S_{4}^{\mu\nu} = \frac{\partial \ln \kappa \cdot \zeta_0}{\partial x_{\mu}} \frac{\partial \ln \kappa \cdot \zeta_0}{\partial y_{\nu}}. \]

(31)

The contribution of spin 2 in the \( t \) channel has the form

\[ \left( 1 + \frac{\alpha_s}{\pi} \right) J_{2,\alpha\beta}(x, y; z_0, \nu) + J_{2,\alpha\beta}^{(NLO)}(x, y; z_1, z_2) \]
\[ = \int \frac{d^2 z_1}{z_{12}} \frac{d^2 z_2}{z_{20}} \frac{\partial \kappa^2}{\partial x^\mu} \frac{\partial \kappa^2}{\partial y^\nu} \left( 1 + \frac{\alpha_s}{\pi} \right) J_{2,\alpha\beta}(x, y; z_1, z_2) \]
\[ + J_{2,\alpha\beta}^{(NLO)}(x, y; z_1, z_2) \left( \frac{z_{12}}{z_{10}z_{20}} \right)^\gamma \frac{z_{12}}{z_{10}} \frac{z_{20}}{z_{10}} \]
\[ = - \frac{1}{2\pi^2(x - y)^2} B(2 - \gamma, 2) \Gamma(\gamma + 2) \]
\[ \times \Gamma(3 - \gamma) \left[ 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s N_c}{2\pi} F_6(\gamma) \right] S_{2}^{\mu\nu}, \]

(32)

where

\[ F_6(\gamma) = F(\gamma) + \frac{2C}{\gamma\gamma} - \frac{2}{\gamma\gamma} + \frac{1}{\gamma\gamma^2} + \frac{1}{\gamma\gamma^2} - \frac{1}{\gamma\gamma} \]

(33)

and

\[ S_{5}^{\mu\nu} = \left[ g^{\mu 1} - ig^{\mu 2} - 2(x - z_0)^\mu \frac{\partial}{\partial z_0} \ln \kappa \cdot \zeta_0 \right] \]
\[ + \frac{4p_{2}^{\mu}}{\sqrt{s}} \left( \frac{\kappa_x \cdot \zeta_0}{(\kappa_y \cdot \zeta_0)\delta} \frac{\partial}{\partial z_0} \ln \kappa_y \cdot \zeta_0 \right) \]
\[ \times \left[ g^{\mu 1} - ig^{\mu 2} - 2(y - z_0)^\mu \frac{\partial}{\partial z_0} \ln \kappa \cdot \zeta_0 \right] \]
\[ + \frac{4p_{2}^{\mu}}{\sqrt{s}} \left( \frac{\kappa_x \cdot \zeta_0}{(\kappa_y \cdot \zeta_0)\delta} \frac{\partial}{\partial z_0} \ln \kappa_y \cdot \zeta_0 \right). \]

(34)

Using the decomposition of the product of the transverse \( \delta \) functions in conformal three-point functions (28)

\[ \delta^{(2)}(z_1 - z_3) \delta^{(2)}(z_2 - z_4) \]
\[ = \sum_{n=-\infty}^{\infty} \frac{d^3 p}{\pi^3} \frac{p^2}{z_{12}^2 z_{34}^2} \int d^2 z_0 E_{\nu,n}(z_{10}, z_{20}) E^{*}_{\nu,n}(z_{30}, z_{40}). \]

(35)

we obtain

\[ \hat{U}(z_1, z_2) = \int \frac{d\nu}{\pi} \int d^2 z_0 \left( \frac{z_{12}}{z_{10}z_{20}} \right)^\gamma \left[ \nu^2 \hat{U}(z_0, \nu) + (\nu^2 + 1) \right] \]
\[ \times \left[ \frac{z_{12}}{z_{10}z_{20}} \hat{U}^{(2)}(z_0, \nu) \right], \]

(36)

where

\[ \hat{U}_{a}(\nu, z_0) = \int \frac{d^2 z_1}{\pi^2} \hat{U}(z_1, z_2), \quad \hat{U}_{a}^{(2)}(\nu, z_0) = \int \frac{d^2 z_1}{\pi^2} \hat{U}^{(2)}(z_1, z_2), \]

(37)

is a composite dipole.

Substituting the decomposition (35) in Eq. (21) we get the high-energy OPE in the form

\[ \frac{1}{N_c} (x - y)^4 T[j^\mu(x) j^\nu(y)] \]
\[ = \int \frac{d\nu}{\pi} \int d^2 z_0 \left\{ \left( 1 + \frac{\alpha_s}{\pi} \right) J_{a\beta}(x, y; z_0, \nu) \right. \]
\[ + J_{a\beta}^{(NLO)}(x, y; z_0, \nu) \left[ \hat{U}_{a}(\nu, z_0) \right. \]
\[ + (\nu^2 + 1) \left[ \left( 1 + \frac{\alpha_s}{\pi} \right) J_{a\beta}(x, y; z_0, \nu) \right. \]
\[ + J_{a\beta}^{(NLO)}(x, y; z_0, \nu)] \hat{U}_{a}(\nu, z_0) \right. \}

(38)

Equation (38) and its Fourier transform (46) are the main results of this paper.

At this point it is instructive to check again the photon gauge invariance \( \frac{\partial}{\partial x^\mu} T[j^\mu(x) j^\nu(y)] = 0 \). Since \( a_0 = -\kappa^2 + i\epsilon \) we need to differentiate \( \hat{U}_{a}(\nu, z_0) \) too:

\[ \frac{\partial}{\partial x^\mu} \hat{U}_{a}(\nu, z_0) = a_0 \frac{d}{da} \hat{U}_{a}(\nu, z_0) |_{a=a_0} \frac{\partial}{\partial x^\mu} \ln a_0. \]

(39)

Let us start with a spin-2 contribution. It is easy to see that

\[ \frac{\partial}{\partial x^\mu} S_{5}^{\mu\nu} = 0, \quad S_{5}^{\mu\nu} \frac{\partial}{\partial x^\mu} \ln a_0 = 0, \]

and therefore the second term in the right-hand side of Eq. (38) is gauge invariant (recall that \( a_0 = -\kappa^2 + i\epsilon \)).

For the spin-0 part we need to use Eq. (39). Since \( \hat{U}_{a}(\nu, z_0) \) are the projections of color dipoles on the eigenfunctions (28) of the BFKL equation, the evolution equation (26) simplifies to
where $\omega(\nu) = \frac{a_sN_c}{\pi} \chi_\gamma$ is the BFKL pomeron intercept (as usual $\gamma = \frac{1}{2} + i \nu$). We obtain

$$\frac{\partial}{\partial x^\mu} J_{a\alpha}(x, y; z_0, \nu) = -\frac{\omega(\nu)}{2} \frac{\partial}{\partial x^\mu} \ln k^2.$$  \hspace{1cm} (41)

In the leading order, the derivative (41) does not contribute so the formula for gauge invariance is simply

$$\frac{\partial}{\partial x^\mu} J_{a\alpha}^{(LO)}(x, y; z_0, \nu) = 0.$$  \hspace{1cm} (42)

It is easy to demonstrate that $J_{a\alpha}^{(LO)}$ in the right-hand side of Eq. (29) satisfies this requirement.

In the NLO, we need both $J_{a\alpha}^{(NLO)}$ and $\omega J_{a\alpha}^{(LO)}$ parts so the requirement for electromagnetic gauge invariance takes the form

$$\frac{\partial}{\partial x^\mu} J_{a\alpha}^{(NLO)}(x, y; z_0, \nu) = \frac{\omega(\nu)}{2} J_{a\alpha}^{(LO)}(x, y; z_0, \nu) \frac{\partial}{\partial x^\mu} \ln k^2.$$  \hspace{1cm} (43)

We have checked that the right-hand side of Eq. (29) satisfies this equation.

**V. PHOTON IMPACT FACTOR IN THE MOMENTUM SPACE**

In general, the rapidity evolution of color dipoles is nonlinear but in this paper we assume that we can linearize it to the dipole form of the BFKL equation, like in the case of scattering of two virtual photons. Moreover, we will consider only the forward case which corresponds to deep inelastic scattering. In this case, one may write down the high-energy OPE in the form of a $k_T$-factorization formula

$$\int d^4x e^{i q \cdot x} \langle p | T [ \hat{J}_\mu(x) \hat{J}_\nu(0)] | p \rangle$$

$$= \frac{1}{2} \int d^2 k \lambda_{\mu\nu} \langle \hat{U}(k_\perp) | p \rangle.$$  \hspace{1cm} (44)

where

$$\langle \hat{U}(k_\perp) \rangle = \int d x_L e^{-i(k_\perp \cdot x)} \langle \hat{U}(x_\perp, 0) \rangle,$$

$q = p_1 + \frac{s}{2} p_2$ and $p = p_2 + \frac{m^2}{T} p_1$ is the target’s momentum. The reduced matrix element $\langle \langle p | \hat{U}(k) | p \rangle \rangle$ is defined as

$$\langle \langle p | \hat{U}(k) | p \rangle \rangle = \int d^2 z e^{-i(k_\perp \cdot z)} \langle \langle p | \hat{U}(z, 0) | p \rangle \rangle.$$  \hspace{1cm} (45)

where the factor $2 \pi \delta(\beta)$ reflects the fact that the forward matrix element of the operator $U \gamma^\gamma$ contains an unrestricted integration along the $p_1$. Our goal in this section is to find the impact factor $I_{\mu\nu}(q, k_\perp)$ in the next-to-leading order.

Since our “energy scale” $a_0 = -\kappa^{-2}$ for color dipoles depends on $x$ and $y$, to perform the Fourier transformation of the OPE (38) one should express $\hat{U}_{a_0}$ in terms of $\hat{U}_{a_0}$ with $a_m$ independent of coordinates $x$ and $y$. A suitable choice is $a_m = 1/x_B$. With this choice, the impact factor does not scale with $s$ and all the energy dependence is included in matrix elements of color dipoles. This is similar to the choice $\mu^2 = Q^2$ for the DGLAP evolution: the coefficient functions in front of the light-ray operators will not depend on $Q^2$ [except for $\alpha_s(Q^2)$ of course] and all the $Q^2$ dependence is shifted to parton densities. The leading-order evolution of a color dipole $\hat{U}_a$ is given by Eq. (40)

$$\hat{U}_{a_0}(\nu, z_0) = \hat{U}_{a_0}(\nu, z_0) |_{a_0=1/x_B}$$

$$\hat{U}_{a_0}^{(2)}(\nu, z_0) = \hat{U}_{a_0}^{(2)}(\nu, z_0) |_{a_0=1/x_B},$$  \hspace{1cm} (46)

so the Fourier transform of Eq. (38) yields
where

\[ P_{1}^{\mu \nu} = g^{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^{2}}, \]

\[ p_{2}^{\mu \nu} = \frac{1}{q^{2}} \left( q^{\mu} - \frac{p_{c}^{\mu} q^{2}}{q \cdot p_{2}} \right) \left( q^{\nu} - \frac{p_{c}^{\nu} q^{2}}{q \cdot p_{2}} \right), \]

\[ \tilde{p}_{\mu \nu} = (g^{\mu 1} - ig^{\mu 2})(g^{\nu 1} - ig^{\nu 2}), \]

\[ \tilde{p}_{\mu \nu} = (g^{\mu 1} + ig^{\mu 2})(g^{\nu 1} + ig^{\nu 2}), \]

and

\[ \langle \hat{U}(z_{0}, \nu) \rangle = \frac{\Gamma(1 - \gamma)\Gamma(1 - 2\gamma)\Gamma(\gamma)}{4\gamma\Gamma(2 - \gamma)\Gamma(2\gamma)} \int \frac{d^{2}k}{4\pi^{2}} k^{2\gamma} \langle \hat{U}(k) \rangle, \]

\[ \langle \hat{U}^{(2)}(z_{0}, \nu) \rangle = -\frac{\Gamma(\gamma)\Gamma(3 - 2\gamma)\Gamma(1 + \gamma)}{4\gamma\Gamma(3 - \gamma)\Gamma(2 - \gamma)\Gamma(1 + 2\gamma)} \int \frac{d^{2}k}{4\pi^{2}} k^{2\gamma} \langle \hat{U}^{(2)}(k) \rangle, \]

we get

\[ \int d^{4}x d\zeta e^{iq \zeta} \langle p | T \left[ j_{\mu} \left( x + \frac{2}{s} \zeta + p_{1} \right) j_{\nu} \left( \frac{2}{s} \zeta + p_{1} \right) \right] | p \rangle \]

\[ = N_{c} \frac{s}{2} \int \frac{d\nu}{32\pi^{2}} \int \frac{d^{2}k}{4\pi^{2}} (k^{2})^{\gamma} \Gamma(2\gamma) \langle \hat{U}(k) | p \rangle \]

\[ \times \left[ (\gamma + 2) P_{1}^{\mu \nu} \left( 1 + \frac{\alpha_{s}}{\pi} + \frac{\alpha_{N_{c}}}{2\pi} \Phi_{1}(\nu) \right) + (3\gamma + 2) P_{2}^{\mu \nu} \left( 1 + \frac{\alpha_{s}}{\pi} + \frac{\alpha_{N_{c}}}{2\pi} \Phi_{2}(\nu) \right) \right] \]

\[ + \frac{\Gamma(\gamma + \omega(\nu) / 2\gamma)}{\Gamma(\gamma)\Gamma(2 - \gamma)\Gamma(4 - 2\gamma)\Gamma(2 + \gamma) + \omega(\nu)} \frac{\gamma^{2}}{2} p_{3}^{\mu \nu} \left( 1 + \frac{\alpha_{s}}{\pi} + \frac{\alpha_{N_{c}}}{2\pi} F_{3}(\nu) \right) \]

where (as usual, \( \gamma = \frac{1}{2} + i\nu \))

\[ F_{1(2)}(\nu) = \Phi_{1(2)}(\nu) + \chi_{\gamma} \Psi_{\nu}(\nu), \]

\[ F_{3}(\nu) = F_{3}(\nu) + \left( \chi_{\gamma} - \frac{1}{\gamma} \right) \Psi_{\nu}(\nu), \]

\[ \Psi_{\nu}(\nu) = \psi(\nu) + 2\psi(2 - \gamma) - 2\psi(4 - 2\gamma) - \psi(2 + \gamma). \]

The structures \( P_{1} \) and \( P_{2} \) correspond to unpolarized structure functions \( F_{1}(x_{B}) \) and \( F_{2}(x_{B}) \). The third term vanishes for nucleon structure function but contributes to the polarized structure functions of a vector meson (or photon).

It is instructive to compare Eq. (51) with the well-known double integral representation of the leading-order impact factor (see, e.g., Refs. [3,21]):

\[ I^{\mu \nu}(q, k_{\perp}) = \frac{N_{c}}{32} \int_{0}^{\gamma} d\nu d\omega \frac{k_{1}^{2}}{Q^{2} \omega + k_{1}^{2} \nu} \left[ (1 - 2\bar{u}u - 2\bar{v}v) P_{1}^{\mu \nu} + (1 - 2\bar{u}u - 2\bar{v}v) \left( \tilde{p}_{\mu \nu} k_{2} + \tilde{p}_{\mu \nu} k_{2}^{2} \right) \right]. \]
VI. NLO BFKL FOR COLOR DIPOLES

For completeness, in this section we present the (linearized) evolution equation for composite color dipoles and discuss how it is related to the usual NLO BFKL approach [2]. The evolution equation for forward matrix elements of color dipoles $U(z_{12}) = \langle U(z_1, z_2) \rangle$ reads [11,12]

\[
2a \frac{d}{da} U_a(z) = \frac{\alpha_s N_c}{2 \pi^2} \int d^2 z' \left[ \frac{z'^2}{z'^2 - z^2} \right] \left[ 1 + \frac{\alpha_s}{4 \pi} \left( b \left( \frac{\ln z'^2}{4} + 2C \right) - b \left( \frac{z - z'}{z^2 - z'^2} \right) \ln \frac{z^2 - z'^2}{z^2} + b \left( -\frac{\ln z'^2}{4} - \frac{\pi^2}{3} \right) N_c \right.ight.
\]

\[
- \left. \frac{10n_f}{9} \right] \left[ [U_a(z')] + U_a(z - z') - U_a(z) \right] + \frac{\alpha_s^2 N_c^2}{4 \pi^2} \int d^2 z' \frac{z'^2}{z'^2 - z^2} \left[ - \frac{1}{(z - z')^2} \ln \frac{z^2 - z'^2}{z^2} + F(z, z') \right.
\]

\[
+ \Phi(z, z') \right] U_a(z') + \frac{3 \alpha_s^2 N_c^2}{2 \pi^2} \zeta(3) U_a(z),
\]

\[
(54)
\]

where

\[
F(z, z') = \left( 1 + \frac{n_f}{N_c^2} \right) \frac{3(z, z')^2 - 2z^2 z'^2}{16 z^2 z'^2} \left( \frac{2}{z^2} + \frac{z^2 - z'^2}{z^2 z'^2} \right.
\]

\[
\times \ln \frac{z^2}{z'^2} - \left[ 3 \left( 1 + \frac{n_f}{N_c^2} \right) \left( 1 - \frac{(z^2 + z'^2)^2}{8z^2 z'^2} \right. \right.
\]

\[
\left. + \frac{3z^4 + 3z'^4 - 2z^2 z'^2}{16z^4 z'^4} \right] \left( z, z' \right)^2 \right]
\]

\[
\times \int_0^\infty \frac{dt}{1 + t} \ln \frac{1 + t}{1 - t},
\]

\[
(55)
\]

and

\[
\Phi(z, z') = \left( \frac{z^2 - z'^2}{(z - z')^2 (z + z')} \right) \left[ \ln \frac{z^2}{z'^2} \ln \frac{z^2 z'^2 (z - z')^2}{(z^2 + z'^2)^2} \right.
\]

\[
+ 2 \text{Li}_2 \left( \frac{z'^2}{z^2} \right) - 2 \text{Li}_2 \left( \frac{z^2}{z'^2} \right) \right]
\]

\[
\left. - \left( 1 - \frac{(z^2 - z'^2)^2}{(z - z')^2 (z + z'^2)} \right) \left[ \int_0^1 - \int_1^\infty \right] \right]
\]

\[
\times \frac{du}{z - u} \ln \frac{u^2 z'^2}{z^2}.
\]

\[
(56)
\]

Note that the kernel is a sum of the “running-coupling” part proportional to $b = \frac{\alpha_s}{N_c} + \frac{\pi}{9} n_f$ and the conformal part; see the discussion in Ref. [4]. Here $\alpha_s = \alpha_s(\mu)$ and $\mu$ is the normalization point in the $\overline{\text{MS}}$ scheme.

With the $k_T$ factorization in view, let us rewrite the evolution equation (54) in terms of

\[
\gamma_{a_n}(z_{\perp}) = -\partial_{z_{\perp}}^2 \langle \hat{U}_{a_n}(z_{\perp}, 0) \rangle,
\]

\[
(57)
\]

proportional to the dipole unintegrated gluon distribution $D(x_B, z_{\perp}, \mu)$,

\[
\gamma_{x_n}(z_{\perp}, \mu) = \frac{4\pi x_B}{N_c} \alpha_s(\mu) D(x_B, z_{\perp}, \mu),
\]

\[
(58)
\]

where

\[
D(x_B, z_{\perp}, \mu) = \left( \frac{2}{x_B} \right) \int \frac{dz_n}{\pi x_B} \left[ p \right] \left| \left[ \alpha_p \alpha_n \frac{2}{\bar{s} z_n p_1 + z_{\perp}} \right. \right.
\]

\[
\times \left. \left[ -\alpha_p \alpha_n \frac{2}{\bar{s} z_n p_1 + z_{\perp}} \right] \right| p \right] a_n = x_n.
\]

\[
(59)
\]

Hereafter we use the notation

\[
[x, y] = P \exp \left\{ i \int_0^1 du (x - y) \mu A_\mu (ux + uy) \right\}
\]

for the gauge link connecting points $x$ and $y$. The color dipole is renorm-invariant so $D$ depends on $\mu$ to compensate $g^2(\mu)$ dependence.

The Fourier transform

\[
D(x_B, k_{\perp}) = \int d^2 z_{\perp} e^{ik_{\perp} z_{\perp}} D(x_B, z_{\perp})
\]

is called the dipole gluon TMD (transverse momentum dependent distribution). Note, however, that the dipole gluon TMD defined above differs from the definition

\[
D(x_B, z_{\perp}, \mu, \eta) = \left( \frac{2}{\bar{s}} \right) \int \frac{dz_n}{\pi x_B} e^{-i s z_n} \left[ p \right] \left| \left[ \alpha_p \alpha_n \frac{2}{\bar{s} z_n p_1 + z_{\perp}} \right. \right.
\]

\[
\times \left. \left[ -\alpha_p \alpha_n \frac{2}{\bar{s} z_n p_1 + z_{\perp}} \right] \right| p \right] \eta,
\]

\[
(60)
\]

which reduces to the usual parton density at $z_{\perp} = 0$. It should be emphasized that Eq. (60) is a more complex operator than (59). The difference is especially clear in the case of $\mathcal{N} = 4$ theory: the dipole gluon TMD (59) is UV finite while Eq. (60) is UV divergent so it needs additional UV counterterms; see the discussion in [22]. [These UV divergent terms are directly proportional to $x_B$ so they vanish for the definition (59).] Also, the role of parameter $x_B$ is different in the two definitions: in Eq. (59) it is defined as a rapidity cutoff $a_m$ while in Eq. (60) the rapidity cutoff $\eta$ should be imposed separately from $x_B$.

Differentiating Eq. (54) two times with respect to $z$ we obtain the NLO BFKL evolution for dipole gluon TMD (59) in the form
\[
2a \frac{d}{da} \mathcal{V}_a(z) = \frac{\alpha_s N_c}{\pi^2} \int d^2 \ell \left[ \left( 1 - \frac{\alpha_s b}{4\pi} \ln \frac{z^2 \mu^2}{4} + 2C + \frac{67 N_c}{9b} - \frac{\pi^2 N_c}{3b} - \frac{10 n_f}{9b} \right) \frac{\mathcal{V}_a(z')}{(z - z')^2} \frac{\mathcal{V}_a(z)}{z'^2(z - z')^2} \right. \\
+ \frac{\alpha_s b}{2\pi} \mathcal{V}_a'(z') - \mathcal{V}_a(z) \ln \frac{(z - z')^2}{\ell^2 \ell' \ell''} + \frac{\alpha_s N_c}{4\pi} \left[ - \frac{\ln \left( z^2 / \ell^2 \right)}{(z - z')^2} + F(z, z') + \Phi(z, z') \right] \mathcal{V}_a(z') \\
+ \frac{3 \alpha_s^2 N_c^2}{2 \pi^2} \zeta(3) \mathcal{V}_a(z),
\]

(61)

Next we need to perform the Fourier transformation of Eq. (61). It can be demonstrated that

\[
\int \frac{d^2 q' d^2 q}{4\pi^2} e^{i(q,q') - (q', z)} \left[ - \frac{\ln \left( q'^2 / q^2 \right)}{(q - q')^2} + F(q, q') + \Phi(q, q') \right] = - \frac{\ln \left( z^2 / \ell^2 \right)}{(z - z')^2} + F(z, z') + \Phi(z, z')
\]

(62)

so the conformal part of the kernel looks the same in coordinate and momentum representations.

Performing also the Fourier transformation of the running-coupling part, one obtains the momentum-representation kernel in the form

\[
2a \frac{d}{da} \mathcal{V}_a(k) = \frac{\alpha_s N_c}{\pi} \int d^2 k \left[ \frac{\mathcal{V}_a(k')}{(k - k')^2} \left( 1 + \frac{\alpha_s b}{4\pi} \left[ \ln \frac{\mu^2}{k^2} + \frac{N_c}{b} \left( \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 n_f}{9N_c} \right) \right] \right) \\
\left. - \frac{b \alpha_s}{4\pi} \left[ \frac{\mathcal{V}_a(k')}{(k - k')^2} \ln \frac{k^2}{k'^2} - \frac{k^2 \mathcal{V}_a(k)}{k'^2 (k - k')^2} \ln \frac{k^2}{k'^2} \right] \right. \\
\left. + \frac{\alpha_s N_c}{4\pi} \left[ - \frac{\ln \left( k^2 / k'^2 \right)}{(k - k')^2} + F(k, k') + \Phi(k, k') \right] \mathcal{V}_a(k') \right] + \frac{3 \alpha_s^2 N_c^2}{2 \pi^2} \zeta(3) \mathcal{V}_a(k),
\]

(63)

where \( \mathcal{V}(k) = \int dz e^{-i(k,z)} \mathcal{V}(z) \).

In terms of Mellin projections (\( \gamma = \frac{1}{t} + i \nu \) as usual)

\[
\mathcal{V}(k) = \sum_{\nu=0}^{\infty} \int \frac{d\nu}{2\pi^2} \mathcal{V}(n, \nu)(k^2)^{\gamma - 1}(\ell / k)^{\nu/2}
\]

\[
\mathcal{V}(n, \nu) = \int d^2 k (k^2)^{\gamma - \gamma}(\ell / k)^{\nu/2} \mathcal{V}(k)
\]

\[
= 4\pi^2 \frac{\Gamma(\gamma + \frac{\nu}{2})}{\Gamma(\gamma + \frac{\nu}{2})} \int d^2 z (z^2)^{\gamma - 1}(\ell / z)^{\nu/2} \mathcal{V}(z),
\]

(64)

the kernel (63) takes the form

\[
2a \frac{d}{da} \mathcal{V}_a(n, \nu)
\]

\[
= \frac{\alpha_s N_c}{\pi} \left\{ \chi(n, \gamma) + \frac{\alpha_s N_c}{4\pi} \left[ \frac{b}{N_c} \chi(n, \gamma) \left( \ln \frac{\mu^2}{d} + \frac{d}{d\gamma} \right) \\
- \frac{\chi(n, \gamma)}{2} + \frac{\chi'(n, \gamma)}{2\chi(n, \gamma)} + f(n, \gamma) \right] \right\} \mathcal{V}_a(n, \nu).
\]

(65)

Here

\[
f(n, \gamma) = \left[ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10 n_f}{9N_c} \right] \chi(n, \gamma) - \chi''(n, \gamma)
\]

\[
+ F(n, \gamma) - 2\Phi(n, \gamma) - 2\Phi(n, 1 - \gamma) + 6 \zeta(3),
\]

(66)

and

\[
F(n, \gamma) = \left\{ -3 + \left[ 1 + \frac{n_f}{N_c} \right] \frac{2 + 3 \gamma \bar{\gamma}}{(3 - 2\gamma)(1 + 2\gamma)} \delta_{2n} \\
+ \left[ 1 + \frac{n_f}{N_c} \right] \frac{\gamma \bar{\gamma}}{2(3 - 2\gamma)(1 + 2\gamma)} \delta_{2n} \right\}
\]

\[
\times \frac{\pi^2 \cos \bar{\gamma} \gamma}{(1 - 2\gamma) \sin^2 \gamma \bar{\gamma}}
\]

\[
\Phi(n, \gamma) = \int_0^1 dt \frac{dt}{1 + t} \gamma^{-1 + \frac{1}{2}} \left[ \frac{\pi^2}{12} - \frac{1}{2} \psi(n + 1) - \psi(1 + n) + \ln(1 + t) \\
- \psi(n + 1) - \psi(1) + \ln(1 + t) \\
+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k + n} \ln t - \sum_{k=1}^{\infty} \frac{t^k}{(k + n)^2} (1 - (-1)^k) \right].
\]

(67)

To compare to NLO BFKL from Ref. [2], one should rewrite the above equation in terms of

\[
\mathcal{L}(k) = \frac{1}{g^2(k)} \mathcal{V}(k)
\]

(68)

since two gluons in the dipole \( \mathcal{U}(k) \) come with an extra \( g^2 \) factor. Equation (65) turns into
\[ \frac{2 \alpha_s d}{da} \mathcal{L}_a(k) = \frac{\alpha_s(k^2) N_c}{\pi^2} \int d^2k \left[ \frac{L_a(k')}{(k-k')^2} - \frac{k^2 L_a(k)}{k^2(k-k')^2} \right] \ln \frac{k^2}{k^2(k-k')^2} + \frac{\alpha_s N_c}{4 \pi} \left[ - \ln^2(k^2/k^2) + F(k, k') \Phi(k, k') \right] \mathcal{L}_a(k') \] 

(69)

with the eigenvalues

\[
\frac{\alpha_s(k^2) N_c}{\pi^2} \int d^2k \left[ \frac{(k^2/k^2)^{-\gamma} e^{in\phi}}{(k-k')^2} - \frac{(k, k')}{k^2(k-k')^2} \right] \left[ 1 + \frac{\alpha_s N_c}{4 \pi} \left( \frac{67}{9} - \frac{\pi^2}{3} - \frac{10n_f}{9N_c} \right) \right] - \frac{b \alpha_s}{4 \pi} \left[ \frac{(k^2/k^2)^{-\gamma} e^{in\phi}}{(k-k')^2} \right] - \frac{k^2}{k^2(k-k')^2} \ln \frac{k^2}{k^2(k-k')^2} + \frac{\alpha_s N_c}{4 \pi} \left[ - \ln^2(k^2/k^2) + F(k, k') \Phi(k, k') \right] \left( k^2/k^2 \right)^{-\gamma} e^{in\phi} + \frac{3 \alpha_s^2 N_c^2}{2 \pi^2} \zeta(3) 
\]

\[
= \frac{\alpha_s(k^2) N_c}{\pi} \left\{ \chi(n, \gamma) + \frac{\alpha_s N_c}{4 \pi} \left[ - \frac{b}{2N_c} [\chi^2(n, \gamma) + \chi'(n, \gamma)] + f(n, \gamma) \right] \right\}. 
\] 

(70)

which coincide with the eigenvalues of the kernel [2] of the partial wave of the forward Reggeized gluon scattering amplitude

\[ \omega G_\omega(q, q') = \delta^{(2)}(q - q') + \int d^2p K(q, p) G_\omega(p, q'), \]

(71)

\[
\int d^2p \left( \frac{n^2}{q} \right)^{\gamma-1} e^{in\phi} K(q, p) = \frac{\alpha_s(q)}{\pi} N_c \left[ \chi(n, \gamma) + \frac{\alpha_s N_c}{4 \pi} \left( f(n, \gamma) - \frac{b}{2N_c} [\chi^2(n, \gamma) + \chi'(n, \gamma)] \right) \right]. 
\] 

(72)

This is somewhat surprising since the evolution of the composite (in $N = 4$ SYM—conformal) dipole with respect to $a$ gives the evolution of a forward Reggeized gluon scattering amplitude with respect to rapidity $\eta$ (of which $\omega$ is the Mellin transform). To illustrate the transition between the two evolutions let us consider the calculation of the dipole evolution directly from the NLO BFKL for Reggeized gluons.

The impact factor $\Phi_B(q')$ for the color dipole $\mathcal{U}(x, y)$ is proportional to $\alpha_s(q)(e^{iqx} - e^{-iqx})(e^{-iqy} - e^{iqy})$ so one obtains the cross section of the scattering of color dipole in the form

\[ \mathcal{U}^\eta(x) = \int \frac{d^2q}{q^2} \frac{d^2q'}{q'^2} \frac{\alpha_s(q)}{4 \pi^2} (e^{iqx} - 1)(e^{-iqx} - 1) \Phi_B(q') \]

\[ \times \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left( \frac{se^n}{q^2} \right)^\omega \tilde{G}_\omega(q, q'), \]

(74)

where $\tilde{G}_\omega(q, q')$ is the modified kernel with eigenvalues shifted by $2\chi(n, \nu) \chi'(n, \nu) \frac{q^2}{4\pi s}$; see Ref. [2]. The corresponding equation for $\mathcal{V}^\eta(n, \nu)$ takes the form [11]

\[ \frac{d}{d\eta} \mathcal{V}^\eta(n, \nu) = \frac{\alpha_s N_c}{\pi} \left[ \chi(n, \gamma) + \frac{\alpha_s N_c}{4 \pi} \left[ \frac{b}{N_c} \chi(n, \gamma) \right. \right. \]

\[ \left. \times \left( \ln \mu^2 + \frac{d}{d\gamma} - \frac{\chi(n, \gamma)}{2} + \frac{\chi'(n, \gamma)}{2\chi(n, \gamma)} + f(n, \gamma) + 2\chi(n, \nu) \chi'(n, \nu) \right) \right] \mathcal{V}^\eta(n, \nu). \]

(75)

Let us demonstrate that it agrees with Eq. (65). The Mellin projection of composite dipole can be obtained from Eq. (17):

\[ \Phi_B(q') = \text{the target impact factor.} \]

\[ \text{To get the evolution equation with respect to rapidity one should change the energy scale to } q'^2. \]
The first-order term can be derived from Eq. (17) while the term $\sim \alpha_s^2$ can be restored from the condition $\frac{d}{dn} \mathcal{V}^n(n, \nu) = 0$ up to an unknown function $X(n, \nu)$ which requires a next-to-next-to-leading order calculation (and does not contribute to the NLO evolution). Now one can see that the derivative with respect to $a$ gives Eq. (65). Thus, the transition between the evolution of the composite dipole $\mathcal{V}^n(n, \nu)$ with respect to $a$ and the rapidity evolution of the dipole $\mathcal{V}^n(n, \nu)$ correspond to the shift in eigenvalues on the function $2\chi(n, \nu)\chi'(n, \nu)\alpha_s N_c^2/4\pi$ — the same transition that describes the shift of eigenvalues when going from energy scale $q \rightarrow q^2$ in formula (74).

VII. CONCLUSIONS

Let us present again the $k_T$-factorization formula for DIS in the next-to-leading order:

$$\int d^4x e^{iqx} \langle p|T[j_\mu(x)\bar{j}_\nu(0)]|p\rangle = \frac{s}{2} \int \frac{d^2k_\perp}{k_\perp^2} I_{\mu\nu}(q, k_\perp) \mathcal{V}_{a_\mu a_\nu}^n(k_\perp).$$

(77)

where $I_{\mu\nu}(q, k_\perp)$ is given by Eq. (51) and the evolution equation for $\mathcal{V}_{a_\mu a_\nu}^n(k_\perp)$ by Eq. (63) [or Eq. (65) in the Mellin representation].

The analytic NLO photon impact factor in momentum space for the pomeron contribution (51) and the NLO $k_T$-factorization formula (77) for the deep inelastic scattering are the main results of this paper.

Since the composite dipole (68) obeys the same equation as the forward scattering amplitude of two Reggeized gluons (71) the impact factor (51) may be obtained as a NLO amplitude of emission of two Reggeized gluons by the virtual photon. There were several attempts in the literature to obtain this amplitude [23], but at present such an impact factor is known only as a combination of analytical and numerical results [24]. Indeed, in Ref. [25] the $\gamma^* - \gamma^*$ cross section has been calculated using only the LO impact factor and the LO and NLO BFKL amplitude for two Reggeized gluons. The authors explain in the paper that the NLO impact factor known at present in Ref. [24] is difficult to handle for a numerical calculation since it is not in a full analytic form. On the other hand, the result of this paper, provided that one knows the solution of the NLO BFKL with the running-coupling constant, allows us to compute the full NLO total cross section for the $\gamma^* - \gamma^*$ scattering process.

An attempt to calculate the NLO impact factor in an analytic form using an approach based on the analytic properties of the amplitude can be found in Ref. [26].

In the past few years, there has been some activity on the calculation of the NLO impact factor of other processes as well: in Ref. [27] the calculation of the NLO impact factor for Mueller-Navelet jets has been performed, while the impact factor for the virtual photon to light vector meson transition has been performed in Ref. [28].

It would be also instructive to compare our result (19) for the coefficient in front of the four-Wilson-line operator (relevant for the structure functions of DIS off a large nucleus) to similar results for the NLO impact factor obtained recently in Ref. [29] using the dipole model. However, as we already mentioned, our final NLO result (51) is defined as a coefficient function in front of a composite operator (17) defined with a counterterm which restores the conformal invariance in $\mathcal{N} = 4$ amplitudes and in our case leads to the conformal impact factor (since the impact factor is given by tree diagrams it should be conformally invariant even in QCD). As a consequence, the impact factor depends on a new parameter $a$ (an analog of the factorization scale $\mu$ in the usual OPE) which we chose in such a way that all the energy dependence is shifted in to the matrix element, leaving the impact factor energy scale invariant. To compare with the result of Ref. [29] representing the coefficient function of a usual dipole (without the counterterm subtraction), we should trace one step back and look at the impact factor $i^{\mu\nu}_{a_\mu a_\nu}(z_1, z_2, z_3; \eta)$ given by (16). One should then perform Fourier transformation to momentum space with respect to the positions $x$ and $y$ of the two electromagnetic currents in formula (7) and compare it to the result (58) from Ref. [29] integrated over $z_1$ and $cz_2$ when appropriate. Hopefully, after these integrations the two results will coincide.

ACKNOWLEDGMENTS

This work was supported by Contract No. DE-AC05-06OR23177 under which the Jefferson Science Associates, LLC operate the Thomas Jefferson National Accelerator Facility, and by Grant No. DE-AC02-05CH11231.
APPENDIX

There is a subtle point in the Fourier transformation of Eq. (38) in the forward case (cf. Ref. [20]). To illustrate it, consider the simplest term in the right-hand side of Eq. (23)

$$- \frac{\partial \kappa^a}{\partial x^\mu} \frac{\partial \kappa^b}{\partial y^\nu} 2 \left( g_{\alpha \beta} - 2 \frac{\kappa^\alpha \kappa^\beta}{\kappa^2} \right) R^2 = \frac{2R^2}{(x-y)^2} \left( g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right).$$

(A1)

The corresponding contribution to the T product of currents (21) is proportional to

$$\int \frac{dz_1 dz_2}{z_{12}^4} f(z_{12}^2) \left( z_{12}^2 \right)^{\frac{1+i
u}{2}} \frac{R^2}{(x-y)^6} \left[ g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] U(z_{12}).$$

(A2)

Consider now the Fourier transform of this equation for the case of forward scattering

$$\int d^4x d^4y \delta(y^*) e^{iq(x-y)} \int \frac{dz_1 dz_2}{z_{12}^4} \frac{R^2}{(x-y)^6} \left[ g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] U(z_{12}),$$

where $U(z_{12}) = \langle \hat{U}(z_1, z_2) \rangle$. We have calculated such integrals by using the representation of the type

$$R^2 = \int \frac{d\nu}{\pi^2} \nu^2 r(\nu) \int \frac{dz_0}{z_{12}^2} \frac{(z_{12}^2)}{(z_{10}^2 z_{20}^2)^{\gamma}} \left[ \frac{\kappa^2}{(2 \kappa^2 \cdot \xi_0^2)} \right]^\gamma$$

based on the decomposition (35) of transverse $\delta$ functions. [For this example $r(\nu) = B(\gamma) \Gamma(1+\gamma) \Gamma(2-\gamma)$.] The integral over $z_1$ and $z_2$ in Eq. (A3) is of the form

$$\int \frac{dz_1 dz_2}{z_{12}^4} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{\frac{1+i
u}{2}} f(z_{12}^2) \left( z_{12}^2 \right)^{1+i\mu} f(z_{12}^2).$$

(A4)

where $f(\mu) = \int dz_{12}^2 (z_{12}^2)^{1+i\mu} f(z_{12}^2)$. To calculate the integral in the right-hand side this equation, we take the orthogonality condition for conformal eigenfunctions [20]

$$\int \frac{dz_1 dz_2}{z_{12}^4} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{1+i\mu} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{1-i\mu} = \left[ \delta(\nu - \mu) \delta^{(2)}(z_0^2) + \delta(\nu + \mu)(z_0^2)^{-1+2i\mu} \right] \times \frac{2i\mu B(\frac{1}{2} + i \mu)}{\pi B(\frac{1}{2} - i \mu)} \frac{\pi^4}{2\nu^2},$$

(A5)

and perform the inversion $z_i \rightarrow \frac{z_0}{z_i}$. We obtain

$$\int \frac{dz_1 dz_2}{z_{12}^4} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{1+i\mu} f(z_{12}^2) = \frac{\pi^4}{2\nu^2} \left[ \delta(\nu - \mu) (z_0^2)^{-1-2i\nu} \delta^{(2)}(\frac{1}{z_0^2}) + \delta(\nu + \mu) \frac{2i\mu B(\frac{1}{2} + i \mu)}{\pi B(\frac{1}{2} - i \mu)} \right].$$

(A6)

so Eq. (A4) turns into

$$\int \frac{dz_1 dz_2}{z_{12}^4} \left( \frac{z_{12}^2}{z_{10}^2 z_{20}^2} \right)^{1+i\nu} f(z_{12}^2) = \frac{\pi^3}{4\nu^3} (z_0^2)^{-1-2i\nu} f(\nu) \delta^{(2)}(\frac{1}{z_0^2}) - i\pi^2 B(\frac{1}{2} - i \nu) \frac{2\nu B(\frac{1}{2} + i \nu)}{2f(-\nu)}.$$

(A7)

Substituting this equation with $f(z_{12}^2) = U(z_{12}) = z_{12}^2 V(z_{12})$ into Eq. (A3) we get

$$\frac{\pi}{4} \int d^4x d^4y \delta(y^*) e^{iq(x-y)} \left[ g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] \times \left( (x-y)^2 x_s y_s (x_s - y_s)^2 \right)^\gamma V(\nu) + \frac{B(\gamma)}{B(\gamma)} \left( (x-y)^2 x_s y_s (x_s - y_s)^2 \right)^\gamma V(-\nu).$$

(A8)

where $V(\nu) \equiv \frac{1}{2} \int d^2z (z^2)^{-1+i\nu} V(z)$. Here the first term comes from $z_0 = \infty$ while the second comes from finite $z_0$. It is easy to see that the two terms coincide after a change of the integration variable $\nu \leftrightarrow -\nu$ so effectively the contribution of the integral over finite $z_0$ is doubled:

$$\int d^4x d^4y \delta(y^*) e^{iq(x-y)} \int \frac{dz_1 dz_2}{z_{12}^4} \frac{R^2}{(x-y)^6} \left[ g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] U(z_{12})$$

$$= \frac{\pi}{2} \int d^4x d^4y \delta(y^*) \left[ g_{\mu \nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right] \times \left( (x-y)^2 x_s y_s (x_s - y_s)^2 \right)^\gamma V(-\nu).$$

(A9)

Since we have not used the explicit form of the Lorentz structure in $\mu$ and $\nu$ indices, it is clear that the doubling effect is general for any contribution to the forward Fourier transform of Eq. (46) (see also the discussion of zero transfer momentum limit in Ref. [20]).
PHOTON IMPACT FACTOR AND $k_T$ FACTORIZATION