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Ian Balitsky
Old Dominion University, ibalitsk@odu.edu

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EVOLUTION OF CONFORMAL COLOR DIPOLES AND HIGH-ENERGY AMPLITUDES IN $\mathcal{N} = 4$ SYM

IAN BALITSKY

Physics Dept., ODU, Norfolk VA 23529
and
Theory Group, Jlab, 12000 Jefferson Ave, Newport News, VA 23606, USA
E-mail: balitsky@jlab.org

The high-energy behavior of the $\mathcal{N}$=4 SYM amplitudes in the Regge limit can be calculated order by order in perturbation theory using the high-energy operator expansion in Wilson lines. At large N_c, a typical four-point amplitude is determined by a single BFKL pomeron. The conformal structure of the four-point amplitude is fixed in terms of two functions: pomeron intercept and the coefficient function in front of the pomeron (the product of two residues). The pomeron intercept is universal while the coefficient function depends on the correlator in question. The intercept is known in the first two orders in coupling constant: BFKL intercept and NLO BFKL intercept calculated in Ref. [1]. As an example of using the Wilson-line OPE, we calculate the coefficient function in front of the pomeron for the correlator of four Z^2 currents in the first two orders in perturbation theory.

Keywords: High energy; conformal invariance; Wilson lines.

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1. Introduction

The high-energy behavior of the amplitudes can be studied in the framework of the rapidity evolution of Wilson-line operators forming color dipoles. The main idea is factorization in rapidity[2,3,4]: we separate a typical functional integral describing scattering of two particles into (**i**) the integral over the gluon (and gluino) fields with rapidity close to the rapidity of the spectator Y_A, (**ii**) the integral over the gluons with rapidity close to the rapidity of the target Y_B, and (**iii**) the integral over the intermediate region of rapidities $Y_A > Y > Y_B$, see Fig. 1. The result of the first integration is a certain coefficient function ("impact factor") times color dipole (ordered in the direction of spectator's velocity) with rapidities up to Y_A. Similarly, the result of second integration is again the impact factor times color dipole ordered in the direction of target's velocity with rapidities greater than Y_B. The result of last integration is the correlation function of two dipoles which can be calculated using the evolution equation for color dipoles which is known in the leading and next-to-leading order. As an example of practical use of this factorization scheme in the NLO approximation, in present paper we calculate the high-energy behavior of the "scattering amplitude of scalar particles" (the four-point correlation function of scalar currents).

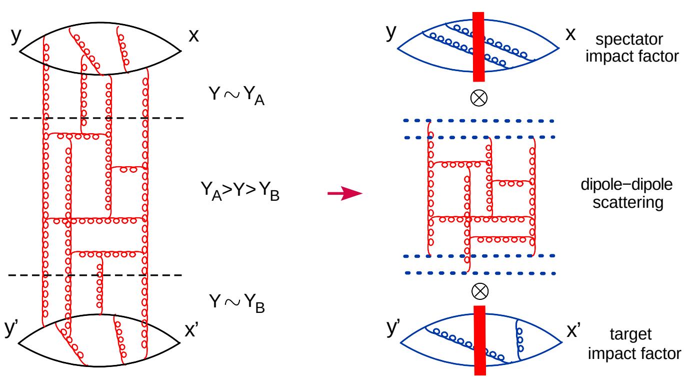

Fig. 1. High-energy factorization

The high-energy (Regge) limit of a four-point amplitude $A(x, y; x', y')$ in the coordinate space can be achieved as

$$x = \rho x_* \frac{2}{s} p_1 + x_\perp, \quad y = \rho y_* \frac{2}{s} p_1 + y_\perp,$$
$$x' = \rho' x_\bullet \frac{2}{s} p_2 + x'_\perp, \quad y' = \rho' y'_\bullet \frac{2}{s} p_2 + y'_\perp \tag{1}$$

with $\rho, \rho' \to \infty$ and $x_* > 0 > y_*$, $x'_\bullet > 0 > y'_\bullet$. (Strictly speaking, $\rho \to \infty$ or $\rho' \to \infty$ would be sufficient to reach the Regge limit). Hereafter we use the notations $x_\bullet = -p_1^\mu x_\mu$, $x_* = -p_2^\mu x_\mu$ where p_1 and p_2 are light-like vectors normalized by

$-2(p_1,p_2)=s$. These "Sudakov variables" are related to the usual light-cone coordinates $x^\pm=\frac{1}{\sqrt{2}}(x^0\pm x^3)$ by $x_*=x^+\sqrt{s/2}$, $x_\bullet=x^-\sqrt{s/2}$ so $x=\frac{2}{s}x_*p_1+\frac{2}{s}x_\bullet p_2+x_\perp$. We use the (-1,1,1,1) metric so $x^2=-\frac{4}{s}x_\bullet x_*+\vec{x}_\perp^2$. In the Regge limit (1) the full conformal group reduces to Möbius subgroup SL(2,C) leaving the transverse plane $(0,0,z_\perp)$ invariant.

For simplicity, let us consider correlation function of four scalar currents

$$(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle \tag{2}$$

where $\mathcal{O}\equiv\frac{4\pi^2\sqrt{2}}{\sqrt{N_c^2-1}}\mathrm{Tr}\{\mathrm{Z}^2\}$ $(Z=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2))$ is a renorm-invariant chiral primary operator.

In a conformal theory this four-point amplitude $A(x,y;x',y')$ depends on two conformal ratios which can be chosen as

$$R=\frac{(x-x')^2(y-y')^2}{(x-y)^2(x'-y')^2},\qquad r=R\Big[1-\frac{(x-y')^2(y-x')^2}{(x-x')^2(y-y')^2}+\frac{1}{R}\Big]^2 \tag{3}$$

In the Regge limit (1) the conformal ratio R scales as $\rho^2{\rho'}^2$ while r does not depend on ρ or ρ'.

As demonstrated in Ref.[5], the pomeron contribution in a conformal theory can be represented as an integral over one real variable ν

$$\begin{aligned}&(x-y)^4(x'-y')^4\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\mathcal{O}(x')\mathcal{O}^\dagger(y')\rangle\\&=\frac{i}{2}\int d\nu\ \tilde{f}_+(\nu)\frac{\tanh\pi\nu}{\nu}F(\nu)\Omega(r,\nu)R^{\frac{1}{2}\omega(\nu)}\end{aligned} \tag{4}$$

Here $\omega(\nu)\equiv\omega(0,\nu)$ is the pomeron intercept, $\tilde{f}_+(\nu)\equiv\tilde{f}_+(\omega(\nu))$ where $\tilde{f}_+(\omega)=(e^{i\pi\omega}-1)/\sin\pi\omega$ is the signature factor in the coordinate space, and $F(\nu)$ is the "pomeron residue" (strictly speaking, the product of two pomeron residues). The conformal function $\Omega(r,\nu)$ is given by[6]

$$\Omega(r,\nu)=\frac{\nu^2}{\pi^3}\int d^2z\Big[\frac{-\kappa^2}{(-2\kappa\cdot\zeta)^2}\Big]^{\frac{1}{2}+i\nu}\Big[\frac{-{\kappa'}^2}{(-2\kappa'\cdot\zeta)^2}\Big]^{\frac{1}{2}-i\nu} \tag{5}$$

where $\zeta\equiv\frac{p_1}{s}+z_\perp^2p_2+z_\perp$ and

$$\begin{aligned}\kappa&=\frac{\sqrt{s}}{2x_*}(\frac{p_1}{s}+x^2p_2+x_\perp)-\frac{\sqrt{s}}{2y_*}(\frac{p_1}{s}+y^2p_2+y_\perp)\\ \kappa'&=\frac{\sqrt{s}}{2x'_\bullet}(\frac{p_1}{s}+{x'}^2p_2+x'_\perp)-\frac{\sqrt{s}}{2y'_\bullet}(\frac{p_1}{s}+{y'}^2p_2+y'_\perp)\end{aligned} \tag{6}$$

are two SL(2,C)-invariant vectors[6] such that $\kappa^2=\frac{s(x-y)^2}{4x_*y_*}$, ${\kappa'}^2=\frac{s(x'-y')^2}{4x'_\bullet y'_\bullet}$ and therefore

$$\kappa^2{\kappa'}^2=\frac{1}{R},\qquad 4(\kappa\cdot\kappa')^2=\frac{r}{R} \tag{7}$$

In our limit (1) $x^2=x_\perp^2$, ${x'}^2={x'}_\perp^2$ and similarly for y. Note that all the dependence on large energy ($\equiv$ large ρ,ρ') is contained in $R^{\frac{1}{2}\omega(\nu)}$.

The dynamical information about the conformal theory is encoded in two functions: pomeron intercept and pomeron residue. At small α_s[7]

$$\begin{aligned}\omega(\nu) &= \frac{\alpha_s}{\pi}N_c\Big[\chi(\nu)+\frac{\alpha_s N_c}{4\pi}\delta(\nu)\Big],\\ \delta(\nu) &= 6\zeta(3)-\frac{\pi^2}{3}\chi(\nu)+\chi''(\nu)-\ 2\Phi(\nu)-2\Phi(-\nu)\end{aligned}\tag{8}$$

where $\chi(\nu)=2\psi(1)-\psi(\frac{1}{2}+i\nu)-\psi(\frac{1}{2}-i\nu)$ and[1]

$$\Phi(\nu) = -\int_0^1\frac{dt}{1+t}\ t^{-\frac{1}{2}+i\nu}\Big[\frac{\pi^2}{6}+2\mathrm{Li}_2(t)\Big]\tag{9}$$

The pomeron residue $F(\nu)$ is known in the leading order[6]

$$F(\nu)\ \stackrel{\lambda\to 0}{\to}\ \lambda^2\frac{\pi\sin\pi\nu}{4\nu\cos^3\pi\nu}\tag{10}$$

To find the NLO amplitude, we must calculate the "pomeron residue" $F(\nu)$ in the next-to-leading order. In the rest of the paper we will do this using the high-energy operator product expansion in Wilson lines[2].

2. Operator Expansion in Conformal Dipoles

As we discussed above, the main idea behind the high-energy operator expansion is the rapidity factorization. At the first step, we integrate over gluons with rapidities $Y>\eta$ and leave the integration over $Y<\eta$ for later time, see Fig. 2. The result of

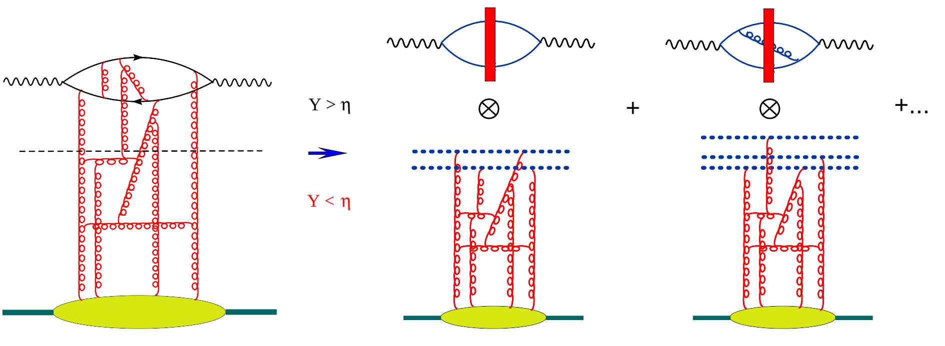

Fig. 2. High-energy operator expansion in Wilson lines

the integration is the coefficient function ("impact factor") in front of the Wilson-line operators with rapidities up to $\eta=\ln\sigma$:

$$\begin{aligned}U^\sigma_x &= \mathrm{Pexp}\Big[-ig\int_{-\infty}^{\infty}du\ p_1^\mu A^\sigma_\mu(up_1+x_\perp)\Big]\\ A^\sigma_\mu(x) &= \int d^4k\ \theta(\sigma-|\alpha_k|)e^{ik\cdot x}A_\mu(k)\end{aligned}\tag{11}$$

where the Sudakov variable α_k is defined as usual, $k = \alpha_k p_1 + \beta_k p_2 + k_\perp$. For the T-product of scalar currents currents $\mathcal{O}$ this coefficient function has the form[8]:

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} \;=\; \frac{1}{\pi^2(N_c^2-1)}\int \frac{d^2z_1 d^2z_2}{z_{12}^4}\,\mathcal{R}^2\,\Big\{\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} - \frac{\alpha_s}{\pi^2} \tag{12}$$
$$\times\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\ln\frac{s}{4}\sigma\mathcal{Z}_3 - \frac{i\pi}{2} + C\Big][\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\Big\}$$

Hereafter we use the notations $\zeta_i \equiv \frac{p_1}{s} + z_i^2 p_2 + z_i^\perp$, $\mathcal{Z}_i \equiv -\frac{4}{\sqrt{s}}(\kappa\cdot\zeta_i) = \frac{(x-z_i)^2}{x_*} - \frac{(y-z_i)^2}{y_*}$, and

$$\mathcal{R} \;=\; -\frac{(x-y)^2 z_{12}^2}{x_* y_* \mathcal{Z}_1\mathcal{Z}_2} \;=\; \frac{\kappa^2(\zeta_1\cdot\zeta_2)}{2(\kappa\cdot\zeta_1)(\kappa\cdot\zeta_2)} \tag{13}$$

Note that the l.h.s. of the Eq. (12) is conformally invariant while the coefficient function in the r.h.s. is not. The reason for that is the cutoff in the longitudinal direction (11). Indeed, we consider the light-like dipoles (in the p_1 direction) and impose the cutoff on the maximal α emitted by any gluon from the Wilson lines. Formally, the light-like Wilson lines are Möbius invariant. Unfortunately, the light-like Wilson lines are divergent in the longitudinal direction and moreover, it is exactly the evolution equation with respect to this longitudinal cutoff which governs the high-energy behavior of amplitudes. At present, it is not known how to find the conformally invariant cutoff in the longitudinal direction. When we use the non-invariant cutoff we expect, as usual, the invariance to hold in the leading order but to be violated in higher orders in perturbation theory. In our calculation we restrict the longitudinal momentum of the gluons composing Wilson lines, and with this non-invariant cutoff the NLO evolution equation in QCD has extra non-conformal parts not related to the running of coupling constant. Similarly, there will be non-conformal parts coming from the longitudinal cutoff of Wilson lines in the $\mathcal{N}=4$ SYM equation. We will demonstrate below that it is possible to construct the "composite conformal dipole operator" (order by order in perturbation theory) which mimics the conformal cutoff in the longitudinal direction so the corresponding evolution equation has no extra non-conformal parts. This is similar to the construction of the composite renormalized local operator in the case when the UV cutoff does not respect the symmetries of the bare operator - in this case the symmetry of the UV-regularized operator is preserved order by order in perturbation theory by subtraction of the symmetry-restoring counterterms. Following Ref. [8] we choose the conformal composite operator in the form

$$[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_a^{\rm conf} \;=\; \mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\} + \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \tag{14}$$
$$\times\,[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\ln\frac{4az_{12}^2}{\sigma^2 s z_{13}^2 z_{23}^2} \;+\; O(\alpha_s^2)$$

so that the $[\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]_a^{\rm conf}$ does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is encoded into a-dependence (see Ref. [9] for details).

Rewritten in terms of conformal dipoles (14), the operator expansion (12) takes the form:

$$\begin{aligned}(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} \;&=\; \frac{1}{\pi^2(N_c^2-1)}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\;\mathcal{R}^2\;\Big\{[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_a^{\rm conf}\\ &-\frac{\alpha_s}{2\pi^2}\int d^2z_3\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big(\ln\frac{asz_{12}^2}{4z_{13}^2z_{23}^2}\mathcal{Z}_3^2-i\pi+2C\Big)\\ &\times\,[\mathrm{Tr}\{T^n\hat{U}_{z_1}\hat{U}^\dagger_{z_3}T^n\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]_a\Big\}. \qquad (15)\end{aligned}$$

We need to choose the new "rapidity cutoff" a in such a way that all the energy dependence is included in the matrix element(s) of Wilson-line operators so the impact factor should not depend on energy ($\equiv$ should not scale with ρ as $\rho\to\infty$). A suitable choice of a is given by $a_0=\kappa^{-2}+i\epsilon=\frac{4x_*y_*}{s(x-y)^2}+i\epsilon$ so we obtain

$$\begin{aligned}(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} \;&=\; \frac{1}{(N_c^2-1)}\int\frac{d^2z_1d^2z_2}{\pi^2z_{12}^4}\;\mathcal{R}^2\;\Big\{[\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]^{\rm conf}_{a_0}-\frac{\alpha_s}{2\pi^2}\\ \int\frac{z_{12}^2d^2z_3}{z_{13}^2z_{23}^2}\Big[\ln\frac{-x_*y_*z_{12}^2\mathcal{Z}_3^2}{(x-y)^2z_{13}^2z_{23}^2}+2C\Big]&[\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\}-N_c\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\Big\}\end{aligned}$$

where the conformal dipole $[\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]^{\rm conf}$ is given by Eq. (14) with $a_0=\frac{4x_*y_*}{s(x-y)^2}$.

Now it is evident that the impact factor in the r.h.s. of this equation is Möbius invariant and does not scale with ρ so Eq. (14) gives conformally invariant operator up to α_s^2 order. In higher orders, one should expect the correction terms with more Wilson lines. This procedure of finding the dipole with conformally regularized rapidity divergence is analogous to the construction of the composite renormalized local operator by adding the appropriate counterterms order by order in perturbation theory.

To find the amplitude (2) in the next-to-leading order it is sufficient to take into account only the linear evolution of Wilson-line operators which corresponds to taking into account only two gluons in the t-channel. The non-linear effects in the evolution (and the production) of t-channel gluons enter the four-current amplitude (2) in the form of so-called "pomeron loops" which start from the NNLO BFKL order. It is convenient to define the "color dipole in the adjoint representation"

$$\hat{\mathcal{U}}^\sigma(x,y)\;=\;1-\frac{1}{N_c^2-1}\mathrm{Tr}\{\hat{U}^\sigma_x\hat{U}^{\dagger\sigma}_y\}. \qquad (16)$$

With this two-gluon accuracy

$$\begin{aligned}&[\frac{1}{N_c}\mathrm{Tr}\{T^n\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_3}T^n\hat{U}^\sigma_{z_3}\hat{U}^{\dagger\sigma}_{z_2}\}-\mathrm{Tr}\{\hat{U}^\sigma_{z_1}\hat{U}^{\dagger\sigma}_{z_2}\}]\;\simeq\\ &\quad-\frac{1}{2}(N_c^2-1)\big[\hat{\mathcal{U}}^\sigma_{\rm conf}(z_1,z_3)+\hat{\mathcal{U}}^\sigma_{\rm conf}(z_2,z_3)-\hat{\mathcal{U}}^\sigma_{\rm conf}(z_1,z_2)\big].\end{aligned}$$

The conformal dipole operator (14) in the BFKL approximation has the form:

$$\begin{aligned}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2)\;&=\;\hat{\mathcal{U}}^\sigma(z_1,z_2) \qquad (17)\\ &+\frac{\alpha_sN_c}{4\pi^2}\int d^2z\frac{z_{12}^2}{z_{13}^2z_{23}^2}\ln\frac{4az_{12}^2}{\sigma^2sz_{13}^2z_{23}^2}[\hat{\mathcal{U}}^\sigma(z_1,z_3)+\hat{\mathcal{U}}^\sigma(z_2,z_3)-\hat{\mathcal{U}}^\sigma(z_1,z_2)].\end{aligned}$$

With the two-gluon accuracy one more integration in the r.h.s. of Eq. (16) can be performed (see Ref.[9]): and the result takes the form

$$(x-y)^4 T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} = -\frac{1}{\pi^2}\int\frac{dz_1dz_2}{z_{12}^4}\,\hat{\mathcal{U}}^{a_0}_{\rm conf}(z_1,z_2)\mathcal{R}^2 \tag{18}$$
$$\times\Big\{1-\frac{\alpha_sN_c}{2\pi}\Big[\ln^2\mathcal{R}-\frac{\ln\mathcal{R}}{\mathcal{R}}-2C\big(\ln\mathcal{R}-\frac{1}{\mathcal{R}}+2\big)+\ 2\mathrm{Li}_2(1-\mathcal{R})-\frac{\pi^2}{3}\Big]\Big\}.$$

We need the projection of the T-product in the l.h.s. of this equation onto the conformal eigenfunctions of the BFKL equation[10]

$$E_{\nu,n}(z_{10},z_{20}) = \Big[\frac{\tilde{z}_{12}}{\tilde{z}_{10}\tilde{z}_{20}}\Big]^{\frac12+i\nu+\frac{n}{2}}\Big[\frac{\bar{z}_{12}}{\bar{z}_{10}\bar{z}_{20}}\Big]^{\frac12+i\nu-\frac{n}{2}} \tag{19}$$

(here $\tilde{z}=z_x+iz_y,\bar{z}=z_x-iz_y,\ z_{10}\equiv z_1-z_0$ etc.). Since $\hat{\mathcal{O}}$'s are scalar operators, the only non-vanishing contribution comes from projection on the eigenfunctions with spin 0:

$$\int\frac{dz_1dz_2}{\pi^2z_{12}^4}\,\mathcal{R}^2\Big\{1-\frac{\alpha_sN_c}{2\pi}\Big[\ln^2\mathcal{R}-\frac{\ln\mathcal{R}}{\mathcal{R}}-2C\big(\ln\mathcal{R}-\frac{1}{\mathcal{R}}+2\big)+\ 2\mathrm{Li}_2(1-\mathcal{R})$$
$$-\frac{\pi^2}{3}\Big]\Big\}\Big[\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big]^{\frac12+i\nu} = \Big[\frac{-\kappa^2}{(-2\kappa\cdot\zeta_0)^2}\Big]^{\frac12+i\nu}\frac{\Gamma^2\big(\frac12-i\nu\big)}{\Gamma(1-2i\nu)}\frac{\big(\frac14+\nu^2\big)\pi}{\cosh\pi\nu}\Big\{1+\frac{\alpha_sN_c}{2\pi}\Phi_1(\nu)\Big\}$$

where

$$\Phi_1(\nu) = -2\psi'\big(\frac12+i\nu\big)-2\psi'\big(\frac12-i\nu\big)+\frac{2\pi^2}{3}+\frac{\chi(\nu)-2}{\nu^2+\frac14}-2C\chi(\nu) \tag{20}$$

and $\zeta_0\equiv\frac{p_1}{s}+z_{0\perp}^2p_2+z_{0\perp}$.

Now, using the decomposition of the product of the transverse δ-functions in conformal 3-point functions $E_{\nu,n}(z_{10},z_{20})$ (see Ref[10]) we obtain

$$(x-y)^4T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\} = -\int d\nu\int d^2z_0\ \frac{\nu^2(1+4\nu^2)}{4\pi\cosh\pi\nu}\frac{\Gamma^2\big(\frac12-i\nu\big)}{\Gamma(1-2i\nu)}$$
$$\times\Big(\frac{-\kappa^2}{(-2\kappa\cdot\zeta_0)^2}\Big)^{\frac12+i\nu}\big[1+\frac{\alpha_sN_c}{2\pi}\Phi_1(\nu)\big]\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0) \tag{21}$$

where

$$\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0) \equiv \frac{1}{\pi^2}\int\frac{d^2z_1d^2z_2}{z_{12}^4}\ \Big(\frac{z_{12}^2}{z_{10}^2z_{20}^2}\Big)^{\frac12-i\nu}\ \hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \tag{22}$$

is a conformal dipole in the z_0,ν representation.

Similarly, one can write down the expansion of the bottom part of the diagram in color dipoles:

$$(x'-y')^4T\{\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\} = -\int d\nu'\int d^2z_0' \tag{23}$$
$$\frac{{\nu'}^2(1+4{\nu'}^2)}{4\pi\cosh\pi\nu'}\frac{\Gamma^2\big(\frac12-i\nu'\big)}{\Gamma(1-2i\nu')}\ \Big(\frac{-{\kappa'}^2}{(-2\kappa'\cdot\zeta_0')^2}\Big)^{\frac12+i\nu'}\big[1+\frac{\alpha_sN_c}{2\pi}\Phi_1(\nu')\big]\hat{\mathcal{V}}^{b_0}_{\rm conf}(\nu',z_0').$$

Here $\zeta'_0 \equiv p_1 + \frac{z'^2_0}{s}p_2 + z'_{0\perp}$, $b_0 = \kappa'^{-2} + i\epsilon = \frac{4x'_\bullet y'_\bullet}{s(x'-y')^2} + i\epsilon$, and

$$\hat{\mathcal{V}}^b_{\rm conf}(\nu', z'_0) \;=\; \frac{1}{\pi^2}\int \frac{d^2z_1 d^2z_2}{z^4_{12}}\;\Big(\frac{z^2_{12}}{z^2_{10}z^2_{20}}\Big)^{\frac12 - i\nu'}\;\hat{\mathcal{V}}^b_{\rm conf}(z_1,z_2), \tag{24}$$

where the conformal operator

$$\begin{aligned}&\hat{\mathcal{V}}^b_{\rm conf}(z_1,z_2) \qquad\qquad (25)\\ &= \hat{\mathcal{V}}^\sigma(z_1,z_2) + \frac{\alpha_sN_c}{4\pi^2}\int d^2z\frac{z^2_{12}}{z^2_{13}z^2_{23}}\ln\frac{4b\sigma^2z^2_{12}}{sz^2_{13}z^2_{23}}[\hat{\mathcal{V}}^\sigma(z_1,z_3) + \hat{\mathcal{V}}^\sigma(z_2,z_3) - \hat{\mathcal{V}}^\sigma(z_1,z_2)]\end{aligned}$$

is made from the dipoles $\hat{\mathcal{V}}^\sigma(x,y) \;=\; 1 - \frac{1}{N_c^2-1}{\rm Tr}\{\hat{V}^\sigma_x\hat{V}^{\dagger\sigma}_y\}$ (cf. Eq. (16)) ordered along the straight line $\parallel\; p_2$ with the rapidity restriction

$$\begin{aligned}V^\sigma_x \;&=\; {\rm Pexp}\Big[-ig\int_{-\infty}^{\infty}du\; p_1^\mu A^\sigma_\mu(up_2 + x_\perp)\Big]\\ A^\sigma_\mu(x) \;&=\; \int d^4k\;\theta(\sigma - |\beta_k|)e^{ik\cdot x}A_\mu(k)\end{aligned} \tag{26}$$

If we substitute both operator expansions (21) and (23) into the correlation function (2), it takes the form

$$\begin{aligned}&(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle \;=\; \int d\nu d\nu'\int d^2z_0d^2z'_0\;\frac{\nu^2(1+4\nu^2)}{4\pi\cosh\pi\nu}\\ &\times\frac{\Gamma^2\big(\frac12 - i\nu\big)}{\Gamma(1-2i\nu)}\Big(\frac{-\kappa^2}{(-2\kappa\cdot\zeta_0)^2}\Big)^{\frac12+i\nu}\Big[1 + \frac{\alpha_sN_c}{2\pi}\Phi_1(\nu)\Big]\frac{\nu'^2(1+4\nu'^2)}{4\pi\cosh\pi\nu'}\frac{\Gamma^2\big(\frac12 - i\nu'\big)}{\Gamma(1-2i\nu')}\\ &\times\Big(\frac{-\kappa'^2}{(-2\kappa'\cdot\zeta'_0)^2}\Big)^{\frac12+i\nu'}\Big[1 + \frac{\alpha_sN_c}{2\pi}\Phi_1(\nu')\Big]\langle\hat{\mathcal{U}}^{a_0}_{\rm conf}(\nu,z_0)\hat{\mathcal{V}}^{b_0}_{\rm conf}(\nu',z'_0)\rangle\end{aligned} \tag{27}$$

3. NLO Scattering of Conformal Dipoles and the NLO Amplitude

The last step is to find the NLO amplitude of the scattering of conformal dipoles $\hat{\mathcal{U}}^a_{\rm conf}(z_0,\nu)$ and $\hat{\mathcal{V}}^b_{\rm conf}(z'_0,\nu')$. First we need to write down the NLO BFKL evolution (as we discussed above the rapidity dependence is now encoded in the a-evolution):

$$2a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(z_1,z_2) \;=\; \int d^2z_3d^2z_4K(z_1,z_2;z_3,z_4)\;\hat{\mathcal{U}}^a_{\rm conf}(z_3,z_4) \tag{28}$$

where the kernel $K(z_1,z_2;z_3,z_4)$ in the first two orders has the form[11,12] (see also Ref.[13])

$$K_{\rm LO}(z_1,z_2;z_3,z_4) \;=\; \frac{\alpha_sN_c}{2\pi^2}\Big[\frac{z^2_{12}\delta^2(z_{13})}{z^2_{14}z^2_{24}} + \frac{z^2_{12}\delta^2(z_{24})}{z^2_{13}z^2_{23}} - \int d^2z\;\frac{z^2_{12}\delta^2(z_{13})\delta^2(z_{24})}{(z_1-z)^2(z_2-z)^2}\Big]$$

$$\begin{aligned}K_{\rm NLO}(z_1,z_2;z_3,z_4) \;&=\; -\frac{\alpha_sN_c}{4\pi}\frac{\pi^2}{3}K_{\rm LO}(z_1,z_2;z_3,z_4) + \frac{\alpha_s^2N_c^2}{8\pi^4z^4_{34}}\;\Big[\frac{z^2_{12}z^2_{34}}{z^2_{13}z^2_{24}}\Big\{\Big(1\\ &+ \frac{z^2_{12}z^2_{34}}{z^2_{13}z^2_{24} - z^2_{14}z^2_{23}}\Big)\ln\frac{z^2_{13}z^2_{24}}{z^2_{14}z^2_{23}} + 2\ln\frac{z^2_{12}z^2_{34}}{z^2_{14}z^2_{23}}\Big\} + 12\pi^2\zeta(3)z^4_{34}\delta(z_{13})\delta(z_{24})\Big]\end{aligned} \tag{29}$$

The eigenfunctions of the kernel K are given by Eq. (19) and the eigenvalues by the pomeron intercept (8).

$$\int d^2z_3 d^2z_4\ K(z_1,z_2;z_3,z_4)E_{\nu,n}(z_{30},z_{40}) \;=\; \omega(n,\nu)E_{\nu,n}(z_{10},z_{20}) \tag{30}$$

For the composite operators with definite conformal spin (22) the evolution equation (28) takes the simple form

$$2a\frac{d}{da}\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0) \;=\; \omega(\nu)\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0) \tag{31}$$

(Since Eq. (2) is a correlation functions of scalar currents, we need only the $n=0$ projection of this evolution).

The result of the evolution is

$$\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0) \;=\; (a/\tilde{a})^{\frac12\omega(\nu)}\hat{\mathcal{U}}^{a_0}_{\rm conf}(\nu,z_0) \tag{32}$$

where the endpoint of the evolution $\tilde{a}$ should be taken from the requirement that the amplitude of scattering of conformal dipoles with "normalization points" $\tilde{a}$ and b should not contain large logarithms of energy so it will serve as the initial point of the evolution. (This is similar to taking μ^2 around 1 GeV for the initial point of the DGLAP evolution). The amplitude of scattering of two conformal dipoles can be taken from Ref.[9]:

$$\begin{aligned}\langle\hat{\mathcal{U}}^{\tilde{a}}_{\rm conf}(\nu,z_0)\hat{\mathcal{V}}^b_{\rm conf}(\nu',z'_0)\rangle \;&=\; -\frac{\alpha_s^2N_c^2}{N_c^2-1}\frac{16\pi^2\nu^{-2}}{(1+4\nu^2)^2}\Big[\delta(z_0-z'_0)\delta(\nu+\nu')\frac{\Gamma\big(\frac12+i\nu\big)}{\Gamma\big(\frac12-i\nu\big)}\\ &+\frac{2^{1-4i\nu}\delta(\nu-\nu')}{\pi|z_0-z'_0|^{2-4i\nu}}\frac{\Gamma(1-i\nu)}{\Gamma(i\nu)}\Big]\Big[1+\frac{\alpha_sN_c}{2\pi}\Big(\chi(\nu)\big[\ln\tilde{a}b-i\pi+4C-\frac{2}{\nu^2+\frac14}\big]-\frac{\pi^2}{3}\Big)\Big].\end{aligned}$$

Using the above equation as an initial condition for the evolution (32) we get the following amplitude of scattering of two conformal dipoles:

$$\begin{aligned}&\langle\hat{\mathcal{U}}^a_{\rm conf}(\nu,z_0)\hat{\mathcal{V}}^b_{\rm conf}(\nu',z'_0)\rangle\\ &= i\frac{\alpha_s^2N_c^2}{N_c^2-1}\frac{(a+i\epsilon)^{\frac{\omega(\nu)}{2}}(b+i\epsilon)^{\frac{\omega(\nu)}{2}}-c.c.}{\sin\pi\omega(\nu)}\Big[1-\frac{\alpha_sN_c}{2\pi}\Big(\chi(\gamma)\big[\frac{8}{1+4\nu^2}-4C\big]+\frac{\pi^2}{3}\Big)\Big]\\ &\times\frac{16\pi^2}{\nu^2(1+4\nu^2)^2}\Big[\delta(z_0-z'_0)\delta(\nu+\nu')+\frac{2^{1-4i\nu}\delta(\nu-\nu')}{\pi|z_0-z'_0|^{2-4i\nu}}\frac{\Gamma\big(\frac12+i\nu\big)\Gamma(1-i\nu)}{\Gamma(i\nu)\Gamma\big(\frac12-i\nu\big)}\Big].\end{aligned} \tag{33}$$

Finally, substituting this amplitude in Eq. (27) we obtain

$$\begin{aligned}&(x-y)^4(x'-y')^4\langle T\{\hat{\mathcal{O}}(x)\hat{\mathcal{O}}^\dagger(y)\hat{\mathcal{O}}(x')\hat{\mathcal{O}}^\dagger(y')\}\rangle\\ &= i\frac{\alpha_s^2N_c^2}{N_c^2-1}\int d\nu\frac{(a_0+i\epsilon)^{\frac{\omega(\nu)}{2}}(b_0+i\epsilon)^{\frac{\omega(\nu)}{2}}-(-a_0-i\epsilon)^{\frac{\omega(\nu)}{2}}(-b_0-i\epsilon)^{\frac{\omega(\nu)}{2}}}{\sin\pi\omega(\nu)}\\ &\times\Big\{1-\frac{\alpha_sN_c}{2\pi}\Big(\chi(\nu)\Big[\frac{8}{1+4\nu^2}-4C\Big]+\frac{\pi^2}{3}\Big)\Big\}2\pi\nu\frac{\tanh\pi\nu}{\cosh^2\pi\nu}\int d^2z_0\\ &\times\Big(\frac{-\kappa^2}{(-2\kappa\cdot\zeta_0)^2}\Big)^{\frac12+i\nu}\Big(\frac{-\kappa'^2}{(-2\kappa'\cdot\zeta'_0)^2}\Big)^{\frac12-i\nu}\big[1+\frac{\alpha_sN_c}{2\pi}\Phi_1(\nu)\big]\big[1+\frac{\alpha_sN_c}{2\pi}\Phi_1(\nu)\big].\end{aligned} \tag{34}$$

Now it is easy to see that Eq. (34) coincides with Eq. (4) with

$$F(\nu) \;=\; \frac{N_c^2}{N_c^2-1}\frac{4\pi^4\alpha_s^2}{\cosh^2\pi\nu}\Big\{1+\frac{\alpha_s N_c}{\pi}\Big[-\frac{2\pi^2}{\cosh^2\pi\nu}+\frac{\pi^2}{2}-\frac{8}{1+4\nu^2}\Big]+O(\alpha_s^2)\Big\}$$

which gives the pomeron residue in the next-to-leading order(recall that $a_0=\kappa^{-2}+i\epsilon$ and $b_0={\kappa'}^{-2}+i\epsilon$). The lowest-order term in this formula agrees with the leading-order impact factor (10) calculated in Ref. [6].

4. Conclusions

The main result of the paper is that the rapidity factorization and high-energy operator expansion in color dipoles works at the NLO level. There are many examples of the factorization which are fine at the leading order but fail at the NLO level. We believe that the high-energy OPE has the same status as usual light-cone expansion in light-ray operators so one can calculate the high-energy amplitudes level by level in perturbation theory. As an outlook we intend to apply the NLO high-energy operator expansion for the description of QCD amplitudes. There are many papers devoted to analysis of the high-energy amplitudes in QCD at the NLO level (see e.g Refs. [14, 15, 16]) but all of them use traditional calculation of Feynman diagrams in momentum space. In our opinion, the high-energy OPE in color dipoles is technically more simple and gives us an opportunity to use an approximate tree-level conformal invariance in QCD. Although our composite dipole (14) is no longer conformal in QCD, we believe that the effects due to the running coupling calculated in Refs. [17, 18] can be incorporated in some sort of structure resembling the formula (4) for $\mathcal{N}=4$ SYM. As an application of the machinery developed here we calculated the photon impact factor[19] for the structure function $F_2(x)$ of deep inelastic scattering which competes the calculation of small-x structure functions at the NLO level.

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