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Rapidity-only TMD factorization at one loop

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Abstract: Typically, a production of a particle with a small transverse momentum in hadron-hadron collisions is described by CSS-based TMD factorization at moderate Bjorken $x_B \sim 1$ and by k_T-factorization at small x_B. A uniform description valid for all x_B is provided by rapidity-only TMD factorization developed in a series of recent papers at the tree level. In this paper the rapidity-only TMD factorization for particle production by gluon fusion is extended to the one-loop level.

Keywords: Deep Inelastic Scattering or Small-x Physics, Resummation

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JHEP03(2023)029

Contents

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1 Introduction

Rapidity factorization and rapidity evolution are main tools for study of QCD processes at small x [1]. On the other hand, at moderate x conventional methods are based on CSS equation [2] and closely related SCET approach (see refs. [3] and [4] reviews). However, with the advent of EIC accelerator the region of energies intermediate between low and moderate x needs to be investigated. One of the ideas is to extend rapidity factorization methods beyond the "pure" small-x region. In a series of recent papers A. Tarasov, G.A. Chirilli and the author applied the method of rapidity-only factorization to processes of particle production in hadron-hadron collisions in the so-called Sudakov region where transverse momentum of produced particle(s) is much smaller than their invariant mass. The typical examples of such processes are the Drell-Yan process or Higgs production by gluon fusion. At moderate x such processes are studied by CSS-based TMD factorization [3, 5]

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp}\mathcal{D}_{f/A}(x_A,b_\perp,\eta_a)\mathcal{D}_{f/B}(x_B,b_\perp,\eta_2)$$
$$\times\,\sigma_{ff\to X}(\eta,\eta_a,\eta_b) + \text{power corrections} + \text{Y}-\text{terms} \tag{1.1}$$

where $\eta = \frac{1}{2}\ln\frac{q^+}{q^-}$ is the rapidity, $\mathcal{D}_{f/h}(x,z_\perp,\eta_i)$ is the TMD density of a parton f in hadron h with rapidity cutoff η_i, and $\sigma_{ff\to X}(\eta,\eta_a,\eta_b)$ is the cross section of production of particle(s) X of invariant mass $m_X^2 = q^2 \equiv Q^2 \gg q_\perp^2$ in the scattering of two partons. The TMD parton densities are regularized with a combination of UV and rapidity cutoffs and the relevant Sudakov logarithms are obtain by solving double evolution with respect to μ_{UV} and the rapidity cutoffs η_i [3].

It should be emphasized that the CSS approach and hence the formula (1.1) are valid at $x_A \sim x_B \sim 1$. At small x_A and/or x_B one should resort to other factorization methods. As I mentioned above, a perspective approach is to apply methods based on rapidity-only factorization used in small-x/BFKL physics. In a series of papers [6–9] A. Tarasov and the author applied rapidity-only factorization approach to get for the first time power corrections $\sim \frac{q_\perp^2}{Q^2}$ restoring EM gauge invariance of DY hadronic tensor both at moderate and small x. Also, in recent papers [9, 10] G.A. Chirilli and the author calculated the rapidity-only evolution of TMD operators, again both at moderate and small x. In the present paper I calculate coefficient function multiplying two TMD distributions at the one-loop level. This completes the task of performing the rapidity-only factorization at the one-loop accuracy.

Apart from requirement $Q^2 = x_A x_B s \gg q_\perp^2$, in this paper it is assumed that

$$\frac{Q^2}{q_\perp^2} \gg \frac{q_\perp^2}{m_N^2} \tag{1.2}$$

The region (1.2) can be understood in terms of rescaling $s \to \zeta s_0$, $\zeta \to \infty$ with $q_\perp^2$ fixed:

$$s \sim \zeta s_0, \qquad q_\perp \sim O(\zeta^0) \tag{1.3}$$

It should be emphasized that we will not use the small-x approximation $s \gg Q^2$ so our formulas are correct both at $x \ll 1$ and $x \sim 1$ provided that the condition (1.2) is satisfied. Thus, at $x \sim 1$ our rapidity-evolution formulas should be equivalent to usual CSS approach, although the exact relation between our rapidity evolution and CSS double evolution in rapidity and UV cutoff remains to be established, see the discussion in the Conclusions section.

The rapidity evolution of TMDs $\mathcal{D}_{f/A}(x_A, b_\perp, \eta_a), \mathcal{D}_{f/B}(x_B, b_\perp, \eta_b)$ should match the one-loop rapidity evolution of the coefficient function $\sigma_{ff\to H}(\eta, \eta_a, \eta_b)$ so that the cutoffs η_a and η_b disappear from physical amplitude. I will demonstrate that the result for the coefficient function in eq. (1.1) for rapidity-only gluon TMD factorization is proportional to ($\gamma \equiv \gamma_E \simeq 0.577$)

$$\exp\left\{\frac{\alpha_s N_c}{2\pi}\left[\left(\ln\frac{b_\perp^2 s}{4} + \eta_a + \eta_b\right)^2 - 2(\ln x_A - \eta_a + \gamma)(\ln x_B - \eta_b + \gamma) + \pi^2\right]\right\} \tag{1.4}$$

and check that η_a, η_b dependence matches the rapidity-only TMD evolution obtained in refs. [9, 10].

The paper is organized as follows. In section 2 I define hadronic tensor and TMD operators for particle production by gluon fusion. In section 3 I discuss separation of functional integral for particle production in three integrals according to the rapidity of the fields involved. In section 4 I set up the calculation of the coefficient function in front of TMD operators by computing diagrams in two background fields. Sections 5, 6, 7, and 8 are devoted to calculation of these diagrams at the one-loop level. The result of the calculation and check of matching to evolution of TMD operators are presented in section 9. The necessary technical and sidelined results are presented in the appendix.

2 TMD factorization for particle production by gluon fusion

Let us consider production of an (imaginary) scalar particle Φ by gluon fusion in proton-proton scattering. The particle is connected to gluons by the vertex

$$\mathcal{L}_\Phi = g_\Phi \int d^4x\ \Phi(x) g^2 F^2(x), \qquad F^2(x) \equiv F^a_{\mu\nu}(x) F^{a\mu\nu}(x) \tag{2.1}$$

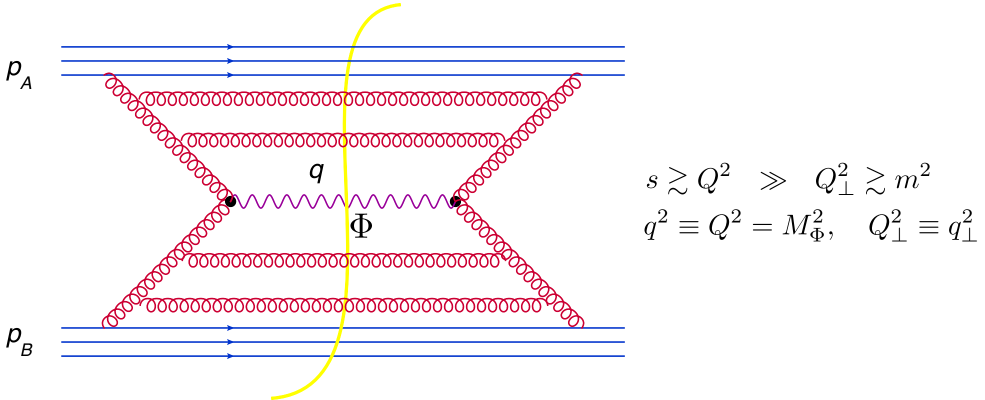

Figure 1. Particle production by gluon-gluon fusion.

This is a $\frac{m_H}{m_t} \ll 1$ approximation for Higgs production via gluon fusion at the LHC. The differential cross section of Φ production is defined by the "hadronic tensor" $W(p_A, p_B, q)$

$$\begin{aligned} W(p_A, p_B, q) &\stackrel{\text{def}}{=} \frac{N_c^2 - 1}{16} \sum_X \int d^4x\ e^{-iqx} \langle p_A, p_B | g^2 F^2(x) | X \rangle \langle X | g^2 F^2(0) | p_A, p_B \rangle \\ &= \frac{N_c^2 - 1}{16} \int d^4x\ e^{-iqx} \langle p_A, p_B | g^4 F^2(x) F^2(0) | p_A, p_B \rangle \end{aligned} \tag{2.2}$$

where the factor $\frac{N_c^2-1}{16}$ is added to simplify factorization formulas. As usual, $\sum_X$ denotes the sum over full set of "out" states.

We will study the hadronic tensor (2.2) with non-zero momentum transfer in t-channel defined as a matrix element of the operator

$$\hat{W}(x_1, x_2) \ \equiv \ \frac{N_c^2 - 1}{16} g^4 F^a_{\mu\nu}(x_2) F^{\mu\nu;a}(x_2) F^b_{\lambda\rho}(x_1) F^{\lambda\rho;b}(x_1) \tag{2.3}$$

between initial and final states with slightly non-equal momenta

$$W(p_A, p_B, p'_A, p'_B; x_1, x_2) = \langle p'_A, p'_B | \hat{W}(x_1, x_2) | p_A, p_B \rangle \tag{2.4}$$

where

$$\begin{aligned} p_A &= p_1 + \frac{m^2 + l_\perp^2}{s} p_2 - \frac{l_\perp}{2}, \qquad & p'_A &= p_1 + \frac{m^2 + l_\perp^2}{s} p_2 + \frac{l_\perp}{2}, \\ p_B &= p_2 + \frac{m^2 + l_\perp^2}{s} p_1 + \frac{l_\perp}{2}, \qquad & p'_B &= p_2 + \frac{m^2 + l_\perp^2}{s} p_1 - \frac{l_\perp}{2} \end{aligned} \tag{2.5}$$

Here p_1 and p_2 are light-like vectors close to p_A and p_B, respectively.[1] We will use light-cone coordinates with respect to the frame where $p_1 = (\frac{\sqrt{s}}{2}, 0, 0, \frac{\sqrt{s}}{2})$ and $p_2 = (\frac{\sqrt{s}}{2}, 0, 0, -\frac{\sqrt{s}}{2})$ so that $p_1^+ = p_2^- = \sqrt{\frac{s}{2}}$, $p_2^+ = p_1^- = 0$ and $p_{1_\perp} = p_{2_\perp} = 0$.

[1] We assume that $t = -l_\perp^2 \sim m_N^2$. If there is a longitudinal component of momentum transfer, one can redefine p_1 and p_2 in such a way that with respect to new p'_1 and p'_2 the formulas are those of eq. (2.5).

JHEP03(2023)029

The kinematical region (1.2) in the coordinate space translates to

$$x_\parallel^2 \ll x_{12_\perp}^4 m_N^2 \tag{2.6}$$

where $x_\parallel^2 \equiv 2x_{12}^+ x_{12}^-$. Also, we must assume $x_{12_\perp}^2 \le m_N^{-2}$ so that the coupling constant $\alpha_s(x_\perp)$ is a valid perturbative parameter.

In the coordinate space, TMD factorization (1.1) for hadronic tensor in eq. (2.4) should look like

$$\begin{aligned}
&\frac{g^4}{16}(N_c^2-1)\langle p_A', p_B'|F^a_{\mu\nu}F^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle \\
&\quad = \int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \mathfrak{C}(x_2,x_1;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t) \\
&\qquad \times \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(w_2^+,w_{2_\perp};w_1^+,w_{1_\perp})|p_B\rangle + \ldots
\end{aligned} \tag{2.7}$$

where the dots stand for power corrections $\sim \frac{q_\perp^2}{Q^2}$ and

$$\hat{\mathcal{O}}_{ij}(z_2^+,z_{2_\perp};z_1^+,z_{1_\perp}) = \mathcal{F}_i^a(z_2)[z_2-\infty^+,z_1-\infty^+]^{ab}\mathcal{F}_j^b(z_1)\Big|_{z_2^-=z_1^-=0}, \tag{2.8}$$

$$\hat{\mathcal{O}}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp}) = \mathcal{F}_i^a(z_2)[z_2-\infty^-,z_1-\infty^-]^{ab}\mathcal{F}_j^b(z_1)\Big|_{z_2^+=z_1^+=0},$$

$$\mathcal{F}^{i,a}(z_\perp,z^+) \equiv gF^{-i,m}(z)[z^+,-\infty^+]_z^{ma}\Big|_{z^-=0},$$

$$\mathcal{F}^{i,a}(z_\perp,z^-) \equiv gF^{+i,m}(z)[z^-,-\infty^-]^{ma}\Big|_{z^+=0}$$

are gluon TMD operators (the precise definitions of rapidity-only cutoffs $\sigma_a = e^{\eta_a}$ and $\sigma_b = e^{\eta_b}$ for gluon TMDs will be given later). Hereafter, we use the notation

$$[x,y] \equiv \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+y-uy)} \tag{2.9}$$

for the straight-line ordered gauge link between points x and y, and space-saving notations

$$[x^+,y^+]_z \equiv [x^+ + z_\perp, y^+ + z_\perp], \qquad [x^-,y^-]_z \equiv [x^- + z_\perp, y^- + z_\perp] \tag{2.10}$$

The coefficient function $\mathfrak{C}$ represents a Fourier transform of $\sigma_{ff\to H}(\eta,\eta_1,\eta_2)$ in eq. (1.1) with $\eta_i = \ln\sigma_i$. The normalization in the l.h.s. of eq. (2.7) is chosen in such a way that $\mathfrak{C} = 1 + \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1 + O(\alpha_s^2)$. The goal of this paper is to find the one-loop coefficient function $\mathfrak{C}_1(x_2,x_1;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_a,\sigma_b)$ and check that the evolution of this coefficient function matches the evolutions of TMD operators.

3 TMD factorization from functional integral

The hadronic tensor (2.2) can be represented by double functional integral

$$\begin{aligned}
&W(p_A,p_B,p_A',p_B';x_2,x_1) = \sum_X \langle p_A',p_B'|g^2F^2(x_2)|X\rangle\langle X|g^2F^2(x_1)|p_A,p_B\rangle \\
&= \lim_{\substack{t_f\to\infty \\ t_i\to-\infty}} g^4 \int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu \int^{\tilde{\psi}(t_f)=\psi(t_f)} D\tilde{\bar{\psi}} D\tilde{\psi} D\bar{\psi} D\psi\ e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})} e^{iS_{\rm QCD}(A,\psi)} \\
&\quad \times \Psi^*_{p_A'}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i))\Psi^*_{p_B'}(\vec{\tilde{A}}(t_i),\tilde{\psi}(t_i))\tilde{F}^2(x_2)F^2(x_1)\Psi_{p_A}(\vec{A}(t_i),\psi(t_i))\Psi_{p_B}(\vec{A}(t_i),\psi(t_i))
\end{aligned} \tag{3.1}$$

Here the fields A, ψ correspond to the amplitude $\langle X|F^2(x_1)|p_A, p_B\rangle$, the fields $\tilde{A}, \tilde{\psi}$ correspond to complex conjugate amplitude $\langle p'_A, p'_B|F^2(x_2)|X\rangle$ and $\Psi_p(\vec{A}(t_i), \psi(t_i))$ denote the proton wave function at the initial time t_i. The boundary conditions $\tilde{A}(t_f) = A(t_f)$ and $\tilde{\psi}(t_f) = \psi(t_f)$ reflect the sum over all states X, cf. refs. [11–13]. We will also use the notation

$$\{x, y\} \equiv \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu \tilde{A}_\mu(ux+y-uy)} \tag{3.2}$$

and similar notations like eq. (2.10) for gauge links in the left sector.

For calculations in the momentum space we will use Sudakov variables related to light-cone components $p^+, p^-, p_\perp$ by $\alpha \equiv p^+/\varrho$ and $\beta \equiv p^-/\varrho$ where $\varrho \equiv \sqrt{s/2}$. In terms of Sudakov variables $p \cdot q = (\alpha_p\beta_q + \alpha_q\beta_p)\frac{s}{2} - (p, q)_\perp$ where $(p, q)_\perp \equiv -p_i q^i$. Throughout the paper, the sum over the Latin indices $i, j \ldots$ runs over the two transverse components while the sum over Greek indices runs over the four components as usual. Also, since we use Sudakov variables it is convenient to change the notations of gluon momentum fractions to

$$\alpha_a \equiv x_A, \quad \beta_b \equiv x_B \tag{3.3}$$

to avoid confusion with coordinates.

Following refs. [6, 7], to derive the factorization formula we separate gluon (and quark) fields in the functional integral (3.1) into three sectors: "projectile" fields A_μ, ψ_a with $|\beta| < \sigma_p \equiv \sigma_a$, " target" fields B_μ, ψ_b with $|\alpha| < \sigma_t \equiv \sigma_b$ and "central rapidity" fields C_μ, ψ with $|\alpha| > \sigma_t$ and $|\beta| > \sigma_p$. Let us specify the values of the TMD cutoffs σ_p and σ_t in our factorization. Needless to say, we should take $\sigma_t \ll \alpha_a$ and $\sigma_p \ll \beta_b$. Moreover, as discussed in ref. [9], power corrections to rapidity evolution of TMDs are $\sim \frac{Q_\perp^2}{\beta_b \sigma_t s}$ so we need to assume $\sigma_t \beta_b s \gg Q_\perp^2$, and similarly $\sigma_p \alpha_a s \gg Q_\perp^2$ for the projectile. Next, as we shall see below, it is convenient to calculate coefficient function (4.19) at $m_N^2 \gg \mu_\sigma^2 \equiv \sigma_p \sigma_t s$ so finally we take the region of σ_p and σ_t as follows

$$\alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s}, \quad \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s}, \quad m_N^2 \gg \mu_\sigma^2 \equiv \sigma_p\sigma_t s \gg \frac{Q_\perp^4}{Q^2} \tag{3.4}$$

Note that due to eq. (1.2) we can choose μ_σ^2 between m_N^2 and parametrically small $\frac{Q_\perp^4}{Q^2}$. In terms of rescaling (1.3) this means that we can choose $\sigma_p, \sigma_t \sim \zeta^{-\frac{3}{4}} \sim (\frac{Q_\perp}{Q})^{3/2}$ so that

$$\mu_\sigma^2 \sim \zeta^{-1/2} \quad \Leftrightarrow \quad 1 \gg \frac{\mu_\sigma^2}{Q_\perp^2} \gg \frac{Q_\perp^2}{Q^2} \sim \zeta^{-1} \tag{3.5}$$

and both conditions in eq. (3.4) are satisfied.

In this paper we are calculating logarithmical corrections so the power corrections due to the small parameters $\frac{Q_\perp^2}{\sigma_p \alpha'_a s}, \frac{Q_\perp^2}{\sigma_t \beta'_b s}$ will be systematically neglected. The convenient notations for small parameters are

$$\lambda \equiv \frac{Q_\perp^2}{Q^2} \sim \frac{1}{\zeta} \ll 1, \quad \lambda_p \equiv \frac{Q_\perp^2}{|\alpha_a|\sigma_p s} \sim \zeta^{-\frac{1}{4}} \ll 1, \quad \lambda_t \equiv \frac{Q_\perp^2}{\sigma_t|\beta_b|s} \sim \zeta^{-\frac{1}{4}} \ll 1 \tag{3.6}$$

In these notations the last condition in eq. (3.4) translates to

$$\lambda_p\lambda_t \gg \lambda \tag{3.7}$$

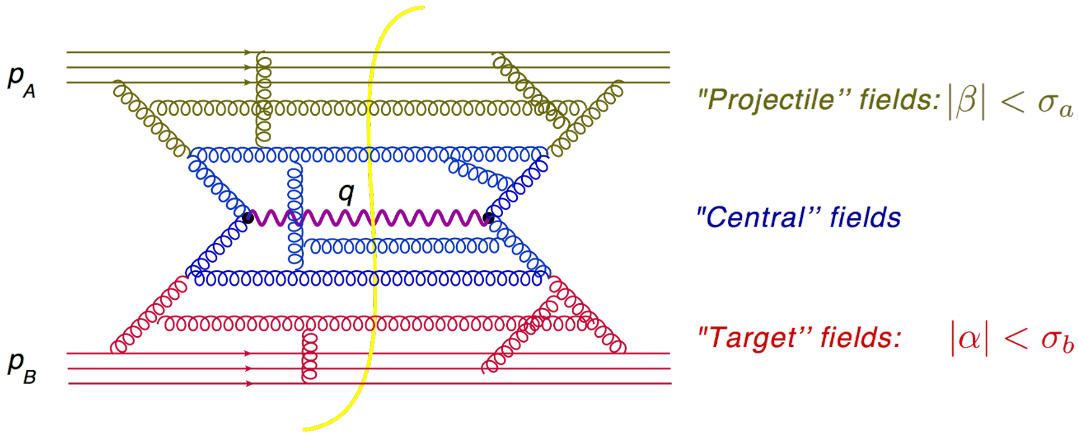

Figure 2. Rapidity factorization for particle production.

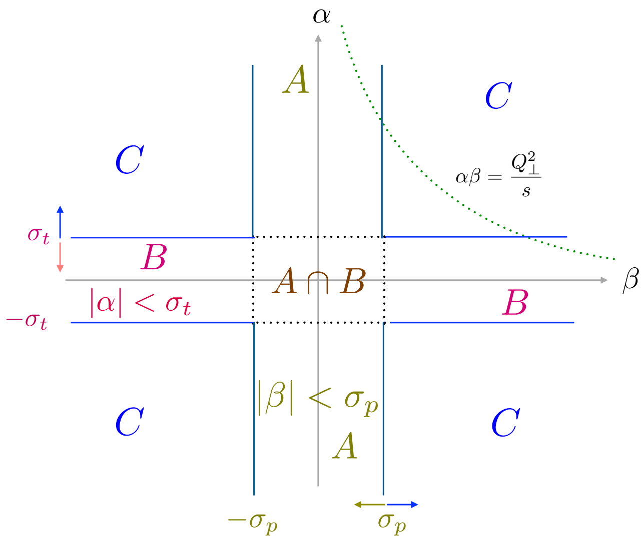

Figure 3. Regions of factorization in the momentum space.

Thus, in terms of rescaling (1.3) we neglect all power corrections $\sim 1/\zeta^{1/4}$ or smaller.

Note that while central fields are well separated from projectile and target fields, the latter have an intersection when both α and β are small, $\alpha \leq \sigma_t \sim \zeta^{-\frac{3}{4}}$ and $\beta \leq \sigma_p \sim \zeta^{-\frac{3}{4}}$, see figure 3. Depending on the scale of characteristic transverse momenta they are called Glauber gluons (if $p_\perp \gg p^+, p^- \sim \zeta^{-\frac{1}{4}}$) or soft gluons (if $p_\perp \sim p^+, p^-$). We will denote both of them by notation $\mathscr{C} = A \cap B$ and call them soft/Glauber (sG) gluons. We will discuss later that Glauber gluons do not contribute to factorization (1.1) [2, 3] and soft gluons form a soft factor which is a power correction with our rapidity cutoffs.

To discuss interaction of central gluons with either A, B, or $\mathscr{C}$ fields it is convenient do denote all the latter by notation $\mathscr{A} = A \cup B$ which means all fields with $|\alpha| < \sigma_a$ and/or $|\beta| < \sigma_b$.

JHEP03(2023)029

We get

$$W(p_A,p_B,p'_A,p'_B;x_1,x_2) = \lim_{t_i\to-\infty} g^4 \int \mathscr{D}\Phi_{\mathscr{A}} \mathscr{D}\Phi_C \; \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i) \\ \times \Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)(\tilde{F}^{\mathscr{A}}+\tilde{F}^C)^2(x_2)(F^{\mathscr{A}}+F^C)^2(x_1) \tag{3.8}$$

where $F^{\mathscr{A}}$ is the usual field tensor for $\mathscr{A}$ field and $F^C_{\mu\nu} \equiv F_{\mu\nu}(\mathscr{A}+C) - F_{\mu\nu}(\mathscr{A})$. Also, we use shorthand notations

$$\begin{aligned} &\Psi_{p_A}(t_i) \equiv \Psi_{p_A}(\vec{A}(t_i),\psi(t_i)), && \Psi^*_{p_A}(t_i) \equiv \Psi^*_{p_A}(\vec{\tilde{A}}(t_i),\tilde{\psi}_a(t_i)) \\ &\Psi_{p_B}(t_i) \equiv \Psi_{p_B}(\vec{B}(t_i),\psi_b(t_i)), && \Psi^*_{p_B}(t_i) \equiv \Psi^*_{p_B}(\vec{\tilde{B}}(t_i),\tilde{\psi}_b(t_i)), \end{aligned} \tag{3.9}$$

for projectile and target protons' wave functions, and

$$\int \mathscr{D}\Phi_{\mathscr{A}} \equiv \int^{\tilde{\mathscr{A}}(t_f)=\mathscr{A}(t_f)} D\tilde{\mathscr{A}}_\mu D\mathscr{A}_\mu \\ \times \int^{\tilde{\psi}_{\mathscr{A}}(t_f)=\psi_{\mathscr{A}}(t_f)} D\tilde{\psi}_{\mathscr{A}} D\psi_{\mathscr{A}} e^{-iS_{\mathrm{QCD}}(\tilde{\mathscr{A}},\tilde{\psi}_{\mathscr{A}})+iS_{\mathrm{QCD}}(\mathscr{A},\psi_{\mathscr{A}})}, \tag{3.10}$$

$$\int \mathscr{D}\Phi_C \equiv \int^{\tilde{C}(t_f)=C(t_f)} D\tilde{C}_\mu DC_\mu \int^{\tilde{\psi}_C(t_f)=\psi_C(t_f)} D\psi_C D\tilde{\psi}_C \; e^{-i\tilde{S}_C+iS_C} \tag{3.11}$$

for functional integrals. In the last line $S_C \equiv S_{\mathrm{QCD}}(\mathscr{A}+C) - S_{\mathrm{QCD}}(\mathscr{A})$.

Our goal is to integrate over central fields and get the amplitude in the factorized form, as a (sum of) products of functional integrals over A fields representing projectile matrix elements (TMDs) and functional integrals over B fields representing target matrix elements. In the spirit of background-field method, we "freeze" projectile and target fields and get a sum of diagrams in these external fields. Since $|\beta| < \sigma_p$ in the projectile fields and $|\alpha| < \sigma_t$ in the target fields, at the tree-level one can set with power accuracy $\beta = 0$ for the projectile fields and $\alpha = 0$ for the target fields - the corrections will be $O(\frac{m^2}{\sigma_p s})$ and $O(\frac{m^2}{\sigma_t s})$.[2] Beyond the tree level, the integration over C fields will produce the logarithms of the cutoffs σ_p and σ_t which cancel with the corresponding logs in gluon TMDs of the projectile and the target, see the discussion in section 9.

[2]Indeed, suppose an "A" gluon with momentum k_a interacts with a "C" gluon with momentum k_C. The resulting propagator is $[(\alpha'_a+\alpha_c)(\beta_a+\beta_c)s - (k_a+k_c)^2_\perp]^{-1}$ and one can neglect β_a with $\beta_a/\beta_c \leq \frac{m^2/s}{\sigma_p}$ accuracy. Similarly, for the target fields one gets the accuracy $\alpha_b/\alpha_c \leq \frac{m^2/s}{\sigma_t}$.

4 Coefficient function from background-field diagrams

We will calculate the coefficient function $\mathfrak{C}$ in the first non-trivial order in perturbation theory: $\mathfrak{C} = 1 + \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1$. The desired formula looks like that

$$\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p_A', p_B'|g^2 F^a_{\mu\nu}F^{a\mu\nu}(x_2) g^2 F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A, p_B\rangle \\
&= \int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p_A'}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p_B'}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \\
&\quad + \int dz_1^- dz_{1_\perp} dz_2^- dz_{2_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t) \\
&\quad\ \times \mathcal{O}^{\sigma_p}_{ij}(z_2^-, z_{2_\perp}; z_1^-, z_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+, z_{2_\perp}; z_1^+, z_{1_\perp}) + \dots\Big]
\end{aligned} \tag{4.1}$$

where dots stand for higher orders in perturbation theory and/or power corrections.

The standard way to obtain a coefficient function is to rewrite eq. (4.1) as an operator formula

$$\begin{aligned}
&\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t) \\
&\times \hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-, z_{2_\perp}; z_1^-, z_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+, z_{2_\perp}; z_1^+, z_{1_\perp}) + \dots \qquad (4.2)\\
&= \frac{g^4}{16}(N_c^2-1)F^a_{\mu\nu}F^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1) - \hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}),
\end{aligned}$$

calculate the l.h.s. and r.h.s. between two initial and two final gluon states and compare. However, the amplitudes with real gluons have infrared divergencies so we will consider amplitudes with virtual gluon tails instead. As is well known, a gauge-invariant way to write down matrix elements "between virtual gluons" is to consider l.h.s. and r.h.s. of eq. (4.2) in a suitable background field.

Following the analysis of rapidity factorization (3.8) in refs. [6, 7], we choose the background field as a result of interaction of "projectile" field $\bar{A}$ and "target" field $\bar{B}$ where the "projectile" field $\bar{A}(z)$ depends only on $z_\perp, z_-$ (corresponding to β_a=0) and "target" field $\bar{B}(z)$ depends only on $z_\perp, z_+$ (corresponding to α_b=0).[3] As demonstrated in ref. [6], in this case one can always choose the gauge where $\bar{A}^- = \bar{B}^+ = 0$. Moreover, since we are after logarithmical corrections to coefficient in front of the operators (2.8), it is convenient to take $\bar{A}^+ = \bar{B}^- = 0$.[4] Thus, we choose "projectile" and "target" fields in the form

$$\begin{aligned}
g\bar{A}_i &= U_i(x^-, x_\perp), \qquad A_+ = A_- = 0, \qquad g\bar{B}_i = V_i(x^+, x_\perp), \qquad B_+ = B_- = 0 \\
g\bar{F}^{+i}(A) &= \partial^+ U_i \equiv U^{+i}(x^-, x_\perp), \qquad g\bar{F}^{+i}(B) = \partial^- V_i \equiv V^{-i}(x^+, x_\perp)
\end{aligned} \tag{4.3}$$

We assume that the "projectile" and "target" fields $A(z_-, z_\perp)$ and $B(z_+, z_\perp)$ satisfy standard YM equations

$$(\partial_i - i[U_i,)U^{-i} = g^2\bar{\psi}_A\gamma^-\psi_A, \qquad (\partial_i - i[V_i,)V^{+i} = g^2\bar{\psi}_B\gamma\gamma^+\psi_B \tag{4.4}$$

[3] See ref. [14] for similar approach.

[4] The general case with background fields $\bar{A}^-, \bar{B}^+ \neq 0$ is relevant for obtaining power corrections to eq. (2.7), see the discussion in refs. [6, 7].

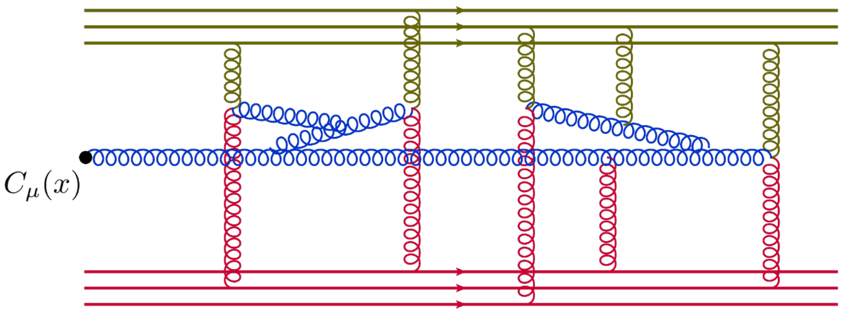

Figure 4. Typical diagram for the classical field with projectile/target sources. The Green functions of the central fields are given by retarded propagators.

and only "good" components of background quark fields $\gamma^-\psi_A(z_-,z_\perp)$ and $\gamma^+\psi_B(z_+,z_\perp)$ do exist.

The "interaction" field $\mathbb{A}$ is defined as is a classical field solving classical YM equations

$$\mathbb{D}^\nu \mathbb{F}^a_{\mu\nu} = \sum_f g\bar{\Psi}^f t^a \gamma_\mu \Psi^f, \qquad (\not{P} + m_f)\Psi^f = 0 \tag{4.5}$$

with boundary conditions[5]

$$\begin{aligned} \mathbb{A}_\mu(x) &\overset{x^+\to-\infty}{=} \bar{A}_\mu(x^-,x_\perp), \qquad & \Psi(x) &\overset{x^+\to-\infty}{=} \psi_A(x^-,x_\perp) \\ \mathbb{A}_\mu(x) &\overset{x^-\to-\infty}{=} \bar{B}_\mu(x^+,x_\perp), \qquad & \Psi(x) &\overset{x^-\to-\infty}{=} \psi_B(x^+,x_\perp) \end{aligned} \tag{4.6}$$

reflecting the fact that at $t \to -\infty$ we have only incoming hadrons with "A" and "B" fields. An important property of the functional integral (3.8) is that since the projectile fields $\bar{\tilde{A}}$ and $\bar{A}$ should coincide at $t \to \infty$ (see eq. (3.10)) and since they do not depend on x^+, they coincide everywhere. Similar property is valid for the target fields so we have the condition

$$\bar{\tilde{A}} = \bar{A}, \quad \bar{\tilde{B}} = \bar{B} \tag{4.7}$$

As proved in refs. [6, 7] the solution of classical equations (4.5) has the same property: $\tilde{\mathbb{A}} = \mathbb{A}$. In terms of perturbative diagrams the solution of eq. (4.5) is given by the sum $\bar{A} + \bar{B} + \bar{C}$ where the "correction" field $\bar{C}$ is given by the sum diagrams of the type shown in figure 4 with *retarded* propagators.

The solution of YM equations (4.5) in general case is yet unsolved problem, especially important for description of scattering of two heavy nuclei in semiclassical approximation. Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ one can construct the "correction" field $\bar{C}$ as a series in this small parameter. The explicit form of the expansion of correction field $\bar{C}$ in powers of this small parameter is presented in refs. [6, 7]. We will need only one term in this expansion shown in eq. (E.1) below.

[5]It is convenient to fix redundant gauge transformations by requirements $\bar{A}_i(-\infty_\bullet, x_\perp) = 0$ for the projectile and $\bar{B}_i(-\infty_*, x_\perp) = 0$ for the target, see the discussion in ref. [15].

JHEP03(2023)029

First, let us present the estimates of relative strength of different components of projectile and target fields in our $Q^2 \gg q_\perp^2$ kinematics from ref. [6]

$$\begin{aligned} &U^i, V^i \sim Q_\perp, \qquad U^{+i} = \partial^+ U^i \sim Q_\perp \sqrt{s}, \qquad V^{-i} = \partial^- V^i \sim Q_\perp \sqrt{s} \\ &U^{ij} = \partial^i U^j - \partial^j U_i - i[U^i, U^j] \sim Q_\perp^2, \qquad V^{ij} = \partial^i V^j - \partial^j V_i - i[V^i, V^j] \sim Q_\perp^2 \end{aligned} \tag{4.8}$$

Note that eq. (4.8) means that characteristic scales of projectile fields are such that extra ∂^+ brings $\sqrt{s}$ and extra $\bar{D}_i$ is $\sim q_\perp$ so $\partial^+ \gg \bar{D}^i$ for the projectile fields. Similarly, for the target fields $\partial^- \gg \bar{D}^i$. The characteristic longitudinal distances are $z^+ \sim 1/\sqrt{s}$ for the projectile fields and $z^- \sim 1/\sqrt{s}$ for the target ones while the characteristic transverse distances are $\sim 1/Q_\perp$ for both of them. Also, in sorting out power corrections according to rescaling parameter ζ in eq. (1.3), we do not distinguish between $\frac{m_N^2}{Q^2}$, $\frac{q_\perp^2}{Q^2}$, $\frac{m_N^2}{s}$, and $\frac{q_\perp^2}{s}$ and use the common notation

$$O\left(\frac{m_\perp^2}{s}\right) \sim O\left(\frac{m_N^2}{Q^2}, \frac{q_\perp^2}{Q^2}, \frac{m_N^2}{s}, \frac{q_\perp^2}{s}\right)$$

and similarly for other ratios.

The relevant terms in the expansion of correction fields $\bar{C}$ are [6]

$$\begin{aligned} \bar{C}^-(x) &= \frac{1}{2\varrho} \int dz \left(x|\frac{1}{\alpha + i\epsilon}|z\right) [U_k(z^-, z_\perp), V^k(z^+, z_\perp)] \\ &= -\frac{i}{2g} \int_{-\infty}^{x^+} dx'^+ \int_{-\infty}^{x^-} dx'^- (x - x')^- [U_k^+(x'^-, x_\perp), V^{-k}(x'^+, x_\perp)] \sim \frac{m_\perp^2}{\sqrt{s}}, \\ \bar{C}^+(x) &= -\frac{1}{2\varrho} \int dz \left(x|\frac{1}{\beta + i\epsilon}|z\right) [U_k(z^-, z_\perp), V^k(z^+, z_\perp)] \\ &= \frac{i}{2g} \int_{-\infty}^{x^-} dx'^- \int_{-\infty}^{x^+} dx'^+ (x - x')^+ [U_k^+(x'^-, x_\perp), V^{-k}(x'^+, x_\perp)] \sim \frac{m_\perp^2}{\sqrt{s}} \\ \bar{C}^i(x) &= \frac{-i}{2g} \int_{-\infty}^{x^-} dx'^- \int_{-\infty}^{x^+} dx'^+ \Big([U^j(x'^-, x_\perp), V_{ij}(x'^+, x_\perp)] + \partial^j [U_i(x'^-, x_\perp), V_j(x'^+, x_\perp)] \\ &\quad + [V^j(x'^+, x_\perp), U_{ij}(x'^-, x_\perp)] + \partial^j [V_i(x'^+, x_\perp), U_j(x'^-, x_\perp)] \Big) \sim \frac{m_\perp^3}{s} \end{aligned} \tag{4.9}$$

where we used

$$U^i(x^-, x_\perp) = \int_{-\infty}^{x^-} dx'^- U^{+i}(x'^-, x_\perp), \qquad V^i(x^+, x_\perp) = \int_{-\infty}^{x^+} dx'^+ V^{+i}(x'^+, x_\perp) \tag{4.10}$$

It should be noted that in expressions (4.9), (E.1) we neglected terms $\sim U_i U_j V_k$ and $U_i V_j V_k$ since they are proportional to $F_{\xi\eta}^3$ and hence cannot contribute to our coefficient function.

Thus, we need to calculate l.h.s. and r.h.s. of eq. (4.2) in the background of the field

$$\mathbb{A} \simeq \bar{A} + \bar{B} + \bar{C} \tag{4.11}$$

given by eqs. (4.3) and (E.1)

$$\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)$$
$$\times \langle \hat{\mathcal{O}}_{ij}^{\sigma_p}(z_2^-, z_{2_\perp}; z_1^-, z_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+, z_{2_\perp}; z_1^+, z_{1_\perp})\rangle_{\mathbb{A}} + \ldots$$
$$= \frac{N_c^2-1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$$
$$- \langle \hat{\mathcal{O}}_{ij}^{\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})\rangle_{\mathbb{A}} \tag{4.12}$$

Since we are after first order of perturbation theory, operators in the l.h.s. of this equation can be replaced in the leading order by corresponding classical fields

$$\langle \hat{\mathcal{O}}^{ij,\sigma_p}(z_2^-, z_{2_\perp}; z_1^-, z_{1_\perp})\rangle_{\bar{A}} = U^{+i,a}(z_2^-, z_{2_\perp}) U^{+j,a}(z_1^-, z_{1_\perp}) + O(\alpha_s)$$
$$\langle \hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+, z_{2_\perp}; z_1^+, z_{1_\perp})\rangle_{\bar{B}} = V^{-i,a}(z_2^+, z_{2_\perp}) V^{-j,a}(z_1^+, z_{1_\perp}) + O(\alpha_s) \tag{4.13}$$

so the master formula (4.12) takes the form

$$\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)$$
$$\times U^{-i,a}(z_2^+, z_{2_\perp}) U^{-j,a}(z_1^+, z_{1_\perp}) V^{+i,a}(z_2^-, z_{2_\perp}) V^{+j,a}(z_1^-, z_{1_\perp})$$
$$= \frac{N_c^2-1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$$
$$- \langle \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})\rangle_{\mathbb{A}} \tag{4.14}$$

In what follows we will calculate the r.h.s. of this equation in the background field (4.3), (E.1) and get the coefficient function.

The double functional integral for the r.h.s. of eq. (4.14) has the form

$$\frac{N_c^2-1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) - \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})\rangle_{\mathbb{A}}$$
$$= \int D\tilde{A} D A D\tilde{\psi} D\psi \; e^{-iS_{\mathbb{A},\Psi}(\tilde{A},\tilde{\psi}) + iS_{\mathbb{A},\Psi}(A,\psi)} \tag{4.15}$$
$$\times \Big(\frac{N_c^2-1}{16} g^4 \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) - \hat{\mathcal{O}}_{ij}^{\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \Big)$$

where $S_{\mathbb{A},\Psi}(A,\psi)$ is a standard QCD action in a background field $\mathbb{A}$ in the background-Feynman gauge:

$$S_{\mathbb{A},\Psi}(A,\psi) \tag{4.16}$$
$$= S_{\rm cl}(\mathbb{A},\Psi) + \int dz \; 2\mathrm{tr}\Big(A^\mu (\mathbb{D}^2 g_{\mu\nu} + 2i\mathbb{G}_{\mu\nu}) A^\nu + ig\mathbb{D}_\mu A_\nu [A^\mu, A^\nu] + \frac{g^2}{2}[A^\mu, A^\nu]^2 \Big) + \ldots$$

where dots stand for quark terms which are not relevant for our calculation. Hereafter "Tr" means color trace in the adjoint representation. Note that the term $S_{\rm cl}(\mathbb{A},\Psi)$ cancels in the exponent in eq. (4.15) so we will ignore it in what follows. The propagators in the background field $\mathbb{A}$ can be obtained as an expansion in "correction field" $\bar{C}$ since it is down by one power of $\frac{m_\perp^2}{s}$ in comparison to $\bar{A}$ and $\bar{B}$. As to propagator in $\bar{A} + \bar{B}$ background, it can be in principle obtained as a cluster expansion, but fortunately we will need only a couple of terms $\sim U^{-i}$ and $\sim V^{+i}$ which will be easily identified.

Most frequently we will perform this calculation in the momentum space, so we introduce Fourier transforms of projectile and target fields

$$
\begin{aligned}
V^{-i}(z^+, z_\perp) &= \int d\!\!^{-}\beta_b d\!\!^{-} k_{b_\perp} V^{-i}(\beta_b, k_{b_\perp}) e^{-i\varrho\beta_b z^+ + i(k_a, z)_\perp}, \\
U^{+i}(z^-, z_\perp) &= \int d\!\!^{-}\alpha_a d\!\!^{-} k_{a_\perp} U^{+i}(\alpha_a, k_{a_\perp}) e^{-i\varrho\alpha_a z^- + i(k_a, z)_\perp}, \\
V^{-i}(\beta_b, k_{b_\perp}) &= \varrho \int dz^+ dz_\perp \, U^{-i}(z^+, z_\perp) e^{i\varrho\beta_b z^+ - i(k_a, z)_\perp}, \\
U^{+i}(\alpha_a, k_{a_\perp}) &= \varrho \int dz^- dz_\perp \, U^{+i}(z^-, z_\perp) e^{i\varrho\alpha_a z^- - i(k_a, z)_\perp}
\end{aligned}
\tag{4.17}
$$

To avoid cluttering of our formulas, throughout the paper we use the $\hbar$-inspired notation

$$
\int d\!\!^{-n} p \equiv \int \frac{d^n p}{(2\pi)^n} \tag{4.18}
$$

where n is the dimension of corresponding momentum space.

Thus, the object of our calculations is the Fourier transform of eq. (4.14)

$$
\begin{aligned}
&\mathfrak{C}_1(x_1, x_2; \alpha'_a, \alpha_a, k_{a_\perp}, k'_{a_\perp}, \beta'_b, \beta_b, k_{b_\perp}, k'_{b_\perp}; \sigma_p, \sigma_t) \\
&= \int dz_1^- dz_2^- dw_1^+ dw_2^+ dz_{1_\perp} dz_{2_\perp} dw_{1_\perp} dw_{2_\perp} e^{-i\varrho\alpha'_a z_2^- + i(k'_a, z_2)_\perp} e^{-i\varrho\alpha_a z_1^- + i(k_a, z_1)_\perp} \\
&\quad \times e^{-i\varrho\beta'_b z_2^+ + i(k'_b, z_2)_\perp} e^{-i\varrho\beta_b z_1^+ + i(k_b, z_1)_\perp} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)
\end{aligned}
\tag{4.19}
$$

For one-loop calculations in the background field $\mathbb{A}$ it is convenient to multiply the hadronic tensor (2.2) by additional factor $\frac{N_c^2-1}{2g^2 N_c}\pi^2$. We define

$$
\begin{aligned}
\mathcal{W}(x_1, x_2) &= \frac{N_c^2 - 1}{2g^2 N_c} \pi^2 \langle W^{\text{one-loop}}(x_1, x_2) \rangle_{\mathbb{A}} \\
&= \frac{N_c^2 - 1}{N_c} 8\pi^2 g^2 \langle F^{-i,a}(x_2) F_i^{+a}(x_2) F^{-j,b}(x_1) F_j^{+b}(x_1) \rangle_{\mathbb{A}}^{\text{one-loop}}
\end{aligned}
\tag{4.20}
$$

The contributions to $\mathcal{W}(x_1, x_2)$ will be parametrized as follows

$$
\begin{aligned}
\mathcal{W}(x_1, x_2) - \mathcal{W}_{\text{eik}}^{\sigma_p, \sigma_t}(x_1, x_2) &= \int d\!\!^{-}\alpha'_a d\!\!^{-} k'_{a_\perp} d\!\!^{-}\beta'_b d\!\!^{-} k'_{b_\perp} d\!\!^{-}\alpha_a d\!\!^{-} k_{a_\perp} d\!\!^{-}\beta_b d\!\!^{-} k_{b_\perp} \\
&\quad \times e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+} e^{-i(k'_a + k'_a, x_2)_\perp - i(k_a + k_b, x_1)_\perp} \\
&\quad \times U_i^{+,b}(\alpha'_a, k'_{a_\perp}) V^{-i,a}(\beta'_b, k'_{b_\perp}) U_j^{+,b}(\alpha_a, k_{a_\perp}) V^{-j,a}(\beta_b, k_{b_\perp}) \\
&\quad \times [I - I_{\text{eik}}^{\sigma_p, \sigma_t}](\alpha'_a, k'_{a_\perp}, \alpha_a, k_{a_\perp}, \beta'_b, k'_{b_\perp}, \beta_b, k_{b_\perp}, x_2, x_1)
\end{aligned}
\tag{4.21}
$$

where $\mathcal{W}_{\text{eik}}^{\sigma_p, \sigma_t}(x_1, x_2)$ is the contribution of eikonal TMD operators $\langle \hat{\mathcal{O}}^{ij, \sigma_p} \hat{\mathcal{O}}^{ij; \sigma_t} \rangle_{\mathbb{A}}$ which has to be subtracted according to eq. (4.15). The coefficient function $\mathfrak{C}_1$ is then a Fourier transform of $[I - I_{\text{eik}}^{\sigma_p, \sigma_t}]$.

JHEP03(2023)029

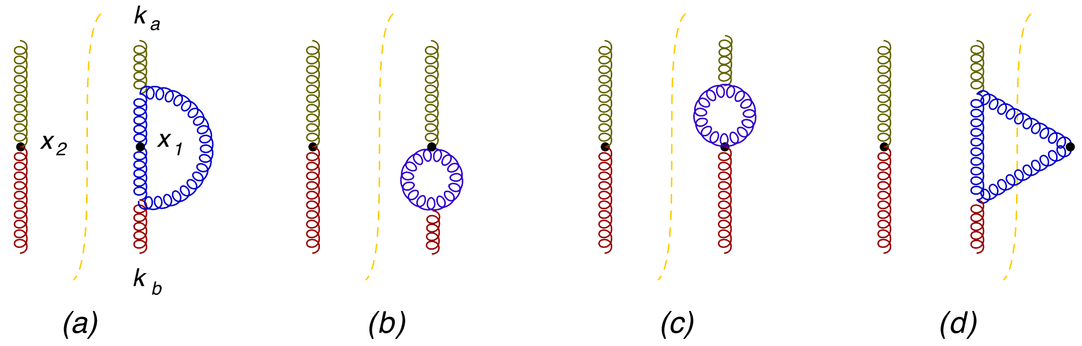

Figure 5. Diagrams (a)-(c): virtual diagrams in the right sector. Diagram (d): related diagram with two-gluon production.

Recall that the kinematical region for hadronic tensor (1.2) translates to eq. (2.6) in the non-forward case so in our approximation longitudinal distances are smaller than the transverse ones.[6]

In order to have parameters of the Fourier transformations with "natural" scales resembling those of forward case, in the formulas (4.17) we should take the origin somewhere between x_2 and x_1, so the kinematical region where we calculate eq. (4.19) is

$$\alpha'_a \sim \alpha_a,\ \beta'_b \sim \beta_b,\ k_{a_\perp} \sim k'_{a_\perp} \sim k_{b_\perp} \sim k'_{b_\perp} \sim Q_\perp, \qquad \alpha_a\beta_b, \alpha'_a\beta_b, \alpha\beta'_b, \alpha'_a\beta'_b \sim \frac{Q^2}{s} \gg \frac{Q_\perp^2}{s} \tag{4.22}$$

which corresponds to

$$x_2^+ \sim x_1^+,\ x_2^- \sim x_1^-,\ x_{2_\perp} \sim x_{1_\perp}, \qquad x_2^+x_2^- \sim x_1^+x_1^- \sim O(\frac{1}{\zeta}) \ll x_{2_\perp}^2 \sim x_{1_\perp}^2 \sim O(\zeta^0) \tag{4.23}$$

in the coordinate space because $x_i^+ \sim \frac{1}{\beta_b \varrho}$ and $x_i^- \sim \frac{1}{\alpha_a \varrho}$ and $\rho \sim (\frac{1}{\sqrt{\zeta}})$.

5 Virtual contributions

It is convenient to start calculation of functional integral (4.15) from the so-called "virtual" contribution to the first term given by diagrams in figure 5 a-c. (The reason for "production" diagram (d) appearing on this figure is explained in the end of this section). Let us start with the diagram in figure 5a.

In an arbitrary background field $\mathcal{A}$ we get

$$F_{\mu\nu}(A + \mathcal{A}) = \mathcal{F}_{\mu\nu}(\mathcal{A}) + (\mathcal{D}_\mu A_\nu - \mathcal{D}_\nu A_\mu) - ig[A_\mu, A_\nu]$$

The diagram in figure 5a corresponds to

$$\langle(\mathcal{D}_\mu A_\nu - \mathcal{D}_\nu A_\mu)(x_1)(\mathcal{D}_\mu A_\nu - \mathcal{D}_\nu A_\mu)(x_1)\rangle \tag{5.1}$$

[6]It is worth noting that for tree-level calculations, the parameter $\frac{q_\perp^2}{Q^2}$ is sufficient. A more restrictive parameter (1.2) is necessary for our calculation of logarithmical corrections. If $\frac{q_\perp^2}{Q^2} \ll 1$ but eq. (1.2) is not satisfied, the calculation of power corrections in refs. [6, 7] is still valid but the logarithmical corrections will probably be more complicated than our result (9.2).

expanded up to two $\mathcal{F}_{\mu\nu}(\mathcal{A})$. Using the background-field propagator in Schwinger's notations

$$i\langle T\{A_\mu(x)A_\nu(y)\}\rangle_{\mathcal{A}} = (x|\frac{1}{\mathcal{P}^2 g_{\mu\nu}+2i\mathcal{F}_{\mu\nu}+i\epsilon}|y) \tag{5.2}$$

and identities

$$\begin{aligned}\mathcal{P}^\mu\frac{1}{\mathcal{P}^2}\mathcal{P}_\mu &= 1-g^2\frac{1}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\frac{1}{P^2}\\ &\quad -g^2\frac{1}{\mathcal{P}^2}\mathcal{P}^\eta\mathcal{D}^\xi\mathcal{F}_{\xi\eta}\frac{1}{\mathcal{P}^2}+g^2\frac{1}{\mathcal{P}^2}\{\mathcal{P}_\alpha,\mathcal{F}^{\alpha\xi}\}\frac{1}{\mathcal{P}^2}\{\mathcal{P}^\beta,\mathcal{F}_{\beta\xi}\}\frac{1}{\mathcal{P}^2}\\ \mathcal{P}^\mu\frac{1}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}\frac{1}{\mathcal{P}^2}\mathcal{P}^\nu &= \frac{ig^2}{2}\frac{1}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}+ig^2\frac{1}{\mathcal{P}^2}\mathcal{P}^\nu\mathcal{D}^\mu\mathcal{F}_{\mu\nu}\frac{1}{\mathcal{P}^2}\\ &\quad -ig^2\frac{1}{\mathcal{P}^2}\{\mathcal{P}_\alpha,\mathcal{F}^{\alpha\xi}\}\frac{1}{\mathcal{P}^2}\mathcal{F}_{\beta\xi}\mathcal{P}^\beta\frac{1}{\mathcal{P}^2}-ig^2\frac{1}{\mathcal{P}^2}\mathcal{P}_\alpha\mathcal{F}^{\alpha\xi}\frac{1}{\mathcal{P}^2}\{\mathcal{F}_{\beta\xi},\mathcal{P}^\beta\}\frac{1}{\mathcal{P}^2}+O(\mathcal{F}^3_{\xi\eta})\end{aligned} \tag{5.3}$$

one obtains after some algebra

$$\begin{aligned}&(\mathcal{P}_\mu\delta^\xi_\nu-\mathcal{P}_\nu\delta^\xi_\mu)\frac{1}{\mathcal{P}^2 g_{\xi\eta}+2i\mathcal{F}_{\xi\eta}+i\epsilon}(\mathcal{P}_\mu\delta^\eta_\nu-\mathcal{P}_\nu\delta^\eta_\mu) = 6-4\frac{g^2}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}\\ &\quad +4\mathcal{F}_{\mu\nu}\frac{g^2}{\mathcal{P}^2}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}+4\frac{g^2}{\mathcal{P}^2}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}-2\frac{g^2}{\mathcal{P}^2}\mathcal{P}^\eta\mathcal{D}^\xi\mathcal{F}_{\xi\eta}\frac{1}{\mathcal{P}^2}+8\frac{g^2}{\mathcal{P}^2}\mathcal{D}_\mu\mathcal{F}_{\lambda\rho}\frac{1}{\mathcal{P}^2}\mathcal{D}^\mu\mathcal{F}^{\lambda\rho}\frac{1}{\mathcal{P}^2}\\ &\quad -2i\frac{g^2}{\mathcal{P}^2}\mathcal{D}_\alpha\mathcal{F}_{\beta\xi}\frac{1}{\mathcal{P}^2}\{\mathcal{P}^\alpha,\mathcal{F}^{\beta\xi}\}\frac{1}{\mathcal{P}^2}+2i\frac{g^2}{\mathcal{P}^2}\{\mathcal{F}_{\alpha\xi},\mathcal{P}_\beta\}\frac{1}{\mathcal{P}^2}\mathcal{D}^\beta\mathcal{F}^{\alpha\xi}\frac{1}{\mathcal{P}^2}+O(\mathcal{F}^3_{\xi\eta},(\mathcal{D}^\xi\mathcal{F}_{\xi\eta})^2)\\ &= 6+2\mathcal{F}_{\mu\nu}\frac{g^2}{\mathcal{P}^2}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}+2\frac{g^2}{\mathcal{P}^2}\mathcal{F}^{\mu\nu}\frac{1}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}-2\frac{g^2}{\mathcal{P}^2}\mathcal{P}^\eta\mathcal{D}^\xi\mathcal{F}_{\xi\eta}\frac{1}{\mathcal{P}^2}+4\frac{g^2}{\mathcal{P}^2}\mathcal{D}_\mu\mathcal{F}_{\lambda\rho}\frac{1}{\mathcal{P}^2}\mathcal{D}^\mu\mathcal{F}^{\lambda\rho}\frac{1}{\mathcal{P}^2}\\ &\quad +O(\mathcal{F}^3_{\xi\eta},(\mathcal{D}^\xi\mathcal{F}_{\xi\eta})^2)\end{aligned} \tag{5.4}$$

(here all $\mathcal{P}^2$ are $\mathcal{P}^2+i\epsilon$).

Next, it is convenient to add to eq. (5.4) the contribution of diagrams in figure 5b,c which has the form

$$\begin{aligned}2\langle igf^{abc}A^b_\mu A^c_\nu(x)\mathcal{F}^{c\mu\nu}(x)\rangle &= -4ig^2f^{abc}(x|\frac{1}{\mathcal{P}^2}G_{\mu\nu}\frac{1}{\mathcal{P}^2}|x)^{ab}G^{c\mu\nu}(x)+O(\mathcal{F}^3_{\xi\eta},(\mathcal{D}^\xi\mathcal{F}_{\xi\eta})^2)\\ &= (x|-4\frac{g^2}{\mathcal{P}^2}\mathcal{F}_{\mu\nu}\frac{1}{\mathcal{P}^2}\mathcal{F}^{\mu\nu}|x)^{aa}+O(\mathcal{F}^3_{\xi\eta},(\mathcal{D}^\xi\mathcal{F}_{\xi\eta})^2)\end{aligned} \tag{5.5}$$

We get

$$\begin{aligned}&2g^2\langle i(\mathcal{D}_\mu A^a_\nu-\mathcal{D}_\nu A^a_\mu)(x_1)\mathcal{D}^\mu A^{a,\nu}(x_1)+igf^{abc}A^b_\mu A^c_\nu(x_1)\mathcal{F}^{c\mu\nu}(x_1)\rangle\\ &\quad = g^4(x_1|-2\frac{1}{\mathcal{P}^2}\mathcal{P}^\eta\mathcal{D}^\xi\mathcal{F}_{\xi\eta}\frac{1}{\mathcal{P}^2}+4\frac{1}{\mathcal{P}^2}\mathcal{D}_\mu\mathcal{F}_{\lambda\rho}\frac{1}{\mathcal{P}^2}\mathcal{D}^\mu\mathcal{F}^{\lambda\rho}\frac{1}{\mathcal{P}^2}|x_1)^{aa}+O(\mathcal{F}^3_{\xi\eta},(\mathcal{D}^\xi\mathcal{F}_{\xi\eta})^2)\end{aligned} \tag{5.6}$$

Note that the absence of UV divergent terms follows from one-loop renorm-invariance of local operator $g^2F^a_{\mu\nu}F^{a\mu\nu}$.

Next, the expansion of propagators $(p^2+2\{p,\mathcal{A}\}+\mathcal{A}^2)^{-1}$ in powers of external field $\mathcal{A}$ will bring more powers of $\mathcal{F}_{\xi\eta}$ in the numerators leading to power corrections to eq. (4.1) rather than the logarithmical ones. Thus, one can replace all $(\mathcal{P}^2+i\epsilon)^{-1}$ in the r.h.s. of eq. (5.4) by usual propagators $(p^2+i\epsilon)^{-1}$. Also, for our background field $\mathbb{A}$

$$\mathcal{F}_{\mu\nu}(\mathbb{A}) = \mathcal{F}_{\mu\nu}(\bar{A})+\mathcal{F}_{\mu\nu}(\bar{B})+\mathcal{F}_{\mu\nu}(\bar{C})-ig([\bar{A}_\mu,\bar{B}_\nu]+[\bar{A}_\mu+\bar{B}_\mu,\bar{C}_\nu]-\mu\leftrightarrow\nu) \tag{5.7}$$

Since the correction field $\bar{C}$ is proportional to commutators of U^{-i} and V^{+j} we can neglect it in the r.h.s. of eq. (5.4) - it will lead to the terms with two U^{-i} and one V^{+j}, or vice versa. By the same token, one can disregard $[\bar{A}_\mu, \bar{B}_\nu]$ since its Lorentz components are either zero or made from $[U^{-i}, V^{+j}]$. Finally, we are interested only in terms with one U^{-i} and one V^{+j} in the r.h.s. of eq. (5.4) corresponding to diagrams in figure 5[7]

Thus, in our approximation

$$
\begin{aligned}
&2g^2\langle(\mathcal{D}_\mu A^a_\nu - \mathcal{D}_\nu A^a_\mu)(x_1)\mathcal{D}^\mu A^{a,\nu}(x_1) + igf^{abc}A^b_\mu A^c_\nu(x_1)\mathcal{F}^{c\mu\nu}(x_1)\rangle_{\mathbb{A}} \\
&\quad = -16i(x_1|\frac{1}{p^2+i\epsilon}\bar{D}_\mu U^{+i}\frac{1}{p^2+i\epsilon}\bar{D}^\mu V^-_i\frac{1}{p^2+i\epsilon}|x_1)^{aa}
\end{aligned}
\tag{5.8}
$$

This term is proportional to $\bar{D}_\mu U^{+i}\otimes\bar{D}^\mu V^-_i = \partial^+U^{+i}\otimes\partial^-V^-_i + \bar{D}_k U^{-i}\otimes\bar{D}^k V^+_i$. As we discussed after eq. (4.8), $\partial^+\otimes\partial^-$ brings extra factor of s and $\bar{D}_k\otimes\bar{D}^k$ only $q_\perp^2$ so we can disregard it. We get

$$
\begin{aligned}
&-16i(x_1|\frac{1}{p^2+i\epsilon}\partial^-U^{+i}\frac{1}{p^2+i\epsilon}\partial^+V^-_i\frac{1}{p^2+i\epsilon}|x_1)^{aa} \\
&= -8N_c\int d\!\!{}^-\alpha_a d\!\!{}^- k_{a_\perp}\int d\!\!{}^-\beta_b d\!\!{}^- k_{b_\perp} U^{+i,a}(\alpha_a,k_{a_\perp})V^{-,a}_i(\beta_b,k_{b_\perp})e^{-i\varrho\alpha_a x_1^- - i\varrho\beta_b x_1^+ + i(k_a+k_a,x_1)_\perp} \\
&\quad\times\int\frac{d\!\!{}^-{}^4p}{i}\frac{s\alpha_a\beta_b}{[(p+k_a)^2+i\epsilon](p^2+i\epsilon)[(p-k_a)^2+i\epsilon]} \\
&= -\frac{g^2N_c}{2\pi^2}\int d\!\!{}^-\alpha_a d\!\!{}^- k_{a_\perp}\int d\!\!{}^-\beta_b d\!\!{}^- k_{b_\perp} U^{+i,a}(\alpha_a,k_{a_\perp})V^{-,a}_i(\beta_b,k_{b_\perp})e^{-i\varrho\alpha_a x_1^- - i\varrho\beta_b x_1^+ + i(k_a+k_a,x_1)_\perp} \\
&\quad\times\Big(\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{a_\perp}}\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{b_\perp}} + \frac{\pi^2}{3}\Big)
\end{aligned}
\tag{5.9}
$$

where we used eq. (F.4) in the last line. Finally, we obtain

$$
\begin{aligned}
&g^2\langle T\{F^a_{\mu\nu}(x_1)F^{\mu\nu,a}(x_1)\}\rangle_{\text{fig.5a+b+c}} \\
&\quad = -\int d\!\!{}^-\alpha_a d\!\!{}^- k_{a_\perp}\int d\!\!{}^-\beta_b d\!\!{}^- k_{b_\perp} U^{+i,a}(\alpha_a,k_{a_\perp})V^{-,a}_i(\beta_b,k_{b_\perp})e^{-i\varrho\alpha_a x_1^- - i\varrho\beta_b x_1^+ + i(k_a+k_a,x_1)_\perp} \\
&\qquad\times\frac{g^2N_c}{2\pi^2}\Big(\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{a_\perp}}\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{b_\perp}} + \frac{\pi^2}{3}\Big)
\end{aligned}
\tag{5.10}
$$

Next, let us consider the diagram in figure 5d. This diagram and its left-right permutation are the only diagrams with two-gluon cut since gluon with k_a or k_b cannot solely produce two gluons. As we will see below, the diagram in figure 5d will change Feynman-type singularities in logarithms in eq. (5.10) to causal-type singularities.

The structure of the diagram in figure 5d is the same as in figure 5a with two Feynman propagators replaced by cut propagators and the left-sector propagator between U^{-i} and V^{+i} being $\frac{i}{p^2-i\epsilon}$ instead of $\frac{-i}{p^2+i\epsilon}$. Note also that the first three terms in the r.h.s. of eq. (5.4) do not contribute since our background fields cannot produce particles. We get

$$
\begin{aligned}
&g^2\langle T\{F^a_{\mu\nu}(x_1)F^{a,\mu\nu}(x_1)\}\rangle_{\text{fig.5d}} \\
&\quad = -16i(x_1|\tilde{\delta}_-(p)\partial^-U^{+i}\frac{1}{p^2-i\epsilon}\partial^+V^-_i\tilde{\delta}_+(p)|x_1)^{aa}
\end{aligned}
\tag{5.11}
$$

[7]The terms with zero U^{-i}'s and two V^{+j}'s will lead to zero color trace after integration over projectile fields in the integral (3.8), and similarly for terms with U^{+j}'s.

Hereafter we introduce space-saving notations

$$\tilde{\delta}_+(p) \equiv 2\pi\delta(p^2)\theta(p_0), \qquad \tilde{\delta}_-(p) \equiv 2\pi\delta(p^2)\theta(-p_0) \tag{5.12}$$

Using integral (F.3) from appendix F one easily obtains

$$\begin{aligned}
&g^2\langle T\{F^a_{\mu\nu}(x_1)F^{\mu\nu,a}(x_1)\}\rangle_{\text{fig.5d}}\\
&=\int d\!\!{}^-\alpha_a d\!\!{}^-k_{a_\perp}\int d\!\!{}^-\beta_b d\!\!{}^-k_{b_\perp} U^{+i}(\alpha_a,k_{a_\perp})V^-_i(\beta_b,k_{b_\perp})e^{-i\varrho\alpha_a x_1^- - i\varrho\beta_b x_1^+ + i(k_a+k_a,x_1)_\perp}\\
&\quad\times\theta(-\alpha_a)\theta(-\beta_b)\frac{g^2N_c}{\pi}(-i)\ln\frac{{\alpha_a}^2{\beta_b}^2s^2}{k^2_{a_\perp}k^2_{b_\perp}}
\end{aligned}\tag{5.13}$$

Next, using the identity

$$\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{a_\perp}}\ln\frac{-\alpha_a\beta_b s - i\epsilon}{k^2_{b_\perp}} + 2\pi i\theta(-\alpha_a)\theta(-\beta_b)\ln\frac{{\alpha_a}^2{\beta_b}^2s^2}{k^2_{a_\perp}k^2_{b_\perp}} = \ln\frac{-Q^2_{ab}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab}}{k^2_{b_\perp}} \tag{5.14}$$

where[8]

$$Q^2_{ab} \equiv (\alpha_a + i\epsilon)(\beta_b + i\epsilon)s \tag{5.15}$$

we get the contribution of diagrams in figure 5 in the form

$$\begin{aligned}
&g^2\langle T\{F^a_{\mu\nu}(x_1)F^{\mu\nu,a}(x_1)\}\rangle^{\text{fig.5}} = \frac{g^2N_c}{2\pi^2}\int d\!\!{}^-\alpha_a d\!\!{}^-k_{a_\perp}\int d\!\!{}^-\beta_b d\!\!{}^-k_{b_\perp} \qquad (5.16)\\
&\quad\times U^{+i,a}(\alpha_a,k_{a_\perp})V^{-,a}_i(\beta_b,k_{b_\perp})e^{-i\varrho\alpha_a x_1^- - i\varrho\beta_b x_1^+ + i(k_a+k_b,x_1)_\perp} I^{\text{virt}}_{\text{fig.5}}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp})
\end{aligned}$$

where

$$\begin{aligned}
&I^{\text{virt}}_{\text{fig.5}}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp})\\
&= -16\pi^2\int\frac{d\!\!{}^-{}^4p}{i}\left[\frac{s\alpha_a\beta_b}{[(p+k_a)^2+i\epsilon](p^2+i\epsilon)[(p-k_b)^2+i\epsilon]} + \tilde{\delta}_-(p+k_a)\frac{s\alpha_a\beta_b}{p^2-i\epsilon}\tilde{\delta}_+(p-k_b)\right]\\
&= -\ln\frac{-Q^2_{ab}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab}}{k^2_{b_\perp}} - \frac{\pi^2}{3} + O(\lambda)
\end{aligned}\tag{5.17}$$

and λ is defined in eq. (3.6).

In coordinate space, the singularity (5.15) means that

$$\begin{aligned}
\int d\!\!{}^-\alpha'_a\, e^{-i\varrho\alpha'_a x_1^-} f(-\alpha'_a - i\epsilon)U(\alpha'_a) &= \int_{-\infty}^{x_1^-} dz_1^-\,\tilde{f}(x_1^- - z_1^-)U(z_1^-)\\
\int d\!\!{}^-\beta'_b\, e^{-i\varrho\beta'_b x_1^+} f(-\beta'_b - i\epsilon)V(\beta'_b) &= \int_{-\infty}^{x_1^+} dz_1^+\,\tilde{f}(x_1^+ - z_1^+)V(z_1^-)
\end{aligned}\tag{5.18}$$

Thus, after summation of the diagrams figure 5a,b,c and figure 5d we get the result that the emission of the background-field gluon always preceeds the original point x_1.

[8] Later we will use similar notations $Q^2_{ab'} \equiv (\alpha_a + i\epsilon)(\beta'_b + i\epsilon)s$, $Q^2_{a'b} \equiv (\alpha'_a + i\epsilon)(\beta_b + i\epsilon)s$, and $Q^2_{a'b'} \equiv (\alpha'_a + i\epsilon)(\beta'_b + i\epsilon)s$.

JHEP03(2023)029

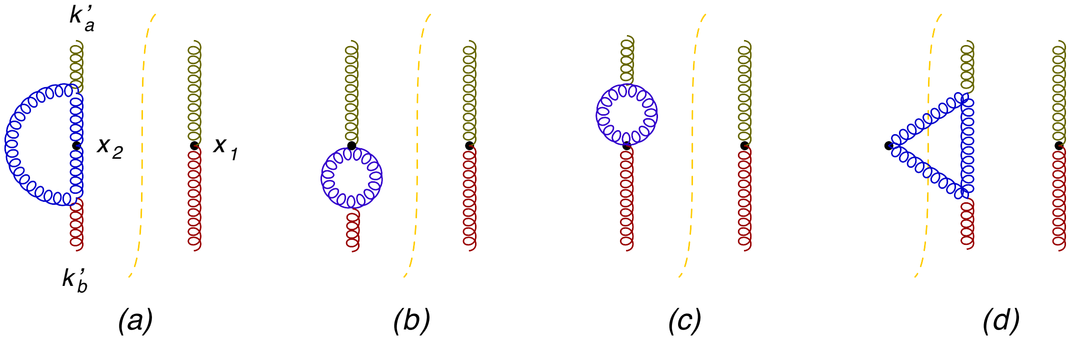

Figure 6. Diagrams (a)-(c): virtual diagrams in the left sector. Diagram (d): related diagram with two-gulon production.

Actually, it can be seen before the calculation of integrals. To this end, consider the identity

$$
\begin{aligned}
&(x|\frac{1}{p^2+i\epsilon}\mathcal{A}\frac{1}{p^2+i\epsilon}\mathcal{B}\frac{1}{p^2+i\epsilon}+\tilde{\delta}_-(p)\mathcal{A}\frac{1}{p^2-i\epsilon}\mathcal{B}\tilde{\delta}_+(p)|y)+\tilde{\delta}_-(p)\mathcal{A}\tilde{\delta}_+(p)\mathcal{B}\frac{1}{p^2+i\epsilon}\\
&+\frac{1}{p^2+i\epsilon}\mathcal{A}\tilde{\delta}_-(p)\mathcal{B}\tilde{\delta}_+(p)=(x|\frac{1}{p^2+i\epsilon p_0}\mathcal{A}\frac{1}{p^2+i\epsilon}\mathcal{B}\frac{1}{p^2-i\epsilon p_0}\\
&-i\frac{1}{p^2+i\epsilon p_0}\mathcal{A}\frac{1}{p^2+i\epsilon p_0}\mathcal{B}\tilde{\delta}_+(p)-i\tilde{\delta}_-(p)\mathcal{A}\frac{1}{p^2-i\epsilon p_0}\mathcal{B}\frac{1}{p^2-i\epsilon p_0}|y)
\end{aligned}
\tag{5.19}
$$

valid for any operators $\mathcal{A}$ and $\mathcal{B}$. Using this formula, the sum of $\partial^-U^{+i}\otimes\partial^+V_i^-$ terms in eqs. (5.8) and (5.11) can be rewritten as

$$
\begin{aligned}
&\frac{1}{p^2+i\epsilon}\partial^-U^{+i}\frac{1}{p^2+i\epsilon}\partial^+V_i^-\frac{1}{p^2+i\epsilon}+\tilde{\delta}_-(p)\partial^-U^{+i}\frac{1}{p^2-i\epsilon}\partial^+V_i^-\tilde{\delta}_+(p)\\
&=\frac{1}{p^2+i\epsilon p_0}\partial^-U^{+i}\frac{1}{p^2+i\epsilon}\partial^+V_i^-\frac{1}{p^2-i\epsilon p_0}\\
&-i\tilde{\delta}_-(p)\partial^-U^{+i}\frac{1}{p^2-i\epsilon p_0}\partial^+V_i^-\frac{1}{p^2-i\epsilon p_0}-i\frac{1}{p^2+i\epsilon p_0}\partial^-U^{+i}\frac{1}{p^2+i\epsilon p_0}\partial^+V_i^-\tilde{\delta}_+(p)
\end{aligned}
\tag{5.20}
$$

from where the causal structure of the result of summation is evident. Note that we used $\tilde{\delta}_-(p)\partial^-U^{+i}\tilde{\delta}_+(p)=\tilde{\delta}_-(p)\partial^-V^{-i}\tilde{\delta}_+(p)=0$ following from the fact that background fields $U^{+i}(x^-,x_\perp)$ and $V^{-i}(x^-,x_\perp)$ cannot produce two real particles. This is similar to the case of tree-level diagrams where the summation of the emissions from both sides of the cut leads to the diagrams with retarded propagators, see ref. [6].

One can calculate diagrams in figure 6 in a similar way. The result for the diagrams in figure 6 a,b,c is obtained by complex conjugation of eq. (5.10)

$$
\begin{aligned}
&g^2\langle\tilde{T}\{F^a_{\mu\nu}(x_2)F^{\mu\nu,a}(x_2)\}\rangle_{\text{fig.6a+b+c}}\\
&=-\int d\!\!{}^-\alpha'_a d\!\!{}^-k'_{a_\perp}\int d\!\!{}^-\beta'_b d\!\!{}^-k'_{b_\perp}U^{+i,a}(\alpha'_a,k'_{a_\perp})V_i^{-,a}(\beta'_b,k'_{b_\perp})e^{-i\varrho\alpha'_a x_2^- -i\varrho\beta'_b x_2^+ +i(k'_a+k'_b,x_2)_\perp}\\
&\times\frac{g^2N_c}{2\pi^2}\ln\frac{-\alpha'_a\beta'_b s+i\epsilon}{k'^2_{a_\perp}}\ln\frac{-\alpha'_a\beta'_b s+i\epsilon}{k^2_{b_\perp}}
\end{aligned}
\tag{5.21}
$$

Next, the diagram in figure 6d

$$g^2\langle\tilde{T}\{F^a_{\mu\nu}(x_2)F^{a,\mu\nu}(x_2)\}\rangle_{\text{fig.6d}} = -16i(x_2|\tilde{\delta}_+(p)\partial^- U^{+i}\frac{1}{p^2+i\epsilon}\partial^+ V^-_i\tilde{\delta}_-(p)|x_2) \tag{5.22}$$

can be obtained from the integrals in eq. (F.4) by the replacements $k_a \to -k'_a,\ k_b \to -k'_b$. The result is

$$\begin{aligned}
&g^2\langle\tilde{T}\{F^a_{\mu\nu}(x_2)F^{\mu\nu,a}(x_2)\}\rangle_{\text{fig.6d}}\\
&=\int đ\,\alpha'_a đ\,k'_{a_\perp}\int đ\,\beta'_b đ\,k'_{b_\perp} U^{+i}(\alpha'_a,k'_{a_\perp})V^-_i(\beta'_b,k'_{b_\perp})e^{-i\varrho\alpha'_a x_2^- - i\varrho\beta'_b x_2^+ + i(k'_a+k'_b,x_2)_\perp}\\
&\quad\times\theta(\alpha'_a)\theta(\beta'_b)\frac{g^2N_c}{\pi}i\ln\frac{{\alpha'_a}^2{\beta'_b}^2 s^2}{{k'}^2_{a_\perp}k^2_{b_\perp}}
\end{aligned}\tag{5.23}$$

Using now

$$\ln\frac{-\alpha'_a\beta'_b s+i\epsilon}{{k'}^2_{a_\perp}}\ln\frac{-\alpha'_a\beta'_b s+i\epsilon}{k'^2_{b_\perp}} - 2\pi i\theta(\alpha'_a)\theta(\beta'_b)\ln\frac{{\alpha'_a}^2{\beta'_b}^2s^2}{k'^2_{a_\perp}k'^2_{b_\perp}} = \ln\frac{-Q^2_{a'b'}}{k'^2_{a_\perp}}\ln\frac{-Q^2_{a'b'}}{k'^2_{b_\perp}} \tag{5.24}$$

we get

$$\begin{aligned}
g^2\langle\tilde{T}\{F^a_{\mu\nu}(x_2)F^{\mu\nu,a}(x_2)\}\rangle^{\text{fig.6}} &= \frac{g^2N_c}{2\pi^2}\int đ\,\alpha'_a đ\,k'_{a_\perp}\int đ\,\beta'_b đ\,k'_{b_\perp}\\
&U^{+i,a}(\alpha'_a,k'_{a_\perp})V^{-,a}_i(\beta'_b,k'_{b_\perp})e^{-i\varrho\alpha'_a x_2^- - i\varrho\beta'_b x_2^+ + i(k'_a+k'_b,x_2)_\perp} I^{\text{virt}}_{\text{fig.6}}
\end{aligned}\tag{5.25}$$

where

$$\begin{aligned}
I^{\text{virt}}_{\text{fig.6}} &= 16\pi^2\int\frac{đ^4p}{i}\Big[\frac{s\alpha'_a\beta'_b}{[(p+k'_a)^2-i\epsilon](p^2-i\epsilon)[(p-k'_b)^2-i\epsilon]}\\
&+\tilde{\delta}_+(p+k'_a)\frac{s\alpha'_a\beta'_b}{p^2+i\epsilon}\tilde{\delta}_-(p-k'_b)\Big] = -\ln\frac{-Q^2_{a'b'}}{k'^2_{a_\perp}}\ln\frac{-Q^2_{a'b'}}{k'^2_{b_\perp}} - \frac{\pi^2}{3}
\end{aligned}\tag{5.26}$$

and Q^2_{ab} is defined in eq. (5.15). Again, the sum of all diagrams in figure 6 reveals causal structure in the coordinate space.

The final result for the virtual contributions can be presented as follows

$$\begin{aligned}
\mathcal{W}^{\text{virt}}(x_1,x_2) &= \frac{N_c^2-1}{N_c}8\pi^2\Big(V^{-i,a}(x_2)U^{+a}_i(x_2)\langle F^{-j,b}(x_1)F^{+b}_j(x_1)\rangle^{\text{fig.5}}_{\mathbb{A}}\\
&\quad+\langle F^{-i,b}(x_2)F^{+b}_i(x_2)\rangle^{\text{fig.6}}_{\mathbb{A}}V^{-j,a}(x_1)U^{+a}_j(x_1)\Big)\\
&=\int đ\,\alpha'_a đ\,k'_{a_\perp}đ\,\beta'_b đ\,k'_{b_\perp}đ\,\alpha_a đ\,k_{a_\perp}đ\,\beta_b đ\,k_{b_\perp}e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times e^{-i(k_a+k_a,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp}\ U^{+,b}_i(\alpha'_a,k'_{a_\perp})V^{-i,a}(\beta'_b,k'_{b_\perp})U^{+,b}_j(\alpha_a,k_{a_\perp})V^{-j,a}(\beta_b,k_{b_\perp})\\
&\quad\times I^{\text{virt}}(\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp},\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp})+O(\lambda)
\end{aligned}\tag{5.27}$$

where

$$\begin{aligned}
I^{\text{virt}}(\alpha'_a,k'_{a_\perp},\beta'_b,k_{b_\perp},\alpha_a,k'_{a_\perp},\beta'_b,k'_{b_\perp}) &= I^{\text{virt}}_{\text{fig.5}}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp})\\
+I^{\text{virt}}_{\text{fig.6}}(\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp}) &= -I^{\text{d.log}}(\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp}) - I^{\text{d.log}}(\alpha_a,k'_{a_\perp},\beta_b,k'_{b_\perp})
\end{aligned}\tag{5.28}$$

For future use, we introduced the notation

$$I^{\rm d.log}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp})=\ln\frac{-Q^2_{ab}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab}}{k^2_{b_\perp}}+\frac{\pi^2}{3}\tag{5.29}$$

for the double-log contributions. The expression for $I^{\rm d.log}(\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp})$ is similar.

6 "Production" diagrams

6.1 Power counting for production terms

From power counting (4.8) it is easy to see that the leading contribution to hadronic tensor (4.20) with one-gluon production comes from the following terms

$$\begin{aligned}&\mathcal{W}(x_1,x_2)\equiv\frac{N_c^2-1}{2N_c}\pi^2g^2\mathbb{F}^a_{\mu\nu}(x_2)\langle F^{\mu\nu,a}(x_2)F^{\alpha\beta,b}(x_1)\rangle_{\mathbb{A}}\mathbb{F}^b_{\alpha\beta}(x_1)=\frac{N_c^2-1}{N_c}8\pi^2\\&\times\big[V^{-i,a}(x_2)\langle F^{+,a}_{i}(x_2)F^{-,b}_{j}(x_1)\rangle_{\mathbb{A}}U^{+j,b}(x_1)+U^{+i,a}(x_2)\langle F^{-,a}_{i}(x_2)F^{+,b}_{j}(x_1)\rangle_{\mathbb{A}}V^{-j,b}(x_1)\\&+U^{+i,a}(x_2)\langle F^{-,a}_{i}(x_2)F^{-,b}_{j}(x_1)\rangle_{\mathbb{A}}U^{+j,b}(x_1)+V^{-i,a}(x_2)\langle F^{+,a}_{i}(x_2)F^{+,b}_{j}(x_1)\rangle_{\mathbb{A}}V^{-j,b}(x_1)\big]\end{aligned}\tag{6.1}$$

In this section we calculate the first term in this equation. The second term is obtained by trivial replacements while the third and the fourth terms correspond to "handbag" diagrams considered in next section.

The gluon propagator in the background field $\mathbb{A}$ is given by eq. (A.5) from appendix A. Also, in appendix E it was proved that the contributions due to background "correction field" $\bar{C}$ can be neglected so we need to compute

$$\begin{aligned}&V^{a,-i}(x_2)\langle(\mathcal{D}^+A_i-\mathcal{D}_iA^+)(x_2)(\mathcal{D}^-A_j-\mathcal{D}_jA^-)(x_1)\rangle^{ab}U^{b,+j}(x_1)=\\&=-V^{a,-i}i(x_2|(\mathcal{P}^+\delta^\alpha_i-\mathcal{P}_ig^{+\alpha})\frac{1}{\mathcal{P}^2g_{\alpha\xi}+2i\mathcal{F}_{\alpha\xi}-i\epsilon}p^2\\&\quad\times\tilde\delta_+(p)p^2\frac{1}{\mathcal{P}^2\delta^\xi_\beta+2i\mathcal{F}^\xi{}_\beta+i\epsilon}(\mathcal{P}^-\delta^\beta_j-\mathcal{P}_jg^{\beta-})|x_1)^{ab}U^{b,+j}\\&=-V^{a,-i}(x_2|(p^+\delta^\alpha_i-\mathcal{P}_ig^{+\alpha})\Big\{g_{\alpha\beta}\frac{1}{\mathcal{P}^2-i\epsilon}p^2\tilde\delta_+(p)p^2\frac{1}{\mathcal{P}^2+i\epsilon}\\&\quad-2i\frac{1}{\mathcal{P}^2-i\epsilon}\Big[p^2\tilde\delta_+(p)p^2\frac{1}{\mathcal{P}^2+i\epsilon}\mathcal{F}_{\alpha\beta}+\mathcal{F}_{\alpha\beta}\frac{1}{\mathcal{P}^2-i\epsilon}p^2\tilde\delta_+(p)p^2\Big]\frac{1}{\mathcal{P}^2+i\epsilon}|y)\\&\quad+4\frac{1}{\mathcal{P}^2-i\epsilon}\Big[\mathcal{F}_{\alpha\xi}\frac{1}{\mathcal{P}^2-i\epsilon}\mathcal{F}_\beta{}^\xi\frac{1}{\mathcal{P}^2-i\epsilon}p^2\tilde\delta_+(p)p^2+\mathcal{F}_{\alpha\xi}\frac{1}{\mathcal{P}^2-i\epsilon}p^2\tilde\delta_+(p)p^2\frac{1}{\mathcal{P}^2+i\epsilon}\mathcal{F}_\beta{}^\xi\\&\quad+p^2\tilde\delta_+(p)p^2\frac{1}{\mathcal{P}^2+i\epsilon}\mathcal{F}_{\alpha\xi}\frac{1}{\mathcal{P}^2+i\epsilon}\mathcal{F}_\beta{}^\xi\Big]\frac{1}{\mathcal{P}^2+i\epsilon}\Big\}(p^-\delta^\beta_j-\mathcal{P}_jg^{\beta-})|x_1)^{ab}U^{b,+j}(x_1)\end{aligned}\tag{6.2}$$

The leading contribution, shown in figure 7,

$$\begin{aligned}&-4V^{-a,i}(x_2)(x_2|\frac{p^+}{p^2-i\epsilon}U^{+i}\frac{1}{p^2-i\epsilon}V^{-j}p^-\tilde\delta_+(p)+\frac{p^+}{p^2-i\epsilon}U^{+i}\tilde\delta_+(p)V^{-j}\frac{p^-}{p^2-i\epsilon}\\&+p^+\tilde\delta_+(p)U^{+i}\frac{1}{p^2+i\epsilon}V^{-j}\frac{p^-}{p^2-i\epsilon}|x_1)^{ab}U^{b,+j}(x_1)\end{aligned}\tag{6.3}$$

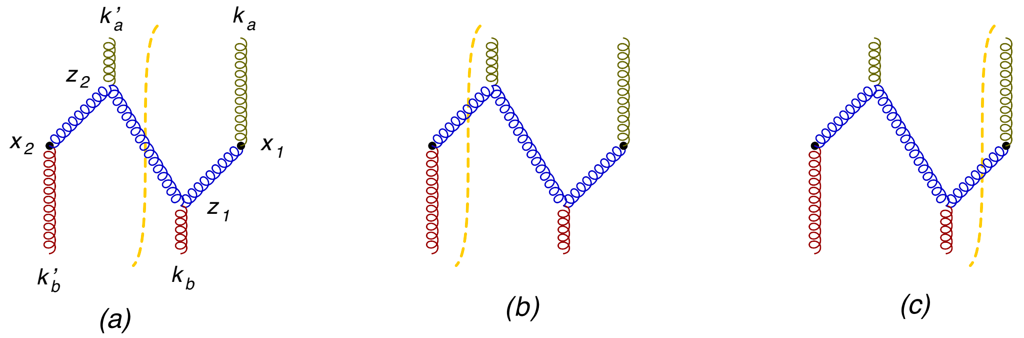

Figure 7. First set of leading diagrams with gluon production. Projectile fields $U^{+i}(z_2)$ are denoted by green tails while target fields $V^{-i}(z_1)$ by red tails.

comes from the last term in the r.h.s. of eq. (6.2). As we will see in the next section, this contribution is logarithmic similarly to eq. (5.9) for virtual diagram.

Next, using power counting (4.8), we demonstrate that all other contributions to the r.h.s. of eq. (6.2) are power corrections with respect to eq. (6.3). To this end, we note that $p^+p^- \sim \alpha'_a\beta_b s = Q^2_{a'b}$ and $U^{+i}V^{-j} \sim Q^2_\perp s$ so the numerator $p^+p^-U^{+i}V^{-j}$ in the integral of eq. (6.3) is of order $Q^2_{ab}Q^2_\perp s$. Now, if we take $\sim P_iP_jU^{+k}V^-_k$ contribution to the last term in eq. (6.2), it is $\sim Q^4_\perp s$ which is down by $\frac{Q^2_\perp}{Q^2_{ab}}$ factor in comparison to $p^+p^-U^{+i}V^{-j}$. As to the term $\sim p^+P_j\mathcal{F}_{ik}\mathcal{F}^{-k} \sim \alpha'_a sQ^4_\perp$, it is $O({\alpha'_a}^2Q^2_\perp/Q^2_{ab})$ in comparison to eq. (6.3).[9]

Let us now consider the terms in the second line in the r.h.s. of eq. (6.2). To get the contribution having four gluon tails, we expand $\frac{1}{P^2}$ once and get terms like

$$V^{a,-i}(x_2|(p^+\delta^\alpha_i - \mathcal{P}_i g^{+\alpha})\frac{1}{p^2-i\epsilon}\{p^k,\mathcal{A}_k\}\tilde{\delta}_+(p)\mathcal{F}_{\alpha\beta}\frac{1}{p^2+i\epsilon}(p^-\delta^\beta_j - \mathcal{P}_j g^{\beta-})|x_1)^{ab}U^{b,+j}(x_1) \tag{6.4}$$

where $\mathcal{A}_k$ is either U_k or V_k. The numerator in the integral of this equation is again $\sim Q^4_\perp s$, so, as we mentioned above, it is a power correction in comparison to the leading term (6.3). Finally, the term in the first line in the r.h.s. of eq. (6.2) should be expanded twice to get four gluons, so we have

$$V^{a,-i}(x_2|(p^+\delta^\alpha_i - \mathcal{P}_i g^{+\alpha})\frac{1}{p^2-i\epsilon}\{p^k,\mathcal{A}_k\}\tilde{\delta}_+(p)\{p^l,\mathcal{A}_l\}\frac{1}{p^2+i\epsilon}(p^-\delta^\beta_j - \mathcal{P}_j g^{\beta-})|y)^{ab}U^{b,+j}(x_1) \tag{6.5}$$

The numerator in this equation is at best $\sim Q^4_\perp s$ which is $O(Q^2_\perp/Q^2_{ab})$ in comparison to the leading term.

Thus, all terms in the r.h.s. of eq. (6.2) are power corrections $\sim O(\frac{m^2_\perp}{s} \sim \zeta^{-1})$ to the logarithmical leading term (6.3) which we calculate in the next section.

[9]Recall that α'_a (and β_b) are either ~ 1 or $\ll 1$ depending on moderate-x or small-x kinematics.

6.2 Calculation of leading production terms

In this section we will calculate the first term in the r.h.s. of term in eq. (6.1) given by eq. (6.3), see figure 7. First, it is convenient to use the identity

$$(x_2|\frac{1}{p^2-i\epsilon}\mathcal{A}\tilde{\delta}_+(p)\mathcal{B}\frac{1}{p^2+i\epsilon}+\tilde{\delta}_+(p)\mathcal{A}\frac{1}{p^2+i\epsilon}\mathcal{B}\frac{1}{p^2+i\epsilon}+\frac{1}{p^2-i\epsilon}\mathcal{A}\frac{1}{p^2-i\epsilon}\mathcal{B}\tilde{\delta}_+(p)$$
$$+\tilde{\delta}_+(p)\mathcal{A}\tilde{\delta}_-(p)\mathcal{B}\tilde{\delta}_+(p)|x_1)=(x_2|\frac{1}{p^2+i\epsilon p_0}\mathcal{A}\tilde{\delta}_+(p)\mathcal{B}\frac{1}{p^2-i\epsilon p_0}$$
$$+\tilde{\delta}_+(p)\mathcal{A}\frac{1}{p^2-i\epsilon p_0}\mathcal{B}\frac{1}{p^2-i\epsilon p_0}+\frac{1}{p^2+i\epsilon p_0}\mathcal{A}\frac{1}{p^2+i\epsilon p_0}\mathcal{B}\tilde{\delta}_+(p)|x_1) \tag{6.6}$$

and rewrite eq. (6.3) changing singularities of propagators accordingly

$$-4V^{-a,i}(x_2)(x_2|\frac{p^+}{p^2+i\epsilon p_0}U^{+i}\frac{1}{p^2+i\epsilon p_0}V^{-j}p^-\tilde{\delta}_+(p)+\frac{p^+}{p^2+i\epsilon p_0}U^{+i}\tilde{\delta}_+(p)V^{-j}\frac{p^-}{p^2-i\epsilon p_0}$$
$$+p^+\tilde{\delta}_+(p)U^{+i}\frac{1}{p^2-i\epsilon p_0}V^{-j}\frac{p^-}{p^2-i\epsilon p_0}|x_1)^{ab}U^{b,+j}(x_1)$$
$$=\frac{N_c}{8\pi^2(N_c^2-1)}\int {d\!\!\!{}^-}\alpha'_a {d\!\!\!{}^-} k'_{a_\perp} {d\!\!\!{}^-}\beta'_b {d\!\!\!{}^-} k'_{b_\perp} {d\!\!\!{}^-}\alpha_a {d\!\!\!{}^-} k_{a_\perp} {d\!\!\!{}^-}\beta_b {d\!\!\!{}^-} k_{b_\perp}$$
$$\times e^{-i\alpha'_a\varrho x_2^- -i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ -i\beta_b\varrho x_1^+}e^{-i(k'_a+k'_b,x_2)_\perp -i(k_a+k_b,x_1)_\perp} \tag{6.7}$$
$$\times U^{+,b}_{i}(\alpha'_a,k'_{a_\perp})V^{-i,a}(\beta'_b,k'_{b_\perp})U^{+,b}_{j}(\alpha_a,k_{a_\perp})V^{-j,a}(\beta_b,k_{b_\perp})I_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)$$

where

$$I_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)=8\pi^2s^2\int {d\!\!\!{}^-}\alpha {d\!\!\!{}^-}\beta {d\!\!\!{}^-} p_\perp\; e^{i\alpha\varrho x_{12}^- +i\beta\varrho x_{12}^+ -i(p,x_{12})_\perp} \tag{6.8}$$
$$\times\Big[\frac{\alpha'_a+\alpha}{(\alpha'_a+\alpha)\beta s-(p+k'_a)^2_\perp+i\epsilon}\tilde{\delta}(\alpha\beta s-p_\perp^2)\theta(\alpha)\frac{\beta-\beta_b}{\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp-i\epsilon}$$
$$+\tilde{\delta}[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2_\perp]\theta(\beta)\frac{1}{\alpha\beta s-p_\perp^2-i\epsilon}\frac{(\alpha+\alpha'_a)(\beta-\beta_b)}{\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp-i\epsilon\alpha}$$
$$+\frac{(\alpha'_a+\alpha)(\beta-\beta_b)}{(\alpha'_a+\alpha)\beta s-(p+k'_a)^2_\perp+i\epsilon(\alpha'_a+\alpha)}\frac{1}{\alpha\beta s-p_\perp^2+i\epsilon}\tilde{\delta}[\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp]\theta(\alpha)\Big]$$

Here we used the fact that after taking matrix elements between nucleon states only the colorless operators survive.

Note that singularities in denominators in there expressions correspond to $\alpha'_a+i\epsilon$ and $\beta_b+i\epsilon$ so it is sufficient to perform calculations at, say, positive α'_a and β_b.

To calculate the integral (6.8) it is convenient to split it in two parts using identity

$$\delta(\alpha\beta s-p_\perp^2)=\delta(\alpha\beta s-p_\perp^2)\Big[\frac{p_\perp^2}{\alpha^2s\xi+p_\perp^2}+\frac{p_\perp^2}{\beta^2s\xi^{-1}+p_\perp^2}\Big] \tag{6.9}$$

where ξ is an arbitrary positive number of order of 1. We get

$$I_1=I_{1a}+I_{1b}, \tag{6.10}$$

where

$$I_{1a}(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)=8\pi^2s^2\int đ\alpha đ\beta đp_\perp\ \theta(\alpha)\ e^{i\alpha\varrho x^-_{12}+i\beta\varrho x^+_{12}-i(p,x_{12})_\perp}$$
$$\times\Big[\frac{\theta(\alpha)(\alpha'_a+\alpha)(\beta-\beta_b)\tilde\delta(\alpha\beta s-p_\perp^2)}{[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2+i\epsilon][\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp-i\epsilon]}\frac{p_\perp^2}{\alpha^2s\xi+p_\perp^2}$$
$$+\frac{(\alpha'_a+\alpha)(\beta-\beta_b)}{[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2_\perp+i\epsilon(\alpha'_a+\alpha)](\alpha\beta s-p^2_\perp+i\epsilon)}\tilde\delta[\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp]\Big] \tag{6.11}$$

and

$$I_{1b}(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)=8\pi^2s^2\int đ\alpha đ\beta đp_\perp\ \theta(\beta)e^{i\alpha\varrho x^-_{12}+i\beta\varrho x^+_{12}-i(p,x_{12})_\perp}$$
$$\times\Big[\frac{(\alpha'_a+\alpha)(\beta-\beta_b)\tilde\delta(\alpha\beta s-p_\perp^2)}{[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2+i\epsilon][\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp-i\epsilon]}\frac{p_\perp^2}{\beta^2s\xi^{-1}+p_\perp^2}$$
$$+\tilde\delta[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2]\frac{1}{\alpha\beta s-p^2_\perp-i\epsilon}\frac{(\alpha+\alpha'_a)(\beta-\beta_b)}{\alpha(\beta-\beta_b)-(p-k_b)^2_\perp-i\epsilon\alpha}\Big] \tag{6.12}$$

As we discussed above, the eq. (6.7) is not yet the contribution to the coefficient function. According to eq. (4.2), one needs to subtract relevant matrix elements of gluon TMDs from the result of calculation in the background fields $\mathcal{A}$ and $\mathcal{B}$. The "projectile" matrix elements of operator $\hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})$ are given by the diagrams shown in figure 12 and "target" matrix elements of operator $\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})$ are given by the diagrams shown in figure 11. Consequently, to get the contribution of eq. (6.8) to the coefficient function one should subtract from the integral (6.8) two eikonal-type contributions of TMD matrix elements coming from diagrams shown in figure 11a,b and figure 12d,e.

The first contribution, coming from the "projectile" eikonals in figure 11a,b, corresponds to the $\alpha\ll\alpha'_a$ asymptotics in eq. (6.8) cut from above according to "smooth" cutoff $e^{-i\frac{\alpha}{\sigma_t}}$ discussed in ref. [10] (see also appendix B)

$$I^{\rm eik}_{\rm fig.11a,b}(\beta_b,k_{b_\perp},x_1^+,x_{1_\perp},x_2^+,x_{2_\perp}) \tag{6.13}$$
$$=8\pi^2s\int_0^\infty đ\alpha\ e^{-i\frac{\alpha}{\sigma_t}}\frac{đ\beta}{\beta+i\epsilon}đp_\perp\ e^{i\beta\varrho x^+_{12}-i(p,x_{12})_\perp}$$
$$\times\left[\tilde\delta(\alpha\beta s-p^2_\perp)\frac{\beta-\beta_b}{\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp-i\epsilon}+\frac{(\beta-\beta_b)}{\alpha\beta s-p^2_\perp+i\epsilon}\tilde\delta[\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp]\right]$$

The second contribution coming from figure 12e,f corresponds to $\beta\ll\beta_b$ asymptotics of the integrand in eq. (6.8) integrated with the upper "smooth cutoff" factor $e^{i\frac{\beta}{\sigma_p}}$

$$I^{\rm eik}_{\rm fig.12e,f}(\alpha'_a,k'_{a_\perp},x_1^-,x_{1_\perp},x_2^-,x_{2_\perp}) \tag{6.14}$$
$$=8\pi^2s\int_0^\infty đ\beta\ e^{i\frac{\beta}{\sigma_p}}\frac{đ\alpha}{\alpha-i\epsilon}đp_\perp\ e^{i\alpha\varrho x^-_{12}-i(p,x_{12})_\perp}$$
$$\times\left[\frac{\alpha'_a+\alpha}{(\alpha'_a+\alpha)\beta s-(p+k'_a)^2+i\epsilon}\tilde\delta(\alpha\beta s-p^2_\perp)+\tilde\delta[(\alpha'_a+\alpha)\beta s-(p+k'_a)^2]\frac{\alpha+\alpha'_a}{\alpha\beta s-p^2_\perp-i\epsilon}\right]$$

To get the contribution to the coefficient function we need to calculate

$$J_1(\alpha'_a,k'{}_{a_\perp},\beta_b,k_{b_\perp},x_2,x_1) \tag{6.15}$$
$$=I_1(\alpha'_a,k'{}_{a_\perp},\beta_b,k_{b_\perp},z_2,z_1)-I^{\rm eik}_{\rm fig.11a,b}(\beta_b,k_{b_\perp},x_2,x_1)-I^{\rm eik}_{\rm fig.12e,f}(\alpha'_a,k'_{a_\perp},x_2x_1)$$

As demonstrated in the appendix D, one can neglect x_{12}^+ and x_{12}^- in the difference

$$I_{1a}(\alpha'_a, {k'}_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) - I^{\rm eik}_{\rm fig.11a,b}(\beta_b, k_{b_\perp}, x_1, x_2) \tag{6.16}$$

and similarly in

$$I_{1b}(\alpha'_a, {k'}_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) - I^{\rm eik}_{\rm fig.12e,f}(\alpha'_a, k'_{a_\perp}, x_1, x_2) \tag{6.17}$$

Qualitatively, the argument is as follows. Consider eq. (6.12). In order for $e^{i\alpha\varrho x_{12}^-}$ to be essential, α should be of order of α'_a due to eq. (4.23). Due to δ-functions in eq. (6.12), this means that β should be small, of order of $\frac{p_\perp^2}{\alpha'_a s} \ll 1$ since $p_\perp \sim \frac{1}{x_{12_\perp}} \sim Q_\perp$. However, the contribution of small β to eq. (6.8) is subtracted by small-β eikonal (6.14) so the resulting difference $I_{1b} - I^{\rm eik}_{\rm fig.11e,f}$ is small and the factor $e^{i\alpha\varrho x_{12}^-}$ can be neglected. Similarly, the factor $e^{i\beta\varrho x_{12}^+}$ in the difference $I_1 - I^{\rm eik}_{\rm fig.12a,b}$ at small α can be replaced by 1. In appendix D it is demonstrated that the corrections due to these approximations are $\sim \frac{Q_\perp^2}{\sigma_t\beta_b s} \sim \lambda_t$ and $\sim \frac{Q_\perp^2}{\sigma_p\alpha'_a s} \sim \lambda_p$, respectively. As we discussed in section 2, we neglect such power corrections.

We get

$$\begin{aligned}
I_{1a}(\alpha'_a, {k'}_{a_\perp}, \beta_b, k_{b_\perp}, x_{1_\perp}, x_{2_\perp}) &= 4\pi\int_0^\infty \frac{d\alpha}{\alpha}\int {\textit{đ}} p_\perp \,\frac{e^{-i(p,x_{12})_\perp}}{\alpha\beta_b s + (p-k_b)_\perp^2 - p_\perp^2 + i\epsilon} \\
&\times \left[\frac{p_\perp^2}{\alpha^2 s\xi + p_\perp^2}\frac{(\alpha+\alpha'_a)(\alpha\beta_b s - p_\perp^2)}{[(\alpha+\alpha'_a)p_\perp^2 - \alpha(p+k'_a)^2 + i\epsilon]}\right. \\
&\left. + \frac{(p-k_b)_\perp^2(\alpha+\alpha'_a)}{\alpha(\alpha'_a+\alpha)\beta_b s + (\alpha'_a+\alpha)(p-k_b)_\perp^2 - \alpha(p+k'_a)_\perp^2 + i\epsilon(\alpha'_a+\alpha)}\right]
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
I_{1b}(\alpha'_a, {k'}_{a_\perp}, \beta_b, {k'}_{b_\perp}, x_{1_\perp}, x_{2_\perp}) &= 4\pi\int_0^\infty \frac{d\beta}{\beta}\int {\textit{đ}} p_\perp \,\frac{e^{-i(p,x_{12})_\perp}}{\alpha'_a\beta s - (p+k'_a)^2 + p_\perp^2 + i\epsilon} \\
&\times \left[\int_0^\infty \frac{{\textit{đ}}\beta}{\beta}\frac{p_\perp^2\xi}{\beta^2 s + p_\perp^2\xi}\frac{(\beta_b-\beta)(\alpha'_a\beta s + p_\perp^2)}{((\beta_b-\beta)p_\perp^2 + \beta(p-k_b)^2 + i\epsilon)}\right. \\
&\left. + \frac{(\beta_b-\beta)(p+k'_a)_\perp^2}{(\alpha'_a+i\epsilon)(\beta_b-\beta)\beta s - (\beta_b-\beta)(p+k'_a)_\perp^2 - \beta(p-k_b)_\perp^2}\right]
\end{aligned} \tag{6.19}$$

To calculate the sum of eqs. (6.18) and (6.19), it is convenient to perform change of variables $\alpha = \frac{p_\perp^2}{\beta s}$ in the first term in square brackets in eq. (6.18) and change $\alpha = \frac{(p-k'_b)_\perp^2}{(\beta-\beta_b)s}$ in the second term. Since this change affects cancellation of logarithmic divergences at $\alpha \to 0$, before the change we replace $\int_0^\infty \frac{d\alpha}{\alpha}$ by $\int_\varepsilon^\infty \frac{d\alpha}{\alpha}$. After some algebra one obtains

$$\begin{aligned}
I_1(\alpha'_a, {k'}_{a_\perp}, \beta_b, k_{b_\perp}, x_{1_\perp}, x_{2_\perp}) &= 4\pi\int {\textit{đ}} p_\perp \, e^{-i(p,x_{12})_\perp}\int_0^\infty d\beta\,\Big[\theta(\beta-\beta_b) - \theta(\beta)\Big] \\
&\times \frac{(\beta-\beta_b)[(p-k_b)_\perp^2 - \alpha'_a(\beta_b-\beta)s]\,[\beta(p-k_b)_\perp^2 + p_\perp^2(\beta_b-\beta)s + i\epsilon]^{-1}}{(-(\alpha'_a+i\epsilon)\beta(\beta_b-\beta)s + (p-k_b)_\perp^2\beta + (p+k'_a)_\perp^2(\beta_b-\beta))} \\
&+ \lim_{\varepsilon\to 0}\frac{1}{\beta[(p-k_b)_\perp^2 - p_\perp^2 + i\epsilon]}\Big[\theta\Big(\beta - \frac{p_\perp^2}{\varepsilon s}\Big) - \theta\Big(\beta - \beta_b - \frac{(p-k_b)_\perp^2}{\varepsilon s}\Big)\Big]\Big]
\end{aligned} \tag{6.20}$$

JHEP03(2023)029

Let us take $\beta_b > 0$, then

$$I_1(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{1_\perp}, x_{2_\perp}) = 4\pi \int d\!\!\!{}^-p_\perp \; e^{-i(p,x_{12})_\perp} \tag{6.21}$$
$$\times \left[\int_0^1 du \; \frac{\bar u[(p-k_b)_\perp^2 - Q^2_{a'b}\bar u]}{[\bar u(p+k'_a)_\perp^2 + u(p-k_b)_\perp^2 - Q^2_{a'b}\bar u u][p_\perp^2 \bar u + u(p-k_b)_\perp^2]} + \frac{\ln(p-k_b)_\perp^2/p_\perp^2}{(p-k_b)_\perp^2 - p_\perp^2} \right]$$
$$= -4\pi \int d\!\!\!{}^-p_\perp \int_0^1 du \; \frac{e^{-i(p,x_{12})_\perp} \bar u Q^2_{ab'}}{[\bar u(p+k'_a)_\perp^2 + u(p-k_b)_\perp^2 - Q^2_{a'b}\bar u u][p_\perp^2 \bar u + u(p-k_b)_\perp^2]} + O(\lambda)$$
$$= -4\pi \int d\!\!\!{}^-p_\perp \int_0^1 du \; \frac{e^{-i(p,x_{12})_\perp} Q^2_{ab'}}{[\bar u(p+k'_a)_\perp^2 - Q^2_{a'b} u][p_\perp^2 \bar u + u(p-k_b)_\perp^2]} + O(\lambda)$$
$$= 4\pi \int d\!\!\!{}^-p_\perp \; e^{-i(p,x_{12})_\perp} \frac{Q^2_{a'b}}{Q^2_{a'b}p_\perp^2 + (p+k'_a)_\perp^2 (p-k_b)_\perp^2} \ln \frac{-Q^2_{a'b}p_\perp^2}{(p+k'_a)_\perp^2 (p-k_b)_\perp^2} + O(\lambda)$$

where $\bar u \equiv 1-u$ and $Q^2_{a'b} \equiv (\alpha'_a + i\epsilon)(\beta_b + i\epsilon)s$ (recall that the analytical properties of integrals over α'_a and β_b are determined by the integral (6.8) to be $\alpha'_a + i\epsilon$ and $\beta_b + i\epsilon$).

This integral is calculated in the appendix F.2, see eq. (F.9):

$$4\pi \int d\!\!\!{}^-p_\perp \; \frac{e^{-i(p,x)_\perp} Q^2_{ab}}{Q^2_{ab}p_\perp^2 + (p+k_a)_\perp^2 (p-k_b)_\perp^2} \ln \frac{-Q^2_{ab}p_\perp^2}{(p+k_a)_\perp^2 (p-k_b)_\perp^2} \tag{6.22}$$
$$= \ln \frac{-Q^2_{ab}}{k^2_{a_\perp}} \ln \frac{-Q^2_{ab}}{k^2_{b_\perp}} - \frac{1}{2}\Big(\ln \frac{-Q^2_{ab}x_\perp^2}{4} + 2\gamma \Big)^2 + \frac{\pi^2}{3}$$
$$+ \int_0^1 \frac{du}{u} \left[\ln \frac{k^2_{a_\perp} x_\perp^2 \bar u u}{4} + 2\gamma + 2e^{iu(k,x)_\perp} K_0\big(\sqrt{k^2_{a_\perp} x_\perp^2 \bar u u}\big) + k_{a_\perp} \leftrightarrow -k_{b_\perp} \right] + O(\lambda)$$

It is convenient to represent it as a sum of the double-log contribution similar to virtual term, and the remainder. We get

$$I_1 = I^{\rm d.log}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}) + I_1^{\rm rem}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) \tag{6.23}$$

where $I^{\rm d.log}$ was defined in eq. (5.29)

$$I^{\rm d.log}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}) \; = \ln \frac{-Q^2_{a'b}}{k'^2_{a_\perp}} \ln \frac{-Q^2_{a'b}}{k^2_{b_\perp}} + \frac{\pi^2}{3} + O(\lambda), \tag{6.24}$$

and

$$I_1^{\rm rem}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) \tag{6.25}$$
$$= -\frac{1}{2}\left(\ln \frac{-Q^2_{a'b}x^2_{12_\perp}}{4} + 2\gamma \right)^2 + \int_0^1 \frac{du}{u} \left[\left(\ln \frac{k'^2_{a_\perp} x^2_{12_\perp} \bar u u}{4} + 2\gamma \right) \right.$$
$$\left. + 2e^{-iu(k'_a,x_{12})_\perp} K_0\big(\sqrt{k'^2_{a_\perp} x^2_{12_\perp} \bar u u}\big) + k'_{a_\perp} \leftrightarrow -k_{b_\perp} \right] + O(\lambda)$$
$$= -\frac{1}{2}\left(\ln \frac{-Q^2_{a'b}x^2_{12_\perp}}{4} + 2\gamma \right)^2 + I_K(k'_{a_\perp}, x_{12_\perp}) + I_K(-k_{b_\perp}, x_{12_\perp}) + O(\lambda)$$

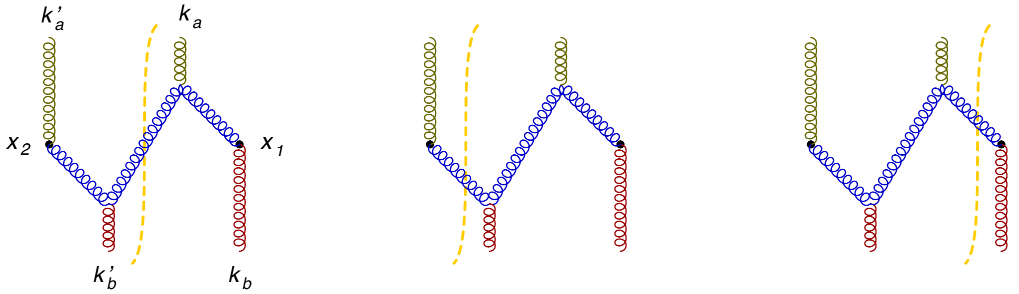

Figure 8. Second set of leading diagrams with gluon production. Projectile fields $U^{-i}(z_2)$ are denoted by green tails while target fields $V^{+i}(z_1)$ by red tails.

where

$$I_K(k_\perp, x_\perp) \equiv \int_0^1 \frac{du}{u}\left[\ln\frac{k_\perp^2 x_\perp^2 \bar{u}u}{4} + 2\gamma + 2e^{iu(k,x)_\perp}K_0\Big(\sqrt{k_\perp^2 x_\perp^2 \bar{u}u}\Big)\right] \tag{6.26}$$

and K_0 is the Macdonald function.

The second leading contribution to hadronic tensor (4.20) comes from the second term in the r.h.s. of eq. (6.1)

$$\begin{aligned}
&U^{+i,a}(x_2)\langle g^2 F_i^{-,a}(x_2)F_j^{+,b}(x_1)\rangle_{\mathbb{A}} V^{-j,b}(x_1)\\
&= -4g^2 U^{+i,a}(x_2)(x_2|\frac{p^-}{p^2+i\epsilon p_0}V^{-i}\frac{1}{p^2+i\epsilon p_0}U^{+j}p^+\tilde{\delta}_+(p)\\
&\quad + \frac{p^-}{p^2+i\epsilon p_0}V^{-i}\tilde{\delta}_+(p)U^{+j}\frac{p^+}{p^2-i\epsilon p_0} + p^-\tilde{\delta}_+(p)V^{-i}\frac{1}{p^2-i\epsilon p_0}U^{+j}\frac{p^+}{p^2-i\epsilon p_0}|x_1)^{ab}V^{-j,b}(x_1)\\
&= \frac{g^2 N_c}{8\pi^2(N_c^2-1)}\int đ\alpha_a đk_{a_\perp} đ\beta_b đk_{b_\perp} đ\alpha'_a đk'_{a_\perp} đ\beta'_b đk'_{b_\perp}\\
&\quad\times e^{-i\alpha_a\varrho x_1^- - i\alpha'_a\varrho x_2^-} e^{-i\beta_b\varrho x_1^+ - i\beta'_b\varrho x_2^+} e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp}\\
&\quad\times U_i^{+,b}(\alpha_a,k_{a_\perp})V^{-i,a}(\beta_b,k_{b_\perp})U_j^{+,b}(\alpha'_a,k'_{a_\perp})V^{-j,a}(\beta'_b,k'_{b_\perp})I_2(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_1,x_2)
\end{aligned} \tag{6.27}$$

The corresponding diagrams are shown in figure 8. It is clear that they differ from the diagrams in figure 7 by trivial projectile↔target replacements

$$x^+ \leftrightarrow x^-, \qquad \alpha_a \leftrightarrow \beta_b, \qquad \alpha'_a \leftrightarrow \beta'_b, \qquad k_{a_\perp} \leftrightarrow k_{b_\perp}, \qquad k'_{a_\perp} \leftrightarrow k'_{b_\perp} \tag{6.28}$$

so we get

$$\begin{aligned}
&I_2(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_1,x_2) = 8\pi^2 s^2\int đ\alpha đ\beta đp_\perp\; e^{i\alpha\varrho x_{12}^- + i\beta\varrho x_{12}^+ - i(p,x_{12})_\perp}\\
&\quad\times\bigg[\frac{\beta'_b+\beta}{(\beta'_b+\beta)\beta s-(p+k'_b)^2+i\epsilon}\tilde{\delta}(\alpha\beta s-p_\perp^2)\theta(\alpha)\frac{\alpha-\alpha_a}{(\alpha-\alpha_a)\beta s-(p-k_a)_\perp^2-i\epsilon}\\
&\quad+\tilde{\delta}[\alpha(\beta'_b+\beta)s-(p+k'_b)^2]\theta(\alpha)\frac{1}{\alpha\beta s-p_\perp^2-i\epsilon}\frac{(\alpha-\alpha_a)(\beta+\beta'_b)}{(\alpha-\alpha_a)\beta s-(p-k_a)_\perp^2-i\epsilon\beta}\\
&\quad+\frac{(\alpha-\alpha_a)(\beta'_b+\beta)}{\alpha(\beta'_b+\beta)s-(p+k'_b)_\perp^2+i\epsilon(\beta'_b+\beta)}\frac{1}{\alpha\beta s-p_\perp^2+i\epsilon}\tilde{\delta}[\beta(\alpha-\alpha_a)s-(p-k_a)_\perp^2]\theta(\alpha)\bigg]
\end{aligned} \tag{6.29}$$

Similarly to the previous case, after subtraction of the corresponding "projectile" eikonals in figure 12a,b and "projectile" eikonals in figure 11e,f one can set $x_{12}^{\parallel}=0$ and get

$$I_2(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_1,x_2)\stackrel{x_{12}^{\parallel}=0}{=}I^{\rm d.log}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp})+I_2^{\rm rem}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_{12_\perp}) \tag{6.30}$$

from eqs. (6.24) and (6.25) projectile↔target replacements (6.28) so that

$$I^{\rm d.log}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp})=\ln\frac{-Q^2_{ab'}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab'}}{k'^2_{b_\perp}}+\frac{\pi^2}{3}+O(\lambda) \tag{6.31}$$

(cf. eq. (6.24)) and

$$\begin{aligned}&I_2^{\rm rem}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_{2_\perp},x_{1_\perp})\\&=-\frac{1}{2}\left(\ln\frac{-Q^2_{ab'}x_\perp^2}{4}+2\gamma\right)^2+I_K(k'_{b_\perp},x_{21_\perp})+I_K(-k_{a_\perp},x_{21_\perp})+O(\lambda)\end{aligned} \tag{6.32}$$

where I_K is given by eq. (6.26).

The final result for "production" contributions (at $x_{12}^+=x_{12}^-=0$) can be presented as follows

$$\begin{aligned}\mathcal{W}^{\rm prod}(x_1,x_2)=&\frac{N_c^2-1}{N_c}8\pi^2\Big(V^{-i,a}(x_2)\langle F_i^{+,a}(x_2)F_j^{-,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.7}U^{+j,b}(x_1)\\&+U^{+i,a}(x_2)\langle F_i^{-,a}(x_2)F_j^{+,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.8}V^{-j,b}(x_1)\Big)\\\stackrel{x_{12}^{\parallel}=0}{=}&\int {d\!\!\!{}^-}\alpha_a {d\!\!\!{}^-}k_{a_\perp}{d\!\!\!{}^-}\beta_b {d\!\!\!{}^-}k_{b_\perp}{d\!\!\!{}^-}\alpha'_a {d\!\!\!{}^-}k'_{a_\perp}{d\!\!\!{}^-}\beta_b {d\!\!\!{}^-}k'_{b_\perp}\,e^{-i\alpha'_a\varrho x_2^- -i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ -i\beta_b\varrho x_1^+}\\&\times e^{-i(k_a+k_a,x_1)_\perp -i(k'_a+k'_b,x_2)_\perp}\;U_i^{+,b}(\alpha'_a,k'_{a_\perp})V^{-i,a}(\beta'_b,k_{b_\perp})U_j^{+,b}(\alpha_a,k_{a_\perp})V^{-j,a}(\beta_b,k_{b_\perp})\\&\times I^{\rm prod}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp},\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp})+O(\lambda)\end{aligned} \tag{6.33}$$

where

$$\begin{aligned}&I^{\rm prod}(\alpha_a,k_{a_\perp},\beta_b,k_{b_\perp},\alpha'_a,k'_{a_\perp},\beta'_b,k'_{b_\perp})\\&=I^{\rm d.log}(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp})+I_1^{\rm rem}(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_{21_\perp})\\&\quad+I^{\rm d.log}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp})+I_2^{\rm rem}(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_{21_\perp})\end{aligned} \tag{6.34}$$

where $I_1^{\rm rem}$ and $I_2^{\rm rem}$ are given by eqs. (6.25) and (6.32), respectively.

6.3 Handbag diagrams

Let us start with the third term in the r.h.s. of eq. (6.1) given by "target handbag" diagrams in figure 9. We need to subtract from these diagrams the corresponding diagrams coming from "target" TMD eikonals in figure 9. As was mentioned above (see appendix B for details), we use "point-splitting" regularization of integrals over α in these contributions. The subtracted "eikonal" diagrams then look the same as those in figure 9 with the only difference in points where gluon fields $V^{-i,a}$, $F_i^{+,a}$, $F_j^{-,b}$, and $U^{+j,b}$ are located.

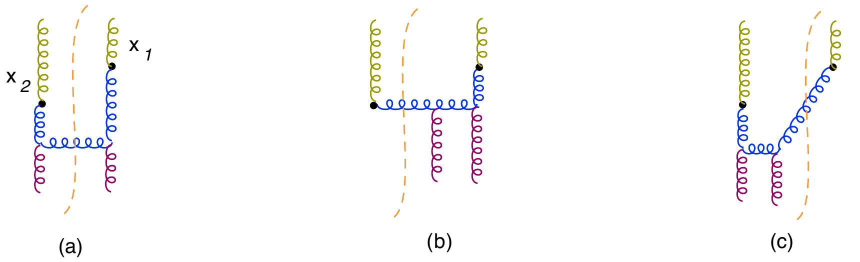

Figure 9. “Target” handbag diagrams.

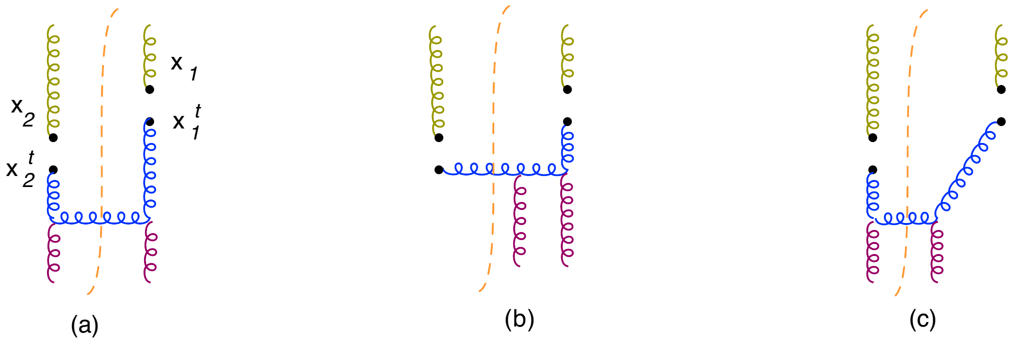

Figure 10. “Target eikonal” handbag diagrams. Here $x_1^t = x_{1_\perp} + x_1^+$, $x_1' = x_1^t + \delta^+$ and similarly for x_2.

Using Wightman “cut” propagator (A.5) in the background field one easily obtains

$$\begin{aligned}
&\langle g^2 \tilde{F}^{-i,a}(x_2) F^{-j,b}(x_1)\rangle = -4(x_2|\frac{p_i}{p^2+i\epsilon p_0} V^-_{\ \xi}\tilde{\delta}_+(p) V^{-\xi}\frac{p_j}{p^2-i\epsilon p_0} \\
&- p_i\tilde{\delta}_+(p) V^-_{\ \xi}\frac{1}{p^2-i\epsilon p_0} V^{-\xi}\frac{p_j}{p^2-i\epsilon p_0} - \frac{p_i}{p^2+i\epsilon p_0} V^-_{\ \xi}\frac{1}{p^2+i\epsilon p_0} V^{-\xi} p_j \tilde{\delta}_+(p)|x_1)^{ab} g^2 \\
&= -2g^2 \int \text{đ}\beta_b \text{đ}\beta_b' \text{đ}k_{b_\perp} \text{đ}k'_{b_\perp} e^{-ik_a x_1 - ik_b' x_2} V^{-,ac}_{\ k}(\beta_b', k'_{b_\perp}) V^{-k,cb}(\beta_b, k_{b_\perp}) \int \text{đ}p_\perp \\
&\times \int_0^\infty \frac{\text{đ}\alpha}{\alpha}\Bigg\{ \frac{(p+k_b')_i (p-k_b)_j \left(e^{-i\beta_b'\varrho x_{12}^+ + i\frac{(p+k_b')_\perp^2}{\alpha s}\varrho x_{12}^+} - e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\right) e^{i\alpha\varrho x_{12}^- - i(p,x_{12})_\perp}}{[\alpha\beta_b' s - (p+k_b')_\perp^2 + p_\perp^2 + i\epsilon][\alpha(\beta_b'+\beta_b)s - (p+k_b')_\perp^2 + (p-k_b)_\perp^2 + i\epsilon]} \\
&+ \frac{(p+k_b')_i (p-k_b)_j \left(e^{i\beta_b\varrho x_{12}^+ + i\frac{(p-k_b)_\perp^2}{\alpha s}\varrho x_{12}^+} - e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\right) e^{i\alpha\varrho x_{12}^- - i(p,x_{12})_\perp}}{[\alpha(\beta_b'+\beta_b)s + (p-k_b)_\perp^2 - (p+k_b')_\perp^2 + i\epsilon][\alpha\beta_b s + (p-k_b)_\perp^2 - p_\perp^2 + i\epsilon]}\Bigg\} \qquad (6.35)
\end{aligned}$$

so we get (see the definition (4.20))

$$\begin{aligned}
&\mathcal{W}^{\text{handbag}}(x_1,x_2) - \mathcal{W}^{\text{handbag}}_{\text{eik}}(x_1,x_2) = 8\pi^2 \int \text{đ}\alpha_a' \text{đ}k'_{a_\perp} \text{đ}\beta_b' \text{đ}k_{b_\perp} \text{đ}\alpha_a \text{đ}k'_{a_\perp} \text{đ}\beta_b \text{đ}k'_{b_\perp} \\
&\times e^{-i\alpha_a'\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta_b'\varrho x_2^+ - i\beta_b\varrho x_1^+} e^{-i(k_a+k_a,x_2)_\perp - i(k_a'+k_b',x_1)_\perp} U^{+,b}_{\ i}(\alpha_a', k'_{a_\perp}) \qquad (6.36) \\
&\times V^{-i,a}(\beta_b', k_{b_\perp}) U^{+,b}_{\ j}(\alpha_a, p'_{A_\perp}) V^{-j,a}(\beta_b, k'_{b_\perp}) I^{h1}(\beta_b, \beta_b', k_{b_\perp}, k'_{b_\perp}, x_1, x_2)
\end{aligned}$$

JHEP03(2023)029

where

$$I^{h1}(\beta_b',k_{b_\perp},\beta_b,k'_{b_\perp},x_1,x_2) = -2\int đp_\perp\, e^{-i(p,x_{12})_\perp}\int_0^\infty \frac{đ\alpha}{\alpha}\Big(e^{i\alpha\varrho x_{12}^-}-1\Big)$$
$$\times\Bigg\{\frac{(p+k_b')_i(p-k_b)_j\Big(e^{-i\beta_b'\varrho x_{12}^+ + i\frac{(p+k_b')_\perp^2}{\alpha s}\varrho x_{12}^+}-e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\Big)}{[\alpha\beta_b's-(p+k_b')_\perp^2+p_\perp^2+i\epsilon][\alpha(\beta_b'+\beta_b)s-(p+k_b')_\perp^2+(p-k_b)_\perp^2+i\epsilon]}$$
$$+\frac{(p+k_b')_i(p-k_b)_j\Big(e^{i\beta_b\varrho x_{12}^+ + i\frac{(p-k_b)_\perp^2}{\alpha s}\varrho x_{12}^+}-e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\Big)}{[\alpha(\beta_b'+\beta_b)s+(p-k_b)_\perp^2-(p+k_b')_\perp^2+i\epsilon][\alpha\beta_b s+(p-k_b)_\perp^2-p_\perp^2+i\epsilon]}\Bigg\} \tag{6.37}$$

The term "-1" in the parentheses in the first line comes from subtraction of "eikonal" diagrams in figure 10. Note that there is no need for the additional cutoff for α integrals in those diagrams since the integral (6.37) is convergent.

Now, in is easy to see that the integral over $p_\perp$ is convergent so the characteristic $p_\perp\sim Q_\perp\sim x_{12_\perp}^{-1}$. Moreover, the integral over α is convergent at $\alpha\sim\frac{1}{\varrho x_{12}^-}\sim\alpha_a'$ which means that even at $\beta_b'+\beta_b=0$ the integral in the eq. (6.37) is $\sim\frac{Q_\perp^2}{Q^2}$ which is a power correction. Similarly, the fourth term in eq. (6.1) is given by the same set of diagrams with projectile↔target reflection so after subtractions of "projectile eikonal" handbag diagrams of figure 12 g-i it becomes a power correction.

7 Result for the sum of diagrams in figures 5, 6, 7, 8 minus TMD matrix elements in figures 11, 12

Assembling eqs. (5.27), (5.28), (6.33), (6.34) and subtracting "eikonal" TMD matrix elements given by eq. (B.11), we get

$$\mathcal{W}(x_1,x_2)-\mathcal{W}^{\rm eik}(x_1,x_2)=\frac{N_c^2-1}{N_c}8\pi^2g^2$$
$$\times\Big\{V^{-i,a}(x_2)\langle F_i^{+a}(x_2)F^{-j,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.7}U_j^{+b}(x_1)+U^{+i,a}(x_2)\langle F_i^{-,a}(x_2)F_j^{+,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.8}V^{-j,b}(x_1)$$
$$+V^{-i,a}(x_2)U_i^{+a}(x_2)\langle F^{-j,b}(x_1)F_j^{+b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.5}+\langle F^{-i,a}(x_2)F_i^{+a}(x_2)\rangle_{\mathbb{A}}^{\rm fig.6}V^{-j,b}(x_1)U_j^{+b}(x_1)$$
$$-U_i^{+a}(x_2)V^{-i,n}(x_2)\langle[x_2^+,-\infty]_{x_{2_\perp}+\delta^-}^{na}[-\infty,x_1^+]_{x_{1_\perp}+\delta^-}^{bc}F^{-j,c}(x_1^+,x_{1_\perp})\rangle_{\mathcal{A}}^{\rm fig.11a-c}U^{+j,b}(x_1)$$
$$-U_i^{+a}(x_2)\langle F^{-i,n}(x_2^+,x_{2_\perp})[x_2^+,-\infty]_{x_{2_\perp}+\delta^-}^{na}[-\infty,x_1^+]_{x_{1_\perp}+\delta^-}^{bc}\rangle_{\mathcal{A}}^{\rm fig.11d-f}V^{-j,c}(x_1)U^{+j,b}(x_1)$$
$$-U_i^{+n}(x_2)V^{-i,a}(x_2)\langle[x_2^-,-\infty]_{x_{2_\perp}+\delta^+}^{na}[-\infty,x_1^-]_{x_{1_\perp}+\delta^+}^{bc}F^{+j,c}(x_1^-,x_{1_\perp})\rangle_{\mathcal{A}}^{\rm fig.12a-c}V^{-j,c}(x_1)$$
$$-V^{-i,a}(x_2)\langle F^{+i,n}(x_2^-,x_{2_\perp})[x_2^-,-\infty]_{x_{2_\perp}+\delta^+}^{na}[-\infty,x_1^-]_{x_{1_\perp}+\delta^+}^{bc}\rangle_{\mathcal{A}}^{\rm fig.12d-f}V^{-j,c}(x_1)U_j^{+b}(x_1)\Big\}$$
$$=\int đ\alpha_a'đk'_{a_\perp}đ\beta_b'đk_{b_\perp}đ\alpha_ađk'_{a_\perp}đ\beta_bđk'_{b_\perp}e^{-i\alpha_a'\varrho x_2^- -i\alpha_a\varrho x_1^-}e^{-i\beta_b'\varrho x_2^+ -i\beta_b\varrho x_1^+}$$
$$\times e^{-i(k_a+k_b,x_1)_\perp - i(k_a'+k_b',x_2)_\perp}U_i^{+,b}(\alpha_a',k'_{a_\perp})V^{-i,a}(\beta_b',k'_{b_\perp})U_j^{+,b}(\alpha_a,k_{a_\perp})V^{-j,a}(\beta_b,k_{b_\perp})$$
$$\times g^2[I-I_{\rm eik}^{\sigma_p,\sigma_t}](\alpha_a,\alpha_a',\beta_b,\beta_b',k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2) \tag{7.1}$$

with

$$
\begin{aligned}
&[I-I^{\sigma_p,\sigma_t}_{\rm eik}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)\\
&=-I^{\rm d.log}(\alpha_a,\beta_b,k_{a_\perp},k_{b_\perp})-I^{\rm d.log}(\alpha'_a,\beta'_b,k'_{a_\perp},k'_{b_\perp})+I^{\rm d.log}(\alpha'_a,\beta_b,k'_{a_\perp},k_{b_\perp})\\
&\quad+I^{\rm d.log}(\alpha_a,\beta'_b,k_{a_\perp},k'_{b_\perp})+I^{\rm rem}_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_{12_\perp})+I^{\rm rem}_2(\alpha_a,k_{a_\perp},\beta'_b,k'_{b_\perp},x_{12_\perp})\\
&\quad-I^{\sigma_p,\sigma_t}_{\rm eik}(\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_{12_\perp})\\
&=-\ln\frac{-Q^2_{ab}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab}}{k^2_{b_\perp}}-\ln\frac{-Q^2_{a'b'}}{k'^2_{a_\perp}}\ln\frac{-Q^2_{a'b'}}{k'^2_{b_\perp}}+\ln\frac{-Q^2_{a'b}}{k'^2_{a_\perp}}\ln\frac{-Q^2_{a'b}}{k^2_{b_\perp}}\\
&\quad+\ln\frac{-Q^2_{ab'}}{k^2_{a_\perp}}\ln\frac{-Q^2_{ab'}}{k'^2_{b_\perp}}-\frac{1}{2}\Big(\ln\frac{-Q^2_{a'b}x^2_\perp}{4}+2\gamma\Big)^2-\frac{1}{2}\Big(\ln\frac{-Q^2_{ab'}x^2_\perp}{4}+2\gamma\Big)^2\\
&\quad+\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha'_a+i\epsilon)\sigma_p s x^2_\perp e^\gamma\Big)+\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha_a+i\epsilon)\sigma_p s x^2_\perp e^\gamma\Big)\\
&\quad+\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta'_b+i\epsilon)\sigma_t s x^2_\perp e^\gamma\Big)+\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta_b+i\epsilon)\sigma_t s x^2_\perp e^\gamma\Big)+\pi^2
\end{aligned}
\tag{7.2}
$$

Note that the contribution proportional to integral (6.26) canceled. After some algebra this result can be represented as

$$
\begin{aligned}
&[I-I^{\sigma_p,\sigma_t}_{\rm eik}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1)\\
&=-\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}+\ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}\\
&\quad-\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p}-\ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2
\end{aligned}
\tag{7.3}
$$

However, this formula is not the final result for the coefficient function (4.2) since the integrals $I^{\rm virt}$ get contributions form soft/Glauber gluons (sG-gluons) which need to be subtracted. Indeed, the coefficient function (4.2) was defined as a result of integration over C-fields with $\alpha>\sigma_t$ and $\beta>\sigma_p$. Since we did not impose these restrictions while calculating the loop integrals like eq. (5.9) and eq. (6.8), we need to subtract $\alpha<\sigma_t,\beta<\sigma_p$ contributions to these integrals. This will be done in the next section.

8 Subtraction of soft/Glauber contributions

As we mentioned above, the coefficient function $\mathfrak{C}_1$ in eq. (4.1) was defined as an integral over large $|\alpha|>\sigma_t$ and $|\beta|>\sigma_p$ so the contributions to our background-field diagrams with $|\alpha|<\sigma_t$ and/or $|\beta|<\sigma_p$ should be subtracted from the result (7.3). We have already subtracted the TMD matrix elements: "target eikonals" with $|\alpha|<\sigma_t$ and "projectile eikonals" with $|\beta|<\sigma_p$. In section 8.4 below we prove that sG-contributions to TMD matrix elements are power corrections so there is no double counting. Still, we need to subtract sG-contributions from $\mathcal{W}(x_1,x_2)$ itself.

As for the case of subtractions of eikonal TMD matrix elements, we use smooth α and β cutoffs which do not change the analytical properties of the diagrams. Let us again start with virtual contributions.

8.1 sG-contributions to virtual diagrams

The virtual contribution of diagrams in figure 5 is given by eqs. (5.16) and (5.17). Let us now calculate contribution of sG-gluons to virtual diagram. Integral (5.17) with "smooth" α and β restrictions has the form

$$I^{\text{virt sG}}_{\text{fig.5}} = -16\pi^2 \int \frac{d\!\!{}^-{}^4 p}{i} \Big[\frac{s\alpha_a\beta_b}{[(p+k_a)^2+i\epsilon](p^2+i\epsilon)[(p-k_b)^2+i\epsilon]} + \tilde{\delta}_-(p+k_a)\frac{s\alpha_a\beta_b}{p^2-i\epsilon}\tilde{\delta}_+(p-k_b)\Big] e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}} \tag{8.1}$$

Note that the choice of signs of exponential cutoffs in sG contributions should be correlated with the choice in TMD matrix elements in order to have the same sG subtractions in eq. (4.2). In addition, as we will see below, the above choice of signs of the exponential cutoffs agrees with analytical properties of the original uncut diagram, namely that the background fields are emitted before the point x_1, see eq. (5.18).

Let us perform the calculation for the most complicated case $\alpha_a, \beta_b < 0$ where we need both terms in the r.h.s. of eq. (8.1).

$$\begin{aligned} I^{\text{virt sG}}_{\text{fig.5}} =& -16\pi^2 \int \frac{d\!\!{}^-{}^4 p}{i} e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}} \Big\{ \frac{\alpha_a\beta_b s[\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp+i\epsilon]^{-1}}{[(\alpha+\alpha_a)\beta s-(p+k_a)^2_\perp+i\epsilon](\alpha\beta s-p^2_\perp+i\epsilon)} \\ &+ \tilde{\delta}[(\alpha_a+\alpha)\beta s-(p+k_a)^2]\theta(-\beta)\frac{\alpha_a\beta_b s}{\alpha\beta s-p^2_\perp-i\epsilon}\tilde{\delta}[\alpha(\beta-\beta_b)s-(p-k_b)^2_\perp]\theta(\alpha)\Big\} \\ \simeq&\ 16\pi^2 \int \frac{d\!\!{}^-{}p}{i}\Big[\frac{\alpha_a}{\alpha_a\beta s-(p+k_a)^2_\perp+i\epsilon}\frac{s}{\alpha\beta s-p^2_\perp+i\epsilon}\frac{\beta_b}{\alpha\beta_b s+(p-k_b)^2_\perp-i\epsilon} \\ &- \tilde{\delta}[\alpha_a\beta s-(p+k_a)^2_\perp]\theta(-\beta)\frac{\alpha_a\beta_b s}{\alpha\beta s-p^2_\perp-i\epsilon}\tilde{\delta}[\alpha\beta_b s+(p-k_b)^2_\perp]\theta(\alpha)\Big]e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}} \end{aligned} \tag{8.2}$$

Here we neglected $\alpha \sim \sigma_t$ in comparison to α_a and $\beta \sim \sigma_p$ in comparison to β_b.

Next, we take residue over α and obtain

$$\begin{aligned} \text{Eq. (8.2)} \stackrel{\beta_b<0}{=}& -8\pi^2 \int d\!\!{}^-{}^2 p \int d\!\!{}^-\beta \Big[\frac{\alpha_a\beta_b s\theta(\beta)}{p^2_\perp\beta_b+(p-k_b)^2_\perp\beta-i\epsilon}\frac{e^{-i\frac{p^2_\perp}{\beta\sigma_t s}+i\frac{\beta}{\sigma_p}}}{\alpha_a\beta s-(p+k_a)^2_\perp+i\epsilon} \\ &- \frac{\alpha_a\beta_b s}{p^2_\perp\beta_b+(p-k_b)^2_\perp\beta-i\epsilon}\frac{\theta(\alpha_a)}{\alpha_a\beta s-(p+k_a)^2_\perp+i\epsilon}e^{i\frac{(p-k_b)^2_\perp}{\beta_b\sigma_t s}+i\frac{\beta}{\sigma_p}} \\ &- 2\pi i\theta(-\alpha_a)|\alpha_a\beta_b|s\ \tilde{\delta}[(p-k_b)^2_\perp\beta-p^2_\perp|\beta_b|]\frac{e^{-i\frac{p^2_\perp}{\beta\sigma_t s}+i\frac{\beta}{\sigma_p}}}{\alpha_a\beta s-(p+k_a)^2_\perp+i\epsilon}\Big] \\ =& -8\pi^2 \int d\!\!{}^-{}^2 p \int d\!\!{}^-\beta \frac{-\alpha_a|\beta_b|s\theta(\beta)}{-p^2_\perp|\beta_b|+(p-k_b)^2_\perp\beta+i\epsilon}\frac{1}{\alpha_a\beta s-(p+k_a)^2_\perp+i\epsilon}e^{-i\frac{p^2_\perp}{\beta\sigma_t s}+i\frac{\beta}{\sigma_p}} \\ \stackrel{\beta=v^2|\beta_b|}{=}&\ 4\pi\int d\!\!{}^-{}^2 p\int_0^\infty dv^2 \frac{\alpha_a|\beta_b|s}{p^2_\perp-(p-k_b)^2_\perp v^2-i\epsilon}\frac{e^{-i\frac{p^2_\perp}{v^2|\beta_b|\sigma_t s}+iv^2\frac{|\beta_b|}{\sigma_p}}}{(p+k_a)^2_\perp-\alpha_a|\beta_b|sv^2-i\epsilon} \\ \stackrel{p_\perp=k_\perp v}{=}&\ 4\pi\int d\!\!{}^-{}^2 k\int_0^\infty dv^2 \frac{\alpha_a|\beta_b|s}{k^2_\perp-(k_b-kv)^2_\perp-i\epsilon}\frac{e^{-i\frac{k^2_\perp}{|\beta_b|\sigma_t s}+iv^2\frac{|\beta_b|}{\sigma_p}}}{(k_a+kv)^2_\perp-\alpha_a|\beta_b|sv^2-i\epsilon} \end{aligned} \tag{8.3}$$

This integral can be rescaled by change $l_\perp^2 = \frac{k_\perp^2}{|\beta_b|\sigma_t s}$ and $t^2 = v^2\frac{|\beta_b|}{\sigma_p}$ as follows

$$-4\pi\int d\!\!{}^-{}^2 l_\perp \int_0^\infty dt^2 \frac{e^{-il^2}}{l_\perp^2 - \frac{(k_b - lt\mu_\sigma)_\perp^2}{|\beta_b|\sigma_t s} - i\epsilon} \frac{e^{it^2}}{t^2 - \frac{(k_a + lt\mu_\sigma)_\perp^2}{\alpha_a\sigma_p s} + i\epsilon} \tag{8.4}$$

where $\mu_\sigma \equiv \sqrt{\sigma_p\sigma_t s}$. As we assumed, $\mu_\sigma \ll q_\perp$ (see eq. (3.5)) so one can neglect $lt\mu_\sigma$ in the denominators and get

$$\text{Eq. (8.2)} \overset{\beta_b\le 0}{=} -\Big(\ln\frac{-i(\alpha_a + i\epsilon)\sigma_p s}{k_{a_\perp}^2} - \gamma\Big)\Big(\ln\frac{i|\beta_b|\sigma_t s}{k_{b_\perp}^2} - \gamma\Big) \tag{8.5}$$

where we used integral

$$\int_0^\infty dx\,\frac{e^{-x}}{x+a} = \ln\frac{1}{a} - \gamma + O(a) \tag{8.6}$$

Performing similar calculation at $\beta_b > 0$ we get sG-contribution to the virtual diagram in the form

$$I_{\text{fig.5}}^{\text{virt sG}} = -\Big(\ln\frac{-i(\alpha_a + i\epsilon)\sigma_p s}{k_{a_\perp}^2} - \gamma\Big)\Big(\ln\frac{-i(\beta_b + i\epsilon)\sigma_t s}{k_{b_\perp}^2} - \gamma\Big) \tag{8.7}$$

Note that this double-log contribution comes from the region $1 \gg l^2 \gg \frac{k_{b_\perp}^2}{\sigma_t|\beta_b'|s}$ and $1 \gg t^2 \gg \frac{k'^2_{a_\perp}}{\sigma_p|\alpha_a'|s}$ in the integral (8.4) which corresponds to the region

$$\sigma_p \gg \beta \gg \frac{k'^2_{a_\perp}}{|\alpha_a'|s}, \qquad \sigma_t \gg \alpha \gg \frac{k_{b_\perp}^2}{|\beta_b'|s}, \qquad \sigma_p\sigma_t s \gg p_\perp^2 \gg \frac{k'^2_{a_\perp}k_{b_\perp}^2}{|\alpha_a'\beta_b'|s} \tag{8.8}$$

in the original integral (8.3).

The result for figure 6 integral (5.26) is obtained from figure 5 result (5.17) by complex conjugation and $k_a \leftrightarrow -k_a'$, $k_b \leftrightarrow -k_b'$ replacement so the sG-contribution can be obtained in a similar way

$$\begin{aligned}
I_{\text{fig.6}}^{\text{virt sG}} &= 16\pi^2\int\frac{d\!\!{}^-{}^4 p}{i}\,\Big[\frac{s\alpha_a'\beta_b'}{[(p+k_a')^2 - i\epsilon](p^2 - i\epsilon)[(p-k_b')^2 - i\epsilon]} \\
&\quad + \tilde\delta_+(p+k_a')\frac{s\alpha_a'\beta_b'}{p^2 + i\epsilon}\tilde\delta_-(p-k_b')\Big]e^{-i\frac{\alpha'}{\sigma_t} + i\frac{\beta'}{\sigma_p}} \\
&= -16\pi^2\int\frac{d\!\!{}^-p}{i}\Big[\frac{\alpha_a'}{\alpha_a'\beta s - (p+k_a')_\perp^2 - i\epsilon}\,\frac{s}{\alpha\beta s - p_\perp^2 - i\epsilon}\,\frac{\beta_b'}{\alpha\beta_b' s + (p-k_b')_\perp^2 + i\epsilon} \\
&\quad - \tilde\delta[\alpha_a'\beta s - (p+k_a')_\perp^2]\theta(\beta)\frac{\alpha_a'\beta_b' s}{\alpha\beta s - p_\perp^2 + i\epsilon}\tilde\delta[\alpha\beta_b' s + (p-k_b')_\perp^2]\theta(-\alpha)\Big]e^{-i\frac{\alpha}{\sigma_t} + i\frac{\beta}{\sigma_p}} \\
&= -\Big(\ln\frac{-i(\alpha_a' + i\epsilon)\sigma_p s}{k'^2_{a_\perp}} - \gamma\Big)\Big(\ln\frac{-i(\beta_b' + i\epsilon)\sigma_t s}{k_{b_\perp}^2} - \gamma\Big)
\end{aligned} \tag{8.9}$$

To get the last line, we performed complex conjugation of eq. (8.2), replaced $k_a \to -k_a'$, $k_b \to -k_b'$, and changed the sign of p.

8.2 sG-contributions to production diagrams

The sG-contribution to the integral (6.8) has the form

$$I_1^{\rm sG}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12}) = 8\pi^2 s^2 \int đ\alpha đ\beta đp_\perp \; e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}-i(p,x_{12})_\perp} \tag{8.10}$$
$$\times \Big[\frac{\alpha'_a}{\alpha'_a\beta s-(p+k'_a)^2+i\epsilon}\tilde\delta(\alpha\beta s-p_\perp^2)\theta(\alpha)\frac{\beta_b}{\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon}$$
$$+\tilde\delta[\alpha'_a\beta s-(p+k'_a)^2]\theta(\beta)\frac{1}{\alpha\beta s-p_\perp^2-i\epsilon}\frac{\alpha'_a\beta_b}{\alpha\beta_b s+(p-k_b)_\perp^2-i\epsilon\beta_b}$$
$$+\frac{-\alpha'_a\beta_b}{\alpha'_a\beta s-(p+k'_a)_\perp^2+i\epsilon\alpha'_a}\frac{1}{\alpha\beta s-p_\perp^2+i\epsilon}\tilde\delta[\alpha\beta_b s+(p-k_b)_\perp^2]\theta(\alpha)\Big]$$

To get the above equation, we neglected $\alpha \sim \sigma_t$ in comparison to α'_a and $\beta \sim \sigma_p$ in comparison to β_b in the denominators in eq. (6.8). Also, in the exponent in eq. (6.8) we neglected $\alpha\varrho x_{12}^-$ in comparison to $\frac{\alpha}{\sigma_t} = \alpha\varrho\delta^-$ and $\beta\varrho x_{12}^+$ in comparison to $\frac{\beta}{\sigma_p} = \beta\varrho\delta^+$. It is easy to see that the integral over α in the second term and over β in the last term in the above equation vanish so we get

$$I_1^{\rm sG}(\alpha'_a, k'_{a\perp}, \beta_b, k_{b_\perp}, x_2, x_1) = 8\pi^2 s\int_0^\infty đ\beta đp_\perp \; e^{-i\frac{p_\perp^2}{\beta\sigma_t s}+i\frac{\beta}{\sigma_p}-i(p,x_{12})_\perp} \tag{8.11}$$
$$\times \frac{\alpha'_a\beta_b s}{[p_\perp^2\beta_b+(p-k_b)\beta+i\epsilon][\alpha'_a\beta s-(p+k'_a)^2+i\epsilon]}$$

Let us again consider $\beta_b<0$, then change of variables $\beta = v^2|\beta_b|$ yields

$$\text{Eq. (8.11)} \overset{\beta_b\le 0}{=} -4\pi\int_0^\infty dv^2\int đp_\perp \; e^{-i\frac{p_\perp^2}{v^2|\beta_b|\sigma_t s}+iv^2\frac{|\beta_b|}{\sigma_p}-i(p,x_{12})_\perp}$$
$$\times\frac{\alpha'_a|\beta_b|s}{[p_\perp^2-(p-k_b)v^2-i\epsilon][(p+k'_a)^2-\alpha'_a|\beta_b|sv^2-i\epsilon]} \tag{8.12}$$

This integral differs from the fifth line in eq. (8.3) by extra factor $e^{-i(p,x_{12})_\perp}$. Doing the same rescaling, we obtain

$$\text{Eq. (8.11)} \overset{\beta_b\le 0}{=} 4\pi\int đ^2 l_\perp\int_0^\infty dt^2\frac{e^{-it(l,x_{12_\perp})\mu_\sigma}}{l_\perp^2-\frac{(k'_a-lt\mu_\sigma)_\perp^2}{|\beta_b|\sigma_t s}-i\epsilon}\,\frac{e^{-il^2+it^2}}{t^2-\frac{(k'_a+lt\mu_\sigma)_\perp^2}{\alpha'_a\sigma_p s}+i\epsilon} \tag{8.13}$$

Again, since $\mu_\sigma \ll q_\perp$, we can neglect all $lt\mu_\sigma$ terms and get

$$I^{\rm sG}_{\text{fig.7}} = \Big(\ln\frac{-i(\alpha'_a+i\epsilon)\sigma_p s}{k'^2_{a_\perp}}-\gamma\Big)\Big(\ln\frac{-i(\beta_b+i\epsilon)\sigma_t s}{k^2_{b_\perp}}-\gamma\Big) \tag{8.14}$$

Similarly to the integral (6.29) itself, the sG-contribution to the integral (6.29) can be obtained from the result for the integral (8.14) by trivial projectile↔target replacements

$$I^{\rm sG}_{\text{fig.8}} = \Big(\ln\frac{-i(\alpha_a+i\epsilon)\sigma_p s}{k'^2_{a_\perp}}-\gamma\Big)\Big(\ln\frac{-i(\beta'_b+i\epsilon)\sigma_t s}{k^2_{b_\perp}}-\gamma\Big) \tag{8.15}$$

8.3 The sum of sG-terms

Assembling eqs. (8.7), (8.1), (8.14), and (8.15) we get

$$\begin{aligned}\mathcal{W}(x_1,x_2)^{\rm sG} \;&=\; \frac{N_c^2-1}{N_c}16\pi^2 g^2\Big\{V^{-i,a}(x_2)\langle F_i^{+a}(x_2)F^{-j,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.7}U_j^{+b}(x_1)\\ &\quad + U^{+i,a}(x_2)\langle F_i^{-,a}(x_2)F_j^{+,b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.8}V^{-j,b}(x_1)\\ &\quad + V^{-i,a}(x_2)U_i^{+a}(x_2)\langle F^{-j,b}(x_1)F_j^{+b}(x_1)\rangle_{\mathbb{A}}^{\rm fig.5}\\ &\quad + \langle F^{-i,a}(x_2)F_i^{+a}(x_2)\rangle_{\mathbb{A}}^{\rm fig.6}V^{-j,b}(x_1)U_j^{+b}(x_1)\Big\}^{\rm sG}\\ =&\int đ\,\alpha_a đ\,k_{a_\perp}đ\,\beta_b đ\,k_{b_\perp}đ\,\alpha'_a đ\,k_{a_\perp}đ\,\beta'_b đ\,k'_{b_\perp}\,e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^- - i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\ &\times e^{-i(k_a+k_a,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp}U_i^{+,b}(\alpha'_a,k'_{a_\perp})V^{-i,a}(\beta'_b,k'_{b_\perp})U_j^{+,b}(\alpha_a,k_{a_\perp})V^{-j,a}(\beta_b,k_{b_\perp})\\ &\times g^2 I^{\rm sG}(\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp})\end{aligned}\tag{8.16}$$

where

$$\begin{aligned}I_{\rm sG}^{\sigma_p,\sigma_t}(\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp}) &= I_{\rm fig.5}^{\rm virt\ sG}+I_{\rm fig.6}^{\rm virt\ sG}+I_{\rm fig.7}^{\rm sG}+I_{\rm fig.8}^{\rm sG}\\ &= -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}+O\left(\frac{\mu_\sigma^2}{Q_\perp^2}\right)\end{aligned}\tag{8.17}$$

It is worth noting that the soft-Glauber contribution (8.17) is actually a soft contribution with characteristic transverse momenta $p_\perp^2 \sim \frac{k^2_{a_\perp}k^2_{b_\perp}}{|\alpha_a\beta_b|s}$. Indeed, while the momenta in the individual integrals for $I^{\rm sG}$ are given by eq. (8.8), the characteristic transverse momenta in their sum (8.17) are of the order of low limit in eq. (8.8). To see that, we rewrite the sum (8.17) as follows

$$\begin{aligned}I_{\rm fig.5}^{\rm virt\ sG}+I_{\rm fig.6}^{\rm virt\ sG}+I_{\rm fig.7}^{\rm sG}+I_{\rm fig.8}^{\rm sG} &= -\int_0^\infty dl^2\;e^{-il^2}\left[\frac{1}{l_\perp^2-\frac{k^2_{b_\perp}}{|\beta_b|\sigma_t s}-i\epsilon}-\frac{1}{l_\perp^2-\frac{k'^2_{b_\perp}}{|\beta'_b|\sigma_t s}-i\epsilon}\right]\\ \times\int_0^\infty dt^2\;e^{it^2}&\left[\frac{1}{t^2-\frac{k^2_{a_\perp}}{\alpha_a\sigma_p s}+i\epsilon}-\frac{1}{t^2-\frac{k'^2_{a_\perp}}{\alpha'_a\sigma_p s}+i\epsilon}\right]+O\left(\frac{\mu_\sigma^2}{Q_\perp^2}\right)\end{aligned}\tag{8.18}$$

It is easy to see that the integral over l^2 is determined by $l^2 \sim \frac{k^2_{b_\perp}}{|\beta_b|\sigma_t s}$ and the integral over t^2 by $t^2 \sim \frac{k^2_{a_\perp}}{|\alpha_a|\sigma_p s}$ which translates to

$$\beta \sim \frac{k^2_{a_\perp}}{|\alpha_a|s},\qquad \alpha\sim\frac{k^2_{b_\perp}}{|\beta_b|s},\qquad p_\perp^2\sim\frac{k^2_{a_\perp}k^2_{b_\perp}}{|\alpha_a\beta_b|s}\tag{8.19}$$

which is a soft contribution since eq. (8.19) means that

$$p^+\sim p^-\sim p_\perp\sim O\left(\frac{1}{\lambda}\right)\tag{8.20}$$

in terms of rescaling (1.3). Thus, the soft-Glauber contribution (8.17) is actually a soft contribution in accordance with general statement that contributions from Glauber gluons cancel.

Actually, the statement that the soft-Glauber contribution (8.17) is a soft contribution can be checked independently. Let us calculate the contribution of small $p_\perp \ll k_{i_\perp}$ to a non-restricted integrals of figure 5. Neglecting $p_\perp$ in comparison to $k_{i_\perp}$ and using dimensional regularization for UV integrals obtained as a result of this approximation, we get instead of eq. (8.2)

$$I^{\text{virt soft}}_{\text{fig.5}} = 16\pi^2 \int \frac{d\!\!{}^-{}^{2+2\varepsilon}p}{i}\Big[\frac{\alpha_a}{\alpha_a\beta s - k^2_{a_\perp} + i\epsilon}\frac{s}{\alpha\beta s - p^2_\perp + i\epsilon}\frac{\beta_b}{\alpha\beta_b s + k^2_{b_\perp} - i\epsilon} \\ - \tilde{\delta}[\alpha_a\beta s - k^2_{a_\perp}]\theta(-\beta)\frac{\alpha_a\beta_b s}{\alpha\beta s - p^2_\perp - i\epsilon}\tilde{\delta}[\alpha\beta_b s + k^2_{b_\perp}]\theta(\alpha)\Big] \tag{8.21}$$

Repeating all the steps in derivation of eq. (8.3) we get

$$\text{Eq. (8.21)} \overset{\beta_b\leq 0}{=} -4\pi \int d\!\!{}^-{}^{2+2\varepsilon}k \frac{1}{k^2_\perp - k^2_{b_\perp} + i\epsilon}\int_0^\infty dv^2 \frac{v^{2\varepsilon}}{v^2 - \frac{k^2_{a_\perp}}{(\alpha_a+i\epsilon)|\beta_b|s}} \\ = -\frac{\Gamma^2(-\varepsilon)\Gamma(1+\varepsilon)}{(4\pi)^\varepsilon}(-k^2_{b_\perp} + i\epsilon)^\varepsilon \left(-\frac{k^2_{a_\perp}}{(\alpha_a + i\epsilon)|\beta_b|s}\right)^\varepsilon \tag{8.22} \\ = -\frac{1}{\varepsilon^2} - \frac{\pi^2}{4} - \gamma^2 - \left(\frac{1}{\varepsilon} + \gamma\right)\left[\ln k^2_{a_\perp} \ln k^2_{b_\perp} - \ln(\alpha_a + i\epsilon)|\beta_b|s\right] - \frac{1}{2}\ln^2 \frac{(\alpha_a + i\epsilon)|\beta_b s}{k^2_{a_\perp} k^2_{b_\perp}}$$

The similar contribution of diagrams in figure 6 is obtained by replacement $\alpha_a \leftrightarrow \alpha'_a$, $\beta_b \leftrightarrow \beta'_b$ and $k_{a,b_\perp} \leftrightarrow k'_{a,b_\perp}$. Moreover, for soft contributions one can neglect $e^{-ipx_{12}}$ in the "production" terms and obtain

$$I^{\text{soft}} = \text{Eq. (8.22)} - (\alpha_a \to \alpha'_a, k_{a_\perp} \to k'_{a_\perp}) - (\beta_b \to \beta'_b, k_{b_\perp} \to k'_{b_\perp}) \tag{8.23} \\ + (\alpha_a \to \alpha'_a, \beta_b \to \beta'_b, k_{a_\perp} \to k'_{a_\perp}, k_{b_\perp} \to k'_{b_\perp}) = -\ln\frac{(i\alpha'_a)k^2_{a_\perp}}{(i\alpha_a)k'^2_{a_\perp}} \ln\frac{(i\beta'_b)k'^2_{b_\perp}}{(i\beta_b)k'^2_{b_\perp}}$$

which coincides with eq. (8.17).

As we mentioned above, in appendix C it is demonstrated that in the first perturbative order such soft contributions cancel in the sum of all diagrams. Non-perturbatively, soft contributions form wave functions of hadrons and also presumably lead to non-perturbative power corrections to scattering amplitude $\sim x^2_{12_\perp}\Lambda^2_{\text{QCD}}$.

8.4 sG-contributions to TMD matrix elements

In this section we demonstrate that sG-contributions to TMD matrix elements are power corrections. Let us start with the "target eikonals" of figure 11 given by eq. (B.2). A "smooth" cutoff $|\beta|<\sigma_p$ is obtained by inserting an extra $e^{i\frac{\beta}{\sigma_p}}=e^{i\beta\varrho\delta^+}$ in the integrand

$$\begin{aligned}
&I^{\text{eik},\beta<\sigma_p}_{\text{fig.11a--d}}(\beta_b,k_{b_\perp},x_1^+,x_{1_\perp},x_2^+,x_{2_\perp})=8\pi^2 s\int_0^\infty d\!\!{}^-\alpha\; e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}}\frac{d\!\!{}^-\beta}{\beta+i\epsilon}d\!\!{}^-p_\perp\; e^{i\beta\varrho x_{12}^+-i(p,x_{12})_\perp}\\
&\times\Big[\tilde\delta(\alpha\beta s-p_\perp^2)\frac{\beta-\beta_b}{\alpha(\beta-\beta_b)s-(p-k_b)_\perp^2-i\epsilon}+\frac{(\beta-\beta_b)}{\alpha\beta s-p_\perp^2+i\epsilon}\tilde\delta[\alpha(\beta-\beta_b)s-(p-k_b)_\perp^2]\Big]\\
&-8\pi^2 i\int d\!\!{}^-\alpha d\!\!{}^-\beta d\!\!{}^-p_\perp\frac{e^{-i\frac{\alpha}{\sigma_t}+i\frac{\beta}{\sigma_p}}s(\beta-\beta_b)}{(\beta+i\epsilon)(\alpha\beta s-p_\perp^2+i\epsilon)[\alpha(\beta-\beta_b)s-(p-k_b)_\perp^2+i\epsilon]}\\
&=8\pi^2\int_0^\infty d\!\!{}^-\alpha\; e^{-i\frac{\alpha}{\sigma_t}}\int d\!\!{}^-p_\perp\Big(\frac{\beta_b s}{p_\perp^2}\frac{e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+-i(p,x_{12})_\perp}-1}{\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon}e^{i\frac{p_\perp^2}{\alpha s}\varrho\delta^+}\\
&+\frac{(p-k_b)_\perp^2e^{i(p,x_{12})_\perp}\big[e^{i(\beta_b+\frac{(p-k_b)_\perp^2}{\alpha s})\varrho(x_{12}^++\delta^+)}-e^{i\frac{p_\perp^2}{\alpha s}\varrho(x_{12}^++\delta^+)}\big]}{\alpha[\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon][\alpha\beta_b s+(p-k_b)_\perp^2-p_\perp^2]}\Big)
\end{aligned}\tag{8.24}$$

Since $x_{12}^+\ll\delta^+$ we can neglect x_{12}^+ and get

$$\begin{aligned}
I^{\text{eik}}_{\text{fig.11a--d}}(\beta_b,k_{b_\perp},x_{12_\perp})&=8\pi^2\int_0^\infty d\!\!{}^-\alpha\int\frac{d\!\!{}^-p_\perp}{p_\perp^2}\frac{\beta_b s e^{-i\frac{\alpha}{\sigma_t}+i\frac{p_\perp^2}{\alpha\sigma_p s}}(e^{-i(p,x_{12})_\perp}-1)}{\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon}\\
&\stackrel{\alpha=v^2\alpha_a}{=}4\pi\int_0^\infty dv^2\int\frac{d\!\!{}^-p_\perp}{p_\perp^2}\frac{\alpha_a\beta_b s e^{-i\frac{\alpha_a}{\sigma_t}v^2+i\frac{p_\perp^2}{v^2\alpha_a\sigma_p s}}(e^{-i(p,x_{12})_\perp}-1)}{v^2\alpha_a\beta_b s+(p-k_b)_\perp^2+i\epsilon}\\
&\stackrel{p_\perp=k_\perp v}{=}4\pi\int_0^\infty dv^2\int\frac{d\!\!{}^-{}^2k}{k_\perp^2}(e^{-iv(k,x_{12})_\perp}-1)\;\frac{\alpha_a\beta_b s e^{-i\frac{\alpha_a}{\sigma_t}v^2+i\frac{k_\perp^2}{\alpha_a\sigma_p s}}}{v^2\alpha_a\beta_b s+(k_\perp v-k_b)_\perp^2+i\epsilon}
\end{aligned}\tag{8.25}$$

Performing the same rescaling as in eq. (8.4) we get

$$\begin{aligned}
I^{\text{eik}}_{\text{fig.11a--d}}(\beta_b,k_{b_\perp},x_{12_\perp})&=4\pi\int_0^\infty dt^2\int\frac{d^2l_\perp}{l_\perp^2}(e^{-it\mu_\sigma(l,x_{12})_\perp}-1)\frac{e^{-it^2+il^2}}{t^2+\frac{(k_b-\mu_\sigma tl)_\perp^2}{\sigma_t\beta_b s}}\\
&\simeq\;4\pi\int_0^\infty dt^2\frac{e^{-it^2}}{t^2+\frac{k_{b_\perp}^2}{\sigma_t\beta_b s}}\int\frac{d^2l_\perp}{l_\perp^2}(e^{-it\mu_\sigma(l,x_{12})_\perp}-1)e^{il_\perp^2}
\end{aligned}\tag{8.26}$$

where we again used $\mu_\sigma\ll q_\perp\sim k_{b_\perp}$. The first integral in the r.h.s. is $\ln\frac{-i\beta_b\sigma_t s}{k_{b_\perp}^2}-\gamma$ (see eq. (8.6) but the second is obviously $O(\mu_\sigma^2x_{12_\perp}^2)\sim O(\frac{\mu_\sigma^2}{q_\perp^2})$ so the sG-contribution to "target" eikonal TMD matrix elements is a power correction. Similarly, one can demonstrate that the sG-contribution to "projectile" eikonal TMD matrix elements of figure 11 is a power correction $O(\frac{\mu_\sigma^2}{q_\perp^2})$. Thus, with power accuracy there is no double counting and we should subtract from the amplitude (7.3) only the sG-contributions (8.17).

9 Result for the coefficient function

According to eq. (4.2), the coefficient function is given by the functional integral over central fields with $\alpha > \sigma_t, \beta > \sigma_p$ minus the eikonal contributions. It is determined by

$$
\begin{aligned}
&\mathcal{W}(x_1,x_2) - \mathcal{W}^{\rm sG}(x_1,x_2) - \mathcal{W}^{\rm eik}(x_1,x_2) \qquad (9.1)\\
&= \int d\!\!\bar{}\,\alpha_a d\!\!\bar{}\,k_{a_\perp} d\!\!\bar{}\,\beta_b d\!\!\bar{}\,k_{b_\perp} d\!\!\bar{}\,\alpha'_a d\!\!\bar{}\,k'_{a_\perp} d\!\!\bar{}\,\beta'_b d\!\!\bar{}\,k'_{b_\perp}\\
&\quad \times e^{-i\alpha_a\varrho x_1^- - i\alpha'_a\varrho x_2^-} e^{-i\beta_b\varrho x_1^+ - i\beta'_b\varrho x_2^+} e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} U^{+,b}_{\;i}(\alpha_a,k_{a_\perp})\\
&\quad \times V^{-i,a}(\beta_b,k_{b_\perp}) U^{+,b}_{\;j}(\alpha'_a,k'_{a_\perp}) V^{-j,a}(\beta'_b,k'_{b_\perp}) \mathfrak{C}_1(\alpha_a,\alpha'_a,\beta_b,\beta'_b,x_{12_\perp};\sigma_p,\sigma_t)\\
&= \int d\!\!\bar{}\,\alpha_a d\!\!\bar{}\,\beta_b d\!\!\bar{}\,\alpha'_a d\!\!\bar{}\,\beta'_b d\!\!\bar{}\,k'_{b_\perp}\; e^{-i\alpha_a\varrho x_1^- - i\alpha'_a\varrho x_2^-} e^{-i\beta_b\varrho x_1^+ - i\beta'_b\varrho x_2^+} U^{+,b}_{\;i}(\alpha_a,x_{1_\perp})\\
&\quad \times V^{-i,a}(\beta_b,x_{1_\perp}) U^{+,b}_{\;j}(\alpha'_a,x_{2_\perp}) V^{-j,a}(\beta'_b,x_{2_\perp}) \mathfrak{C}_1(\alpha_a,\alpha'_a,\beta_b,\beta'_b,x_{12_\perp};\sigma_p,\sigma_t)
\end{aligned}
$$

where the coefficient function in the momentum representation is

$$\mathfrak{C}_1(\alpha_a,\alpha'_a,\beta_b,\beta'_b,x_{12_\perp};\sigma_p,\sigma_t) = I - I^{\sigma_p,\sigma_t}_{\rm eik} - I^{\sigma_p,\sigma_t}_{\rm sG}$$

The explicit form of $\mathfrak{C}_1$ is easily found from eq. (7.3) and eq. (8.17)

$$
\begin{aligned}
&\mathfrak{C}_1(\alpha_a,\alpha'_a,\beta_b,\beta'_b,x_{12_\perp};\sigma_p,\sigma_t) = \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4} \qquad (9.2)\\
&- \ln\frac{(-i\alpha_a+\epsilon)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b+\epsilon)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha'_a+\epsilon)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b+\epsilon)e^\gamma}{\sigma_p} + \pi^2 + O(\lambda_p,\lambda_t)
\end{aligned}
$$

This formula is the main technical result of the paper.

The very important property of the coefficient function $\mathfrak{C}_1$ is that the r.h.s. (9.2) does not actually depend on transverse momenta so all the dynamics at the one-loop level proceeds in the longitudinal direction. This fact can be used to check the algebra and approximations leading to the result (9.2). In the appendix G the coefficient function is calculated using the on-shell background fields with zero transverse momenta

$$
\begin{aligned}
U^{+i}(z^-) &= \int d\!\!\bar{}\,\alpha_a\; U^{+i}(\alpha_a)\; e^{-i\varrho\alpha_a z^-} \quad \Leftrightarrow \quad U^{+i}(\alpha_a) = \varrho\int dz^- dz_\perp\; U^{+i}(z^-)\; e^{i\varrho\alpha_a z^-},\\
V^{-i}(z^+) &= \int d\!\!\bar{}\,\beta_b\; V^{-i}(\beta_b)\; e^{-i\varrho\beta_b z^+} \quad \Leftrightarrow \quad V^{-i}(\beta_b) = \varrho\int dz^+ dz_\perp\; V^{-i}(z^+)\; e^{i\varrho\beta_b z^+}
\end{aligned}
\qquad (9.3)
$$

and the result (9.2) is confirmed.

In the coordinate space our result reads

$$
\begin{aligned}
\langle \hat{W}(x_1,x_2)\rangle_{\mathbb{A}} &= \langle \hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\rangle_{\mathbb{A}}\\
&\quad + \int dz_2^- dz_1^- dw_1^+ dw_2^+ \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{2_\perp},x_{1_\perp};z_2^-,z_1^-,z_2^+,z_1^+;\sigma_p,\sigma_t)\\
&\quad \times U^{+,b}_{\;i}(z_2^-,x_{2_\perp})V^{-i,a}(z_2^+,x_{2_\perp})U^{+,b}_{\;j}(z_1^-,x_{1_\perp})V^{-j,a}(z_1^+,x_{1_\perp}) + \ldots \qquad (9.4)
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathfrak{C}_1(x_{2_\perp}, x_{1_\perp}; z_2^-, z_1^-, z_2^+, z_1^+; \sigma_p, \sigma_t) \qquad (9.5)\\
&= \left[\ln^2 \frac{2\delta^+\delta^-}{x_{12_\perp}^2} + \pi^2\right]\delta(x_2-z_2)^-\delta(x_1-z_1)^-\delta(x_2-z_2)^+\delta(x_1-z_1)^+\\
&\quad - \delta(x_1-z_1)^-\delta(x_1-z_1)^+\\
&\quad\times \left[\frac{\theta(x_2-z_2)^-}{(x_2-z_2)^-} - \delta(x_2-z_2)^-\int_0^{\delta^-}\frac{dz_2^-}{z_2^-}\right]\left[\frac{\theta(x_2-z_2)^+}{(x_2-z_2)^+} - \delta(x_2-z_2)^+\int_0^{\delta^+}\frac{dz_2^+}{z_2^+}\right]\\
&\quad - \delta(x_2-z_2)^-\delta(x_2-z_2)^+\\
&\quad\times \left[\frac{\theta(x_1-z_1)^-}{(x_1-z_1)^-} - \delta(x_1-z_1)^-\int_0^{\delta^-}\frac{dz_1^-}{z_1^-}\right]\left[\frac{\theta(x_1-z_1)^+}{(x_1-w_2)^+} - \delta(x_1-z_1)^+\int_0^{\delta^+}\frac{dz_1^+}{z_1^+}\right]
\end{aligned}
$$

Let us check matching of the cutoffs, namely that the r.h.s. of eq. (9.4) does not depend on σ_p and σ_t. We start with $\sigma_t\frac{d}{d\sigma_t} = -\delta^-\frac{d}{\delta^-}$. Since

$$
\begin{aligned}
&-\delta^-\frac{d}{\delta^-}\mathfrak{C}_1(x_{2_\perp}, x_{1_\perp}; z_i^-, z_{i_\perp}, z_i^+, z_{i_\perp}; \sigma_p, \sigma_t) \qquad (9.6)\\
&= -\delta(x_2-z_2)^-\delta(x_1-z_1)^-\left\{2\ln\frac{2\delta^+\delta^-}{x_{12_\perp}^2}\delta(x_2-z_2)^+\delta(x_1-z_1)^+\right.\\
&\quad + \delta(x_1-z_1)^+\left[\frac{\theta(x_2-z_2)^+}{(x_2-z_2)^+} - \delta(x_2-z_2)^+\int_0^{\delta^+}\frac{dz_2^+}{z_2^+}\right]\\
&\quad \left. + \delta(x_2-z_2)^+\left[\frac{\theta(x_1-z_1)^+}{(x_1-z_1)^+} - \delta(x_1-z_1)^+\int_0^{\delta^+}\frac{dz_1^+}{z_1^+}\right]\right\}\\
&= \delta(x_2-z_2)^-\delta(x_1-z_1)^-\left\{2\ln\frac{sx_{12_\perp}^2}{4}\delta(x_2-z_2)^+\delta(x_1-z_1)^+\right.\\
&\quad - \delta(x_1-z_1)^+\left[\frac{\theta(x_2-z_2)^+}{(x_2-z_2)^+} - \delta(x_2-z_2)^+\int_0^{\frac{2}{s\delta^-}}\frac{dz_2^+}{z_2^+}\right]\\
&\quad \left. - \delta(x_2-z_2)^+\left[\frac{\theta(x_1-z_1)^+}{(x_1-z_1)^+} - \delta(x_1-z_1)^+\int_0^{\frac{2}{s\delta^-}}\frac{dz_1^+}{z_1^+}\right]\right\}
\end{aligned}
$$

we get

$$
\begin{aligned}
&\sigma_t\frac{d}{d\sigma_t}[\text{r.h.s. of Eq. (9.4)}] = U_i^{+,b}(x_2^-, x_{2_\perp})U_j^{+,b}(x_1^-, x_{1_\perp}) \qquad (9.7)\\
&\times\left[\sigma_t\frac{d}{d\sigma_t}\langle\hat{O}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})\rangle_B + \frac{\alpha_s N_c}{2\pi}\left\{2\ln\frac{sx_{12_\perp}^2}{4}V^{-i,a}(x_2^+, x_{2_\perp})V^{-j,a}(x_1^+, x_{1_\perp})\right.\right.\\
&-\int dz_2^+\left[\frac{\theta(x_2-z_2)^+}{(x_2-z_2)^+} - \delta(x_2-z_2)^+\int_0^{\sigma_t/\varrho}\frac{dz_2^+}{z_2^+}\right]V_i^{-,b}(z_2^+, x_{2_\perp})V_j^{-,b}(x_1^+, x_{1_\perp})\\
&\left.\left.-\int dz_1^+\left[\frac{\theta(x_1-z_1)^+}{(x_1-z_1)^+} - \delta(x_1-z_1)^+\int_0^{\sigma_t/\varrho}\frac{dz_1^+}{z_1^+}\right]V_i^{-,b}(x_2^+, x_{2_\perp})V_j^{-,b}(z_1^+, x_{1_\perp})\right\}\right] = 0
\end{aligned}
$$

due to eq. (B.16). Similarly, the r.h.s. of eq. (9.4) does not depend on σ_p cutoff.

It is possible to represent our result for the coefficient function as evolution equations with respect to σ_t and σ_p. Since the differentiation over σ_t is represented by the integration

JHEP03(2023)029

operator in the coordinate space and by simple multiplication in the momentum space, we will use the latter. We define the Fourier transform of the operator $\hat{W}(x_1,x_2)$ in eq. (2.3) as follows

$$\begin{aligned}&\hat{W}(\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_{1_\perp},x_{2_\perp})\\&=\rho^4\int dx_2^-dx_1^-dx_2^+dx_1^+\ e^{i\alpha'_a\varrho x_2^-+i\alpha'_a\varrho x_2^-+i\beta'_b\varrho x_2^++i\beta'_b\varrho x_2^+}\hat{W}(p_A,p'_A,p_B,p'_B;x_{1_\perp},x_{2_\perp})\end{aligned}\tag{9.8}$$

The general TMD factorization formula (2.7) for $\hat{W}(\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_{1_\perp},x_{2_\perp})$ can be written as

$$\begin{aligned}\hat{W}(\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_{1_\perp},x_{2_\perp})=\int &đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)\\&\times\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{2_\perp},x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{2_\perp},x_{1_\perp})+\dots\end{aligned}\tag{9.9}$$

where

$$\begin{aligned}\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{2_\perp},x_{1_\perp})&\equiv\rho^2\int dx_2^-dx_1^-e^{i\alpha'_a\varrho x_2^-+i\alpha_a\varrho x_1^-}\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\\ \hat{\mathcal{O}}^{\sigma_t}_{ij}(\beta'_b,\beta_b,x_{2_\perp},x_{1_\perp})&\equiv\rho^2\int dx_2^+dx_1^+e^{i\beta'_b\varrho x_2^++i\beta_b\varrho x_1^+}\hat{\mathcal{O}}^{\sigma_t}_{ij}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\end{aligned}\tag{9.10}$$

Here we took into account the absence of dynamics in the transverse space (and tacitly assumed that such property survives in higher orders of perturbation theory). Since the evolution equations for TMD operators in the momentum space are given by eqs. (B.14) and (B.15), the coefficient function should satisfy matching evolution equations

$$\begin{aligned}&\sigma_t\frac{d}{d\sigma_t}\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)=\frac{\alpha_sN_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4}\\&\quad+\ln(-i\beta'_b\sigma_t+\epsilon)+\ln(-i\beta_b\sigma_t+\epsilon)+2\gamma\Big]\mathfrak{C}(x_1,x_2;\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)\\&\sigma_p\frac{d}{d\sigma_p}\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)=\frac{\alpha_sN_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4}\\&\quad+\ln(-i\alpha'_a\sigma_p+\epsilon)+\ln(-i\alpha_a\sigma_p+\epsilon)+2\gamma\Big]\ \mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)\end{aligned}\tag{9.11}$$

The solution of this equations compatible with first-order result (9.2) is

$$\begin{aligned}\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)&=e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)}\\&+O\Big(\alpha_s^2\times\Big[\ln\frac{\alpha'_a}{\sigma_t}\ln\frac{\beta'_b}{\sigma_p},\ln\frac{\alpha'_a}{\sigma_t},\ln\frac{\beta'_b}{\sigma_p},\text{const}\Big]\Big)\end{aligned}\tag{9.12}$$

and the final form of "double operator expansion" reads

$$\begin{aligned}\hat{W}(\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_{1_\perp},x_{2_\perp})=\int &đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b\ e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)}\\&\times\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{2_\perp},x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{2_\perp},x_{1_\perp})+\dots\end{aligned}\tag{9.13}$$

JHEP03(2023)029

In the next section we will consider matrix elements of the operator equation (9.13) between initial and final protons' states and demonstrate that

$$\langle p'_A, p'_B|\hat{\mathcal{O}}^{\sigma_p}_{ij}\hat{\mathcal{O}}^{ij;\sigma_t}|p_A, p_B\rangle = \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}|p_B\rangle \tag{9.14}$$

To prove the above equation, we need to check that the contribution of sG-gluons cancel up to power corrections terms.

9.1 Factorization of integral over $A \cup B$ fields

9.1.1 Cancellation of soft and Glauber gluons

The functional integral form of our result for hadronic tensor (2.4) reads

$$\begin{aligned}
&\frac{1}{16}(N_c^2-1)\rho^4\int dx_1^- dx_2^- dx_1^+ dx_2^+\ e^{i\alpha_a\varrho x_1^- + i\alpha'_a\varrho x_2^- + i\beta_b\varrho x_1^+ + i\beta'_b\varrho x_2^+} \\
&\times\langle p'_A, p'_B|g^2 F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A, p_B\rangle = e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha_a,\alpha'_a,\beta_b,\beta'_b;\sigma_p,\sigma_t)} \\
&\times\int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\mathcal{O}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{12_\perp})\mathcal{O}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{12_\perp}) + \ldots
\end{aligned} \tag{9.15}$$

where the TMD operators $\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{12_\perp})$ and $\hat{\mathcal{O}}^{\sigma_t}_{ij}(\beta'_b,\beta_b,x_{12_\perp})$ are made of A and B fields, respectively. However, as we mentioned above (see figure 3), there are $\mathscr{C} = A\cap B$ fields with both $\alpha < \sigma_t$ and $\beta < \sigma_p$ so to get the desired factorization (2.7) we need to discuss the interactions between A and B fields. The integrals over A and B fields give matrix elements of TMD operators (2.8) between projectile and target fields while the integral over $\mathscr{C}$ fields cancels due to unitarity with power corrections accuracy. To understand this, let us discuss the $\mathscr{C}$ fields which are defined as gluons (and, in principle, quarks) with both $|\alpha| < \sigma_t$ and $|\beta| < \sigma_p$, see figure 3. As we mentioned above, depending on the scale of characteristic transverse momenta they may be Glauber gluons with $p_\perp \sim q_\perp$ or soft gluons with $p_\perp \ll q_\perp$).

The cancellation of Glauber gluons is proved in ref. [2] (see also ref. [16] for recent discussion).[10] As to soft gluons, it is demonstrated in refs. [3, 17, 18] that they form the correlation function of four Wilson lines going form points x_1 and x_2 in the light-like directions so the result of the integration over soft gluons in eq. (9.15) is

$$\begin{aligned}
&\int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\mathcal{O}^{\mathscr{A}}_{ij}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\mathcal{O}^{\mathscr{A}}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp}) \\
&= S(x_2,x_1;\sigma_p,\sigma_t)\int \mathscr{D}\Phi_A\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\mathcal{O}^A_{ij}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp}) \\
&\quad\times\int \mathscr{D}\Phi_B \Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\mathcal{O}^B_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp}) \\
&= S(x_2,x_1;\sigma_p,\sigma_t)\langle p'_A|\hat{\mathcal{O}}_{ij}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_B\rangle
\end{aligned} \tag{9.16}$$

where

$$S(x_2,x_1;\sigma_p,\sigma_t) = \frac{1}{N_c^2-1}\mathrm{Tr}\int \mathscr{D}\mathscr{C}\ \{-\infty^+,x_2^+\}_{x_2}\{x_2^+,-\infty^+\}_{x_2}[-\infty^+,z_1^+]_{z_1}[z_1^+,-\infty^+]_{z_1} \tag{9.17}$$

[10] The discussion of this cancellation in the functional-integral language used in this paper is presented in refs. [6, 7].

JHEP03(2023)029

is the soft factor. Here $\mathscr{D}\mathscr{C}$ is defined as in eq. (3.10) with $\mathscr{A} \to \mathscr{C}$ replacement. As we will see in section C, with the rapidity-only cutoff the dependence of the soft factor on σ_p and σ_t gives power corrections $\sim \frac{Q_\perp^2}{\sigma_p s}$ and/or $\frac{Q_\perp^2}{\sigma_t s}$ hence, contrary to CSS approach based on double UV+rapidity cutoff, there is no logarithmic dependence on the cutoffs in the soft factor. Of course, there may be the non-perturbative power corrections $\sim \Lambda_{\rm QCD}^2 x_{12_\perp}^2$ which should be studied by some non-perturbative methods, but the claim is that the soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution.

Thus, we can neglect the integration over $\mathscr{C}$ fields in eq. (9.15) and get the factorized result

$$\rho^4 \int dx_2^- dx_1^- dx_2^+ dx_1^+ \; e^{i\alpha'_a \varrho x_2^- + i\alpha_a \varrho x_1^- + i\beta'_b \varrho x_2^+ + i\beta_b \varrho x_1^+} W(p_A, p_B, p'_A, p'_B; x_1, x_2) \\ = e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)} \langle p'_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(\alpha'_a, \alpha_a, x_{12_\perp})|p_A\rangle \langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{12_\perp})|p_B\rangle \tag{9.18}$$

9.1.2 Factorization in terms of generalized TMDs

Let us rewrite our result in terms of generalized TMDs (gTMDs). They can be defined as follows [19]

$$\begin{aligned} \mathcal{G}_{ij}^{\sigma_p}(x_A, b_\perp; p'_A, p_A) &= -\frac{g^{-2}}{\pi \varrho x_A}\int dz^- e^{ix_A \varrho z^-} \langle p'_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}\left(-\frac{z^-}{2} - \frac{b_\perp}{2}, \frac{z^-}{2} + \frac{b_\perp}{2}\right)|p_A\rangle, \\ \mathcal{G}_{ij}^{\sigma_t}(x_B, b_\perp; p'_B, p_B) &= -\frac{g^{-2}}{\pi \varrho x_A}\int dz^- e^{ix_B \varrho z^-} \langle p'_B|\hat{\mathcal{O}}_{ij}^{\sigma_p}\left(-\frac{z^-}{2} - \frac{b_\perp}{2}, \frac{z^-}{2} + \frac{b_\perp}{2}\right)|p_B\rangle \end{aligned} \tag{9.19}$$

The above choice of normalization reproduces gluon TMDs for unpolarized hadrons defined in ref. [20] at $p'_A = p_A$.

$$\begin{aligned} \langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-, 0^-, b_\perp)|p_A\rangle &= -g^2\varrho^2 \int_0^1 du\; u\mathcal{G}_{ij}^{\sigma_p}(u, b_\perp)\cos u\varrho z^-, \\ \langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(\alpha_q, b_\perp)|p_A\rangle &\equiv \rho \int dz^- e^{i\alpha_q \varrho z^-} \langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-, 0^-, b_\perp)|p_A\rangle \\ &= -\pi g^2 \varrho^2 |\alpha_q| \mathcal{G}_{ij}^{\sigma_p}(|\alpha_q|, b_\perp, p_A), \\ \mathcal{G}_{ij}^{\sigma_p}(u, b_\perp) = g_{ij} D_g(u, b_\perp; \sigma_p) &+ \frac{1}{2m_N^2}(2\partial_i \partial_j + g_{ij}\partial_\perp^2) H(u, b_\perp; \sigma_p) \end{aligned} \tag{9.20}$$

and similarly for $\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(\beta_q, b_\perp)|p_B\rangle$. Note that at $b_\perp = 0$ the TMD $D_g(x_B, \sigma)$ is the gluon PDF with the rapidity-only cutoff discussed in ref. [21]. At the leading order, this is equivalent to usual UV regularization of light-ray operator $\hat{\mathcal{O}}_{ij}^{\sigma}(z^\pm)$ and reproduces LO DGLAP equation [21]. At the NLO level, the two-loop DGLAP equation should be reproduced by the combination of rapidity-only evolution of light-ray operator $\hat{\mathcal{O}}_{ij}^{\sigma}(z^\pm)$ and usual μ^2 evolution for self-energy and vertex Z-factors.

With the normalization (9.19) we get

$$\begin{aligned} \langle p'_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle &= -2\pi^2 \delta(\alpha_a + \alpha'_a) g^2 \rho^2 |\alpha_a| e^{-\frac{i}{2}(l, x_1 + x_2)_\perp} \mathcal{G}_{ij}^{\sigma_p}(|\alpha_a|, x_{12_\perp}; p_A, p'_A) \\ \langle p'_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle &= -2\pi^2 \delta(\beta_b + \beta'_b) g^2 \rho^2 |\beta_b| e^{\frac{i}{2}(l, x_1 + x_2)_\perp} \mathcal{G}_{ij}^{\sigma_t}(|\beta_b|, x_{12_\perp}; p_B, p'_B) \end{aligned} \tag{9.21}$$

JHEP03(2023)029

The δ-functions in the above expressions for $\langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}|p_A\rangle$ and $\langle p'_B|\hat{\mathcal{O}}^{\sigma_t p}_{ij}|p_B\rangle$ are present also in the l.h.s. of eq. (9.18) because $p'_A + p'_B = p_A + p_B$. Canceling them, one obtains

$$\int dx_{12}^- dx_{12}^+ \; e^{i\alpha_a \varrho x_{12}^- + i\beta_b \varrho x_{12}^+} \frac{N_c^2-1}{16} \langle p'_A, p'_B|F^2\Big(-\frac{x_{12}}{2}\Big)F^2\Big(\frac{x_{12}}{2}\Big)|p_A,p_B\rangle \tag{9.22}$$
$$= \frac{\pi^2}{2} Q^2 e^{\frac{\alpha_s N_c}{2\pi}\left[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\ln\frac{\alpha_a e^\gamma}{\sigma_t}\ln\frac{\beta_b e^\gamma}{\sigma_p} + \frac{\pi^2}{2}\right]} \mathcal{G}^{\sigma_p}_{ij}(\alpha_a, x_{12\perp}; p_A, p'_A)\mathcal{G}^{ij;\sigma_t}(\beta_b, x_{12\perp}; p_B, p'_B)$$

This is the final formula for rapidity-only TMD factorization of hadronic tensor.

10 Conclusions and outlook

In conclusion let us present our final formula (9.22) for the practical case of hadronic tensor (2.2) which corresponds to "forward" matrix element with $p'_A = p_A$ and $p'_B = p_B$. It reads

$$W(p_A, p_B; q) = \int db_\perp \; e^{i(q,b)_\perp} W(p_A, p_B; \alpha_q, \beta_q, b_\perp),$$
$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) = \frac{\pi^2}{2} Q^2 \mathcal{G}^{\sigma_p}_{ij}(\alpha_q, b_\perp; p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q, b_\perp; p_B)$$
$$\times \exp\left\{\frac{\alpha_s N_c}{2\pi}\left[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t} + \gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p} + \gamma\Big) + \frac{\pi^2}{2}\right]\right\}$$
$$+ \text{NLO terms} \sim O(\alpha_s^2) + \text{power corrections} \tag{10.1}$$

where gluon TMDs $\mathcal{G}^{\sigma_p}_{ij}(\alpha_q, b_\perp)$ and $\mathcal{G}^{ij}_{\sigma_t}(\beta_q, b_\perp)$ are defined in eq. (9.20) above. Note that this formula is actually our goal - TMD factorization (1.1) with the coefficient function (1.4) at $\eta_a = \ln\sigma_p$ and $\eta_b = \ln\sigma_t$. Also, note that up to $\frac{\pi^2}{2}$ constant, this formula can be restored from the rapidity-only evolution of gluon TMD calculated in refs. [9, 10]. Since leading-order evolutions of quark and gluon TMDs differ only in replacement of color factors $N_c \to c_F$, one should expect a similar Sudakov-type formula for the Drell-Yan process, probably with a different constant in place of $\frac{\pi^2}{2}$.

Let us discuss now the region of applicability of eq. (10.1). The r.h.s. of the evolution formula (10.1) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_q s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$, see eq. (3.4). Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) = \frac{\pi^2}{2} Q^2 \mathcal{G}^{\tilde{\sigma}_p}_{ij}(\alpha_q, b_\perp; p_A)\mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q, b_\perp; p_B) \tag{10.2}$$
$$\times \exp\left\{-\frac{\alpha_s N_c}{2\pi}\left[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\right]\right\} + O(\alpha_s^2) \text{ terms} + \text{power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region. This is discussed in appendix G of ref. [9] and here I briefly sum up the main points of that discussion. First, if $x_B \sim 1$ and $q_\perp^2 \geq m_N^2$, there is no room for any evolution and one should turn to phenomenological

JHEP03(2023)029

models of TMDs like the replacement of b by b_* in refs. [2, 22]. If $x_B \sim 1$ and $q_\perp^2 \geq m_N^2$, there is a room for DGLAP-type evolution summing logs $\left(\alpha_s \ln q_\perp^2/m_N^2\right)^n$. Similarly, if $x_B = \beta_b \ll 1$, then even at $\beta_b \sigma s = q_\perp^2$ there can be the BFKL-type evolution from $\sigma = \frac{q_\perp^2}{\beta_b' s}$ to $\sigma = \frac{q_\perp^2}{s}$ which sums up logs $(\alpha_s \ln x_B)^n$. The matching between double-log Sudakov evolution (10.1) and single-log DGLAP or BFKL evolutions can in principle be performed by solving general rapidity evolution equations discussed in ref. [21].

There is another issue that should be addressed before matching to BFKL and especially to DGLAP evolutions. As usually for rapidity-only factorization, the argument of coupling constant in eq. (10.1) is undetermined in the leading order and should be obtained from higher orders of perturbative expansion. Typically, argument of coupling constant in the small-x evolution equations is fixed using the BLM/renormalon approach [23], see for example ref. [24] for the BFKL equation and refs. [25, 26] for the BK equation [27–29]. In recent paper [9] G.A. Chirilli and the author used this BLM optimal scale setting [23] to fix the argument of coupling constant in the rapidity-only TMD evolution (B.14). The result is that the effective argument of a coupling constant is halfway in the logarithmical scale between the transverse momentum and energy of TMD distribution. One of the future directions of this research is to use BLM prescription to fix the argument of coupling constant in the coefficient function $\mathfrak{C}(x_{12_\perp}; \alpha_a, \beta_b; \sigma_p, \sigma_t)$ and obtain the running-coupling generalization of Sudakov-type formula (10.1).

Another outlook is to connect to usual CSS/SCET-type evolution of TMDs at moderate x where the two [30–33] and three-loop results are available [34–36]. It should be noted that the "double operator expansion" method recently used in ref. [14] is very similar to the approach of this paper and also uses calculation of Feynman diagrams in two background fields. However, the UV+rapidity cutoff of TMD operators in ref. [14] is very different from the rapidity-only cutoff used here so the hope is to fix the argument of coupling constant in eq. (10.1) and compare the final results for the evolution.

Summarizing, the proposed rapidity-only factorization may serve as a bridge between classical TMD factorization [2] at moderate x and k_T-factorization [37] at small x. It would be interesting to compare this rapidity-only factorization/evolution with other approaches to unification of TMD evolution based on small-x improvement of usual $Q_\perp^2$ evolution (see e.g. [38, 39]) and on various saturation - inspired methods, see e.g. [40, 41] and 'improved TMD" discussed in refs. [42–44]. The study is in progress.

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A Gluon "cut" propagator in the background field $\mathbb{A}$

In general, the "cut" gluon propagator from left to right sector in the background-Feynman gauge is given by the double functional integral

$$\langle A_\alpha(x)A_\beta(y)\rangle_{\mathbb{A}} \tag{A.1}$$
$$= \int^{\tilde{A}(t_f)=A(t_f)} D\tilde{A}_\mu DA_\mu\ \tilde{A}_\alpha(x)A_\beta(y)\ e^{i\int dz\ \frac{1}{2}\left(\tilde{A}^{\mu,a}(\tilde{\mathbb{D}}^2 g_{\mu\nu}-2i\tilde{\mathbb{F}}_{\mu\nu})^{ab}\tilde{A}^{\nu,b}+A^a_\mu(\mathbb{D}^2 g^{\mu\nu}-2i\mathbb{F}^{\mu\nu})A^b_\nu\right)}$$

(recall that in our case background field $\mathbb{A}$ is the same for both left and right sector). In Schwinger's notations, in can be written down as

$$\langle A_\alpha(x)A_\beta(y)\rangle_{\mathbb{A}} = -(x|\Big(\frac{1}{\mathbb{P}^2 g_{\alpha\xi}+2i\mathbb{F}_{\alpha\xi}-i\epsilon}p^2\Big)\tilde{\delta}_+(p)\Big(p^2\frac{1}{\mathbb{P}^2\delta^\xi_\beta+2i\mathbb{F}^\xi{}_\beta+i\epsilon}\Big)|y) \tag{A.2}$$

where expressions in parenthesis in the r.h.s. can be understood as a series

$$p^2\frac{1}{(p+\mathbb{A})^2\delta^\xi_\beta+2i\mathbb{F}^\xi{}_\beta+i\epsilon} = 1-\mathbb{O}^\xi{}_\beta\frac{1}{p^2+i\epsilon}+\mathbb{O}^\xi{}_\eta\frac{1}{p^2+i\epsilon}\mathbb{O}^\eta{}_\beta\frac{1}{p^2+i\epsilon}+\dots$$
$$\frac{1}{(p+\mathbb{A})^2 g_{\alpha\xi}+2i\mathbb{F}_{\alpha\xi}-i\epsilon}p^2 = 1-\frac{1}{p^2+i\epsilon}\mathbb{O}_{\alpha\xi}+\frac{1}{p^2-i\epsilon}\mathbb{O}_{\alpha\eta}\frac{1}{p^2-i\epsilon}\mathbb{O}^\eta{}_\beta+\dots \tag{A.3}$$

with

$$\mathbb{O}^{\alpha\beta} \equiv (\{p^\lambda,\mathbb{A}_\lambda\}+\mathbb{A}^2)g^{\alpha\beta}+2i\mathbb{F}^{\alpha\beta}. \tag{A.4}$$

By rearranging the series, it is possible to prove that

$$\begin{aligned}\langle A^a_\alpha(x)A^b_\beta(y)\rangle_{\mathbb{A}} = (x|&\frac{1}{p^2+i\epsilon p_0}\mathbb{O}\tilde{\delta}_+(p^2)+\tilde{\delta}_+(p^2)\mathbb{O}\frac{1}{p^2-i\epsilon p_0}\\ &+\frac{1}{p^2+i\epsilon p_0}\mathbb{O}\tilde{\delta}_+(p)\mathbb{O}\frac{1}{p^2-i\epsilon p_0}+\tilde{\delta}_+(p)\mathbb{O}\frac{1}{p^2-i\epsilon p_0}\mathbb{O}\frac{1}{p^2-i\epsilon p_0}\\ &+\frac{1}{p^2+i\epsilon p_0}\mathbb{O}\frac{1}{p^2+i\epsilon p_0}\mathbb{O}\tilde{\delta}_+(p)-\tilde{\delta}_+(p)\mathbb{O}\tilde{\delta}_-(p)\mathbb{O}\tilde{\delta}_+(p)+O(\mathbb{O}^3)|y)^{ab}_{\alpha\beta}\end{aligned} \tag{A.5}$$

Thus, the only term which spoils the "retardiness" property is the last term, but it can be proportional only to correction field $\bar{C}$ since neither projectile no target field can solely produce a pair of gluons. However, two fields $\bar{C}$ involve four $\mathbb{F}_{\xi\eta}$ (two U^{+i} and two V^{-j}, see eq. (E.1)) which exceeds our accuracy. Thus, we use formula (A.5) without the last term.

B TMD matrix elements

In this section we list the necessary results for the "eikonal" contributions - one-loop TMD matrix elements calculated in ref. [9]. As discussed there, the cutoffs in α and β respecting analytical properties of Feynman diagrams (and hence IR real-virtual cancellations) are

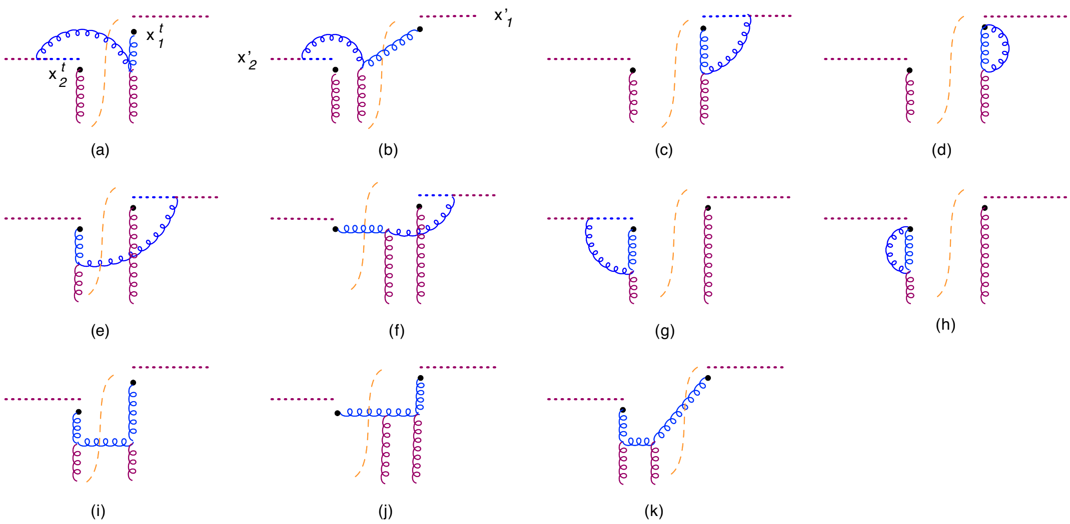

Figure 11. "Target" TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting: F^{-k} shown by dots stand at $x_1^t = x_{1_\perp} + x_1^+$ and $x_2^t = x_{2_\perp} + x_2^+$ while Wilson lines start from $x_1' = x_1^t + \delta^-$ and $x_2' = x_2^t + \delta^-$ where $\delta^- = \frac{1}{\varrho\sigma_t}$.

obtained by "smooth" cuts $e^{\pm i\frac{\alpha}{\sigma_t}}$ and $e^{\pm i\frac{\beta}{\sigma_t}}$. These cutoffs are visualized with "point-splitting" regularization as shown in figures 11 and 12.[11] As demonstrated in ref. [9], δ^+ and δ^- should be positive which follows from the requirement that the distances between the "splitted" operators should be space-like.

The results of calculation of diagrams in figure 11 are:

$$\begin{aligned}
&\langle [x_2^+, -\infty^+]^{ab}_{x_{2_\perp}+\delta^-} [-\infty^+, x_1^+]^{bc}_{x_{1_\perp}+\delta^-} F^{-j,c}(x_1^+, x_{1_\perp})\rangle^{\text{fig.11a-d}}_{\mathcal{A}} \\
&= \frac{g^2 N_c}{8\pi^2} \int đ\beta_b đ k_{b_\perp} V^{-j,a}(\beta_b, k_{b_\perp}) e^{-i\beta_b\varrho x_1^+ + i(k_b', x_1)_\perp} I^{\text{eik}}_{\text{fig.11a-d}}(\beta_b, k_{b_\perp}, x_{12})
\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}
&I^{\text{eik}}_{\text{fig.11a-d}}(\beta_b, k_{b_\perp}, x_1^+, x_{1_\perp}, x_2^+, x_{2_\perp}) = 8\pi^2 s \int_0^\infty đ\alpha\; e^{-i\frac{\alpha}{\sigma_t}} \frac{đ\beta}{\beta + i\epsilon} đ p_\perp\; e^{i\beta\varrho x_{12}^+ - i(p, x_{12})_\perp} \\
&\quad \times \Big[\tilde{\delta}(\alpha\beta s - p_\perp^2) \frac{\beta - \beta_b}{\alpha(\beta - \beta_b)s - (p - k_b)_\perp^2 - i\epsilon} + \frac{(\beta - \beta_b)}{\alpha\beta s - p_\perp^2 + i\epsilon} \tilde{\delta}[\alpha(\beta - \beta_b)s - (p - k_b)_\perp^2]\Big] \\
&\quad - 8\pi^2 i \int đ\alpha đ\beta đ p_\perp \frac{e^{-i\frac{\alpha}{\sigma_t}} s(\beta - \beta_b)}{(\beta + i\epsilon)(\alpha\beta s - p_\perp^2 + i\epsilon)[\alpha(\beta - \beta_b)s - (p - k_b)_\perp^2 + i\epsilon]} \\
&= 8\pi^2 \int_0^\infty đ\alpha\; e^{-i\frac{\alpha}{\sigma_t}} \int đ p_\perp \bigg(\frac{\beta_b s}{p_\perp^2} \frac{e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+ - i(p, x_{12})_\perp} - 1}{\alpha\beta_b s + (p - k_b)_\perp^2 + i\epsilon} \\
&\quad + \frac{(p - k_b)_\perp^2 e^{i(p, x_{12})_\perp} \big[e^{i(\beta_b + \frac{(p - k_b)_\perp^2}{\alpha s})\varrho x_{12}^+} - e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\big]}{\alpha[\alpha\beta_b s + (p - k_b)_\perp^2 + i\epsilon][\alpha\beta_b s + (p - k_b)_\perp^2 - p_\perp^2]}\bigg)
\end{aligned} \tag{B.2}$$

[11] As demonstrated in ref. [9], the violations of gauge invariance due to this point-splitting are power corrections $\sim \lambda_p$ or λ_t.

At $x_{12}^+ = 0$ this integral is simplified to

$$I^{\text{eik}}_{\text{fig.11a-d}}(\beta_b, k_{b_\perp}, x_{12_\perp}) = 8\pi^2 \int_0^\infty d\!\!{}^-\alpha \int \frac{d\!\!{}^- p_\perp}{p_\perp^2} \frac{\beta_b s e^{-i\frac{\alpha}{\sigma_t}}(e^{-i(p,x_{12})_\perp} - 1)}{\alpha\beta_b s + (p-k_b)_\perp^2 + i\epsilon}$$
$$= 8\pi^2 \int_0^\infty d\!\!{}^-\alpha\, e^{-i\frac{\alpha}{\sigma_t}} \int \frac{d\!\!{}^- p_\perp}{p_\perp^2} \frac{\beta_b s(e^{-i(p,x_{12})_\perp} - 1)}{\alpha\beta_b s + p_\perp^2 + i\epsilon}\Big[1 - \frac{p_\perp^2 - (p-k_b)_\perp^2}{\alpha\beta_b s + (p-k_b)_\perp^2 + i\epsilon}\Big]$$
$$= 8\pi^2 \int_0^\infty d\!\!{}^-\alpha\, e^{-i\frac{\alpha}{\sigma_t}} \int \frac{d\!\!{}^- p_\perp}{p_\perp^2} \frac{\beta_b s(e^{-i(p,x_{12})_\perp} - 1)}{\alpha\beta_b s + p_\perp^2 + i\epsilon} + 4\pi \int \frac{d\!\!{}^- p_\perp}{p_\perp^2}(e^{-i(p,x_{12})_\perp} - 1)\ln\frac{p_\perp^2}{(p-k_b)_\perp^2}$$
$$= -\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta_b + i\epsilon)\sigma_t s x_{12}^2 e^\gamma\Big) - \frac{\pi^2}{4} + I_K(-k_{b_\perp}, x_{12_\perp}) \tag{B.3}$$

where I_K is defined in eq. (6.26). Here we neglected the $e^{-i\frac{\alpha}{\sigma}}$ cutoff in the second integral in the third line since it converges at $\alpha \sim \frac{Q_\perp^2}{|\beta_b'|s}$ so $\frac{\alpha}{\sigma_t} \sim \lambda_t$. The first term in the last line is given by eq. (C4) from ref. [9] and the second by eq. (F.11) from appendix F.2.

Similarly, the result for diagrams in figure 11 e-h reads

$$\langle F^{-i,a}(x_2^+, x_{2_\perp})[x_2^+, -\infty]^{ab}_{x_{2_\perp}+\delta^-}[-\infty, x_1^+]^{bc}_{x_{1_\perp}+\delta^-}\rangle^{\text{fig.11e-h}}_{\mathcal{A}} \tag{B.4}$$
$$= \frac{g^2 N_c}{8\pi^2}\int d\!\!{}^-\beta_b' d\!\!{}^- k_{b_\perp}' e^{-i\beta_b' x_{12}^+ + i(k_b', x_{12_\perp})} V^{-i,c}(\beta_b', k_{b_\perp}') I^{\text{eik}}_{\text{fig.11e-h}}(\beta_b', k_{b_\perp}' x_{12_\perp}),$$
$$I^{\text{eik}}_{\text{fig.11e-h}}(\beta_b', k_{b_\perp}', x_{12}) = 8\pi^2 \int_0^\infty d\!\!{}^-\alpha \int d\!\!{}^- p_\perp\, e^{i\frac{\alpha}{\sigma_t}}\Big(\frac{\beta_b' s}{p_\perp^2}\frac{(e^{-i(p,x_{12})_\perp + i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+} - 1)}{\alpha\beta_b' s - (p+k_b')_\perp^2 + i\epsilon}$$
$$+ \frac{(p+k_b')_\perp^2 e^{-i(p,x_{12})_\perp}[e^{i\frac{(p-k_a)_\perp^2}{\alpha s}\varrho x_{12}^+ - i\beta_b \varrho x_{12}^+} - e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}]}{\alpha[\alpha\beta_b' s + p_\perp^2 - (p+k_b')_\perp^2 + i\epsilon][\alpha\beta_b' s - (p+k_b')_\perp^2 + i\epsilon]}\Big)$$

which simplifies to

$$I^{\text{eik}}_{\text{fig.11e-h}}(\beta_b', k_{b_\perp}', x_{12}) = 8\pi^2 \int_0^\infty d\!\!{}^-\alpha \int d\!\!{}^- p_\perp\, e^{i\frac{\alpha}{\sigma_t}} \frac{\beta_b' s}{p_\perp^2}\frac{(e^{-i(p,x_{12})_\perp} - 1)}{\alpha\beta_b' s - (p+k_b')_\perp^2 + i\epsilon} \tag{B.5}$$

at $x_2^+ = x_1^+$. This integral can be obtained from eq. (B.3) by complex conjugation and replacement $k_b \to -k_b'$ so one obtains

$$I^{\text{eik}}_{\text{fig.11e-h}}(\beta_b', k_{b_\perp}', x_{12_\perp}) = -\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta_b' + i\epsilon)\sigma_t s x_{12_\perp}^2 e^\gamma\Big) - \frac{\pi^2}{4} + I_K(k_{b_\perp}', x_{12_\perp}) + O(\lambda_t) \tag{B.6}$$

As to 'handbag" diagrams in figure 11i-k, they were already discussed in section 6.3.

The diagrams in figure 12 are obtained by simple target↔projectile replacements (6.28) in eqs. (B.3) and (B.4). We get

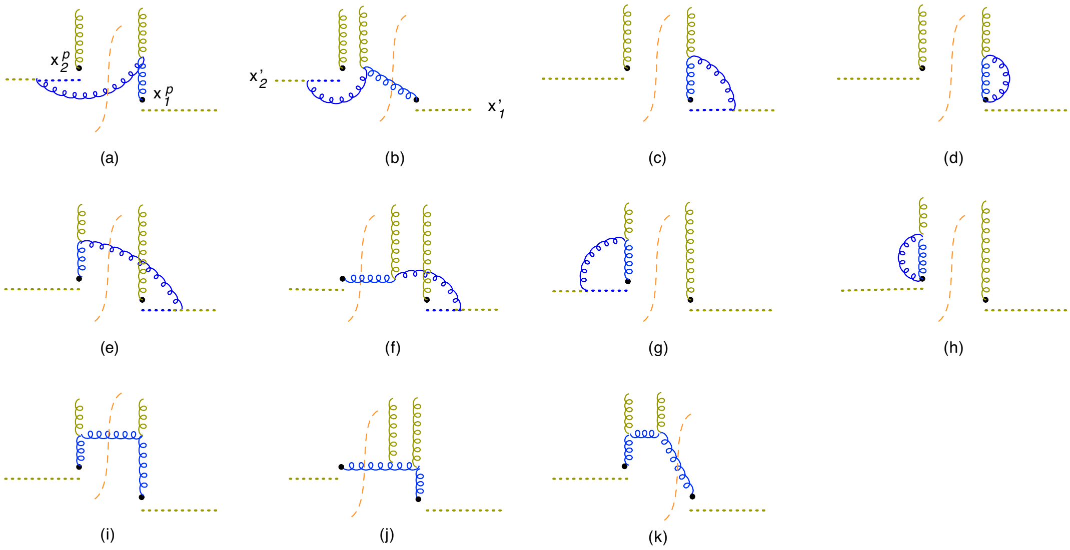

Figure 12. "Projectile" TMD matrix elements. The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: F^{+k} shown by dots stand at $x_1^p = x_{1_\perp} + x_1^-$ and $x_2^p = x_{2_\perp} + x_2^-$ while Wilson lines start from $x_1' = x_2 + \delta^+$ and $x_2' = x_1 + \delta^+$ where $\delta^+ = \frac{1}{\varrho\sigma_p}$.

$$\begin{aligned}
&\langle [x_2^-, -\infty]^{ab}_{x_{2_\perp}+\delta^+}[-\infty, x_1^-]^{bc}_{x_{1_\perp}+\delta^+}F^{+j,c}(x_1^-, x_{1_\perp})\rangle_{\mathcal{A}}^{\text{fig.12a-d}} \\
&\quad = \frac{g^2N_c}{8\pi^2}\int d\!\!{}^-\alpha_a d\!\!{}^-k_{a_\perp} U^{+j,b}(\alpha_a, k_{a_\perp})e^{-i\alpha_a\varrho x_1^- + i(k_a,x_1)_\perp} I^{\text{eik}}_{\text{fig.12a-d}}(\alpha_a, k_{a_\perp}, x_{12}),
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
I^{\text{eik}}_{\text{fig.12a-d}}(\alpha_a, k_{a_\perp}, x_{12}) &= 8\pi^2\int_0^\infty d\!\!{}^-\beta\; e^{-i\frac{\beta}{\sigma_p}}\int d\!\!{}^-p_\perp\bigg(\frac{\alpha_a s}{p_\perp^2}\,\frac{e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- - i(p,x_{12})_\perp} - 1}{\alpha_a\beta s + (p-k_a)_\perp^2 + i\epsilon} \\
&\quad + \frac{(p-k_a)_\perp^2 e^{-i(p,x_{12})_\perp}\big[e^{i(\alpha_a + \frac{(p-k_a)_\perp^2}{\beta s})\varrho x_{12}^-} - e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}\big]}{\alpha[\alpha_a\beta s + (p-k_a)_\perp^2 + i\epsilon][\alpha_a\beta s + (p-k_a)_\perp^2 - p_\perp^2]}\bigg)
\end{aligned}$$

simplified to

$$I^{\text{eik}}_{\text{fig.12a-d}}(\alpha_a, k_{a_\perp}, x_{12_\perp}) = -\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha_a + i\epsilon)\sigma_p s x_{12_\perp}^2 e^\gamma\Big) - \frac{\pi^2}{4} + I_K(-k_{a_\perp}, x_{12_\perp}) \tag{B.8}$$

at $x_{12}^- = 0$, and

$$\begin{aligned}
&\langle F^{+i,a}(x_2^-, x_{2_\perp})[x_2^-, -\infty]^{ab}_{x_{2_\perp}+\delta^+}[-\infty, x_1^-]^{bc}_{x_{1_\perp}+\delta^+}\rangle_{\mathcal{A}}^{\text{fig.12e-h}} \\
&\quad = g^2N_c\int d\!\!{}^-\alpha_a d\!\!{}^-k'_{a_\perp} e^{-i\alpha_a x_{12}^+ + i(k'_a, x_{12_\perp})} U^{+i,c}(\alpha_a, k'_{a_\perp}) I^{\text{eik}}_{\text{fig.12e-h}}(\alpha_a, k'_{a_\perp}, x_2, x_1),
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
I^{\text{eik}}_{\text{fig.12e-h}}(\alpha'_a, k'_{a_\perp}, x_{12}) &= 8\pi^2\int_0^\infty d\!\!{}^-\beta\int d\!\!{}^-p_\perp\; e^{i\frac{\beta}{\sigma_p}}\bigg(\frac{\alpha'_a s}{p_\perp^2}\,\frac{(e^{-i(p,x_{12})_\perp + i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-} - 1)}{\alpha'_a\beta s - (p+k'_a)_\perp^2 + i\epsilon} \\
&\quad + \frac{(p+k'_a)_\perp^2 e^{-i(p,x_{12})_\perp}\big[e^{i\frac{(p+k'_a)_\perp^2}{\alpha s}\varrho x_{12}^- - i\alpha'_a\varrho x_{12}^-} - e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}\big]}{\beta[\alpha'_a\beta s + p_\perp^2 - (p+k'_a)_\perp^2 + i\epsilon][\alpha'_a\beta s - (p+k'_a)_\perp^2 + i\epsilon]}\bigg)
\end{aligned}$$

JHEP03(2023)029

which similarly simplifies to

$$I^{\rm eik}_{\rm fig.12e-h}(\alpha_a,k'_{a\perp},x_{12\perp}) = -\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha'_a+i\epsilon)\sigma_p s x_{12}^2 e^\gamma\Big) - \frac{\pi^2}{4} + I_K(k'_{a\perp},x_{12\perp}) + O(\lambda_p) \tag{B.10}$$

at $x_{12}^+ = 0$. Again, the "handbag" diagrams in fig.12i-k were already accounted for in section 6.3.

Assembling eqs. (B.3), (B.6), (B.8), and (B.10), we get the matrix elements of eikonal TMD operators which have to be subtracted from $\mathcal{W}$ according to eq. (4.2):

$$\begin{aligned}
&\mathcal{W}(x_1,x_2)^{\rm eik} = \frac{N_c^2-1}{N_c}8\pi^2\\
&\times\Big\{U^{+a}_i(x_2)V^{-i,n}(x_2)\langle[x_2^+,-\infty]^{na}_{x_{2\perp}+\delta^-}[-\infty,x^+]^{bc}_{x_{1\perp}+\delta^-}F^{-j,c}(x^+,x_{1\perp})\rangle^{\rm fig.11a-c}_{\mathcal{A}}U^{+j,b}(x_1)\\
&+U^{+a}_i(x_2)\langle F^{-i,n}(x_2^+,x_{2\perp})[x_2^+,-\infty]^{na}_{x_{2\perp}+\delta^-}[-\infty,x_1^+]^{bc}_{x_{1\perp}+\delta^-}\rangle^{\rm fig.11d-f}_{\mathcal{A}}V^{-j,c}(x_1)U^{+j,b}(x_1)\\
&+U^{+n}_i(x_2)V^{-i,a}(x_2)\langle[x^-,-\infty]^{na}_{x_{2\perp}+\delta^+}[-\infty,x_1^-]^{bc}_{x_{1\perp}+\delta^+}F^{+j,c}(x^-,x_{1\perp})\rangle^{\rm fig.12a-c}_{\mathcal{A}}V^{-j,c}(x_1)\\
&+V^{-i,a}(x_2)\langle F^{+i,n}(x_2^-,x_{2\perp})[x_2^-,-\infty]^{na}_{x_{2\perp}+\delta^+}[-\infty,x_1^-]^{bc}_{x_{1\perp}+\delta^+}\rangle^{\rm fig.12d-f}_{\mathcal{A}}V^{-j,c}(x_1)U^{+b}_j(x_1)\Big\}\\
&=\int {d\!\!\!^-}\alpha'_a {d\!\!\!^-}k'_{a\perp}{d\!\!\!^-}\beta'_b{d\!\!\!^-}k_{b\perp}{d\!\!\!^-}\alpha_a{d\!\!\!^-}k'_{a\perp}{d\!\!\!^-}\beta_b{d\!\!\!^-}k'_{b\perp}\ e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times e^{-i(k_a+k_a,x_2)_\perp - i(k'_a+k'_b,x_1)_\perp}U^{+,b}_i(\alpha'_a,k'_{a\perp})V^{-i,a}(\beta'_b,k_{b\perp})U^{+,b}_j(\alpha_a,p'_{A\perp})V^{-j,a}(\beta_b,k'_{b\perp})\\
&\quad\times[I^{\sigma_p,\sigma_t}_{\rm eik}(\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a\perp},k'_{a\perp},k_{b\perp},k'_{b\perp},x_{12\perp})+O(\lambda_p)+O(\lambda_t)]
\end{aligned}\tag{B.11}$$

with

$$\begin{aligned}
&I^{\sigma_p,\sigma_t}_{\rm eik}(\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a\perp},k'_{a\perp},k_{b\perp},k'_{b\perp},x_{12\perp})\\
&=-\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha'_a+i\epsilon)\sigma_p s x_{12\perp}^2 e^\gamma\Big)-\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\alpha_a+i\epsilon)\sigma_p s x_{12\perp}^2 e^\gamma\Big)\\
&\quad-\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta'_b+i\epsilon)\sigma_t s x_{12\perp}^2 e^\gamma\Big)-\frac{1}{2}\ln^2\Big(-\frac{i}{4}(\beta_b+i\epsilon)\sigma_t s x_{12\perp}^2 e^\gamma\Big)-\pi^2\\
&\quad+I_K(-k_{a\perp},x_{12\perp})+I_K(k'_{a\perp},x_{12\perp})+I_K(-k_{b\perp},x_{12\perp})+I_K(k'_{b\perp},x_{12\perp})
\end{aligned}\tag{B.12}$$

where I_K is defined in eq. (6.26).

Let us present also the derivative[12]

$$\begin{aligned}
&\sigma_t\frac{d}{d\sigma_t}I^{\sigma_p,\sigma_t}_{\rm eik}(\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a\perp},k_{a\perp},k_{b\perp},k'_{b\perp},x_{12\perp})\\
&=-\ln\Big(-\frac{i}{4}(\beta'_b+i\epsilon)\sigma_t s x_{12\perp}^2 e^\gamma\Big)-\ln\Big(-\frac{i}{4}(\beta_b+i\epsilon)\sigma_t s x_{12\perp}^2 e^\gamma\Big)
\end{aligned}\tag{B.13}$$

which translates into evolution equation [9, 10]

$$\begin{aligned}
&\sigma_t\frac{d}{d\sigma_t}\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{2\perp},x_{1\perp})\\
&=-\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x_{12\perp}^2}{4}+\ln(-i\beta'_b\sigma_t+\epsilon)+\ln(-i\beta_b\sigma_t+\epsilon)+2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{2\perp},x_{1\perp})
\end{aligned}\tag{B.14}$$

[12] It is worth noting that handbag diagrams in Figs 11i-k and 12i-k do not contribute to evolution equation since they do not need a rapidity cutoff, see the discussion in ref. [9] and in section 6.3.

Similarly, one obtains

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp}) \tag{B.15}$$
$$= -\frac{\alpha_s N_c}{2\pi} \left[2\ln \frac{s x_{12_\perp}^2}{4} + \ln(-i\alpha'_a \sigma_p + \epsilon) + \ln(-i\alpha_a \sigma_p + \epsilon) + 2\gamma \right] \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$

In the coordinate space, the evolution equation (B.14) takes the form

$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \tag{B.16}$$
$$= -\frac{\alpha_s N_c}{2\pi} \Big\{ 2\ln \frac{s x_{12_\perp}^2}{4} \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})$$
$$- \int dz_2^+ \left[\frac{\theta(x_2 - z_2)^+}{(x_2 - z_2)^+} - \delta(x_2 - z_2)^+ \int_0^{\sigma_t/\varrho} \frac{dz_2^+}{z_2^+} \right] \hat{\mathcal{O}}_{ij}^{\sigma_p}(z_2^+, x_{2_\perp}; x_1^+, x_{1_\perp})$$
$$- \int dz_1^+ \left[\frac{\theta(x_1 - z_1)^+}{(x_1 - z_1)^+} - \delta(x_1 - z_1)^+ \int_0^{\sigma_t/\varrho} \frac{dz_1^+}{z_1^+} \right] \hat{\mathcal{O}}_{ij}^{\sigma_p}(x_2^+, x_{2_\perp}; z_1^+, x_{1_\perp}) \Big\}$$

where we used formula

$$\int d\!\!\!{}^-\beta \; e^{-i\beta z} \big[\ln\big(-i\beta\sigma + \epsilon \big) + \gamma \big] = -\frac{\theta(z)}{z} + \delta(z) \int_0^\sigma \frac{dz'}{z'} \tag{B.17}$$

The evolution equation of $\hat{\mathcal{O}}^{ij;\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp})$ looks like eq. (B.16) with trivial replacements $x^+ \to x^-$ and $\sigma_t \to \sigma_p$.

C Soft factor with rapidity-only cutoffs

In this section we demonstrate that the soft factor with rapidity-only cutoffs is a power correction. The soft factor is given by the correlation function of four Wilson lines

$$\frac{1}{N_c^2 - 1} \mathrm{Tr} \langle \{x_2^-, -\infty^-\}_{x_{2_\perp}} \{-\infty^+, x_2^+\}_{x_{2_\perp}} [x_1^+, -\infty^+]_{x_{1_\perp}} [-\infty^-, x_1^-]_{x_{1_\perp}} \rangle_{\mathscr{C}} \tag{C.1}$$

The diagrams for the one-loop soft factor with rapidity-only regularization by “point splitting” is shown in figure 13

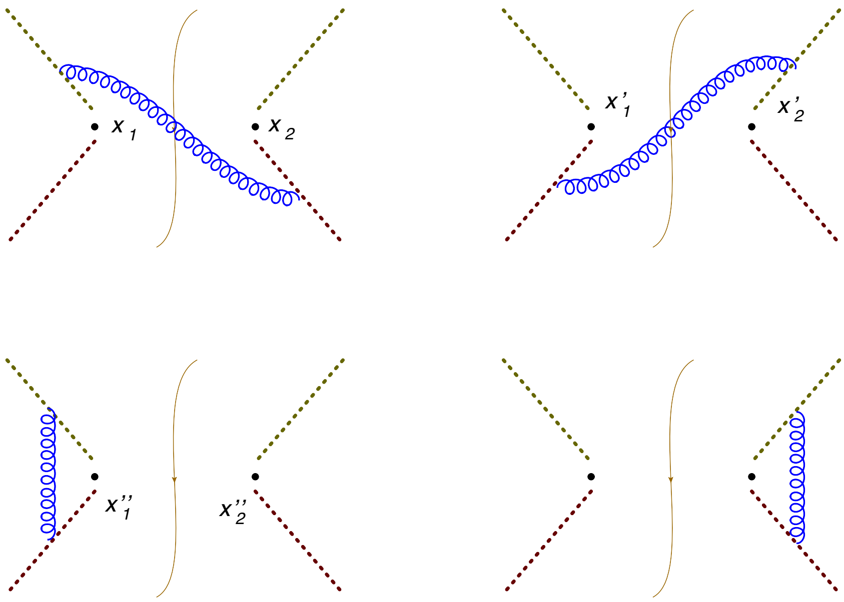

Figure 13. First-order perturbative diagrams for the soft factor. The rapidity regularization is depicted by point splitting: "projectile" Wilson lines start from $x_1' = x_1 + \delta^+$ and $x_2' = x_2 + \delta^+$ while "target" Wilson lines from $x_1'' = x_1 + \delta^-$ and $x_2'' = x_2 + \delta^-$.

$$
\begin{aligned}
&\frac{\mathrm{Tr}}{N_c(N_c^2-1)}\langle\{x_2^-,-\infty^-\}_{x_{2\perp}+\delta^+}\{-\infty^+,x_2^+\}_{x_{2\perp}+\delta^-}[x_1^+,-\infty^+]_{x_{1\perp}+\delta^-}[-\infty^-,x_1^-]_{x_{1\perp}+\delta^+}\rangle \\
&= \frac{1}{N_c(N_c^2-1)}\mathrm{Tr}\Big[\int_{-\infty}^{x_2^-} dx_2'^- \int_{-\infty}^{x_2^+} dx_2'^+ \ \tilde{A}^+(x_2'^- + \delta^+, x_{2\perp})\tilde{A}^-(x_2'^+ + \delta^-, x_{2\perp}) \\
&\quad - \int_{-\infty}^{x_2^-} dx_2'^- \int_{-\infty}^{x_1^+} dx_1'^+ \tilde{A}^+(x_2'^- + \delta^+, x_{2\perp})A^-(x_1'^+ + \delta^-, x_{1\perp}) \\
&\quad - \int_{-\infty}^{x_2^+} dx_2'^+ \int_{-\infty}^{x_1^-} dx_1'^- \ \tilde{A}^-(x_2'^+ + \delta^-, x_{2\perp})A^+(x_1'^- + \delta^+, x_{1\perp}) \\
&\quad + \int_{-\infty}^{x_1^-} dx_1'^- \int_{-\infty}^{x_1^+} dx_1'^+ \ A^+(x_1'^- + \delta^+, x_{1\perp})A^-(x_1'^+ + \delta^-, x_{1\perp})\Big] \\
&= \int \bar{d}^4p \Big[\int_{-\infty}^{x_2^-} dx_2'^- \int_{-\infty}^{x_2^+} dx_2'^+ \ e^{-i\alpha\varrho(x_2'^- - \delta^-) + i\beta\varrho(x_2'^+ - \delta^+)} \frac{i}{\alpha\beta s - p_\perp^2 - i\epsilon} \\
&\quad + \int_{-\infty}^{x_2^-} dx_2'^- \int_{-\infty}^{x_1^+} dx_1'^+ \ e^{-i\alpha\varrho(x_2'^- - \delta^-) + i\beta\varrho(x_1'^+ - \delta^+) - i(p, x_{12})_\perp} \delta_+(\alpha\beta s - p_\perp^2) \\
&\quad + \int_{-\infty}^{x_2^+} dx_2'^+ \int_{-\infty}^{x_1^-} dx_1'^- \ e^{i\alpha\varrho(x_1'^- - \delta^-) - i\beta\varrho(x_2'^+ - \delta^+) - i(p, x_{12})_\perp} \delta_+(\alpha\beta s - p_\perp^2) \\
&\quad - \int_{-\infty}^{x_1^-} dx_1'^- \int_{-\infty}^{x_1^+} dx_1'^+ \ e^{-i\alpha\varrho(x_1'^- - \delta^-) + i\beta\varrho(x_1'^+ - \delta^+)} \frac{i}{\alpha\beta s - p_\perp^2 + i\epsilon}\Big] \qquad \text{(C.2)} \\
&= \int \bar{d}\alpha \bar{d}\beta \bar{d}p_\perp \Big\{ \frac{i}{(\alpha + i\epsilon)(\beta - i\epsilon)} \Big[\frac{e^{i\alpha\varrho(\delta^- - x_2^-) - i\beta\varrho(\delta^+ - x_2^+)}}{\alpha\beta s - p_\perp^2 - i\epsilon} - \frac{e^{i\alpha\varrho(\delta^- - x_1^-) - i\beta\varrho(\delta^+ - x_1^+)}}{\alpha\beta s - p_\perp^2 + i\epsilon} \Big] \\
&\quad + \frac{s}{p_\perp^2} e^{-i(p, x_{12})_\perp} \Big[e^{i\alpha\varrho(\delta^- - x_2^-) - i\beta\varrho(\delta^+ - x_1^+)} + e^{-i\alpha\varrho(\delta^- - x_1^-) + i\beta\varrho(\delta^+ - x_2^+)} \Big] \delta_+(\alpha\beta s - p_\perp^2) \Big\}
\end{aligned}
$$

JHEP03(2023)029

Since the $x_2^-, x_1^- \sim \frac{1}{\varrho\alpha'_a} \ll \delta^-$ and $x_2^+, x_1^+ \sim \frac{1}{\varrho\beta'_b} \ll \delta^+$ we can neglect $x_2^-, x_1^-, x_2^+, x_1^+$ in the above integrals and get

$$
\begin{aligned}
&= s\int đ\alpha đ\beta \frac{đ p_\perp}{p_\perp^2}\Big\{ - e^{i\alpha\varrho\delta^- - i\beta\varrho\delta^+}\tilde{\delta}(\alpha\beta s - p_\perp^2) \\
&\quad + e^{-i(p,x_{12})_\perp}\big[e^{i\alpha\varrho\delta^- - i\beta\varrho\delta^+} + e^{-i\alpha\varrho\delta^- + i\beta\varrho\delta^+}\big]\delta_+(\alpha\beta s - p_\perp^2)\Big\} \\
&= s\int đ\alpha đ\beta \frac{đ p_\perp}{p_\perp^2}\big[e^{i\alpha\varrho\delta^- - i\beta\varrho\delta^+} + e^{-i\alpha\varrho\delta^- + i\beta\varrho\delta^+}\big]\delta_+(\alpha\beta s - p_\perp^2)(e^{-i(p,x_{12})_\perp} - 1) \\
&= \frac{1}{2\pi^2}\int \frac{dp_\perp^2}{p_\perp^2}[J_0(p_\perp\Delta_\perp) - 1]K_0\Big(p_\perp\sqrt{2\delta^+\delta^-}\Big) = \frac{1}{4\pi^2}\mathrm{Li}_2\Big(-\frac{x_{12_\perp}^2}{2\delta^+\delta^-}\Big)
\end{aligned}
\tag{C.3}
$$

Thus, we get the perturbative contribution to rapidity-regularized soft factor in the form

$$
\begin{aligned}
&\langle \{x_2^-, -\infty^-\}_{x_{2_\perp}+\delta^+}\{-\infty^+, x_2^+\}_{x_{2_\perp}+\delta^-}[x_1^+, -\infty^+]_{x_{1_\perp}+\delta^-}[-\infty^-, x_1^-]_{x_{1_\perp}+\delta^+}\rangle \\
&= \frac{1}{4\pi^2}\mathrm{Li}_2\Big(-\frac{x_{12_\perp}^2}{2\delta^+\delta^-}\Big) \sim O\Big(\frac{\Delta_\perp^2}{2\delta^+\delta^-}\Big) \sim O\Big(\frac{\sigma_p\sigma_t s}{Q_\perp^2}\Big) = O\Big(\frac{\mu_\sigma^2}{Q_\perp^2}\Big) \sim \zeta^{-1/2}
\end{aligned}
\tag{C.4}
$$

which is a parametrically small power correction. Of course, there are non-perturbative contributions to the soft factor - power corrections presumably of order of $\Lambda_{\mathrm{QCD}}^2 x_{12_\perp}^2$, but, as we mentioned above, the lesson is that the soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution, quite unlike the usual regularization of the soft factor with "UV+rapidity" cutoff.

D Approximation $x_{12}^\parallel = 0$ for the calculation of coefficient function

In this section we prove that for the calculation of the coefficient function $\mathfrak{C}_1$ in eq. (4.1) one can set $x_{12}^\parallel = 0$ with power accuracy. Let us start with the difference $I_{1a} - I_{1a}^{\mathrm{eik}}$ in eq. (6.16). Since vitual eikonals do not depend on x, it is sufficient to consider

$$
I_{1a}(\alpha'_a, k'_{a_\perp}, \beta'_b, k_{b_\perp}, x_2, x_1) - I^{\mathrm{eik}}_{\mathrm{fig.11a,b}}(\alpha'_a, k'_{a_\perp}, \beta'_b, k_{b_\perp}, x_2, x_1) - (x^\parallel \to 0)
\tag{D.1}
$$

The expression for I_{1a} is given by eq. (6.11)

$$
\begin{aligned}
I_{1a}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{2_\perp}, x_{1_\perp}) &= 8\pi^2\int_0^\infty \frac{đ\alpha}{\alpha}\int đ p_\perp \frac{e^{-i(p,x_{12})_\perp}}{\alpha\beta_b s + (p-k'_b)_\perp^2 - p_\perp^2 + i\epsilon} \\
&\times e^{i\alpha\varrho x_{12}^-}\Big[\frac{p_\perp^2}{\alpha^2 s\xi + p_\perp^2}\frac{(\alpha+\alpha'_a)(\alpha\beta_b s - p_\perp^2)e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}}{[(\alpha+\alpha'_a)p_\perp^2 - \alpha(p+k'_a)^2 + i\epsilon]} \\
&+ \frac{(p-k'_b)_\perp^2(\alpha+\alpha'_a)e^{i\beta_b\varrho x_{12}^+ + i\frac{(p-k'_b)_\perp^2}{\alpha s}\varrho x_{12}^+}}{\alpha(\alpha'_a+\alpha)\beta_b s + (\alpha'_a+\alpha)(p-k_b)_\perp^2 - \alpha(p+k'_a)_\perp^2 + i\epsilon(\alpha'_a+\alpha)}\Big]
\end{aligned}
\tag{D.2}
$$

(cf. eq. (6.18) at $x^\parallel = 0$), and the expression for $I^{\mathrm{eik}}_{\mathrm{fig.11a,b}}$ can be taken as eq. (B.2) without virtual term

$$
\begin{aligned}
I^{\mathrm{eik}}_{\mathrm{fig.11a,b}}(\beta_b, k_{b_\perp}, x_2, x_1) = 8\pi^2\int_0^\infty \frac{đ\alpha}{\alpha}\, e^{-i\frac{\alpha}{\sigma_t}}\int đ p_\perp\Big(&\frac{\alpha\beta_b s - p_\perp^2}{p_\perp^2}e^{-i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+} \\
&+ \frac{(p-k_b)_\perp^2 e^{i(\beta_b + \frac{(p-k_b)_\perp^2}{\alpha s})\varrho x_{12}^+}}{[\alpha\beta_b s + (p-k_b)_\perp^2 + i\epsilon]}\Big)\frac{e^{-i(p,x_{12})_\perp}}{\alpha\beta_b s - p_\perp^2 + (p-k_b)_\perp^2}
\end{aligned}
\tag{D.3}
$$

For definiteness, let us take $\beta_b > 0$ (the case of $\beta_b < 0$ is similar). The difference (D.1) can be represented as a sum of two contributions. The first one is the difference between first terms in eqs. (D.2) and (D.3) minus same difference at $x^{\parallel}_{12} = 0$.

$$
\begin{aligned}
&8\pi^2\int_0^\infty \frac{\text{đ}\alpha}{\alpha}\int \text{đ}p_\perp\,\Bigg[\frac{\frac{p_\perp^2}{\alpha^2 s\xi+p_\perp^2}(\alpha+\alpha'_a)(\alpha\beta_b s-p_\perp^2)\Big(e^{i\alpha\varrho x^-_{12}+i\frac{p_\perp^2}{\alpha s}\varrho x^+_{12}}-1\Big)}{((\alpha+\alpha'_a)p_\perp^2-\alpha(p+k'_a)^2+i\epsilon)(\alpha\beta_b s+(p-k_b)_\perp^2-p_\perp^2+i\epsilon)}\\
&\qquad\qquad-\int_0^\infty\frac{\text{đ}\alpha}{\alpha}\,e^{-i\frac{\alpha}{\sigma_t}}\,\frac{\Big(e^{i\frac{p_\perp^2}{\alpha s}\varrho x^+_{12}}-1\Big)(\alpha\beta_b s-p_\perp^2)}{p_\perp^2[\alpha\beta_b s-p_\perp^2+(p-k_b)_\perp^2+i\epsilon]}\Bigg]e^{-i(p,x_{12})_\perp}\\
&=4\pi\int_0^\infty\frac{dt}{t}\int\text{đ}p_\perp\,\Bigg\{\frac{p_\perp^2}{\frac{t^2}{\alpha'_a\beta_b s}|\frac{\alpha'_a}{\beta_b}|\xi+p_\perp^2}\,\frac{(t+\alpha'_a\beta_b s)\big(e^{i\frac{t\alpha'_a\varrho x^-_{12}}{\alpha'_a\beta_b s}}-1\big)e^{i\frac{p_\perp^2}{t}\beta'_b\varrho x^+_{12}}}{(t+\alpha'_a\beta_b s)p_\perp^2-t(p+k'_a)^2+i\epsilon}\\
&\qquad+\Bigg[\frac{p_\perp^2}{\frac{t^2}{\alpha'_a\beta_b s}|\frac{\alpha'_a}{\beta_b}|\xi+p_\perp^2}\,\frac{(t+\alpha'_a\beta_b s)}{(t+\alpha'_a\beta_b s)p_\perp^2-t(p+k_a)^2+i\epsilon}-\frac{e^{-i\frac{t}{\sigma_t\beta_b s}}}{p_\perp^2}\Bigg]\\
&\qquad\quad\times\Big(e^{-i\frac{p_\perp^2}{t}\beta_b\varrho x^+_{12}}-1\Big)\Bigg\}\frac{t-p_\perp^2}{t+(p-k_b)_\perp^2-p_\perp^2+i\epsilon}=O\left(\frac{Q_\perp^2}{\sigma_t\beta_b s}\right)
\end{aligned}\tag{D.4}
$$

Indeed, integrals over $p_\perp$ and t converge at $p_\perp \sim x_\perp^{-1} \sim Q_\perp$ and t between $Q_\perp^2$ and $\alpha'_a\beta_b s$. At $t \sim \alpha'_a\beta_b s$ the first term is $\sim \frac{Q_\perp^2}{Q^2} = \lambda$ and the second $\sim \frac{Q_\perp^2}{Q^2}\beta_b\varrho x^+_{12} \sim \lambda$. At $t \sim Q_\perp^2$ the first term is $\sim \frac{Q_\perp^2}{Q^2}\alpha'_a\varrho x^-_{12} \sim \frac{Q_\perp^2}{Q^2\lambda}$ while the second can be rewritten as

$$
4\pi\int_0^\infty\frac{dt}{t}\int\frac{\text{đ}p_\perp}{p_\perp^2}\,e^{-i(p,x_{12})_\perp}\Big(e^{i\frac{p_\perp^2}{t}\beta_b\varrho x^+_{12}}-1\Big)\frac{\Big(1-e^{-i\frac{t}{\sigma_t\beta_b s}}\Big)(t-p_\perp^2)}{[t-p_\perp^2+(p-k_b)_\perp^2+i\epsilon]}\sim O(\lambda_t)\tag{D.5}
$$

The second contribution to the eq. (D.1) is the difference between second terms in eqs. (6.18) and (D.3)

$$
\begin{aligned}
&8\pi^2\int_0^\infty\frac{\text{đ}\alpha}{\alpha}\int\text{đ}p_\perp\,\Bigg[\frac{(\alpha+\alpha'_a)\Big(e^{i\alpha\varrho x^-_{12}+i\beta'_b\varrho x^+_{12}+i\frac{(p-k_b)_\perp^2}{\alpha s}\varrho x^+_{12}}-1\Big)}{\alpha(\alpha'_a+\alpha)\beta_b s+(\alpha'_a+\alpha)(p-k_b)_\perp^2-\alpha(p+k'_a)_\perp^2+i\epsilon(\alpha'_a+\alpha)}\\
&\qquad\qquad-e^{-i\frac{\alpha}{\sigma_t}}\,\frac{e^{i(\beta'_b+\frac{(p-k'_b)_\perp^2}{\alpha s})\varrho x^+_{12}}-1}{[\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon]}\Bigg]\frac{(p-k'_b)_\perp^2e^{-i(p,x_{12})_\perp}}{\alpha\beta_b s-p_\perp^2+(p-k_b)_\perp^2+i\epsilon}
\end{aligned}\tag{D.6}
$$

Again, at $\alpha \sim \alpha'_a$ this integral is of order of λ whereas at small $\alpha \ll \alpha'_a$ it turns to

$$
\begin{aligned}
&8\pi^2\int_0^\infty\frac{\text{đ}\alpha}{\alpha}\int\text{đ}p_\perp\Big[e^{i\alpha\varrho x^-_{12}+i\beta_b\varrho x^+_{12}+i\frac{(p-k_b)_\perp^2}{\alpha s}\varrho x^+_{12}}-1-e^{-i\frac{\alpha}{\sigma_t}}\Big(e^{i(\beta_b+\frac{(p-k_b)_\perp^2}{\alpha s})\varrho x^+_{12}}-1\Big)\Big]\\
&\qquad\times\frac{(p-k_b)_\perp^2e^{-i(p,x_{12})_\perp}}{[\alpha\beta_b s+(p-k_b)_\perp^2+i\epsilon][\alpha\beta_b s-p_\perp^2+(p-k_b)_\perp^2+i\epsilon]}\\
&=4\pi\int_0^\infty\frac{dt}{t}\int\text{đ}p_\perp\,\Big[e^{i(1+\frac{(p-k_b)_\perp^2}{t})\beta_b\varrho x^+_{12}}\big[e^{i\frac{t}{\beta_b s}\varrho x^-_{12}}-e^{i\frac{t}{\sigma_t\beta_b s}}\big]+e^{-i\frac{t}{\sigma_t\beta_b s}}-1\Big]\\
&\qquad\times\frac{(p-k_b)_\perp^2e^{i(p,x_{12})_\perp}}{[t+(p-k_b)_\perp^2+i\epsilon][t-p_\perp^2+(p-k_b)_\perp^2+i\epsilon]}=O(\lambda_t)
\end{aligned}\tag{D.7}
$$

since the integral over t converges at $t \sim Q_\perp^2$.

JHEP03(2023)029

Thus,

$$I_{1a}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_2, x_1) - I^{\rm eik}_{\text{fig.11a,b}}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) \\ = I_{1a}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) - I^{\rm eik}_{\text{fig.11a,b}}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) + O(\sigma_t) \tag{D.8}$$

Similarly, one can demonstrate that

$$I_{1b}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) - I^{\rm eik}_{\text{fig.12e,f}}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) \\ = I_{1b}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) - I^{\rm eik}_{\text{fig.12e,f}}(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_{12_\perp}) + O(\sigma_p) \tag{D.9}$$

so we get

$$I_1(\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) - [I^{\rm eik}_{\text{fig.11a,b}} + I^{\rm eik}_{\text{fig.12e,f}}](\alpha'_a, k'_{a_\perp}, \beta_b, k_{b_\perp}, x_1, x_2) - (x^{\parallel}_{12} \to 0) \\ = O\Big(\frac{Q_\perp^2}{\sigma_t \beta_b s}\Big) + O\Big(\frac{Q_\perp^2}{\sigma_p \alpha'_a s}\Big) \sim O(\lambda_p) + O(\lambda_t) \tag{D.10}$$

By projectile$\leftrightarrow$target replacement we get

$$I_2(\alpha_a, k_{a_\perp}, \beta'_b, k'_{b_\perp}, x_1, x_2) - [I^{\rm eik}_{\text{fig.11e,f}} + I^{\rm eik}_{\text{fig.12a,b}}](\alpha_a, k_{a_\perp}, \beta'_b, k'_{b_\perp}, x_1, x_2)] - (x^{\parallel}_{12} \to 0) \\ = O\Big(\frac{Q_\perp^2}{\sigma_t \beta'_b s}\Big) + O\Big(\frac{Q_\perp^2}{\sigma_p \alpha_a s}\Big) \sim O(\lambda_p) + O(\lambda_t) \tag{D.11}$$

This justifies the calculation of the coefficient function in eq. (6.15) at $x^{\parallel}_{12} = 0$.

E Diagrams with correction field $\bar{C}$

In this section we demonstrate that diagrams in the background field $\bar{C}$ lead to power corrections. The correction fields $\bar{C}_\mu$ are given by eq. (4.9) and we will also use the expressions for field strengths[13] from ref. [6]

$$\bar{C}_{\mu\nu} \equiv F_{\mu\nu}(\mathbb{A}) - F_{\mu\nu}(\bar{A}) - F_{\mu\nu}(\bar{B}), \tag{E.1}$$

$$g\bar{C}^{-i}(x^+, x^-) = \frac{i}{2}\int_{-\infty}^{x^+} dx'^+ \int_{-\infty}^{x^-} dx'^- (x - x')^- \\ \times \Big(\partial^i[U^+_k(x'^-), V^{-k}(x'^+)] - \partial_k[U^{+k}(x'^-), V^{-i}(x'^+)] + \partial^k[U^{+i}(x'^-), V^{-k}(x'^+)]\Big) \sim \frac{Q_\perp^3}{\sqrt{s}},$$

$$g\bar{C}^{+i}(x^+, x^-) = -\frac{i}{2}\int_{-\infty}^{x^-} dx'^- \int_{-\infty}^{x^+} dx'^+ (x - x')^+ \\ \times \Big(\partial^i[U^+_k(x'^-), V^{-k}(x'^+)] + \partial_k[U^{+k}(x'^-), V^{-i}(x'^+)] - \partial_k[U^{+i}(x'^-), V^{-k}(x'^+)]\Big) \sim \frac{Q_\perp^3}{\sqrt{s}},$$

$$g\bar{C}^{+-}(x) = -i[U_j(x^-, x_\perp), V^j(x^+, x_\perp)] \sim Q_\perp^2,$$

$$g\bar{C}_{ik}(x) = U_{ik}(x^-, x_\perp) + V_{ik}(x^+, x_\perp) - i[U_i(x^-, x_\perp)V_k(x^+, x_\perp) - i \leftrightarrow k) \sim Q_\perp^2$$

There are two types of diagrams with background field $\bar{C}$ shown in figure 14.

[13] Note that $\bar{C}_{\mu\nu} \neq \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu - ig[\bar{C}_\mu, \bar{C}_\nu]$.

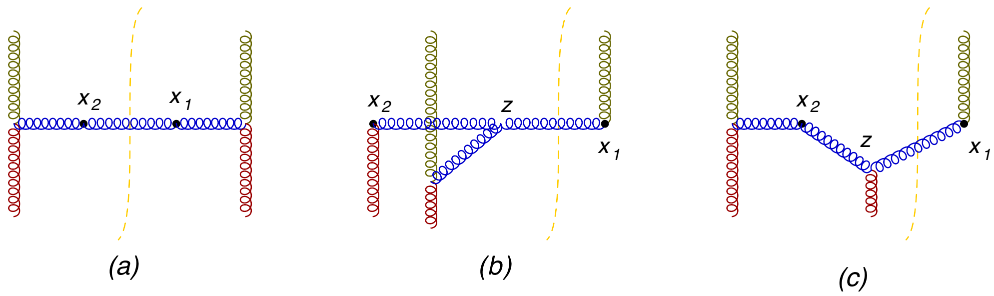

Figure 14. Typical diagrams in the background correction field $\bar{C}$.

Let us start with the first one. We get

$$\bar{C}^{-i}(x_2)\langle F^+_i(x_2)F^-_j(x_1)\rangle\bar{C}^{+j}(x_1) = \frac{x^2_{12\perp}g^{ij}+2x^i_{12}x^j_{12}}{\pi^2(-x^2_{12}-i\epsilon x^0_{12})^3}\bar{C}^-_i(x_2)\bar{C}^+_j(x_1) \tag{E.2}$$
$$\simeq\left(g^{ij}+2\frac{x^i_{12}x^j_{12}}{x^2_{12\perp}}\right)\frac{1}{x^4_{12\perp}}\bar{C}^-_i(x_2)\bar{C}^+_j(x_1)\sim O\left(\frac{Q^6_\perp}{s^3}\right)\times U^{-i}V^{+i}(x_2)U^{-j}V^{+j}(x_1)$$

since $\bar{C}^{-i},\bar{C}^{+j}\sim\frac{Q^3_\perp}{\sqrt{s}}$, see eq. (E.1).

Next we consider diagram in figure 14b where we replaced Feynman propagator by the retarded one according to eq. (A.5).

$$\begin{aligned}&V^{-i}(x_2)\langle(\mathcal{D}^+A_i-\mathcal{D}_iA^+)(x_2)(\mathcal{D}^-A_j-\mathcal{D}_jA^-)(x_1)\rangle U^{+j}(x_1)=\\&=-V^{-i}(x_2)(x_2|(p^+\delta^\alpha_i-p_ig^{+\alpha})\frac{1}{p^2+i\epsilon p_0}\Big([\{p^\lambda,\bar{C}_\lambda\}+\bar{C}^2]g^{\alpha\beta}+2i\bar{C}^{\alpha\beta}\Big)\\&\quad\times\ \tilde{\delta}_+(p)(p^-\delta^\beta_j-p_jg^{\beta-})|x_1)U^{+j}(x_1)\end{aligned} \tag{E.3}$$

Looking at power counting for the correction fields (4.9) and (E.1) we see that the largest contribution to the r.h.s. of eq. (E.3) comes from the term

$$\begin{aligned}&V^{-i}(x_2)\langle(\mathcal{D}^+A_i-\mathcal{D}_iA^+)(x_2)(\mathcal{D}^-A_j-\mathcal{D}_jA^-)(x_1)\rangle U^{+j}(x_1)=\\&=-V^{-i}(x_2|p^+\frac{1}{p^2+i\epsilon p_0}(\{p^\lambda,\bar{C}_\lambda\}g^{ij}+\bar{C}^2g^{ij}+2i\bar{C}^{ij})\tilde{\delta}_+(p)p^-|x_1)U^{+j}(x_1)\end{aligned} \tag{E.4}$$

Let us estimate the term with $\bar{C}^{ij}$. It is similar to eq. (6.3), only instead of $U^{+i}\frac{1}{p^2}V^{-j}\geq m^2_\perp$ we have here $\bar{C}_{ij}\sim\frac{m^4_\perp}{s}$ or $\bar{C}^+\bar{C}^-\sim\frac{m^4_\perp}{s}$, see eq. (4.9). Next, let us consider term with $\{p^\lambda,\bar{C}_\lambda\}=\{p^+,C^-\}+\{p^-,C^+\}+\{p_i,C^i\}$. From eq. (4.9) we see that the last term is $O(\frac{m^2_\perp}{s})$ with respect to the first two terms so we get[14]

$$(x_2|p^+\frac{1}{p^2+i\epsilon p_0}(\{p^+,C^-\}+\{p^-,C^+\})\tilde{\delta}_+(p)p^-|x_1)^{ab} \tag{E.5}$$
$$=(x_2|\frac{1}{p^2}\bigg(\Big(\{p^+,i\partial^+C^-\}+\{p^-,i\partial^+C^+\}\Big)p^-+\Big(\{p^+,C^-\}+\{p^-,C^+\}\Big)\frac{p^2_\perp}{2}\bigg)\tilde{\delta}_+(p)|x_1)^{ab}$$

[14]For the estimate of power corrections the exact form of singularity in the gluon propagator is not important.

Since $\partial^+, \partial^- \sim \sqrt{s}$ the second term is $O(\frac{p_\perp^2}{s} \sim \frac{m_\perp^2}{s})$ in comparison to the first one, the r.h.s. of eq. (E.5) reduces to

$$(x_2|\frac{1}{p^2}\Big(\{p^+, i\partial^+C^-\} + \{p^-, i\partial^+C^+\}\Big)p^-\tilde{\delta}_+(p)|x_1)^{ab} \tag{E.6}$$
$$= \frac{1}{2}\int \text{đ}\alpha'_a \text{đ}k'_{a_\perp}\text{đ}\beta_b \text{đ}k_{b_\perp} \frac{e^{-ik'_a x_2 - ik_b x_1}}{\alpha'_a\beta'_b}[U_i^+(\alpha'_a, k'_{a_\perp}), V^{-i}(\beta_b, k_{b_\perp})]^{ab}\int \text{đ}\alpha \text{đ}\beta \text{đ}p_\perp$$
$$\times\, e^{i\alpha\varrho x_{12}^- + i\beta\varrho x_{12}^+ - i(p,x_{12})_\perp}\frac{\theta(\beta-\beta_b)(-\alpha'_a)}{(\alpha+\alpha'_a)\beta s - (p+k_a)_\perp^2}\Big[1 + \frac{\alpha}{\alpha'_a} - \frac{\beta}{\beta_b}\Big]\tilde{\delta}\Big(\alpha - \frac{(p-k'_a)_\perp}{(\beta-\beta_b)s}\Big)$$
$$= -\int \text{đ}\alpha'_a \text{đ}k'_{a_\perp}\text{đ}\beta_b \text{đ}k_{b_\perp}\frac{e^{-ik'_a x_2 - ik_b x_1}}{2\beta_b}[U_i^+(\alpha'_a, k'_{a_\perp}), V^{-i}(\beta_b, k_{b_\perp})]^{ab}\int \text{đ}\beta \text{đ}p_\perp$$
$$\times\,\frac{\theta(\beta)\beta e^{i\frac{(p-k_b)_\perp^2}{\beta s}\varrho x_{12}^- + i(\beta+\beta_b)\varrho x_{12}^+ - i(p,x_{12})_\perp}}{\alpha'_a\beta(\beta+\beta_b)s + (p-k_b)_\perp^2(\beta+\beta_b) - (p+k'_a)_\perp^2\beta}\Big[\frac{\beta}{\beta_b} - \frac{(p-k_b)_\perp^2}{\alpha'_a\beta s}\Big]$$
$$= \int \text{đ}\alpha'_a \text{đ}k'_{a_\perp}\text{đ}\beta_b \text{đ}k_{b_\perp}[U_i^+(\alpha'_a, k'_{a_\perp}), V^{-i}(\beta_b, k_{b_\perp})]^{ab}e^{-i\alpha'_a\varrho x_2^- + i(k'_a,x_2)_\perp - i\beta_b\varrho x_1^- + i(k_b,x_1)_\perp}$$
$$\times\,\frac{1}{4\pi}\int \text{đ}p_\perp \int_0^\infty dt \frac{[t^2 - Q_{ab}^2(p-k_b)_\perp^2]e^{it^{-1}(p-k_b)_\perp^2\alpha'_a\varrho x_{12}^- + i\left(Q_{a'b}^{-2}t+1\right)\beta_b\varrho x_{12}^+ - i(p,x_{12})_\perp}}{Q_{a'b}^4[t(t+Q_{a'b}^2) - (p-k_b)_\perp^2(t+Q_{a'b}^2) + (p+k'_a)_\perp^2 t]}$$

This should be compared to the leading order contribution (6.7) $\sim U_i^+(\alpha'_a, k'_{a_\perp})V^{-j}(\beta_b, k_{b_\perp})\times$logs. It is clear that contribution from $t \lesssim Q_{a'b}^2$ to the last line in eq. (E.6) is $O(\frac{Q_\perp^2}{Q_{a'b}^2})$, and if $t \gg Q_{a'b}^2$ the last line in the above equation reduces to

$$\frac{1}{4\pi Q_{a'b}^4}\int \text{đ}p_\perp \int_0^\infty dt\; e^{it^{-1}(p-k_b)_\perp^2\alpha'_a\varrho x_{12}^- + iQ_{a'b}^{-2}t\beta_b\varrho x_{12}^+ - i(p,x_{12})_\perp} \tag{E.7}$$
$$= -\frac{ie^{-i(k_b,x_{12})_\perp}}{16\pi^2 Q_{a'b}^4}\int_0^\infty dt\;\frac{t}{\alpha'_a\varrho x_{12}^-}e^{iQ_{a'b}^{-2}t\beta_b\varrho x_{12}^+ - it\frac{x_{12_\perp}^2}{4\alpha'_a\varrho x_{12}^-}} = -\frac{i\alpha'_a\varrho x_{12}^- e^{-i(k_b,x_{12})_\perp}}{\pi^2 Q_{a'b}^4(x_{12_\perp}^2 - x_{12_\parallel}^2)^2}$$

which is a power correction $\sim \frac{Q_\perp^4}{Q_{ab}^4}$ since $\alpha'_a\varrho x_{12}^- \sim 1$. Thus, the contribution of the diagram in figure 14b is a power correction.

Finally, let us consider diagram in figure 14c. We get

$$\bar{C}^{-i}(x_2)\langle(\mathcal{D}^+A_i - \mathcal{D}_iA^+)(x_2)(\mathcal{D}^-A_j - \mathcal{D}_jA^-)(x_1)\rangle_{\bar{B}}U^{+j}(x_1) =$$
$$= -\bar{C}^{-i}(x_2)(x_2|(p^+\delta_i^\alpha - p_i g^{+\alpha})\frac{1}{p^2+i\epsilon p_0}\bar{B}_{\alpha\beta}2i\tilde{\delta}_+(p)(p^-\delta_j^\beta - p_j g^{\beta-})|x_1)U^{+j}(x_1)$$
$$= -2i\bar{C}^{-i}(x_2)(x_2|\frac{p^+}{p^2+i\epsilon p_0}V_{-i}\tilde{\delta}_+(p)p^j|x_1)U_j^+(x_1)$$
$$= -i\bar{C}_i^{-;a}(x_2)U_j^{+;b}(x_1)\int \text{đ}\beta_b \text{đ}k_{b_\perp}V^{-i;ab}(\beta_b, k_{b_\perp})e^{i\beta_b\varrho x_1^+ - i(k_b,x_1)_\perp}$$
$$\times\int \text{đ}\alpha \text{đ}\beta \text{đ}p_\perp\; e^{i\alpha\varrho x_{12}^- + i(\beta+\beta_b)\varrho x_{12}^+ - i(p+k_b,x_{12})_\perp}\frac{\varrho(\beta+\beta_b)(p+k_b)^j}{\alpha(\beta+\beta_b)s - (p+k_b)_\perp^2}\frac{\theta(\beta)}{\beta}\tilde{\delta}\Big(\alpha - \frac{p_\perp^2}{\beta s}\Big)$$
$$= -\frac{i}{2\pi}\varrho\bar{C}_i^{-;a}(x_2)U^{+j;b}(x_1)\int \text{đ}\beta_b \text{đ}k_{b_\perp}V^{-i;ab}(\beta_b, k_{b_\perp})e^{i\beta_b\varrho x_1^+ - i(k_b,x_1)_\perp}$$
$$\times\int_0^\infty d\beta\int \text{đ}p_\perp\; e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- + i(\beta+\beta_b)\varrho x_{12}^+ - i(p+k_b,x_{12})_\perp}\frac{(\beta+\beta_b)(p+k_b)^j}{p_\perp^2(\beta+\beta_b) - (p+k_b)_\perp^2\beta} \tag{E.8}$$

Since $\varrho\bar{C}^{-i} \sim m_\perp^3$ in comparison to $U^{+i}V^{-k} \sim sm_\perp^2$ in the leading term, we need an extra $s/m_\perp$ from the last line. At finite $\beta \sim \beta'_b$ the integral in the last line is $\sim k_a^j \sim m_\perp$ so we need to check this integral at very small or very large β. It is convenient to make change of variables $p_\perp = k_\perp\sqrt{b}$:

$$\int_0^\infty d\beta \int đp_\perp \; e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- + i(\beta+\beta_b)\varrho x_{12}^+ - i(p+k_b,x_{12})_\perp} \frac{(\beta+\beta_b)(p+k_b)^j}{p_\perp^2(\beta+\beta_b) - (p+k_b)_\perp^2\beta} \tag{E.9}$$
$$= \int_0^\infty d\beta \int đk_\perp \; e^{i\frac{k_\perp^2}{s}\varrho x_{12}^- + i(\beta+\beta_b)\varrho x_{12}^+ - i(k\sqrt{\beta}+k_b,x_{12})_\perp} \frac{(\beta+\beta_b)(k\sqrt{\beta}+k_b)^j}{k_\perp^2(\beta+\beta_b) - (k\sqrt{\beta}+k_b)_\perp^2}$$

As $\beta \to 0$ we get

$$e^{i\beta_b\varrho x_{12}^+ - i(k_b,x_{12})_\perp} \int_0^\infty d\beta \; e^{i\beta\varrho x_{12}^-} \int đk_\perp \; \frac{k_b^j e^{i\frac{k_\perp^2}{s}\varrho x_{12}^-}}{k_\perp^2 - \frac{k_{b_\perp}^2}{\beta_b}} \sim \frac{\beta_b k_b^j}{\beta_b \varrho x_{12}^+} \ln\frac{Q_{ab}^2}{k_{b_\perp}^2 \alpha'_a \varrho x_{12}^-} \sim \beta_b m_\perp \tag{E.10}$$

since $\beta_b\varrho x_{12}^+ \sim \alpha_a\varrho x_{12}^- \sim 1$. Conversely, as $\beta \to \infty$ one obtains

$$\int_0^\infty d\beta \int đp_\perp \; e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- + i\beta\varrho x_{12}^+ - i(p+k_b,x_{12})_\perp} \frac{(p+k_b)^j}{k_{b_\perp}^2 + 2(p,k_b)_\perp} \tag{E.11}$$
$$= 2e^{-i(k_b,x_{12})_\perp} \int đp_\perp \sqrt{\frac{p_\perp^2 x_{12}^-}{s x_{12}^+}} K_1\Big(p_\perp\sqrt{-x_{12_\parallel}^2}\Big) \frac{(p+k_b)^j e^{-i(p,x_{12})_\perp}}{k_{b_\perp}^2 + 2(p,k_b)_\perp} \sim \frac{m_\perp}{\rho x_{12}^+} \sim \beta_b m_\perp$$

Thus, we got $m_\perp$ instead of an extra $s/m_\perp$ needed to compensate the smallness in eq. (E.8) so the contribution of the diagram in figure 14c is a power correction $O(\frac{m_\perp^2}{s})$ in comparison to the leading term (6.7).

Summarizing, we demonstrated that the diagrams with correction fields (4.9) are power corrections $\sim O(\frac{1}{\zeta})$.

F Necessary integrals

F.1 Integrals for virtual diagrams

Master integral for virtual diagrams can be taken from integrals (11) - (18) of ref. [45]. At $k_1^2, k_2^2 < 0$ and $(k_1+k_2)^2 > 0$ it reads

$$\int \frac{đ^4p}{i} \frac{1}{[(p+k_1)^2 + i\epsilon][(p-k_2)^2 + i\epsilon](p^2 + i\epsilon)} = \tag{F.1}$$
$$= \frac{1}{16\pi^2\kappa}\Bigg[\mathrm{Li}_2\Big(\frac{-k_1^2}{(k_1,k_2)+\kappa}\Big) + \mathrm{Li}_2\Big(\frac{-k_2^2}{(k_1,k_2)+\kappa}\Big) + \frac{1}{2}\ln\frac{k_2^2}{k_1^2}\ln\frac{k_2^2 + k_1\cdot k_2 + \kappa}{k_1^2 + k_1\cdot k_2 + \kappa}$$
$$+ \frac{1}{2}\ln\Big(\frac{k_1^2}{(k_1,k_2)+\kappa} + i\epsilon\Big)\ln\Big(\frac{k_2^2}{(k_1,k_2)+\kappa} + i\epsilon\Big) + \frac{\pi^2}{6}\Bigg]$$

JHEP03(2023)029

where $\kappa \equiv \sqrt{(k_1\cdot k_2)^2 - k_1^2k_2^2}$. In our kinematics $k_1\cdot k_2 \gg k_1^2, k_2^2$ so

$$\int \frac{{d\!\!\!{}^-}^4p}{i} \frac{1}{[(p+k_1)^2+i\epsilon][(p-k_2)^2+i\epsilon](p^2+i\epsilon)} \tag{F.2}$$
$$= \frac{1}{32\pi^2(k_1\cdot k_2)} \ln\frac{-2(k_1\cdot k_2)-i\epsilon}{-k_1^2} \ln\frac{-2(k_1\cdot k_2)-i\epsilon}{-k_2^2} + O\Big(\frac{k_1^2,k_2^2}{(k_1\cdot k_2)}\Big)$$

We will also need similar integral with cut propagators. The standard calculation yields

$$\int {d\!\!\!{}^-}^4p\ \tilde\delta(p+k_1)^2(p+k_1)_0 \frac{1}{p^2}\tilde\delta(p-k_2)^2\theta(k_2-p)_0 \tag{F.3}$$
$$= -\frac{\theta(k_1+k_2)^2\theta(k_1+k_2)_0}{32\pi\kappa} \ln\frac{k_1\cdot k_2+\kappa}{k_1\cdot k_2-\kappa} \simeq -\frac{\theta(k_1+k_2)^2\theta(k_1+k_2)_0}{32\pi k_1\cdot k_2} \ln\frac{4(k_1\cdot k_2)^2}{k_1^2k_2^2}$$

in agreement with eq. (F.2). Note that at $k_1^2, k_2^2 < 0$ the denominator $1/p^2$ in the l.h.s. of this equation is not singular.

Using eqs. (F.2) and (F.3) it is easy to obtain

$$\int {d\!\!\!{}^-}^4p \frac{\alpha_a}{(p+k_a)^2+i\epsilon} \frac{s}{p^2+i\epsilon} \frac{\beta_b}{(p-k_b)^2+i\epsilon} = \frac{i}{16\pi^2}\Big[\ln\frac{-\alpha_a\beta_b s-i\epsilon}{k_{a_\perp}^2} \ln\frac{-\alpha_a\beta_b s-i\epsilon}{k_{b_\perp}^2} + \frac{\pi^2}{3}\Big]$$
$$\int {d\!\!\!{}^-}^4p\ \tilde\delta(p+k_a)^2\theta(-\beta) \frac{\alpha_a\beta_b s}{p^2-i\epsilon}\tilde\delta(p-k_b)^2\theta(\alpha) = -\frac{\theta(-\alpha_a)\theta(-\beta_b)}{8\pi} \ln\frac{(\alpha_a\beta_b s)^2}{k_{a_\perp}^2 k_{b_\perp}^2} \tag{F.4}$$

To compare to the calculation of the "production" diagrams in section 6, we need also an explicit calculation of the sum of the "virtual" integrals in eq. (5.20). Performing the integration over α, we get

$$16\pi^2 \int \frac{{d\!\!\!{}^-}^4p}{i} \Big\{ \frac{\alpha_a\beta_b s}{[(\alpha+\alpha_a)\beta s-(p+k_a)_\perp^2+i\epsilon](\alpha\beta s-p_\perp^2+i\epsilon)[\alpha(\beta-\beta_b)s-(p-k_b)_\perp^2+i\epsilon]}$$
$$+\tilde\delta[(\alpha_a+\alpha)\beta s-(p+k_a)^2]\theta(-\beta) \frac{\alpha_a\beta_b s}{\alpha\beta s-p_\perp^2-i\epsilon} \tilde\delta[\alpha(\beta-\beta_b)s-(p-k_b)_\perp^2]\theta(\alpha)\Big\}$$
$$= -4\pi \int {d\!\!\!{}^-}p_\perp \int_0^1 du \frac{\bar u Q_{ab}^2}{[\bar u p_\perp^2+u(p-k_b)^2][\bar u(p+k_a)_\perp^2+u(p-k_b)_\perp^2-Q_{ab}^2\bar u u]}$$
$$= \ln\frac{-Q_{ab}^2}{k_{a_\perp}^2} \ln\frac{-Q_{ab}^2}{k_{b_\perp}^2} + \frac{\pi^2}{3} + O(\lambda) \tag{F.5}$$

where Q_{ab}^2 is defined in eq. (5.15). To get the last line in the above equation, we performed the integration over $p_\perp$

$$\text{Eq. (F.5)} = \int_0^1 dudv \frac{-Q_{ab}^2}{k_{a_\perp}^2 v(1-\bar u v)+k_{b_\perp}^2 u-(Q_{ab}^2-2(k_a,k_b)_\perp)uv}$$
$$= \int_0^1 dudv \frac{1}{av(1-\bar u v)+bu+uv} + O(a,b)$$
$$= \int_0^1 dudv \Big[\frac{1}{av+bu+uv} + \frac{a\bar u v^2}{[av+bu+uv][av(1-\bar u v)+bu+uv]}\Big]$$
$$= \int_0^1 dudv \Big[\frac{1}{av+bu+uv} + \frac{a\bar u v^2}{[av+bu+uv][av(1-\bar u v)+bu+uv]}\Big] + O(a,b)$$
$$= \int_0^1 dudv \Big[\frac{1}{av+bu+uv} + \frac{a}{(a+u)(a\bar v+u)}\Big] + O(a,b) = \ln a \ln b + \frac{\pi^2}{3} + O(a,b)$$

where $a = -k_{a_\perp}^2/Q_{ab}^2$ and $b = -k_{b_\perp}^2/Q_{ab}^2$.

F.2 Integrals for "production" diagrams

To calculate the integral (6.21) we will represent it as follows

$$\begin{aligned}
&4\pi\int đp_\perp \frac{e^{-i(p,x)_\perp}Q_{ab}^2}{Q_{ab}^2p_\perp^2+(p+k_a)_\perp^2(p-k_b)_\perp^2}\ln\frac{-Q_{ab}^2p_\perp^2}{(p+k_a)_\perp^2(p-k_b)_\perp^2}\\
&=4\pi\int đp_\perp \frac{e^{-i(p,x)_\perp}Q_{ab}^2}{Q_{ab}^2p_\perp^2+k_{a_\perp}^2k_{b_\perp}^2}\ln\frac{-Q_{ab}^2p_\perp^2}{(p+k_a)_\perp^2(p-k_b)_\perp^2}\\
&\quad+\frac{4\pi}{Q_{ab}^2}\int đp_\perp \frac{e^{-i(p,x)_\perp}[k_{a_\perp}^2k_{b_\perp}^2-(p+k_a)_\perp^2(p-k_b)_\perp^2]}{\left[p_\perp^2+\frac{k_{a_\perp}^2k_{b_\perp}^2}{Q_{ab}^2}\right]\left[p_\perp^2+\frac{(p+k_a)_\perp^2(p-k_b)_\perp^2}{Q_{ab}^2}\right]}\ln\frac{-Q_{ab}^2p_\perp^2}{(p+k_a)_\perp^2(p-k_b)_\perp^2}\\
&=4\pi\int đp_\perp \frac{e^{-i(p,x)_\perp}Q_{ab}^2}{Q_{ab}^2p_\perp^2+k_{a_\perp}^2k_{b_\perp}^2}\ln\frac{-Q_{ab}^2p_\perp^2}{(p+k_a)_\perp^2(p-k_b)_\perp^2}+O(\lambda)
\end{aligned}\tag{F.6}$$

To get the last line we note that the integral in the third line is at best logarithmic at small $p_\perp$. Next, we split the integral in the last line in three parts

$$8\pi\int đp_\perp \frac{e^{-i(p,x)_\perp}}{p_\perp^2+\frac{k_{a_\perp}^2k_{b_\perp}^2}{Q_{ab}^2}}\left[\ln\frac{-Q_{ab}^2p_\perp^2}{k_{a_\perp}^2k_{b_\perp}^2}-\ln\frac{(p+k_a)_\perp^2}{k_{a_\perp}^2}-\ln\frac{(p-k_b)_\perp^2}{k_{b_\perp}^2}\right],\tag{F.7}$$

use integrals $\left(m^2=-\frac{k_{a_\perp}^2k_{b_\perp}^2}{Q_{ab}^2}\right)$

$$\begin{aligned}
4\pi\int đ^2p\frac{e^{-i(p,x)}}{p^2-m^2}\ln\frac{p^2}{m^2}&=\frac{1}{2}\left(\ln\frac{m^2x^2}{4}+2\gamma\right)^2+\frac{\pi^2}{3}+O(m^2x^2),\\
4\pi\int đ^2p\frac{e^{-i(p,x)}}{p^2}\ln\frac{(p+k)^2}{k^2}&=\frac{1}{2}\left(\ln\frac{k^2x^2}{4}+2\gamma\right)^2-I_K(k,x)
\end{aligned}\tag{F.8}$$

where I_K is defined in eq. (6.26), and get

$$\begin{aligned}
&4\pi\int đp_\perp \frac{e^{-i(p,x)_\perp}Q_{ab}^2}{Q_{ab}^2p_\perp^2+(p+k_a)_\perp^2(p-k_b)_\perp^2}\ln\frac{-Q_{ab}^2p_\perp^2}{(p+k_a)_\perp^2(p-k_b)_\perp^2}\\
&=\ln\frac{-Q_{ab}^2}{k_{a_\perp}^2}\ln\frac{-Q_{ab}^2}{k_{b_\perp}^2}-\frac{1}{2}\left(\ln\frac{-Q_{ab}^2x_\perp^2}{4}+2\gamma\right)^2+\frac{\pi^2}{3}+I_K(k_{a_\perp},x_\perp)+I_K(-k_{b_\perp},x_\perp)+O(\lambda)
\end{aligned}\tag{F.9}$$

For the calculation of the integral (B.3) we need also

$$4\pi\int\frac{đ^2p}{p^2}(e^{-ipx}-1)\ln\frac{(p+q)_\perp^2}{p^2}=\left\{4\pi\int đ^2p(e^{-ipx}-1)\frac{1}{(p^2)^{1+\delta}[(q+p)^2]^{-\delta}}\right\}_{(\delta)}\tag{F.10}$$

where $\{\dots\}_{(\delta)}$ denotes the first term of the expansion in power of δ. After some algebra, one obtains

$$
\begin{aligned}
&4\pi\int\frac{d\!\!{}^-{}^2p}{p^2}(e^{-ipx}-1)\ln\frac{(p+q)_\perp^2}{p^2}\\
&=\Big\{4\pi\int d\!\!{}^-p(e^{-ipx}-1)\frac{1}{(p^2)^{1+\delta}[(q+p)^2]^{-\delta}}\Big\}_{(\delta)}\quad=2\int_0^1\frac{du}{u}\,(1-e^{i(qx)u})K_0(qx\sqrt{\bar{u}u})\\
&\quad+\Big[\frac{\delta}{\Gamma(1-\delta)\Gamma(1+\delta)}\int_0^1 du\ u^{-\delta-1}\bar{u}^\delta\{-(\ln q^2x^2\bar{u}u-\ln 4+2\gamma)-2K_0(qx\sqrt{\bar{u}u})\}\Big]_\delta\\
&=2\int_0^1\frac{du}{u}\,(1-e^{i(qx)u})K_0(qx\sqrt{\bar{u}u})-\int_0^1\frac{du}{u}\Big[2K_0(qx\sqrt{\bar{u}u})-\Big(\ln\frac{q^2x^2\bar{u}u}{4}+2\gamma\Big)\Big]\\
&=-\int_0^1\frac{du}{u}\Big[\ln\frac{q^2x^2\bar{u}u}{4}+2\gamma+2e^{i(q,x)u}K_0(qx\sqrt{\bar{u}u})\Big]=-I_K(q_\perp,x_\perp)
\end{aligned}
\tag{F.11}
$$

where I_K is defined in eq. (6.26).

G Coefficient function from the calculation with background gluons on the mass shell

In this section we double-check the calculation of the coefficient function (9.2) using background fields on the mass shell (9.3). For the background fields on the mass shell the hadronic tensor (4.20) is parametrized as follows

$$
\begin{aligned}
&\mathcal{W}(x_2,x_1)-\mathcal{W}_{\rm eik}^{\sigma_p,\sigma_t}(x_2,x_1)=\int d\!\!{}^-\alpha'_a d\!\!{}^-\beta'_b d\!\!{}^-\alpha_a d\!\!{}^-\beta_b e^{-i\alpha'_a\varrho x_2^--i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+-i\beta_b\varrho x_1^+}\\
&\times U^{+,b}_i(\alpha'_a)V^{-i,a}(\beta'_b)U^{+,b}_j(\alpha_a)V^{-j,a}(\beta_b)[I-I_{\rm eik}^{\sigma_p,\sigma_t}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_2,x_1)
\end{aligned}
\tag{G.1}
$$

As demonstrated below (see section G.4), for the massless background fields there are no soft contributions, and Glauber gluons cancel as usual.

G.1 Virtual contributions

Let us again start from the virtual diagram and take $\beta_b<0$. From eq. (8.1) we get

$$
\begin{aligned}
I^{\rm virt\ msh}_{\rm fig.5}&=-16\pi^2\int\frac{d\!\!{}^-p}{i}\Big\{\frac{\alpha_a\beta_b s(\alpha\beta s-p_\perp^2+i\epsilon)^{-1}}{[(\alpha+\alpha_a)\beta s-p_\perp^2+i\epsilon][\alpha(\beta-\beta_b)s-p_\perp^2+i\epsilon]}\\
&\quad+\tilde{\delta}[(\alpha_a+\alpha)\beta s-p^2]\theta(-\beta)\frac{\alpha_a\beta_b s}{\alpha\beta s-p_\perp^2-i\epsilon}\tilde{\delta}[\alpha(\beta-\beta_b)s-p_\perp^2]\theta(\alpha)\Big\}\\
&=-4\pi\int_0^1 du\int\frac{d\!\!{}^-p_\perp}{p_\perp^2}\frac{\bar{u}\alpha_a|\beta_b|s}{\bar{u}u\alpha_a|\beta_b|s+p_\perp^2+i\epsilon}=-\frac{1}{\varepsilon^2}\Big(\frac{(\alpha_a+i\epsilon)|\beta_b|s}{4\pi}\Big)^\varepsilon\frac{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)}{\Gamma(1+2\varepsilon)}\\
&=-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\Big(\ln\frac{(\alpha_a+i\epsilon)|\beta_b|s}{4\pi}+\gamma\Big)-\frac12\Big(\ln\frac{(\alpha_a+i\epsilon)|\beta_b|s}{4\pi}+\gamma\Big)^2+\frac{\pi^2}{12}+\frac{\pi^2}{6}+O(\varepsilon)
\end{aligned}
\tag{G.2}
$$

JHEP03(2023)029

where $\varepsilon = \frac{d_\perp}{2} - 1 = \frac{d}{2} - 2$. It is easy to see that for $\beta_b > 0$ the singularity is changed to $\ln \frac{-(\alpha_a+i\epsilon)\beta_b s}{4\pi}$. Adding the similar contribution of figure 6 diagram, we obtain

$$
\begin{aligned}
&I^{\rm virt}_{\rm mass\ shell} = \qquad\qquad (G.3)\\
&= -\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon}\Big(\ln\frac{-(\alpha_a+i\epsilon)(\beta_b+i\epsilon)s}{4\pi} + \gamma\Big) - \frac{1}{2}\Big(\ln\frac{-(\alpha_a+i\epsilon)(\beta_b+i\epsilon)s}{4\pi} + \gamma\Big)^2\\
&\quad - \frac{1}{\varepsilon}\Big(\ln\frac{-(\alpha'_a+i\epsilon)(\beta'_b+i\epsilon)s}{4\pi} + \gamma\Big) - \frac{1}{2}\Big(\ln\frac{-(\alpha'_a+i\epsilon)(\beta'_b+i\epsilon)s}{4\pi} + \gamma\Big)^2 + O(\varepsilon)
\end{aligned}
$$

Let us consider now virtual eikonals. From eq. (B.3) we get

$$
\begin{aligned}
&I^{\rm eik}_{\rm fig.11c,d}(\beta_b,k_b)\Big|_{k_{b_\perp}=0}\\
&= -8\pi^2\int_0^\infty d\!\!{}^-\alpha\ e^{-i\frac{\alpha}{\sigma_t}}\int\frac{d\!\!{}^-p_\perp}{p_\perp^2}\frac{\beta_b s}{\alpha(\beta_b+i\epsilon)s+p_\perp^2} = -\frac{2}{\varepsilon^2}\Big(\frac{-i\beta_b\sigma_t s}{4\pi}\Big)^\varepsilon\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\\
&= -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}[\ln(-i\beta_b\sigma_t s) - \ln 4\pi] - \frac{1}{2}[\ln(-i\beta_b\sigma_t s) - \ln 4\pi]^2 - \frac{\pi^2}{6} + O(\varepsilon) \qquad (G.4)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I^{\rm eik}_{\rm fig.11g,h}(\beta'_b,k'_b)\Big|_{k_{b_\perp}=0} &= -8\pi^2\int_0^\infty d\!\!{}^-\alpha\int\frac{d\!\!{}^-p_\perp}{p_\perp^2}\ e^{i\frac{\alpha}{\sigma_t}}\frac{\beta'_b s}{\alpha(\beta'_b+i\epsilon)s-p_\perp^2}\\
&= -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}[\ln(-i\beta'_b\sigma_t s) - \ln 4\pi] - \frac{1}{2}[\ln(-i\beta'_b\sigma_t s) - \ln 4\pi]^2 - \frac{\pi^2}{6} + O(\varepsilon)
\end{aligned}
$$

where $-i\beta' \equiv -i\beta' + \epsilon$ etc.

Next, we can get contributions of virtual diagrams in figure 11c,d and figure 11g,h by usual projectile↔target replacements (6.28) so the final result for virtual eikonal TMD contributions on the mass shell reads

$$
\begin{aligned}
I^{\rm virt\ eik}_{\rm mass\ shell} &= I^{\rm eik}_{\rm fig.11c,d}(\beta_b) + I^{\rm eik}_{\rm fig.11g,h}(\beta'_b) + I^{\rm eik}_{\rm fig.12c,d}(\alpha_a) + I^{\rm eik}_{\rm fig.12g,h}(\alpha'_a)\\
&= -\frac{4}{\varepsilon^2} - \frac{1}{\varepsilon}\Big[\ln\frac{-i\alpha'_a\sigma_p s}{4\pi} + \ln\frac{-i\alpha_a\sigma_p s}{4\pi} + \ln\frac{-i\beta'_b\sigma_t s}{4\pi} + \ln\frac{-i\beta_b\sigma_t s}{4\pi}\Big] \qquad (G.5)\\
&\quad - \frac{1}{2}\Big[\ln^2\frac{-i\alpha'_a\sigma_p s}{4\pi} + \ln^2\frac{-i\alpha_a\sigma_p s}{4\pi} + \ln^2\frac{-i\beta'_b\sigma_t s}{4\pi} + \ln^2\frac{-i\beta_b\sigma_t s}{4\pi}\Big] - \frac{2\pi^2}{3} + O(\varepsilon)
\end{aligned}
$$

G.2 Production terms minus TMD matrix elements

Next we will calculate the difference $J_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp}=0,x_1,x_2)$ from eq. (6.15) at $k'_{a_\perp}=0$ and $k_{b_\perp}=0$. To avoid confusing singularities, we keep $x_{12}^{\parallel}\neq 0$ in this calculation.

$$
\begin{aligned}
&J_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)\Big|_{k'_{a_\perp}=k_{b_\perp}=0} = \Big(I_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)\\
&-I^{\rm eik}_{\rm fig.12e,f}(\alpha'_a,k'_{a_\perp},x_1,x_2)-I^{\rm eik}_{\rm fig.11a,b}(\beta_b,k_{b_\perp},x_1,x_2)\Big)_{k'_{a_\perp}=k'_{b_\perp}=0}\\
&=8\pi^2\!\int\! đp_\perp\ e^{-i(p,x_{12})_\perp}\Big\{\int đ\beta\Big[\frac{\theta(\beta)}{\beta^2 s}\frac{(\beta_b-\beta)(\alpha'_a\beta s+p_\perp^2)}{(\alpha'_a+i\epsilon)(\beta_b+i\epsilon)p_\perp^2}e^{i\beta\varrho x_{12}^+ +i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}\\
&+\frac{\theta(\beta)}{\beta^2 s}\frac{(\beta_b-\beta)p_\perp^2}{(\alpha'_a+i\epsilon)[(\alpha'_a+i\epsilon)(\beta_b-\beta)\beta s-\beta_b p_\perp^2]}e^{i\beta\varrho x_{12}^+ -i\alpha'_a\varrho x_{12}^- +i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}\\
&+\frac{\theta(\beta-\beta_b)(\beta-\beta_b)[\alpha'_a(\beta-\beta_b)s+p_\perp^2]}{p_\perp^2[(\alpha'_a+i\epsilon)\beta(\beta-\beta_b)s+\beta_b p_\perp^2](\beta_b+i\epsilon)}e^{i\beta\varrho x_{12}^+ +i\frac{p_\perp^2}{(\beta-\beta_b)s}\varrho x_{12}^-}\Big]\\
&-\frac{1}{p_\perp^2}\int_0^\infty đ\beta\ e^{-i\frac{\beta}{\sigma_p}}\frac{e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}}{\beta^2(\alpha'_a+i\epsilon)s}\Big[\alpha'_a\beta s+p_\perp^2+\frac{p_\perp^4 e^{-i\alpha'_a\varrho x_{12}^-}}{\alpha'_a\beta s-p_\perp^2+i\epsilon}\Big]\\
&-\frac{1}{p_\perp^2}\int_0^\infty đ\alpha\ e^{i\frac{\alpha}{\sigma_t}}\frac{e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}}{\alpha^2(\beta_b+i\epsilon)s}\Big[\alpha\beta_b s-p_\perp^2+\frac{p_\perp^4 e^{i\beta_b\varrho x_{12}^+}}{\alpha\beta_b s+p_\perp^2+i\epsilon}\Big]\Big\}
\end{aligned}\tag{G.6}
$$

It is convenient to rearrange terms in the r.h.s. as follows

$$
J_1(\alpha'_a,k'_{a_\perp},\beta_b,k_{b_\perp},x_1,x_2)\Big|_{k'_{a_\perp}=k_{b_\perp}=0}=J_1^{(1)}+J_1^{(2)}+J_1^{(3)}+J_1^{(4)}+J_1^{(5)}\tag{G.7}
$$

where

$$
\begin{aligned}
&J_1^{(1)}(\alpha'_a,\beta_b,x_1,x_2)\qquad\qquad(\text{G.8})\\
&=\ 8\pi^2\!\int\!\frac{đp_\perp}{p_\perp^2}\ e^{-i(p,x_{12})_\perp}\Big[\int_0^\infty\frac{đ\beta}{\beta}e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}\big(e^{i\beta\varrho x_{12}^+}-e^{i\frac{\beta}{\sigma_p}}\big)-\int_0^\infty\frac{đ\alpha}{\alpha}e^{i\frac{\alpha}{\sigma_t}}e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}\Big],\\
&J_1^{(2)}(\alpha'_a,\beta_b,x_1,x_2)\qquad\qquad(\text{G.9})\\
&=8\pi^2\!\int\! đp_\perp\ e^{-i(p,x_{12})_\perp}\Big[\int_0^\infty đ\alpha\ e^{i\frac{\alpha}{\sigma_t}}\frac{e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+}}{\alpha^2 s(\beta_b+i\epsilon)}-\int_0^\infty đ\beta\ \frac{e^{i\beta\varrho x_{12}^+ +i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}}{(\beta_b+i\epsilon)p_\perp^2}\Big],\\
&J_1^{(3)}(\alpha'_a,\beta_b,x_1,x_2)\qquad\qquad(\text{G.10})\\
&=8\pi^2\!\int\! đp_\perp\ e^{-i(p,x_{12})_\perp}\int_0^\infty đ\beta\Big[\frac{e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}}{(\alpha'_a+i\epsilon)\beta^2 s}\big(e^{i\beta\varrho x_{12}^+}-e^{-i\frac{\beta}{\sigma_p}}\big)-\frac{e^{i\beta\varrho x_{12}^+ +i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}}{\beta s(\beta_b+i\epsilon)(\alpha'_a+i\epsilon)}\Big],\\
&J_1^{(4)}(\alpha'_a,\beta_b,x_1,x_2)=8\pi^2\!\int\! đp_\perp\ e^{-i(p,x_{12})_\perp}\Big[\int đ\beta\ e^{i\beta\varrho x_{12}^+ +i\frac{p_\perp^2}{(\beta-\beta_b)s}\varrho x_{12}^-}\frac{\theta(\beta-\beta_b)}{p_\perp^2(\beta_b+i\epsilon)}\\
&\times\frac{(\beta-\beta_b)[\alpha'_a(\beta-\beta_b)s+p_\perp^2]}{(\alpha'_a+i\epsilon)\beta(\beta-\beta_b)s+\beta_b p_\perp^2}-\int_0^\infty\frac{đ\alpha}{\alpha}\ e^{-i\frac{\alpha}{\sigma_t}}\frac{p_\perp^2 e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+ +i\beta_b\varrho x_{12}^+}}{(\alpha\beta_b s+i\epsilon)(\alpha\beta_b s+p_\perp^2+i\epsilon)}\Big]
\end{aligned}\tag{G.11}
$$

and

$$J_1^{(5)}(\alpha'_a,\beta_b,x_1,x_2) = \frac{8\pi^2}{\alpha'_a+i\epsilon}\int d\!\!\!{}^-p_\perp\int d\!\!\!{}^-\beta\Big[\big(e^{i\beta\varrho x_{12}^+}-e^{i\frac{\beta}{\sigma_p}}\big)\frac{p_\perp^2 e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- - i\alpha'_a\varrho x_{12}^-}}{\beta^2 s[\alpha'_a\beta s-p_\perp^2+i\epsilon]}$$
$$+\frac{p_\perp^4 e^{i\beta\varrho x_{12}^+ - i\alpha'_a\varrho x_{12}^- + i\frac{p_\perp^2}{\beta s}\varrho x_{12}^-}}{\beta s\big[(\alpha'_a+i\epsilon)(\beta_b-\beta)\beta s-\beta_b p_\perp^2\big]\big[(\alpha'_a+i\epsilon)\beta s-p_\perp^2\big]}\Big]\, e^{-i(p,x_{12})_\perp} \tag{G.12}$$

Let us start with $J_1^{(1)}$ term. After changing $\alpha=\frac{p_\perp^2}{\beta s}$ in the last term it takes the form

$$J_1^{(1)}(\alpha'_a,\beta_b,x_2,x_1) \tag{G.13}$$
$$=4\pi\int\frac{d\!\!\!{}^-p_\perp}{p_\perp^2}\,e^{-i(p,x_{12})_\perp}\int_0^\infty\frac{d\beta}{\beta}\Big[e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- + i\beta\varrho x_{12}^+} - e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- - i\beta\varrho\delta^+} - e^{i\frac{p_\perp^2}{\beta s}\varrho\delta^- + i\beta\varrho x_{12}^+}\Big]$$

Using integral

$$4\pi\int\frac{d\!\!\!{}^-p_\perp}{p_\perp^2}\,e^{-i(p,x_{12})_\perp}\int_0^\infty\frac{d\beta}{\beta}e^{i\frac{p_\perp^2}{\beta s}\varrho x_{12}^- + i\beta\varrho x_{12}^+} = (\pi x_{12_\perp}^2)^{-\epsilon}\int_0^1 du\,\frac{\Gamma(\epsilon)u^{\epsilon-1}}{\big(u+\frac{2(ix_{12}^+-\epsilon)(ix_{12}^--\epsilon)}{x_{12_\perp}^2}\big)^\epsilon}$$
$$=\frac{1}{\epsilon^2}-\frac{\ln[2\pi(ix_{12}^++\epsilon)(ix_{12}^-+\epsilon)]+\gamma}{\epsilon}+\frac{1}{2}\big[\ln(\pi x_{12_\perp}^2)+\gamma\big]$$
$$\times\left[\ln(\pi x_{12_\perp}^2)+\gamma+2\ln\frac{2\pi(ix_{12}^+-\epsilon)(ix_{12}^--\epsilon)}{x_{12_\perp}^2}\right]-\frac{\pi^2}{12}+O\left(\frac{x_{12}^+x_{12}^-}{x_{12_\perp}^2}\right)+O(\epsilon) \tag{G.14}$$

we get

$$J_1^{(1)}(\alpha'_a,\beta_b,x_2,x_1) = -\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\big[\ln 2\pi\delta^+\delta^-+\gamma\big] \tag{G.15}$$
$$-\frac{1}{2}\big[\ln(\pi x_{12_\perp}^2)+\gamma\big]\left[\ln(\pi x_{12_\perp}^2)+\gamma+2\ln\frac{2\pi\delta^+\delta^-}{x_{12_\perp}^2}\right]+\frac{\pi^2}{12}+O(\epsilon)+O(\lambda_p)+O(\lambda_t)$$

In the next section we will prove now that all $J_1^{(i)}$ except $J_1^{(1)}$ are power corrections so the result for "production - eikonal" terms is

$$J_1(\alpha'_a,\beta_b,x_2,x_1)+J_2(\alpha_a,\beta'_b,x_2,x_1) = -\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\big[\ln 2\pi\delta^+\delta^-+\gamma\big] \tag{G.16}$$
$$-\big[\ln(\pi x_{12_\perp}^2)+\gamma\big]\left[\ln(\pi x_{12_\perp}^2)+\gamma+2\ln\frac{2\pi\delta^+\delta^-}{x_{12_\perp}^2}\right]+\frac{\pi^2}{6}+O(\epsilon)+O(\lambda_p)+O(\lambda_t)$$

where we have added second term obtained by projectile↔target replacements.

JHEP03(2023)029

Now we are in a position to assemble the result for the coefficient function obtained by on-the-mass-shell calculation. We get

$$\begin{aligned}
&J_1(\alpha'_a,\beta_b,x_2,x_1)+J_2(\alpha_a,\beta'_b,x_2,x_1)+I^{\rm virt}_{\rm mass\ shell}-I^{\rm virt\ eik}_{\rm mass\ shell} \qquad\qquad (G.17)\\
&=-\frac{2}{\epsilon^2}+\frac{2}{\epsilon}\big[\ln 2\pi\delta^+\delta^-+\gamma\big]+\big[\ln(\pi x^2_{12_\perp})+\gamma\big]\Big[\ln x^2_{12_\perp}+2\ln\frac{\sigma_p\sigma_t s}{4}-3\ln\pi-\gamma\Big]+\frac{\pi^2}{6}\\
&\quad-\frac{2}{\varepsilon^2}-\frac{1}{\varepsilon}\Big(\ln\frac{-(\alpha'_a+i\epsilon)(\beta'_b+i\epsilon)s}{4\pi}+\gamma\Big)-\frac{1}{2}\Big(\ln\frac{-(\alpha'_a+i\epsilon)(\beta'_b+i\epsilon)s}{4\pi}+\gamma\Big)^2\\
&\quad-\frac{1}{\varepsilon}\Big(\ln\frac{-(\alpha_a+i\epsilon)(\beta_b+i\epsilon)s}{4\pi}+\gamma\Big)-\frac{1}{2}\Big(\ln\frac{-(\alpha_a+i\epsilon)(\beta_b+i\epsilon)s}{4\pi}+\gamma\Big)^2+\frac{\pi^2}{6}\\
&\quad+\frac{4}{\varepsilon^2}+\frac{1}{\varepsilon}\Big[\ln\frac{-i\alpha'_a\sigma_p s}{4\pi}+\ln\frac{-i\alpha_a\sigma_p s}{4\pi}+\ln\frac{-i\beta'_b\sigma_t s}{4\pi}+\ln\frac{-i\beta_b\sigma_t s}{4\pi}\Big]\\
&\quad+\frac{1}{2}\Big[\ln^2\frac{-i\alpha'_a\sigma_p s}{4\pi}+\ln^2\frac{-i\alpha_a\sigma_p s}{4\pi}+\ln^2\frac{-i\beta'_b\sigma_t s}{4\pi}+\ln^2\frac{-i\beta_b\sigma_t s}{4\pi}\Big]+\frac{2\pi^2}{3}+O(\varepsilon)\\
&=\ \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}-\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p}-\ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2
\end{aligned}$$

which agrees with eq. (9.2).

G.3 $J_1^{(i>1)}$ are power corrections

Here we prove that all terms in the r.h.s. of eq. (G.7) except for $J_1^{(1)}$ are power corrections. Let us start from $J_1^{(2)}$ which can be rewritten as

$$\begin{aligned}
J_1^{(2)}&=\frac{4\pi}{\beta_b}\int {d\!\!\bar{}}p_\perp\ e^{-i(p,x_{12})_\perp}\int_0^\infty\frac{d\alpha}{\alpha^2 s}e^{i\frac{p_\perp^2}{\alpha s}\varrho x^+_{12}}\big[e^{i\frac{\alpha}{\sigma_t}}-e^{i\alpha\varrho x^-_{12}}\big] \qquad (G.18)\\
&=-\frac{i}{\varrho x^+_{12}}\int_0^\infty\frac{d\alpha}{\alpha}e^{-i\frac{x^2_{12_\perp}}{4\varrho x^+_{12}}\alpha s}\big[e^{i\alpha\varrho\delta^-}-e^{i\alpha\varrho x^-_{12}}\big]\\
&=\frac{i}{\beta_b\varrho x^+_{12}}\ln\frac{x^2_{12_\perp}-2x^-_{12}x^+_{12}}{x^2_{12_\perp}+2\delta^- x^+_{12}}=2i\frac{\alpha'_a(\varrho\delta^- -\varrho x^-_{12})}{Q^2_{a'b}x^2_{12_\perp}}\simeq O(\lambda_t)
\end{aligned}$$

Next, after the integration over $p_\perp$ the term $J_1^{(3)}$ turns to

$$\begin{aligned}
J_1^{(3)}&=\frac{i}{\varrho x^-_{12}}\int_0^\infty d\beta\Big[\frac{1}{\alpha'_a\beta}e^{-i\frac{x^2_{12_\perp}}{4\varrho x^-_{12}}\beta s}\big(e^{i\beta\varrho x^+_{12}}-e^{-i\frac{\beta}{\sigma_p}}\big)-\frac{1}{\alpha'_a\beta_b}e^{i\beta\varrho x^+_{12}-i\frac{x^2_{12_\perp}}{4\varrho x^-_{12}}\beta s}\Big]\\
&=\frac{i}{\alpha'_a\varrho x^-_{12}}\ln\frac{x^2_{12_\perp}-2x^+_{12}x^-_{12}}{x^2_{12_\perp}+2\delta^+ x^-_{12}}+\frac{4}{Q^2_{ab'}}\frac{1}{x^2_{12_\perp}-x^2_{12_\parallel}}\simeq O(\lambda_p) \qquad (G.19)
\end{aligned}$$

To calculate $J_1^{(4)}$ term, it is convenient to change variable $\beta=\beta_b+\frac{p_\perp^2}{\alpha s}$ in the first integral. For simplicity, we take $\beta_b > 0$ and get

$$J_1^{(4)} = \frac{8\pi^2}{\beta_b} \int đp_\perp \, e^{-i(p,x_{12})_\perp} \int_0^\infty đ\alpha \Big[\frac{p_\perp^2}{\alpha^2 s}(e^{i\alpha\varrho x_{12}^-} - e^{i\frac{\alpha}{\sigma_t}}) \frac{e^{i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+ + i\beta_b\varrho x_{12}^+}}{\alpha\beta_b s + p_\perp^2 + i\epsilon}$$
$$+ \frac{p_\perp^4}{\alpha s} \frac{e^{i\alpha\varrho x_{12}^- + i\frac{p_\perp^2}{\alpha s}\varrho x_{12}^+ + i\beta_b\varrho x_{12}^+}}{[\alpha\beta_b s + p_\perp^2 + i\epsilon][(\alpha_a' + \alpha)\alpha\beta_b s + \alpha_a' p_\perp^2]} \Big]$$
$$= \int_0^\infty dp_\perp^2 \; J_0(px_{12_\perp}) \int_0^\infty dt \; \Big\{ \frac{t}{t^2(t+p_\perp^2)} e^{i\frac{t+p_\perp^2}{t}\beta_b\varrho x_{12}^+}$$
$$\times \Big(e^{i\frac{t}{Q_{a'b}^2}\alpha_a'\varrho x_{12}^-} - e^{i\frac{t}{\beta_b\sigma_t s}} \Big) + \frac{p_\perp^4}{t} \frac{e^{i\frac{t}{Q_{a'b}^2}\alpha_a'\varrho x_{12}^- + i\frac{p_\perp^2}{t}\beta_b\varrho x_{12}^+ + i\beta_b\varrho x_{12}^+}}{(t+p_\perp^2)[(Q_{a'b}^2 + t)t + Q_{a'b}^2 p_\perp^2]} \Big\} \tag{G.20}$$

The first term in the r.h.s. can be represented as

$$\int_0^\infty dp_\perp^2 \; J_0(px_{12_\perp}) \int_0^\infty dt \frac{p_\perp^2}{t^2(t+p_\perp^2)} e^{i\frac{t+p_\perp^2}{t}\beta_b\varrho x_{12}^+} \Big[e^{i\frac{t}{Q_{a'b}^2}\alpha_a'\varrho x_{12}^-} - e^{i\frac{t}{Q_{a'b}^2}\alpha_a'\varrho\delta^-} \Big] \tag{G.21}$$
$$= 2\int_0^\infty \frac{dv}{v} \; J_0(v) \int_0^\infty dt \frac{1}{t^2(t+1)} e^{i\frac{t+1}{t}\beta_b\varrho x_{12}^+} \Big[e^{i\frac{t}{Q_{a'b}^2 x_{12_\perp}^2}\alpha_a'\varrho x_{12}^- v^2} - (x_{12}^- \to \delta^-) \Big]$$

We need to check two regions: $t \ll 1$ and $t \gg 1$. Assuming essential $t's$ are small, we get

$$2\int_0^\infty \frac{dv}{v} \; J_0(v) \int_0^\infty dt \frac{1}{t^2} e^{i\frac{t+1}{t}\beta_b\varrho x_{12}^+} \Big[e^{i\frac{t}{Q_{a'b}^2 x_{12_\perp}^2}\alpha_a'\varrho x_{12}^- v^2} - (x_{12}^- \to \delta^-) \Big]$$
$$= 2i\frac{e^{i\beta_b\varrho x_{12}^+}}{\beta_b\varrho x_{12}^+} \int_0^\infty dv \; J_0(v) \Big[\sqrt{\frac{-2x_{12}^+ x_{12}^-}{x_{12_\perp}^2}} K_1\Big(v\sqrt{-2x_{12}^+ x_{12}^-/x_{12_\perp}^2}\Big) - (x_{12}^- \to \delta^-) \Big]$$
$$= i\frac{e^{i\beta_b\varrho x_{12}^+}}{\beta_b\varrho x_{12}^+} \ln\frac{x_{12_\perp}^2 - x_{12}^+\delta^-}{x_{12_\perp}^2 - x_{12}^+ x_{12}^-} \simeq \frac{i\delta^-}{\beta_b\varrho x_{12_\perp}^2} = O(\lambda_t) \tag{G.22}$$

which is a power correction. However, quick look at the integral over t shows that $t \sim vQ_{a'b}x_{12_\perp}\sqrt{\frac{\alpha_a'\delta^-}{\beta_b x_{12}^+}} \sim v\frac{\sigma_t\beta_b s}{Q_\perp^2}$ so $t \ll 1$ corresponds to $v \ll \lambda_t$ which gives negligible contribution to the integral (G.22). Thus, characteristic t's in the integral (G.21) are not small. Let us assume they are large and check it *a posteriori*. At $t \gg 1$ we get

$$\text{Eq. (G.21)} = 2\int_0^\infty \frac{dv}{v} \; J_0(v) \int_0^\infty dt \frac{1}{t^3} e^{i\frac{t+1}{t}\beta_b\varrho x_{12}^+} \Big[e^{i\frac{t}{Q_{ab'}^2 x_{12_\perp}^2}\alpha_a'\varrho x_{12}^- v^2} - (x_{12}^- \to \delta^-) \Big]$$
$$= -2\frac{e^{i\beta_b\varrho x_{12}^+}}{(\beta_b\varrho x_{12}^+)^2} \int_0^\infty dv \; vJ_0(v) \Big[\Big(-\frac{2x_{12}^+ x_{12}^-}{x_{12_\perp}^2} \Big) K_2\Big(v\sqrt{-2x_{12}^+ x_{12}^-/x_{12_\perp}^2}\Big) - (x_{12}^- \to \delta^-) \Big]$$
$$= -2\frac{e^{i\beta_b\varrho x_{12}^+}}{(\beta_b\varrho x_{12}^+)^2} \Big[\ln\frac{x_{12_\perp}^2 - x_{12}^+\delta^-}{x_{12_\perp}^2 - x_{12}^+ x_{12}^-} + \frac{x_{12_\perp}^2}{x_{12_\perp}^2 - x_{12}^+ x_{12}^-} - \frac{x_{12_\perp}^2}{x_{12_\perp}^2 - x_{12}^+\delta^-} \Big]$$
$$\simeq -2\frac{e^{i\beta_b\varrho x_{12}^+}}{s{\beta_b}^2} \frac{(\delta^-)^2 - (x_{12}^-)^2}{x_{12_\perp}^4} = O(\lambda_t^2) \tag{G.23}$$

which is smaller than a typical power correction. Also, in this integral $v \sim 1$ and $t \sim vQ_{a'b}x_{12_\perp}\sqrt{\frac{\alpha'_a\delta^-}{\beta_b x^+_{12}}} \sim v\frac{\sigma_t\beta_b s}{Q^2_\perp} \sim \frac{1}{\lambda_t}$ so we justified large-t approximation.

The last term in the r.h.s. of eq. (G.20) is even smaller

$$\begin{aligned}
&\int_0^\infty dp^2_\perp\ J_0(px_{12_\perp})\int_0^\infty dv\ \frac{e^{iv\frac{p^2_\perp}{Q^2_{a'b}}\alpha'_a\varrho x^-_{12}+i\frac{1}{v}\beta_b\varrho x^+_{12}+i\beta_b\varrho x^+_{12}}}{v(v+1)[Q^2_{a'b}(v+1)+v^2p^2_\perp]}\\
&\overset{p_\perp x_{12_\perp}\sim 1}{\simeq}\quad \int_0^\infty dp^2_\perp\ J_0(px_{12_\perp})\int_0^\infty dv\ \frac{e^{i\frac{1}{v}\beta_b\varrho x^+_{12}+i\beta_b\varrho x^+_{12}}}{v(v+1)[Q^2_{a'b}(v+1)+v^2p^2_\perp]}\\
&=2\int_0^\infty dv\ \frac{v^2}{v+1}e^{i(v+1)\beta_b\varrho x^+_{12}}K_0\Big(Q_{a'b}x_{12_\perp}\sqrt{v(v+1)}\Big)\sim\frac{Q^6_\perp}{Q^6}
\end{aligned}\tag{G.24}$$

Thus, $J^{(4)}_1 = O(\lambda^2_t)$ and can be neglected.

Next, it is easy to see that $J^{(5)}_1$ of eq. (G.12) differs from the first two lines in eq. (G.20) for $J^{(4)}_1$ by projectile$\leftrightarrow$ target replacements $\alpha'_a \leftrightarrow \beta_b, \sigma_p \leftrightarrow \sigma_t, x^+_{12} \leftrightarrow x^-_{12}$ so we have $J^{(5)}_1 = O(\lambda^2_p)$.

G.4 sG-contribution

The soft-Glauber contribution to virtual diagram in figure (5) can be easily obtained from eq. (8.4) by setting ${k'}_{a_\perp} = k_{b_\perp} = 0$

$$\begin{aligned}
I^{\text{virt sG ms}}_{\text{fig.5}} &= -\frac{\mu^{2\varepsilon}_\sigma}{(4\pi)^\varepsilon}\int_0^\infty\frac{dl^2}{l^2}l^{2\varepsilon}\frac{e^{-il^2}}{1-\frac{l^2\sigma_t}{\alpha'_a}+i\epsilon}\int_0^\infty\frac{dt^2}{t^2}t^{2\varepsilon}\frac{e^{it^2}}{1-\frac{t^2\sigma_p}{|\beta'_b|}-i\epsilon}\\
&= -\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\Big(2\gamma-\ln\frac{\mu^2_\sigma}{4\pi i}\Big)-\frac{1}{2}\ln^2\frac{\mu^2_\sigma}{4\pi i}-2\gamma\ln\frac{\mu^2_\sigma}{4\pi i}-\gamma^2+O(\lambda_t,\lambda_p)
\end{aligned}\tag{G.25}$$

where we used integral

$$\int_0^\infty dv\ v^{\varepsilon-1}\frac{e^{-v}}{1+\lambda v}=\frac{1}{\varepsilon}-\gamma-\lambda+\varepsilon[\gamma\lambda+\frac{\gamma^2}{2}+\frac{\pi^2}{12}]+O(\lambda^2)+O(\varepsilon^2)\tag{G.26}$$

The expression for $I^{\text{virt sG ms}}_{\text{fig.6}}$ will be the same as seen from eq. (8.9). Moreover, it is easy to see from eqs. (8.14)–(8.15) that the sG-contribution to "production" diagrams differs in sign from virtual contribution (G.25)

$$I^{\text{sG ms}}_{\text{fig.7}} = I^{\text{sG ms}}_{\text{fig.8}} = -I^{\text{virt sG ms}}_{\text{fig.5}} = -I^{\text{virt sG ms}}_{\text{fig.6}}\tag{G.27}$$

and therefore the total sG-contribution to the sum of the diagrams with background fields on the mass shell is a power correction.

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