APPENDICES

1 Numerical illustration of patch disease extinction when $\mathfrak{R}_0 < 1$ in both patch 1 and 2

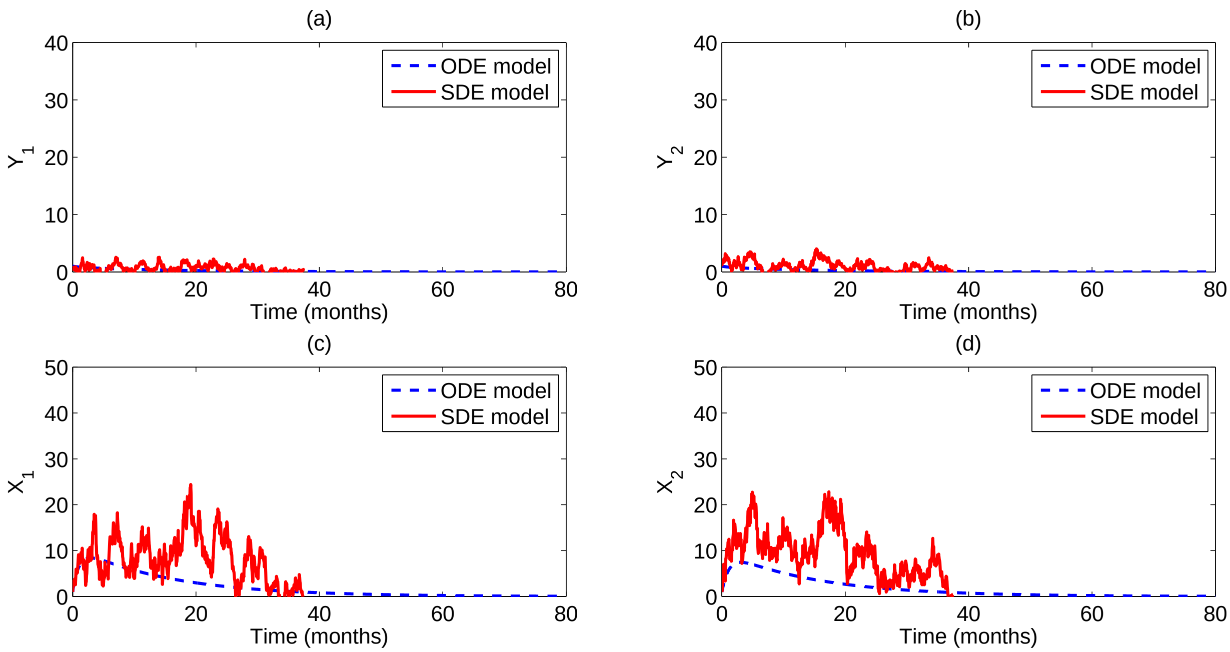

Figure 1: One sample path and ODE solution for the two-patch SDE and ODE models, respectively, for the number of (a) infectious deer in patch 1 (b) infectious deer in patch 2 (c) infectious ticks in patch 1 and (d) infectious ticks in patch 2, illustrating disease extinction in both patches when $\mathfrak{R}_0 < 1$ in both patches. Initial conditions are $Y_1(0) = 1 = Y_2(0)$ and $X_1(0) = 1 = X_2(0)$. Parameter values are given in Table 2 with $K = 120$ and $\hat{A}_1 = 0.012$, $\hat{A}_2 = 0.01$, $A_1 = 0.05$, and $A_2 = 0.04$. The basic reproduction number for patch 1 ($\mathfrak{R}_{01}$) and patch 2 ($\mathfrak{R}_{02}$) is $\mathfrak{R}_{01} = 0.8884$ and $\mathfrak{R}_{02} = 0.6799$, respectively

The migration rate from patch 1 to patch 2 and vice versa is the same, so we use m for migration rate in the following two-patch ODE and SDE models:

2 Two-patch ODE Model

The two-patch ODE model is obtained by letting $j = 1, 2$ in system (1) and is given by the following system of non-linear ODEs:

$$
\begin{aligned}
\frac{\mathrm{d}D_1}{\mathrm{d}t} &= \beta_1 N_1 - \left(\frac{\beta_1 N_1}{K_1} + b_1\right) D_1 - \hat{A}_1 \frac{D_1 X_1}{N_1} + m(D_2 - D_1), \\
\frac{\mathrm{d}Y_1}{\mathrm{d}t} &= \hat{A}_1 \frac{D_1 X_1}{N_1} - \left(\frac{\beta_1 N_1}{K_1} + b_1\right) Y_1 + m(Y_2 - Y_1), \\
\frac{\mathrm{d}S_1}{\mathrm{d}t} &= \hat{\beta}_1 V_1 - \left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) S_1 - A_1 \frac{S_1 Y_1}{N_1} + m(S_2 - S_1), \\
\frac{\mathrm{d}X_1}{\mathrm{d}t} &= A_1 \frac{S_1 Y_1}{N_1} - \left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) X_1 + m(X_2 - X_1), \\
\frac{\mathrm{d}D_2}{\mathrm{d}t} &= \beta_2 N_2 - \left(\frac{\beta_2 N_2}{K_2} + b_2\right) D_2 - \hat{A}_2 \frac{D_2 X_2}{N_2} - m(D_2 - D_1), \\
\frac{\mathrm{d}Y_2}{\mathrm{d}t} &= \hat{A}_2 \frac{D_2 X_2}{N_2} - \left(\frac{\beta_2 N_2}{K_2} + b_2\right) Y_2 - m(Y_2 - Y_1), \\
\frac{\mathrm{d}S_2}{\mathrm{d}t} &= \hat{\beta}_2 V_2 - \left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) S_2 - A_2 \frac{S_2 Y_2}{N_2} - m(S_2 - S_1), \\
\frac{\mathrm{d}X_2}{\mathrm{d}t} &= A_2 \frac{S_2 Y_2}{N_2} - \left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) X_2 - m(X_2 - X_1).
\end{aligned}
$$

3 Two-patch SDE Model

The possible changes, 8×1 expectation vector, $E[\Delta\mathcal{H}(t)]$ and 8×8 covariance matrix C can be easily determined by letting $j = 1,\, 2$ in (18), (19) and (21), respectively.

The diffusion matrix G in equation (23) is of order 8×24, which is given by

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & O \\ G_{21} & G_{22} & G_{23} & G_{24} \end{bmatrix} \tag{1}$$

where

$$G_{11} = \begin{bmatrix} \sqrt{\beta_1 N_1} & -\sqrt{\left(\frac{\beta_1 N_1}{K_1} + b_1\right) D_1} & -\sqrt{\frac{\hat{A}_1 D_1 X_1}{N_1}} & -\sqrt{m D_1} & \sqrt{m D_2} & 0 \\ 0 & 0 & \sqrt{\frac{\hat{A}_1 D_1 X_1}{N_1}} & 0 & 0 & -\sqrt{\left(\frac{\beta_1 N_1}{K_1} + b_1\right) Y_1} \\ 0 & 0 & \sqrt{\hat{\beta}_1 V_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{m Y_1} & \sqrt{m Y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\hat{\beta}_1 V_1} & -\sqrt{\left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) S_1} & -\sqrt{\frac{A_1 S_1 Y_1}{N_1}} & -\sqrt{m S_1} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{A_1 S_1 Y_1}{N_1}} & 0 \end{bmatrix},$$

$$G_{13} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{m S_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) X_1} & -\sqrt{m X_1} & \sqrt{m X_2} & 0 & 0 \end{bmatrix},$$

$$G_{21} = \begin{bmatrix} 0 & 0 & 0 & \sqrt{mD_1} & -\sqrt{mD_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{mY_1} & -\sqrt{mY_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{mS_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{\beta_2 N_2} & -\sqrt{\left(\frac{\beta_2 N_2}{K_2} + b_2\right) D_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{mS_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{mX_1} & -\sqrt{mX_2} & 0 & 0 \end{bmatrix},$$

$$G_{24} = \begin{bmatrix} -\sqrt{\frac{\hat{A}_2 D_2 X_2}{N_2}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{\hat{A}_2 D_2 X_2}{N_2}} & -\sqrt{\left(\frac{\beta_2 N_2}{K_2} + b_2\right) Y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\hat{\beta}_2 V_2} & -\sqrt{\left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) S_2} & -\sqrt{\frac{A_2 S_2 Y_2}{N_2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{A_2 S_2 Y_2}{N_2}} & -\sqrt{\left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) X_2} \end{bmatrix},$$

and O is a 4×6 zero matrix.

Note that the twenty-four column entries of matrix G represent the number of possible changes for the two-patch system as indicated in (18).

The two-patch system of Itô SDEs is explicitly given by

$$\begin{aligned}
\mathrm{d}D_1 &= \left[\beta_1 N_1 - \left(\frac{\beta_1 N_1}{K_1} + b_1\right) D_1 - \hat{A}_1 \frac{D_1 X_1}{N_1} + m(D_2 - D_1)\right] \mathrm{d}t + \sqrt{\beta_1 N_1}\mathrm{d}W_1 \\
&\quad - \sqrt{\left(\frac{\beta_1 N_1}{K_1} + b_1\right) D_1}\mathrm{d}W_2 - \sqrt{\frac{\hat{A}_1 D_1 X_1}{N_1}}\mathrm{d}W_3 - \sqrt{mD_1}\mathrm{d}W_4 + \sqrt{mD_2}\mathrm{d}W_5, \\
\mathrm{d}Y_1 &= \left[\hat{A}_1 \frac{D_1 X_1}{N_1} - \left(\frac{\beta_1 N_1}{K_1} + b_1\right) Y_1 + m(Y_2 - Y_1)\right] \mathrm{d}t + \sqrt{\frac{\hat{A}_1 D_1 X_1}{N_1}}\mathrm{d}W_3 \\
&\quad - \sqrt{\left(\frac{\beta_1 N_1}{K_1} + b_1\right) Y_1}\mathrm{d}W_6 - \sqrt{mY_1}\mathrm{d}W_7 + \sqrt{mY_2}\mathrm{d}W_8, \\
\mathrm{d}S_1 &= \left[\hat{\beta}_1 V_1 - \left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) S_1 - A_1 \frac{S_1 Y_1}{N_1} + m(S_2 - S_1)\right] \mathrm{d}t + \sqrt{\hat{\beta}_1 V_1}\mathrm{d}W_9 \\
&\quad - \sqrt{\left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) S_1}\mathrm{d}W_{10} - \sqrt{\frac{A_1 S_1 Y_1}{N_1}}\mathrm{d}W_{11} - \sqrt{mS_1}\mathrm{d}W_{12} + \sqrt{mS_2}\mathrm{d}W_{13}, \\
\mathrm{d}X_1 &= \left[A_1 \frac{S_1 Y_1}{N_1} - \left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) X_1 + m(X_2 - X_1)\right] \mathrm{d}t + \sqrt{\frac{A_1 S_1 Y_1}{N_1}}\mathrm{d}W_{11} \\
&\quad - \sqrt{\left(\frac{\hat{\beta}_1 V_1}{M_1 N_1} + \hat{b}_1\right) X_1}\mathrm{d}W_{14} - \sqrt{mX_1}\mathrm{d}W_{15} + \sqrt{mX_2}\mathrm{d}W_{16}, \\
\mathrm{d}D_2 &= \left[\beta_2 N_2 - \left(\frac{\beta_2 N_2}{K_2} + b_2\right) D_2 - \hat{A}_2 \frac{D_2 X_2}{N_2} - m(D_2 - D_1)\right] \mathrm{d}t + \sqrt{mD_1}\mathrm{d}W_4 - \sqrt{mD_2}\mathrm{d}W_5 \\
&\quad + \sqrt{\beta_2 N_2}\mathrm{d}W_{17} - \sqrt{\left(\frac{\beta_2 N_2}{K_2} + b_2\right) D_2}\mathrm{d}W_{18} - \sqrt{\frac{\hat{A}_2 D_2 X_2}{N_2}}\mathrm{d}W_{19}, \\
\mathrm{d}Y_2 &= \left[\hat{A}_2 \frac{D_2 X_2}{N_2} - \left(\frac{\beta_2 N_2}{K_2} + b_2\right) Y_2 - m(Y_2 - Y_1)\right] \mathrm{d}t + \sqrt{mY_1}\mathrm{d}W_7 - \sqrt{mY_2}\mathrm{d}W_8 \\
&\quad + \sqrt{\frac{\hat{A}_2 D_2 X_2}{N_2}}\mathrm{d}W_{19} - \sqrt{\left(\frac{\beta_2 N_2}{K_2} + b_2\right) Y_2}\mathrm{d}W_{20}, \\
\mathrm{d}S_2 &= \left[\hat{\beta}_2 V_2 - \left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) S_2 - A_2 \frac{S_2 Y_2}{N_2} - m(S_2 - S_1)\right] \mathrm{d}t + \sqrt{mS_1}\mathrm{d}W_{12} - \sqrt{mS_2}\mathrm{d}W_{13} \\
&\quad + \sqrt{\hat{\beta}_2 V_2}\mathrm{d}W_{21} - \sqrt{\left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) S_2}\mathrm{d}W_{22} - \sqrt{\frac{A_2 S_2 Y_2}{N_2}}\mathrm{d}W_{23}, \\
\mathrm{d}X_2 &= \left[A_2 \frac{S_2 Y_2}{N_2} - \left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) X_2 - m(X_2 - X_1)\right] \mathrm{d}t + \sqrt{mX_1}\mathrm{d}W_{15} - \sqrt{mX_2}\mathrm{d}W_{16} \\
&\quad + \sqrt{\frac{A_2 S_2 Y_2}{N_2}}\mathrm{d}W_{23} - \sqrt{\left(\frac{\hat{\beta}_2 V_2}{M_2 N_2} + \hat{b}_2\right) X_2}\mathrm{d}W_{24}.
\end{aligned}$$