1979

Stable Equilibrium Statistical States for Spheromaks

George Vahala

Linda L. Vahala
Old Dominion University, lvahala@odu.edu

Original Publication Citation

Vahala, G., & Vahala, L. (1979). Stable equilibrium statistical states for spheromaks. *Physics of Fluids, 22*(5), 871-873. doi: 10.1063/1.862675

Stable equilibrium statistical states for spheromaks

George Vahala and Linda Vahala

Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830
and Department of Physics, College of William and Mary, Williamsburg, Virginia 23185
(Received 30 August 1978; final manuscript received 18 October 1978)

Incompressible nondissipative magnetohydrodynamic turbulence is treated for spherical systems. From the absolute equilibrium expectation values of the fields one can investigate those initially quiescent states for which no large mean square velocity will develop. This stable state is force-free and gives rise to the Hill vortex structure for the magnetic flux surfaces.

I. INTRODUCTION

Recently, Montgomery *et al.*[1] introduced a new way of considering the problem of nonlinear incompressible magnetohydrodynamic stability. In this approach, one determines the absolute equilibrium states for initially quiescent systems and considers a state to be stable if no finite level of mean kinetic energy is predicted; otherwise, the state is said to be unstable. Here, we shall apply these techniques to finding stable equilibrium states for spherical (Spheromak[2,3]) geometry.

The basic procedure for finding the absolute equilibrium states of a plasma is to: (i) expand the velocity and magnetic fields in an appropriately chosen eigenfunction basis, (ii) truncate these infinite series expansions and consider the phase space consisting of the generalized Fourier coefficients, (iii) identify those (quadratic) invariants of the motion which remain invariant under any finite level of truncation in the series expansions (the so-called "rugged invariants"), and, (iv) construct the equilibrium Gibbs ensemble (with the rugged invariants and any externally imposed constraints in the exponent). The reader is referred to Ref. 1 and the cited references therein for a more detailed account of the numerical verifications of this statistical approach as well as the connection this model has with turbulence.

II. SPHEROMAK EQUILIBRIA

For ideal incompressible magnetohydrodynamics

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + (\nabla\times\mathbf{B})\times\mathbf{B}\,, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})\,, \tag{2}$$

$$\nabla\cdot\mathbf{B} = 0\,, \tag{3}$$

$$\nabla\cdot\mathbf{v} = 0\,, \tag{4}$$

where $\mathbf{v}$ and $\mathbf{B}$ are the velocity and magnetic fields. Since the velocity field is assumed incompressible, the pressure is a bilinear functional of $\mathbf{v}$ and $\mathbf{B}$, $p = p(\mathbf{v}, \mathbf{B})$. We assume the plasma to be enclosed by a perfectly conducting spherical shell of radius a, so that for the radial components of the fields

$$B_r = v_r = 0, \quad \text{at } r = a\,. \tag{5}$$

From the magnetohydrodynamic action principle,[4] time and gauge invariances lead to the following quadratic integrals of the motion which are the rugged invariants: total energy E:

$$E = \int d^3\mathbf{x}(\mathbf{v}^2 + \mathbf{B}^2)\,, \tag{6}$$

the magnetic helicity H_m:

$$H_m = \int d^3\mathbf{x}\,\mathbf{A}\cdot\mathbf{B}\,, \tag{7}$$

where $\mathbf{A}$ is the magnetic vector potential ($\nabla\times\mathbf{A} = \mathbf{B}$), and the cross helicity H_c

$$H_c = \int d^3\mathbf{x}\,\mathbf{v}\cdot\mathbf{B}\,. \tag{8}$$

We denote by I_c any externally imposed global constraints on the plasma, which will involve the magnetic and not the velocity field.[1]

Because the magnetic helicity is a rugged invariant, it is convenient to expand the magnetic and velocity vector potentials in terms of the eigenfunctions of the curl operator. In spherical polar coordinates (r, θ, ϕ), the eigenfunctions of

$$\nabla\times\mathbf{F} = \lambda\mathbf{F}\,, \tag{9}$$

with

$$\mathbf{F}\cdot\hat{r} = 0, \quad \text{at } r = a \tag{10}$$

are[5]

$$\begin{aligned}\mathbf{F}_{nmq} = c_{nmq}\exp(im\phi)\Bigg(&\hat{r}\left[n(n+1)P_n^m(\cos\theta)\frac{j_n(\lambda_{nq}r)}{\lambda_{nq}r}\right] + \hat{\theta}\left\{\frac{im}{\sin\theta}P_n^m(\cos\theta)j_n(\lambda_{nq}r) + \frac{dP_n^m(\cos\theta)}{d\theta}\frac{1}{\lambda_{nq}r}\frac{d}{dr}[rj_n(\lambda_{nq}r)]\right\}\\ &+\hat{\phi}\left\{-\frac{dP_n^m(\cos\theta)}{d\theta}j_n(\lambda_{nq}r) + \frac{im}{\sin\theta}P_n^m(\cos\theta)\frac{1}{\lambda_{nq}r}\frac{d}{dr}[rj_n(\lambda_{nq}r)]\right\}\Bigg)\,.\end{aligned} \tag{11}$$

P_n^m are the associated Legendre polynomials and the boundedness of $\mathbf{F}\cdot\hat{r}$ at $r=0$ requires the integer $n \geq 0$. The integer $|m| = 0, 1, \ldots, n$ and from the boundary condition, Eq. (10), $\lambda_{nq}a$ is the qth zero of the nth spherical Bessel

 0031-9171/79/050871-03$00.90

function

$$j_n(\lambda_{nq}a)=0\,, \tag{12}$$

with $n=1,2,3,\ldots$ [since for $n=0$, Eq. (11) gives $\mathbf{F}_{00q}\equiv 0$]. Finally, the c_{nmq} are normalization constants such that

$$\int d^3\mathbf{x}\,\mathbf{F}^*_{nmq}\cdot\mathbf{F}_{n'm'q'}=\delta_{nn'}\delta_{mm'}\delta_{qq'}\;. \tag{13}$$

Thus, the spectral expansions for the vector potentials are

$$\mathbf{A}(r,\theta,\phi,t)=\sum_{nmq}\xi_{nmq}(t)\mathbf{F}_{nmq}(r,\theta,\phi)\,, \tag{14}$$

$$\mathbf{A}_v(r,\theta,\phi,t)=\sum_{nmq}\eta_{nmq}(t)\mathbf{F}_{nmq}(r,\theta,\phi)\,, \tag{15}$$

where $\xi_{nmq}(t)$ and $\eta_{nmq}(t)$ are generalized Fourier coefficients.

The equilibrium statistical states to which the plasma would evolve are determined by the Gibbs' canonical ensemble

$$D_{\mathrm{eq}}=\mathrm{const}\times\exp(-\alpha E-\beta H_m-\gamma H_c-\delta I_c)\,, \tag{16}$$

where the Lagrange multipliers α, β, γ, and δ can be interpreted as inverse temperatures and are determined by the requirement that the ensemble average $\langle E\rangle$, $\langle H_m\rangle$, $\langle H_c\rangle$, and $\langle I_c\rangle$, calculated from Eq. (16), are equal to the given values for the problem at hand. It should be noted that for spherical geometry all the eigenvalues λ_{nq} are determined by the boundary condition (10), while for cylindrical geometry,[1,6] one of the eigenvalues is not determined by the boundary condition but must be determined by externally imposed constraints.

For initially quiescent systems ($\mathbf{v}=0$), the cross helicity $H_c=0$. For ensembles for which $\langle H_c\rangle=0$, one can readily show that the corresponding inverse temperature

$$\gamma=0\;. \tag{17}$$

Since $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{v}=\nabla\times\mathbf{A}_v$,

$$\mathbf{B}=\sum_{nmq}\lambda_{nq}\xi_{nmq}\mathbf{F}_{nmq}\,, \tag{18}$$

$$\mathbf{v}=\sum_{nmq}\lambda_{nq}\eta_{nmq}\mathbf{F}_{nmq}\,, \tag{19}$$

so that the rugged invariants take the following simple forms

$$E=\sum\lambda^2_{nq}(|\xi_{nmq}|^2+|\eta_{nmq}|^2)\,, \tag{20}$$

$$H_m=\sum\lambda_{nq}|\xi_{nmq}|^2\,, \tag{21}$$

or using the orthogonality condition (13).

Using Eqs. (17), (20), and (21) one can readily calculate

$$\langle H_m\rangle=\sum\frac{1}{\alpha\lambda_{nq}+\beta}[1+O(\delta)]\,, \tag{22}$$

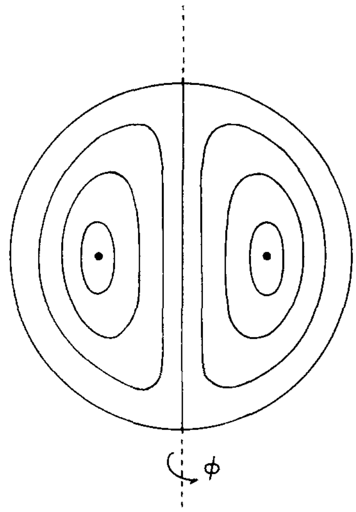

FIG. 1. The magnetic flux surfaces for the stable spherical plasma.

$$\langle E\rangle=\int d^3\mathbf{x}\langle \mathbf{v}^2\rangle+\int d^3\mathbf{x}\langle \mathbf{B}^2\rangle$$

$$=\int d^3\mathbf{x}\langle \mathbf{v}^2\rangle+\lambda_{11}\langle H_m\rangle+\sum\frac{\lambda_{nq}-\lambda_{11}}{\alpha\lambda_{nq}+\beta}\,, \tag{23}$$

where the mean kinetic energy of the plasma is given by an equipartition spectrum in the n,m,q modes

$$\int d^3\mathbf{x}\langle \mathbf{v}^2\rangle=\sum\frac{1}{\alpha} \tag{24}$$

$\langle I_c\rangle=0(\delta)$ with the symbol $O(\delta)$ standing for those terms introduced by the imposition of the external global constraint.

Equation (24) implies that for nearly all choices of $\langle E\rangle$ and $\langle H_m\rangle$, initially quiescent magnetohydrodynamic profiles are unstable. However, for a special choice of $\langle E\rangle$ and $\langle H_m\rangle$, there is a stable limit in which the mean kinetic energy of the equilibrium statistical state is zero. In this limit $\alpha\to+\infty$, $\beta\to-\infty$ with $\alpha/\beta\to-1/\lambda_{11}$, so that $\alpha\lambda_{11}+\beta\to$ constant while $\alpha\lambda_{nq}+\beta\to+\infty$ for all other (n,q) values since λ_{11} is the minimum eigenvalue. This stable state yields a maximum for the ratio of magnetic helicity to energy

$$\langle H_m\rangle/\langle E\rangle=1/\lambda_{11}\,, \tag{25}$$

and has all the excitation locked in the $n=q=1$ purely

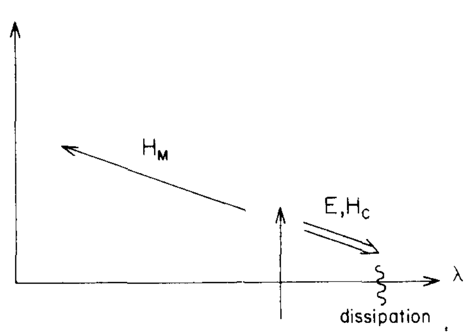

FIG. 2. Conjectured dual cascade process for a dissipative medium.

magnetic mode, which is necessarily force-free. Since the $n=q=1$ mode is degenerate in m, the corresponding magnetic field

$$\mathbf{B}=\lambda_{11}(\xi_{101}\mathbf{F}_{01}+\xi_{111}\mathbf{F}_{111}+\xi_{1,-1,1}\mathbf{F}_{1,-1,1}) \equiv \mathbf{B}_{101}+\mathbf{B}_{111}+\mathbf{B}_{1,-1,1}\,. \tag{26}$$

However, by the addition theorem for Legendre polynomials, there exists a rotation of axes which simplifies Eq. (26) to the structure of $\mathbf{B}_{101}$:

$$\mathbf{B}_{101}=\lambda_{11}\xi_{101}c_{101}\left\{2\frac{j_1(\lambda_{11}r)}{\lambda_{11}r}\cos\theta\hat{r}-\frac{1}{\lambda_{11}r}\frac{d}{dr}[rj_1(\lambda_{11}r)] \times\sin\theta\hat{\theta}+j_1(\lambda_{11}r)\sin\theta\hat{\varphi}\right\}, \tag{27}$$

which gives rise to the Hill vortex structure, Fig. 1, for its magnetic flux surfaces ($\mathbf{B}_{101}\cdot\nabla\psi=0$)

$$\psi(r,\theta)=rj_1(\lambda_{11}r)\sin^2\theta=\text{const}\,. \tag{28}$$

These flux surfaces can be considered to be the limiting case of a low-aspect-ratio, D-shaped tokamak ("spheromak").

III. DISCUSSION

The (fully) nonlinear analysis given here indicates that for initially quiescent spherical plasmas, the stable absolute equilibrium state must be (i) force-free, and (ii) have a ratio of magnetic helicity to energy which is determined by geometry

$$\langle H_m\rangle/\langle E\rangle=1/4.493a\,,$$

where a is the radius of the sphere. That is, any incompressible state with a pressure profile is necessarily unstable in the sense that the plasma will have a finite mean kinetic energy arising from an equipartition spectrum for $\langle v^2\rangle$. This result holds not only for spherical geometry, but for all geometries for which H_m and E are rugged invariants, e.g., cylindrical and toroidal geometries [the invariance of H_m follows immediately from Eqs. (2) and (3) and the boundary condition $\mathbf{B}\cdot\mathbf{n}=0$. H_m ceases to be a rugged invariant if one externally imposes the constraint of constant toroidal magnetic field]. However, in nonspherical geometry the ratio of H_m/E is determined by the external imposed constraint I_c (e.g., toroidal current) with an upper bound to I_c determined by geometry.[1,6] This result rests on the assumption that the eigenfunctions of the curl operator are complete.

Rosenbluth[2] and Bussac *et al.*[3] have employed Taylor's[7] theory (which incidently was first shown by Woltjer[8]) that decay of energy to a minimum value compatible with conserved magnetic helicity leads to a force-free state. The decay process is unspecified. In our approach, E and H_m are given constants and we search for stable (ideal) states. For completeness, we mention[1] a conjectured relaxation mechanism to explain the significance of the Taylor–Woltjer prescription: from the spectral forms for E and H_m, Eqs. (20) and (21), one can immediately conjecture that E should peak at high values of λ while H_m should peak at lower λ's giving rise to the dual cascade process illustrated in Fig. 2 for a dissipative (via resistivity and/or viscosity) medium. Since resistivity/viscosity enter the magnetohydrodynamic equations through the ∇^2 operator on the fields (which corresponds to a λ^2 behavior), dissipation should have important effects predominantly at high λ's. Hence, one can conjecture that energy would decay much faster than the magnetic helicity. (It should be pointed out that all verified dual cascade processes in fluids and plasmas have first been conjectured from the ideal, nondissipative theory and then actually demonstrated numerically from the full dissipative equations.)

ACKNOWLEDGMENTS

This work was supported at Oak Ridge National Laboratory by the Department of Energy under contract No. W-7405-eng-26 and at William and Mary under contract No. E(40-1)-5260.

[1] D. Montgomery, L. Turner, and G. Vahala, Phys. Fluids **21**, 757 (1978).
[2] M. N. Rosenbluth, Nucl. Fusion (to be published).
[3] M. N. Bussac, N. P. Furth, and M. N. Rosenbluth, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1979) (to be published).
[4] M. G. Calkin, Can. J. Phys. **41**, 2241 (1963).
[5] S. Chandrasekhar and P. C. Kendall, Astrophys. J. **126**, 457 (1957).
[6] G. Vahala, Phys. Fluids **21**, 2167 (1978).
[7] J. B. Taylor, in *Pulsed High Beta Plasmas*, edited by D. E. Evans (Pergamon, Oxford, 1976), p. 59.
[8] L. Woltjer, Proc. Natl. Acad. Sci. **44**, 489 (1958); **44**, 833 (1958).