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# Effects of quantum noise on a two-level system in a single-mode cavity

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The effects of quantum noise on a two-level system in the bad-cavity regime are considered perturbatively in the form of closure at the pair-correlation level. It is found that pair-correlation effects can reduce the level of semiclassical chaos. However, under the rotating-wave approximation (RWA), quantum noise can lead to chaos if there is an initial population inversion, while the full RWA Hamiltonian system remains integrable.

One of the most standard problems considered in quantum optics[1–6] is the interaction of a two-level quantum system with a single-mode radiation field. Here, we shall concentrate on laser fields with frequencies $\omega/2\pi$ in the optical range. Since the Pauli spin matrices $\sigma$ can describe the atomic system, the full Hamiltonian in the electric dipole approximation is given by[3]

$$H=\tfrac{1}{2}\hbar\omega_0\sigma_z+\hbar\omega(a^\dagger a+\tfrac{1}{2})+\lambda\hbar\sigma_x(a+a^\dagger)\ , \tag{1}$$

where $\hbar\omega_0$ is the atomic level spacing and $\lambda$ is the interaction strength between the two-level system and the radiation field.

For the Hamiltonian, Eq. (1), there is just one constant of the motion:[6] the spin $S^2$,

$$S^2\equiv\sigma_x^2+\sigma_y^2+\sigma_z^2\ ,$$

besides the energy $H$ and the parity operator $\exp[i\pi(a^\dagger a+\sigma_z+\frac{1}{2})]$. For chaos to be exhibited one needs to consider $N$ such isolated two-level systems.[2] However, as discussed by Fox and Eidson,[2] the effect of these $N$ systems can be handled by rescaling the coupling parameter $\lambda$ in Eq. (1).

In the Heisenberg representation, the coupled nonlinear operator equations for this Hamiltonian are

$$\frac{d\sigma_x}{dt}=-\omega_0\sigma_y,\quad \frac{d\sigma_y}{dt}=\omega_0\sigma_x-2\lambda\sigma_z(a+a^\dagger)\ ,$$
$$\frac{d\sigma_z}{dt}=2\lambda\sigma_y(a+a^\dagger),\quad \frac{d(a+a^\dagger)}{dt}=-i\omega(a-a^\dagger)\ , \tag{2}$$
$$\frac{d(a-a^\dagger)}{dt}=i\omega(a+a^\dagger)-2i\lambda\sigma_x\ .$$

By introducing expectation values

$$\langle\sigma_x\rangle\equiv x,\quad \langle\sigma_y\rangle\equiv y,\quad \langle\sigma_z\rangle\equiv z\ ,$$
$$\langle a+a^\dagger\rangle\equiv A,\quad i\langle a-a^\dagger\rangle\equiv B\ , \tag{3}$$

these equations become an infinite set of coupled nonlinear ordinary differential equations,

$$\dot{x}=-\omega_0 y,\quad y=\omega_0 x-2\lambda\langle\sigma_z(a+a^\dagger)\rangle\ ,$$
$$\dot{z}=2\lambda\langle\sigma_y(a+a^\dagger)\rangle\ , \tag{4}$$
$$\dot{A}=-\omega B,\quad \dot{B}=\omega A+2\lambda x,\ \ldots,$$

where the ellipsis represents equations for the infinite number of expectation values $\langle\sigma_z(a+a^\dagger)\rangle,\langle\sigma_y(a+a^\dagger)\rangle,\ldots$ . The system Eq. (4) is completely equivalent to the full quantum operator system, Eq. (3).

Some closure approximation must be made to turn this system, Eq. (4), into a tractable model. All previous Heisenberg studies[2–5,7] have introduced the semiclassical[8,9] factorization approximation

$$\langle\sigma_z(a+a^\dagger)\rangle\approx zA,\quad \langle\sigma_y(a+a^\dagger)\rangle\approx yA\ , \tag{5}$$

thereby achieving fifth-order closure for the expectation values $x$, $y$, $z$, $A$, and $B$. As a consequence of neglecting quantum noise[8,9] another constant of the motion is introduced into this semiclassical model,

$$\frac{\omega}{4}(A^2+B^2)+\frac{\omega_0}{2}z+\lambda Ax=\text{const}\ , \tag{6}$$

besides the spin expectation value $\langle S^2\rangle=x^2+y^2+z^2$. This reduces the semiclassical model to a third-order nonlinear system and chaos has been found for this mode.[2–5,7] This factorization approximation leads to errors[2] of order $O(n^{-1/2})$, where $n$ is the number of cavity photons (typically $10^{10}<n<10^{18}$).

In the rotating-wave approximation (RWA), where terms of the form $\exp[\pm i(\omega+\omega_0)t]$ are neglected, the Hamiltonian becomes[2]

$$H_{\text{RWA}}=\tfrac{1}{2}\hbar\omega_0\sigma_z+\hbar\omega(aa^\dagger+\tfrac{1}{2})+\hbar\lambda(\sigma_-a^\dagger+\sigma_+a)\ , \tag{7}$$

where $2\sigma_\pm=\sigma_x\pm i\sigma_y$. This is an integrable system and hence nonchaotic.[5,7] In the Heisenberg representation and under the neglect of quantum noise, Eq. (5), a third constant of the motion now exists.[2] The resulting RWA model now reduces to a second-order nonlinear system, which must necessarily be nonchaotic. But the RWA is valid provided $|\omega-\omega_0|\ll\omega_0$ and $\lambda/\omega_0\ll1$ (typically $\lambda\approx2\times10^8\ \text{s}^{-1}$ and $\omega_0\approx10^{15}\ \text{s}^{-1}$). Hence both factorization and the RWA lead to comparable errors.

Here, we examine the comparable effects of the inclusion of quantum noise on the chaotic properties of this two-level system with and without the RWA. We consider the effects of quantum noise perturbatively. In particular, we retain all pair correlations, $P$ between operators

 

$$\langle \sigma_z(a+a^\dagger)\rangle \equiv zA+P_{zA} \ ,$$
$$\langle \sigma_y\sigma_z(a+a^\dagger)\rangle \equiv yzA+yP_{zA}+zP_{yA}+AP_{yz}+T_{yzA} \ , \tag{8}$$

and achieve closure by ignoring all triple correlations $T$; i.e., in Eq. (8) we set $T_{yzA}=0$. One can now readily determine the 20th-order closed sets of equations for both the full Hamiltonian, Eq. (1) (Hamiltonian $H$):

$$\begin{aligned}
&\dot{x}=-\omega_0 y, \quad \dot{y}=\omega_0 x-2\lambda z(A+P_{zA}), \quad \dot{z}=2\lambda(yA+P_{yA}) \ ,\\
&\dot{A}=-\omega B, \quad \dot{B}=\omega A+2\lambda x \ ,\\
&\dot{P}_{xA}=-\omega_0 P_{yA}-\omega P_{xB}, \quad \dot{P}_{xB}=-\omega_0 P_{yB}+\omega P_{xA}+2\lambda P_{xx} \ ,\\
&\dot{P}_{yA}=\omega_0 P_{xA}-\omega P_{yB}-2\lambda(AP_{zA}-zP_{AA}) \ ,\\
&\dot{P}_{yB}=\omega_0 P_{xB}+\omega P_{yA}-2\lambda(AP_{zB}+zP_{AB}-P_{xy}) \ ,\\
&\dot{P}_{zA}=-\omega P_{zB}+2\lambda(yP_{AA}+AP_{yA}) \ ,\\
&\dot{P}_{zB}=\omega P_{zA}+2\lambda(yP_{AB}+AP_{yB}+P_{xz}) \ ,\\
&\dot{P}_{xx}=-2\omega_0 P_{xy}, \quad \dot{P}_{yy}=2\omega_0 P_{xy}-4\lambda(zP_{yA}+AP_{yz}) \ ,\\
&\dot{P}_{zz}=4\lambda(yP_{zA}+AP_{yz}) \ ,\\
&\dot{P}_{xy}=\omega_0(P_{xx}-P_{yy})-2\lambda(zP_{xA}+AP_{xz}) \ ,\\
&\dot{P}_{xz}=-\omega_0 P_{yz}+2\lambda(yP_{xA}+AP_{xy}) \ ,\\
&\dot{P}_{yz}=\omega_0 P_{xz}+2\lambda(yP_{yA}+AP_{yy}-zP_{zA}-AP_{zz}) \ ,\\
&\dot{P}_{AA}=-2\omega_0 P_{AB}, \quad \dot{P}_{AB}=-\omega(P_{BB}-P_{AA})+2\lambda P_{xA} \ ,\\
&\dot{P}_{BB}=2\omega P_{AB}+4\lambda P_{xB} \ ,
\end{aligned} \tag{9}$$

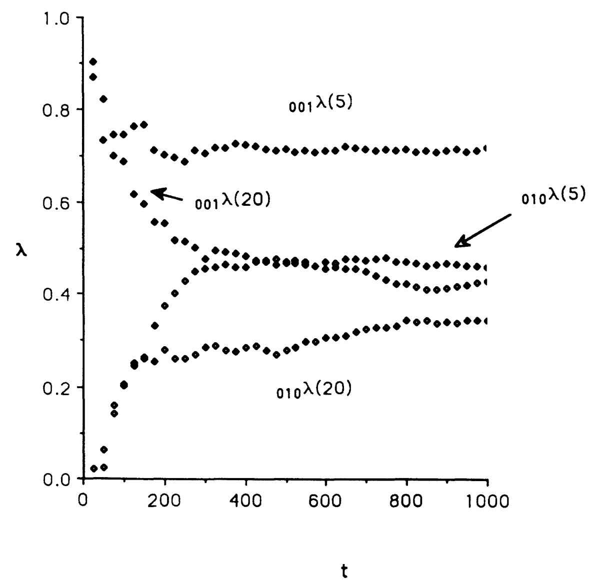


FIG. 1. Effect of pair correlations on the time evolution of the leading Liapunov exponent for the Hamiltonian system, Eq. (9). The fifth-order system results from no initial pair correlations. It is seen that the introduction of initial correlations leads to a decrease in the positive Liapunov exponent. All systems are chaotic. $\omega=0.7$ and $\lambda=1.0$. ${}_{001}\lambda(5)$ and ${}_{010}\lambda(5)$ represent initial conditions [001] and [010] but no initial pair correlations. ${}_{001}\lambda(20)$ and ${}_{010}\lambda(20)$ represent initial conditions [001] and [010] but nonzero initial pair correlations, Eq. (14).

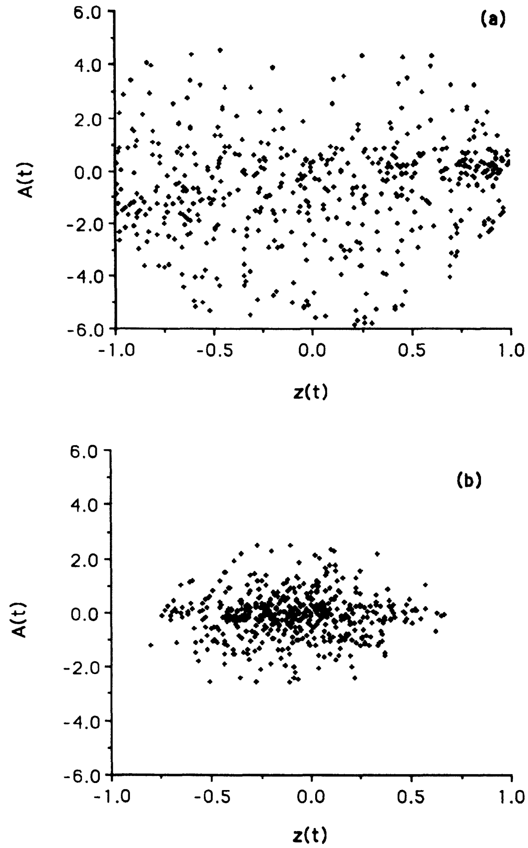


FIG. 2. Chaotic asymptotic Poincaré plots in the $z$-$A$ plane corresponding to Liapunov exponent: (a) ${}_{001}\lambda(5)$, no quantum noise; (b) ${}_{001}\lambda(20)$, quantum noise.

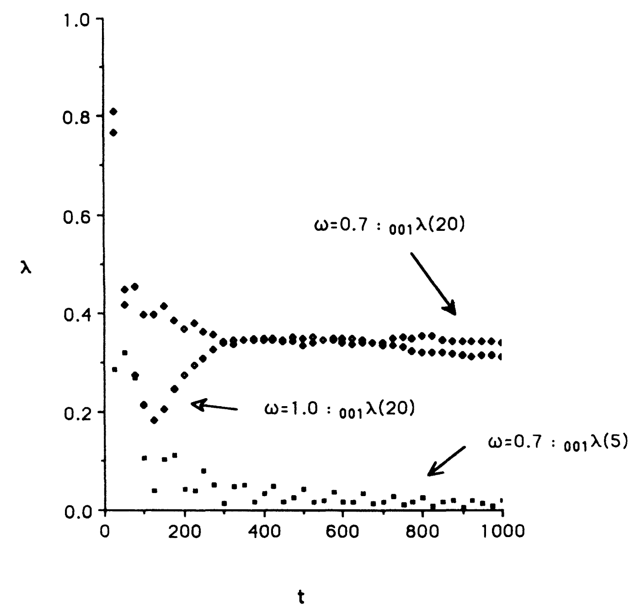


FIG. 3. Time evolution of the leading Liapunov exponent for the RWA Hamiltonian system (10) under pair-correlation closure. Under initial conditions [001], this 20th-order system is chaotic. This is shown for both $\omega=0.7$ and $\omega=1.0$: ${}_{001}\lambda(20)$. Under initial conditions [010] the system is nonchaotic ($\lambda<1.0\times10^{-4}$). Also shown is the nonchaotic case for no quantum noise: ${}_{001}\lambda(5)$.

and a similar set for the RWA Hamiltonian, Eq. (7). The constant of the motion, Eq. (6), for the semiclassical model is no longer a constant of the motion in the "bad-cavity" regime[9] with quantum noise, but the expectation value of the spin remains a constant of the motion

$$x^2+y^2+z^2+P_{xx}+P_{yy}+P_{zz}=\text{const} . \tag{10}$$

Here, we wish to concentrate directly on the effects of nonzero pair correlations and so neglect any other dissipative effects introduced in the bad-cavity limit. If initially *all* the pair correlations are zero $[P_{xA}(0)=0,\ldots,P_{BB}(0)=0]$ then they remain zero for all time and so we recover the semiclassical model.[2-5,7] Of course, this is equivalent to neglecting quantum noise and the bad-cavity regime.[9]

As mentioned earlier, the effects of $N$ noninteracting two-level systems can be taken into account[1] by rescaling $A(0)$ and $\lambda$ such that $2\lambda A(0)=-1.0\times10^{-6}$ (with time measured in units of $1/\omega_0$). In the examples presented here, we consider representative initial conditions; either an initially inverted atomic population state ($z=1$), which we shall denote as [001],

$$x(0)=0,\quad y(0)=0,\quad z(0)=1.0 ,\qquad 2\lambda A(0)=-1.0\times10^{-6},\quad B(0)=0 \tag{11}$$

or an initial state denoted by [010],

$$x(0)=0,\quad y(0)=1.0,\quad z(0)=0 ,\qquad 2\lambda A(0)=-1.0\times10^{-6},\quad B(0)=0 \tag{12}$$

with quantum noise modeled by the nonzero initial pair correlations

$$P_{xx}(0)=P_{yy}(0)=-P_{zz}/2=1.0\times10^{-9} . \tag{13}$$

With this choice of noise, $x^2(t)+y^2(t)+z^2(t)=\text{const}$ for both the semiclassical fifth-order system and the 20th-order correlation model.

In Fig. 1 we consider the effects of quantum noise on the Liapunov exponent for the Hamiltonian $H$, Eq. (9). For the initial conditions [001], Eq. (11), the semiclassical system exhibits a leading Liapunov exponent, ${}_{001}\lambda(5)$, which approaches 0.72 asymptotically: ${}_{001}\lambda(5)\rightarrow0.72$. With quantum noise of the form (13), the leading exponent for this 20th-order system ${}_{001}\lambda(20)\rightarrow0.43$. This reduction in chaotic behavior by the introduction of noise

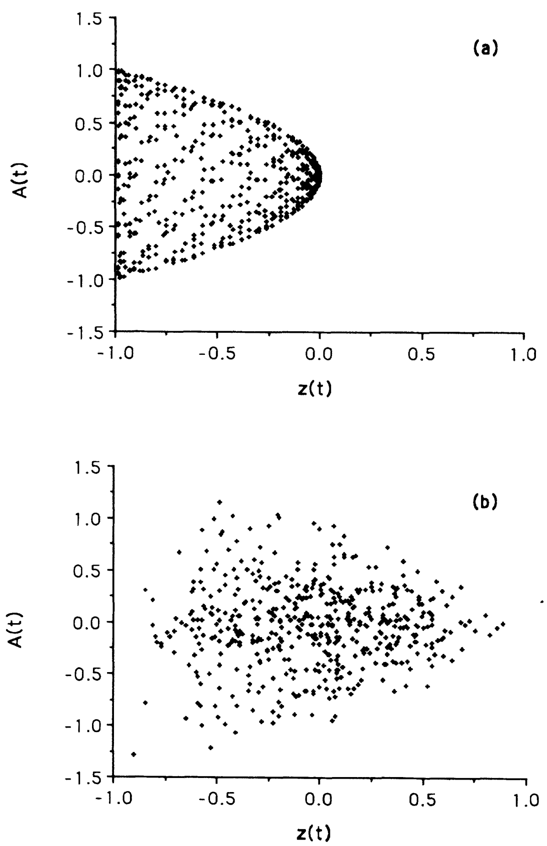


FIG. 4. Asymptotic Poincaré plots in the $z$-$A$ plane for the RWA system, Eq. (10), under correlation closure for $\omega=0.7$ for quantum noise: (a) the nonchaotic case [010], with Liapunov exponent ${}_{010}\lambda(20)<1.0\times10^{-4}$; (b) the chaotic case [001], with positive Liapunov exponent ${}_{001}\lambda(20)\rightarrow0.34$. The Poincaré plot for the integrable ${}_{010}\lambda(5)$ case is just like that of ${}_{010}\lambda(20)$ (a).

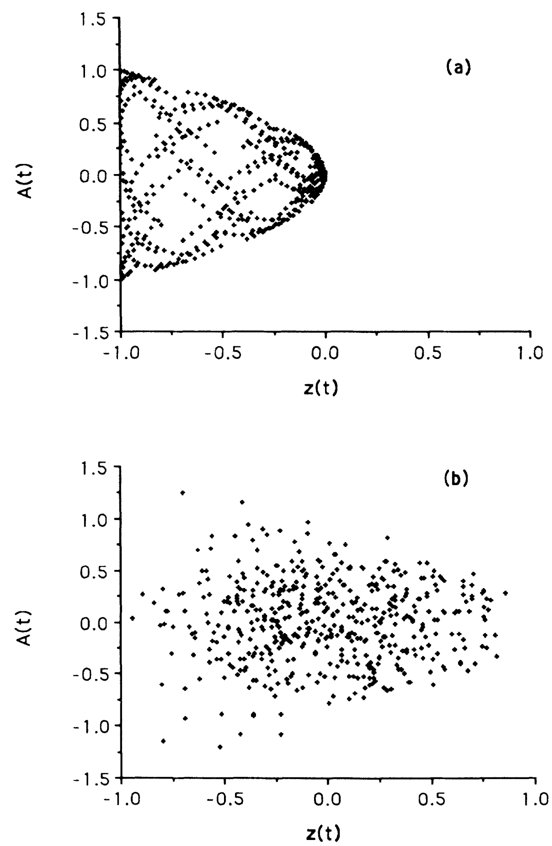


FIG. 5. Asymptotic Poincaré plots in the $z$-$A$ plane for the RWA system, Eq. (10), under correlation closure for $\omega=1.0$ for quantum noise: (a) the nonchaotic case [010], with Liapunov exponent ${}_{010}\lambda(20)<1.0\times10^{-4}$; (b) the chaotic case [001], with positive Liapunov exponent ${}_{001}\lambda(20)\rightarrow0.31$.

is seen in the corresponding Poincaré plots, Fig. 2. For initial conditions [010], Eq. (12), the semiclassical system is chaotic with ${}_{010}\lambda(5)\rightarrow 0.46$, while with quantum noise the resulting 20th-order system again has a reduced asymptotic exponent ${}_{010}\lambda(20)\rightarrow 0.34$.

The RWA Hamiltonian is nonchaotic. In the Heisenberg representation, the semiclassical system is also nonchaotic since it exhibits three constants of motion. This is also verified numerically by calculating the leading Liapunov exponent; for initial conditions [001] the Liapunov exponent ${}_{001}\lambda(5)<0.02$, while for initial conditions [010] ${}_{010}\lambda(5)<1.0\times 10^{-5}$. However, the inclusion of quantum noise under the initial conditions [001] leads to chaos. The asymptotic time development for ${}_{001}\lambda(20)$ is shown in Fig. 3 for two different values of frequency ratios: $\omega=0.7$ and $\omega=1.0$. It is interesting to note that quantum noise for the RWA system under initial conditions [010] is nonchaotic: ${}_{010}\lambda(20)<1.0\times 10^{-4}$. The corresponding Poincaré plots, shown in Figs. 4 and 5, reinforce these Liapunov exponent results.

Thus the inclusion of quantum noise to a two-level system in a single-cavity mode leads to different results, depending on whether the RWA is or is not made. Under the Hamiltonian $H$, Eq. (9), the inclusion of correlation effects reduces the chaos, but under the RWA Hamiltonian, chaos is induced provided there is an initial population inversion.

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