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Analyzing mean transport equations of turbulence and linear disturbances in decaying flows

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The decay of laminar disturbances and turbulence in mean shear-free flows is studied. In laminar flows, such disturbances are linear superpositions of modes governed by the Orr–Sommerfeld equation. In turbulent flows, disturbances are described through transport equations for representative mean quantities. The link between a description based on a deterministic evolution equation and a probability-based mean transport equation is established. Because an uncertainty in initial conditions exists in the laminar as well as the turbulent regime, a probability distribution must be defined even in the laminar case. Using this probability distribution, it is shown that the exponential decay of the linear modes in the laminar regime can be related to a power law decay of both the (ensemble) mean disturbance kinetic energy and the dissipation rate. The evolution of these mean disturbance quantities is then described by transport equations similar to those for the corresponding turbulent decaying flow.
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I. INTRODUCTION

Demands on the range of applicability of turbulence modeling are increasing, and with these increasing demands has come the need to develop transition models within the context of the traditional Reynolds-averaged turbulence modeling. Such transition models would encompass the laminar/transitional regimes, along with the turbulent regime, describing averaged flow properties such as the mean disturbance energy and dissipation rate. In order to develop a transition/turbulence model for wall-bounded flows, which uses the strategy of employing an intermittency function that interpolates between the laminar regime with its linear disturbances and the fully turbulent regime with its stochastic fluctuations, we believe a consistent mathematical description of the two regimes is necessary. This ultimate objective leads us to this initial study.

The study of homogeneous turbulence has become an essential element in the calibration of turbulence closure models. The simplest of the homogeneous flows is decaying, isotropic turbulence, which traditionally has been used to determine the destruction coefficient in the modeled dissipation rate equation used in two-equation and higher-order closure models. As a first step toward the development of a transition model, and in order to establish a commonality between the two flow regimes, homogeneous disturbance fields in the laminar regime are studied and compared to homogeneous, isotropic decaying turbulence.

In all flow regimes, disturbances are defined here as deviations from the ensemble mean. The laminar regime is defined as the region of the flow in which the ensemble mean velocity (zero in the case of the homogeneous flows considered in this study) corresponds to a stationary solution of the Navier–Stokes equation, and disturbances from this mean velocity are small enough in amplitude that their nonlinear interactions can be neglected. In this regime, the evolution of the disturbances is completely predictable from their initial state. Traditionally in laminar stability theory, disturbance fields are studied through the linear Orr–Sommerfeld equation, which describes the evolution of individual infinitesimal disturbance modes. Even when a quantity, such as the disturbance energy, is studied, it is the evolution of the instantaneous quantity rather than an ensemble average that is investigated. In contrast, the turbulent regime is defined as the region where the flow is subject to stochastic fluctuations, arising from nonlinear interactions, which render the behavior of the disturbances unpredictable. In this case the disturbance field is studied through (modeled) transport equations which describe the evolution of mean turbulent correlations. The purpose of this study is to reconcile these two apparently disparate approaches through a common mathematical analysis.

While the Orr–Sommerfeld equation accounts for viscous effects on disturbances, it neglects the influence of any nonlinear interactions. The behavior of finite-amplitude disturbances can be different and is more complex since nonlinear interactions occur. However, the intent here is to analyze, within the framework of (ensemble) mean disturbance

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transport equations, the behavior of the linear disturbance fields which characterize the beginning of a transition process, and ultimately lead to a fully turbulent field. The approach taken here is based on the observation that even in the laminar regime every flow is subject to an inevitable uncertainty in initial conditions. Therefore, although each individual disturbance evolves deterministically, a probability distribution must be introduced for the calculation of ensemble mean properties. This approach is similar to rapid distortion theory (RDT) in that it is based on linearized disturbance equations; however, the physical interpretation is different. The RDT considers flows in which turbulence is fully developed and uses linearized equations to study the behavior of the disturbances under rapid distortion. The RDT is usually applied to short time evolution and the effect of viscosity is neglected. In the approach taken here, there is no limitation on the period of time evolution as long as the disturbances remain small, and the effect of viscosity is essential.

II. BOUNDARY-FREE DISTURBANCE FIELDS

Consider first the case of disturbances in a laminar, zero mean-shear flow with no boundaries. Solid boundaries do indeed play a key role in the dynamics of any laminar disturbances and the transition process itself; however, in order to establish the mathematical framework, this first example will be unbounded, as is the decaying homogeneous turbulent flow. For this initial-value problem, an arbitrary solution of the Orr–Sommerfeld equation (here, the linearized Navier–Stokes equations) is given by

$$u_i(\mathbf{x},t)=\int d^3\mathbf{k}\ u_i^0(\mathbf{k})e^{(i\mathbf{k}\cdot\mathbf{x}-\nu k^2 t)}, \tag{1}$$

where $\mathbf{k}=(k_1,k_2,k_3)$ is the wave number vector in the coordinate directions $\mathbf{x}=(x,y,z)$, respectively, and ν is the kinematic viscosity. Because the initial conditions are uncertain, an ensemble of such disturbances is considered for which a probability distribution can be ascribed to the initial mode amplitudes, $u_i^0(\mathbf{k})$. The mean of this distribution is zero and the covariance is $\langle u_i^0(\mathbf{k})u_j^0(\mathbf{k}')\rangle$. The corresponding two-point correlation function of the disturbance field is then

$$\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle=\int d^3\mathbf{k}\ d^3\mathbf{k}'\langle u_i^0(\mathbf{k})u_j^0(\mathbf{k}')\rangle \times e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')}e^{-\nu(k^2+k'^2)t}, \tag{2}$$

where $\mathbf{x}'=\mathbf{x}+\mathbf{r}$, and the homogeneity of $\langle u_i(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle$ implies that

$$\langle u_i^0(\mathbf{k})u_j^0(\mathbf{k}')\rangle=\delta^3(\mathbf{k}+\mathbf{k}')f_{ij}(\mathbf{k}). \tag{3}$$

The two-point correlation function is then given in the form

$$R_{ij}(\mathbf{r},t)=\int d^3\mathbf{k}\ f_{ij}(\mathbf{k})e^{(i\mathbf{k}\cdot\mathbf{r}-2\nu k^2 t)}, \tag{4}$$

with the corresponding energy spectrum tensor

$$E_{ij}(\mathbf{k},t)=f_{ij}(\mathbf{k})e^{-2\nu k^2 t}, \tag{5}$$

which is assumed to be analytic at the origin.[1] This yields a wave number distribution for f_{ij} which satisfies both isotropy and incompressibility,

$$f_{ij}(\mathbf{k})=(k^2\delta_{ij}-k_ik_j)f(k^2), \tag{6}$$

where $f(k^2)$ is a nonsingular, scalar function which can be related to the mean disturbance kinetic energy through

$$K(t)=\frac{1}{2}\int d^3\mathbf{k}\ E_{ii}(\mathbf{k},t)=\int d^3\mathbf{k}\ f(k^2)k^2e^{-2\nu k^2 t}. \tag{7}$$

In general, $f(k^2)$ must go to zero sufficiently rapidly at infinity so that the integral in (7) is finite at $t=0$. A natural choice for $f(k^2)$ is the normal distribution $\exp(-ak^2)$. With this choice of $f(k^2)$, Eq. (7) then gives the mean disturbance kinetic energy

$$K(t)=K_0\left(1+\frac{2\nu}{a}t\right)^{-5/2}, \tag{8}$$

where K_0 is the initial value of K. Recall that for decaying homogeneous turbulence, the power law behavior for the final period of decay[1,2] is $K(t)\propto t^{-5/2}$. Thus, the decaying linear disturbances have the same temporal character as turbulence in the final period of decay.

The covariance of the initial mode amplitudes given in (3) can then be written explicitly as

$$\langle u_i^0(\mathbf{k})u_j^0(\mathbf{k}')\rangle=\delta^3(\mathbf{k}+\mathbf{k}')\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)K_0\mathcal{P}(\mathbf{k}), \tag{9}$$

where

$$\mathcal{P}(\mathbf{k})=\frac{2}{3}\left(\frac{a^5}{\pi^3}\right)^{1/2}k^2e^{-ak^2} \tag{10}$$

is a probability distribution for the disturbance second-moments, normalized so that $\int d^3\mathbf{k}\ \mathcal{P}(\mathbf{k})=1$, and a is related to the variance of the distribution. With this distribution $\mathcal{P}(\mathbf{k})$, other second moments can be calculated. For example, the ensemble mean energy dissipation rate

$$\varepsilon=\nu\left\langle\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle \tag{11}$$

can be written as

$$\begin{aligned}\varepsilon&=-\nu\int d^3\mathbf{k}\ d^3\mathbf{k}'e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}e^{-\nu(k^2+k'^2)t}\\&\quad\times[\langle u_i^0(\mathbf{k})u_i^0(\mathbf{k}')\rangle k_jk_j'+\langle u_i^0(\mathbf{k})u_j^0(\mathbf{k}')\rangle k_jk_i']\\&=2\nu K_0\int d^3\mathbf{k}\ k^2e^{-2\nu k^2 t}\mathcal{P}(\mathbf{k})\\&=\frac{5\nu}{a}K_0\left(1+\frac{2\nu t}{a}\right)^{-7/2},\end{aligned} \tag{12}$$

which, for this decaying flow, is simply given by the time derivative of the disturbance kinetic energy in (8), that is, $\dot{K}=-\varepsilon$.

It is well known that for any power law decay of the mean disturbance kinetic energy, with power $-p$, the mean disturbance dissipation rate equation

$$\dot{\varepsilon} = -C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{13}$$

is satisfied exactly, and the coefficient $C_{\varepsilon 2}$ is given by

$$C_{\varepsilon 2} = 1 + \frac{1}{p}. \tag{14}$$

For this case of mean, laminar shear-free flow with no boundaries, the coefficient $C_{\varepsilon 2}$ is then 1.40. In the turbulent case, values in the range of 1.80–2.00 have been deduced from experiments,[3–5] although the values more commonly used are delimited by 1.83–1.92.

In the turbulence case, there has been a considerable amount of research associated with the proper choice of decay rate.[6,7] The values obtained have ranged from a power law decay with exponent $-\frac{6}{5}$ [corresponding to a k^2 low wave number behavior of the energy spectrum $E(k) \propto k^2 E_{ii}(\mathbf{k})$] to a power law decay with exponent $-\frac{10}{7}$ (Kolmogorov decay law corresponding to a k^4 low wave number behavior). In the laminar disturbance case, the temporal and the wave number dependence of the energy spectrum tensor are both given explicitly from the disturbance mode solutions of the Orr–Sommerfeld equation together with a probability distribution. The probability distribution enforces analyticity at $\mathbf{k}=\mathbf{0}$ and goes to zero sufficiently rapidly at infinity so that the integrals exist. A simple integration over wave numbers then yields the decay law of the disturbance kinetic energy with exponent $-\frac{5}{2}$, in agreement with the final period of decay for the fully turbulent case. This agreement is not surprising, because in the final period of decay, viscous effects are dominant, with the small scale (high wave number) turbulence already decayed away.

III. WALL-BOUNDED DISTURBANCE FIELDS

While it is encouraging to have identified a relationship between the decaying linear disturbances and decaying turbulence in the boundary-free flow, a wall-bounded flow is more relevant for an analysis of linear disturbances which evolve through a transition stage and then into full turbulence. Shear-free decaying turbulent flows with boundaries have been the focus of experimental,[8,9] theoretical,[10] and numerical simulation[11] studies. The motivation for these studies was to assess the inhibiting effect on the turbulent fluctuations due to the presence of the wall in the absence of any mean shear arising from the relative motion between the mean flow and the wall. In this paper, the corresponding case of disturbances in a laminar, zero mean-shear flow with a boundary will be studied. For laminar disturbances, the wall has the effect of fixing the phase of the Orr–Sommerfeld modes in the wall-normal direction.

The wall-normal direction is the y direction, and the wall boundary is located in the (x,z) plane at $y=0$. In this flow, the modes are given by

$$\hat{u}_i(y,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{15}$$

where now $\mathbf{k}=(k_1,\ 0,\ k_3)$ is the wave number vector in the coordinate directions (x,y,z), respectively, and ω is the complex frequency of the disturbances. An analysis[12] of the Orr–Sommerfeld equation shows that bounded solutions only exist when ω is pure imaginary ($\omega = i\omega_i,\ i=\sqrt{-1}$) and

$$\omega_i = -(1+\lambda)\nu k^2 < -\nu k^2, \tag{16}$$

where the parameter $\lambda(>0)$ has been introduced. For three-dimensional disturbances, Squire's transformation is used, and the Orr–Sommerfeld equation determines the velocity disturbances $\hat{u}_2$ and $k_1\hat{u}_1 + k_3\hat{u}_3$. The individual disturbance components, $\hat{u}_1$ and $\hat{u}_3$, can then be determined in one of two ways. First, the pressure field can be determined from the linearized $\hat{u}_2$ momentum equation and then substituted into both the $\hat{u}_1$ and $\hat{u}_3$ linearized momentum equations. Second, the kinematic equation for the y component of vorticity $i(k_3\hat{u}_1 - k_1\hat{u}_3)$ can be coupled with the relation between $\hat{u}_2$ and $(k_1\hat{u}_1 + k_3\hat{u}_3)$ from the Orr–Sommerfeld equation to determine $\hat{u}_1$ and $\hat{u}_3$ individually. Because the solution for $(k_1\hat{u}_1 + k_3\hat{u}_3)$ and $\hat{u}_2$ obtained from the Orr–Sommerfeld equation is an irrotational wave of arbitrary amplitude, and the solution of the vorticity equation has an independent amplitude, it is necessary to introduce a new parameter, μ, for the ratio of these two amplitudes. The component disturbance modes are then given by

$$\hat{u}_1(y,\mathbf{k}) = i\left[\frac{k_1}{k}\cos(\sqrt{\lambda}ky) + \left(\sqrt{\lambda}\frac{k_1}{k} - \mu\frac{k_3}{k}\right)\times\sin(\sqrt{\lambda}ky) - \frac{k_1}{k}e^{-ky}\right], \tag{17}$$

$$\hat{u}_2(y,\mathbf{k}) = \left[-\cos(\sqrt{\lambda}ky) + \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}ky) + e^{-ky}\right], \tag{18}$$

$$\hat{u}_3(y,\mathbf{k}) = i\left[\frac{k_3}{k}\cos(\sqrt{\lambda}ky) + \left(\sqrt{\lambda}\frac{k_3}{k} + \mu\frac{k_1}{k}\right)\times\sin(\sqrt{\lambda}ky) - \frac{k_3}{k}e^{-ky}\right], \tag{19}$$

where $k=\sqrt{k_1^2+k_3^2}$ because $k_2=0$. At the wall where the no-slip condition is applied, $\hat{u}_i=0$, and far from the wall in the freestream, where $e^{-ky}\to 0$, the modes $\hat{u}_i$ are periodic functions of y. It should be recognized that these expressions for the velocity components, because of the no-slip wall boundary conditions, have fixed the phase of the disturbances in the y direction. In the absence of the wall boundary condition, the exponential terms would not appear and the phase would remain arbitrary in this direction.

The component disturbance modes (17)–(19) form a complete set[13] of non-normalizable eigenfunctions. A disturbance in the laminar regime is a linear combination of these eigenfunctions

$$u_i(\mathbf{x},t) = \int d\lambda\, d\mu\, d^2\mathbf{k}\,\Phi(\lambda,\mu,k_1,k_3)\hat{u}_i(y,\mathbf{k})\times e^{[i\mathbf{k}\cdot\mathbf{x} - (1+\lambda)\nu k^2 t]}, \tag{20}$$

which is normalizable in accordance with the finiteness of the disturbance kinetic energy. In Eq. (20), the initial mode amplitude $\Phi(\lambda,\mu,k_1,k_3)$ is again an element of an ensemble with mean zero and covariance $\langle \Phi^*(\lambda,\mu,k_1,k_3) \times\Phi(\lambda',\mu',k_1',k_3')\rangle$. Since the modes $\hat{u}_i$ are complex, the two-point correlation function is the real part of

$$\langle u_i^*(\mathbf{x},t)u_j(\mathbf{x}',t)\rangle=\int d\lambda\, d\mu\, d\lambda' d\mu' d^2\mathbf{k}\, d^2\mathbf{k}' \times\langle \Phi^*(\lambda,\mu,k_1,k_3)\Phi(\lambda',\mu',k_1',k_3')\rangle \times\hat{u}_i^*(y,\mathbf{k})\hat{u}_j(y',\mathbf{k}')e^{i(\mathbf{k}'\cdot\mathbf{x}'-\mathbf{k}\cdot\mathbf{x})} \times e^{-\nu[(1+\lambda)k^2+(1+\lambda')k'^2]t}, \quad (21)$$

where* quantities denote complex conjugates. Homogeneity in the x_1 and x_3 directions then implies that $\langle \Phi^*(\lambda,\mu,k_1,k_3)\Phi(\lambda',\mu',k_1',k_3')\rangle$ contains $\delta(k_1-k_1')\delta(k_3-k_3')\mathcal{P}(k_1,k_3)$ where, due to invariance under rotations about the y axis, $\mathcal{P}(k_1,k_3)$ is a function of k^2. It is still necessary to fix the dependence of the covariance $\langle \Phi^*(\lambda,\mu,k_1,k_3)\Phi(\lambda',\mu',k_1',k_3')\rangle$ on $(\lambda,\mu,\lambda',\mu')$. This is done by imposing an appropriate isotropy condition in the freestream, where the decaying exponentials can be set equal to zero in the component disturbance mode equations (17)–(19). In the boundary-free case, the disturbance field is isotropic, and the ensemble mean square of the three velocity components and of the three vorticity components are equal at each point throughout the domain. In the bounded case, the wall fixes the phases of the laminar disturbances in the y direction even for large values of y, and it is not possible to impose pointwise isotropy as a condition on an ensemble of such disturbances. However, for large values of y (freestream), a weaker form of isotropy can be imposed on the disturbance field by averaging over the y direction. Specifically, this requires that in the free stream the ensemble mean square of the three velocity components and of the three vorticity components, integrated over y, be equal. The simplifying assumption must also be made that $\langle \Phi^*(\lambda,\mu,k_1,k_3)\Phi(\lambda',\mu',k_1',k_3')\rangle$ is diagonal in λ and μ so that

$$\langle \Phi^*(\lambda,\mu,k_1,k_3)\Phi(\lambda',\mu',k_1',k_3')\rangle =\delta(\lambda'-\lambda)\,\delta(\mu'-\mu)\,\delta^2(\mathbf{k}-\mathbf{k}')\mathcal{P}(k_1,k_3)p(\lambda,\mu), \quad (22)$$

where the function $p(\lambda,\mu)$ remains to be determined from the isotropy conditions. Taking y_1 sufficiently large that $e^{-ky}\approx 0$ for $y\geqslant y_1$, one obtains from (17)–(19)

$$\lim_{L\to\infty}\frac{1}{L}\int_{y_1}^{y_1+L}dy\ \langle |u_1(\mathbf{x},t)|^2\rangle =\int dk_1\, dk_3 d\lambda\, d\mu \mathcal{P}(k_1,k_3)p(\lambda,\mu) \times\frac{1}{2}\left[(1+\lambda)\frac{k_1^2}{k^2}+\mu^2\frac{k_3^2}{k^2}\right]e^{-2(1+\lambda)\nu k^2 t}, \quad (23)$$

$$\lim_{L\to\infty}\frac{1}{L}\int_{y_1}^{y_1+L}dy\ \langle |u_2(\mathbf{x},t)|^2\rangle =\int dk_1\, dk_3 d\lambda\, d\mu \mathcal{P}(k_1,k_3)p(\lambda,\mu) \times\frac{1}{2}\left(1+\frac{1}{\lambda}\right)e^{-2(1+\lambda)\nu k^2 t}, \quad (24)$$

$$\lim_{L\to\infty}\frac{1}{L}\int_{y_1}^{y_1+L}dy\ \langle |u_3(\mathbf{x},t)|^2\rangle =\int dk_1\, dk_3 d\lambda\, d\mu \mathcal{P}(k_1,k_3)p(\lambda,\mu) \times\frac{1}{2}\left[(1+\lambda)\frac{k_3^2}{k^2}+\mu^2\frac{k_1^2}{k^2}\right]e^{-2(1+\lambda)\nu k^2 t}, \quad (25)$$

where the integration over y has eliminated the cross terms $\cos(\sqrt{\lambda}ky)\sin(\sqrt{\lambda}ky)$, while $\sin^2(\sqrt{\lambda}ky)$ and $\cos^2(\sqrt{\lambda}ky)$ have each yielded an average of $\frac{1}{2}$. The terms containing k_1k_3 have also vanished since $\mathcal{P}(k_1,k_3)$ is even in k_1 and in k_3. According to the assumed isotropy, the function $p(\lambda,\mu)$ must have the property that the right-hand sides of Eqs. (23)–(25) must be equal. The condition that, averaged over y, $\langle |u_1|^2\rangle=\langle |u_3|^2\rangle$ is satisfied automatically, while the condition $\langle |u_1|^2\rangle=\langle |u_2|^2\rangle$ yields the relation

$$\frac{1}{2}(1+\lambda+\mu^2)=1+\frac{1}{\lambda}, \quad (26)$$

where use has been made of the identity

$$\int d^2\mathbf{k}\ k_1^2F(k^2)=\int d^2\mathbf{k}\ k_3^2F(k^2)=\frac{1}{2}\int d^2\mathbf{k}\ k^2F(k^2), \quad (27)$$

when F is any function of k^2 for which the integral exists. Calculating the vorticity $\omega_j(\mathbf{x},t)$ from the modes $\hat{u}_i(y,\mathbf{k})e^{(i\mathbf{k}\cdot\mathbf{x}-(1+\lambda)\nu k^2 t)}$, with $e^{-ky}\equiv 0$, one obtains expressions analogous to (23)–(25), for the ensemble mean square vorticity components averaged over y. Isotropy again requires that these three mean square vorticity components are equal. The condition that, averaged over y, $\langle |\omega_1|^2\rangle =\langle |\omega_3|^2\rangle$ is satisfied for any function $p(\lambda,\mu)$, and the demand that $\langle |\omega_1|^2\rangle=\langle |\omega_2|^2\rangle$ yields the relation

$$\frac{1}{2}\left[(1+\lambda)^2+\lambda\mu^2+\lambda+2+\frac{1}{\lambda}\right]=\mu^2. \quad (28)$$

Solving (26) and (28) for positive λ and μ then gives

$$\lambda=\frac{1}{2}\quad\text{and}\quad \mu=\frac{3}{\sqrt{2}}, \quad (29)$$

which is enforced by taking

$$p(\lambda,\mu)=\delta\left(\lambda-\frac{1}{2}\right)\delta\left(\mu-\frac{3}{\sqrt{2}}\right). \quad (30)$$

The remaining function $\mathcal{P}(k_1,k_3)$ in the covariance (22) will be determined from a consideration of the two-point correlation function and the energy spectrum tensor.

Since the disturbance field is partially homogeneous,[14] that is, homogeneous in the x and z directions and inhomogeneous in the y direction, the two-point correlation function is

$$R_{ij}(y,y',\mathbf{r},t)=A\int d^2\mathbf{k}\ \mathcal{P}(k_1,k_3)e^{i\mathbf{k}\cdot\mathbf{r}-3\nu k^2 t}\hat{u}_i^*(y,\mathbf{k})\times\hat{u}_j(y',\mathbf{k}), \tag{31}$$

where $\mathbf{r}=(x'-x,\,0,\,z'-z)$, A is a constant proportional to the initial disturbance kinetic energy, and $\mathcal{P}(k_1,k_3)$ again plays the role of a probability distribution for the disturbance second moments. In order to obtain explicit expressions for the components of $R_{ij}(y,y',\mathbf{r},t)$, the values for the parameters λ and μ in (29) must be substituted into the component disturbance mode equations (17)–(19). In the boundary-free case, an exponential time decay with coefficient $-2\nu k^2$ is shown in (4), whereas, in this wall-bounded case, the coefficient $-3\nu k^2$ is found. This is because of the value of λ, which is zero in the boundary-free case and $\frac{1}{2}$ in this wall-bounded case.

An energy spectrum tensor can also be defined at each fixed y as a Fourier transform in x and z:

$$E_{ij}(y,k_1,k_3,t)=A\mathcal{P}(k_1,k_3)\hat{u}_i^*(y,\mathbf{k})\hat{u}_j(y,\mathbf{k})e^{-3\nu k^2 t}. \tag{32}$$

Once again, it is necessary that the energy spectrum be analytic at $\mathbf{k}=\mathbf{0}$. From (17)–(19), this analyticity requires that the distribution $\mathcal{P}(k_1,k_3)$ have a leading-order behavior of k^2. In addition, the finiteness of the initial disturbance energy implies that the spectrum should go to zero sufficiently rapidly for large k, specifically at $t=0$. These requirements lead to the normalized distribution

$$\mathcal{P}(k_1,k_3)=\frac{a^2}{\pi}k^2e^{-ak^2}, \tag{33}$$

where a is related to the variance of the probability distribution. Substituting the probability distribution (33), and the mode disturbance equations, (17)–(19), into the expression for the energy spectrum tensor (32), and integrating, it is found that in the free stream the mean disturbance energy, averaged over one y wavelength, $2\pi/\sqrt{\lambda}k=2\sqrt{2}\pi/k$, is

$$K(t)=\frac{1}{2}\int d^2\mathbf{k}\ \frac{k}{2\sqrt{2}\pi}\int dy\ E_{ii}(y,k_1,k_3,t) =K_0\left(1+\frac{3\nu}{a}t\right)^{-2}, \tag{34}$$

where K_0 is the initial mean disturbance energy, averaged over y. This then identifies A, introduced in the two-point correlation equation (31), as $A=4K_0/9$.

In the presence of a boundary, the mean disturbance energy decays according to a power law with exponent -2, in contrast to the exponent $-\frac{5}{2}$ obtained for the boundary-free case. The difference between the two cases arises from the fact that for bounded flow only two independent wave numbers are integrated in (34), while the integral in (7), for boundary-free flow, contains three independent wave numbers.

From Eq. (14) with $p=2$, it is seen that for laminar, linear decaying disturbances in the presence of a wall, the coefficient in the mean disturbance dissipation rate equation is $C_{\varepsilon 2}=1.5$. In the turbulent case, the experiment of Thomas and Hancock,[9] with a turbulent Reynolds number Re_T $(=K^2/\nu\varepsilon)$ of 2000, yielded a decay law near the wall of approximately 1, which in turn gave a value for $C_{\varepsilon 2}$ of 2. The recent direct numerical simulation of Perot and Moin,[11] with $\mathrm{Re}_T=137$, also displayed a decay law near the wall of approximately 1.

It may appear surprising that the wall affects the decay rate of linear disturbances infinitely far from the wall. The derivation of this result is based on the assumption that the initial disturbance field is an ensemble consisting of linear superpositions of modes which each satisfy the wall boundary condition $\hat{u}_i(0)=0$. This boundary condition fixes the phases of the disturbances in the direction normal to the wall up to infinity, while the phases in the directions parallel to the wall are random. The situation is different for decaying isotropic turbulence, where, in the initial stages of decay, nonlinear interactions among the modes randomize the phases in the y direction outside of a viscous sublayer next to the wall. Even when the turbulence has decayed to a stage where the nonlinear terms can be neglected, the phases will remain random far from the wall. For this reason, turbulence in the final stage of decay is unaffected by a wall outside of the viscous sublayer.

It is finally necessary to consider the condition for the validity of the linear approximation for decaying disturbances. Since there is no mean flow $\mathbf{U}$, one cannot impose the obvious condition $|\mathbf{u}|/|\mathbf{U}|\ll 1$. In the absence of mean flow, the disturbance velocity obeys the Navier–Stokes equation

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}. \tag{35}$$

The nonlinear terms can be neglected when the inertial forces are much smaller than the viscous forces, that is,

$$\left|u_j\frac{\partial u_i}{\partial x_j}\right|\ll\left|\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}\right|. \tag{36}$$

The scale for the amplitude of the disturbance velocity is set by the initial disturbance kinetic energy, $u_i\sim K_0^{1/2}$. The length scale L is determined from the probability distribution for disturbance second moments, the standard deviation fixing a wave vector scale L^{-1}. For a probability distribution of the form $k^2e^{-ak^2}$, $L\sim a^{1/2}$, and the condition (36) gives a Reynolds number criterion

$$\frac{\sqrt{aK_0}}{\nu}\ll 1. \tag{37}$$

Although the parameter a (or L^2) appearing in the probability distributions (10) and (33) does not affect the value of the destruction of dissipation coefficient $C_{\varepsilon 2}$, it does appear in the crucial inequality (37) which determines the validity of the linear approximation. This inequality can be expressed in terms of the disturbance Reynolds number

$$\mathrm{Re}_d = \frac{K_0^2}{\nu \varepsilon_0}, \tag{38}$$

where the initial disturbance dissipation rate ε_0 is proportional, in both the boundary-free and wall-bounded cases, to $\nu K_0/a$. Thus, $\mathrm{Re}_d \sim a K_0/\nu^2$, and the condition for the validity of the linear approximation for decaying disturbances can be expressed in the form $\mathrm{Re}_d \ll 1$. This inequality gives a quantitative definition of linear disturbances in the laminar regime.

IV. CONCLUSION

This work has introduced an approach to transition modeling in which deterministic solutions of the linearized Navier–Stokes equations are combined with a probability distribution that accounts for the uncertainty in initial conditions to obtain mean transport equations for disturbances in the laminar regime. This approach was applied to the study of decaying disturbances in zero mean-shear flows. The analysis has shown that the linear disturbance field in laminar flow decays at rates which differ from that of the turbulence field. In the boundary-free case, the disturbance kinetic energy decays at a much faster rate of $\frac{5}{2}$ than the corresponding turbulent kinetic energy decay rate of ≈ 1.25. This yields a destruction-of-dissipation rate coefficient $C_{\varepsilon 2}$ in the mean dissipation rate transport equation of 1.4 for the linear disturbances and ≈ 1.8 for the turbulence. In the case with a wall boundary, the disturbance kinetic energy decayed at a rate of 2, which is faster than the corresponding turbulent kinetic energy rate of 1.0. This yields a destruction-of-dissipation rate coefficient $C_{\varepsilon 2}$ of 1.5 for the linear disturbances and ≈ 2 for the turbulence. These results are summarized in Table I which clearly shows that the wall has the effect of reducing the decay rate of the disturbance energy in both the laminar and turbulent regimes.

TABLE I. Decay exponent p and model coefficients $C_{\varepsilon 2}$ for laminar and turbulent cases: Laminar—no wall, Lam—NW; Laminar—wall, Lam—W; Turbulent—no wall, Turb—NW; Turbulent—wall, Turb—W.

Case	p	$C_{\varepsilon 2}$
Lam—NW	2.5	1.4
Lam—W	2.0	1.5
Turb—NW	≈1.10	≈1.9
Turb—W	≈1.0	≈2.0

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