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Prediction of the stochastic behaviour of nonlinear systems by deterministic models as a classical time-passage probabilistic problem

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Abstract. Assuming that the behaviour of a nonlinear stochastic system can be described by a Markovian diffusion approximation and that the evolution equations can be reduced to a system of ordinary differential equations, a method for the calculation of prediction time is developed. In this approach, the prediction time depends upon the accuracy of prediction, the intensity of turbulence, the accuracy of the initial conditions, the physics contained in the mathematical model, the measurement errors, and the number of prediction variables. A numerical application to zonal channel flow illustrates the theory. Some possible generalizations of the theory are also discussed.

1 Introduction

Prediction of the characteristics of nonlinear stochastic systems is of considerable interest in modern theoretical physics. See, for example, Kravtsov (1986) for a discussion of approaches to this problem. In the last few years this topic has become very popular in hydrodynamics, plasma physics, geophysics, astrophysics, etc. As noted in the excellent review of Harms et al. (1992) climate prediction is now emerging as an extremely important issue in geophysics.

A basic problem of prediction is the estimation of the prediction time τ_{pred} . Knowledge of τ_{pred} gives the time period during which a mathematical model might be used for reliable predictions of the evolution of a real system (reality). In principle, τ_{pred} can be calculated if the cross-correlation function between the model prediction and reality is known (Kravtsov, 1986). There are essentially two ways to achieve this. One is to use archived observations as reality and construct cross-correlation functions by comparison with model predictions. In effect, τ_{pred} is assessed from hindcasts. Alternatively, the cross-correlation function can be estimated by direct comparison of model predictions or forecasts and

new observations.

Both approaches have several obvious deficiencies. First, it is difficult to obtain the functional connections between τ_{pred} and the physical factors which limit the models. Secondly, τ_{pred} is usually calculated a posteriori. However, in some cases τ_{pred} can be estimated a priori using special analytical methods that have been developed in the last years. Unfortunately, this approach has been limited to cases of the prediction of the behaviour for linear systems or nonlinear systems with the weak noise intensities (e.g., Kravtsov and Etkin, 1981) and so the application of these methods is limited.

By assuming that the evolution of a nonlinear stochastic system can be described by the Markovian diffusion approximation and that the evolution equations can be reduced to ordinary differential equations, a new method for the calculation of τ_{pred} for deterministic models is proposed. This method does not require the assumption of weak noise; moreover, it is applicable to strongly nonlinear dynamical systems.

The remainder of this paper is organized as follows. In the next section the classical approach to the prediction problem is described. In the third section this problem is reformulated. The reformulation is based on three premises. One is a finite dimensional representation in terms of dynamical systems variables. The second is the use of the Markov diffusion approximation as a description of the stochastic behavior. The third is the utilization of the analogy between the reformulated prediction problem and the classic first passage time problem. These latter two types of prediction problems are discussed in some detail. The mathematical description introduces two special functions P and W , and the formalism for their calculation is discussed. Here these functions are represented as single-layer potentials, as this is more useful in our applications than the traditional double-layer potential representation. Section 5 discusses approximate equations for the calculation of P and W . In Sect. 6 the approach is used to estimate

the prediction time for a three-mode model of flow in a zonal channel. The approach is generalized for more complex cases than the Markovian diffusion approximation in Sect. 7. Finally, in Sect. 8, the results of this investigation are summarized.

2 Classical approach to the prediction problem

Let the evolution of the characteristics of a medium be described by the following stochastic equations

$$\frac{d\hat{x}}{dt} = \hat{\mathcal{L}}(\hat{x}, t) + \hat{\mathcal{L}}_s(\hat{x}, t) \quad (1)$$

$$\hat{x} |_{t=t_0} = \hat{x}_0. \quad (2)$$

Here, \hat{x} is an m -dimensional vector that characterizes the medium, \hat{x}_0 is its initial value, and $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}_s$ are nonlinear operators describing the deterministic and the stochastic evolutions of the medium. Here \hat{x} will denote reality.

The deterministic model of the evolution of \hat{x} is expressed generally as

$$\frac{d\hat{z}}{dt} = \hat{M}(\hat{z}, t) \quad (3)$$

$$\hat{z} |_{t=t_0} = \hat{z}_0 \quad (4)$$

where \hat{M} is a nonlinear vector operator. In the classical statement of the problem for the calculation of the prediction time it is necessary to find a time, τ_{pred} , such that for any time larger than τ_{pred}

$$\alpha_p \overline{[x_p - z_p]^2} \leq \varepsilon^2 \quad p = 1, \dots, m \quad (5)$$

where ε is the prediction accuracy, $\overline{\quad}$ is the ensemble average of the initial position of the vector \hat{x} , and α_p are weighted multipliers which characterize the importance of the contribution of each component of \hat{x} to the prediction.

3 Reformulation of the prediction problem

It is known that the stochastic behaviour of nonlinear dynamical systems can be described by linear probabilistic models (e.g., Klyatskin, 1980). For example, the dynamical problem of the nonlinear Navier-Stokes' equations with stochastic forcing can be solved as a probabilistic problem using the linear Hopf equation. The statistical characteristics of nonlinear stochastic systems that describe ordinary differential equations can be calculated from linear evolution equations for the probability density. We shall use this idea to reformulate the prediction problem.

Obviously, the direct solution of (1)–(5) is possible only in very special cases because $\hat{\mathcal{L}}$, $\hat{\mathcal{L}}_s$ and \hat{M} are nonlinear operators. Therefore, the problem of the calculation of τ_{pred} needs to be reformulated. The reformulation is based on the following three assumptions.

First, consider solutions to (1) and (3) to be given by a truncated eigenfunction expansion. This is consistent with the recent trend in hydrodynamics to use finite dimensional representation of media characteristics and the Galerkin-Bubnov method to treat partial derivatives for the solution of the system equations. See Gledzer et al. (1981) for a summary and discussion.

Secondly, assume that \hat{x} is an m -dimension Markovian diffusion process ($m \geq 3$) and for its description transport coefficients L_q ($q = 1, \dots, m$) and mixing tensor α_{rs} ($r, s = 1, \dots, m$) will be used. In principle, the theory can be developed for non-Markovian processes, but then the methods for calculation of τ_{pred} are more complicated and not justified at this stage.

The third basis for the development of theory is the analogy between the prediction problem and the classic first passage time problem. See Stratonovich (1961, 1963). Apparently, this analogy was first noted by Ivanov and Kirwan (1993). The analogy provides an efficient mathematical theory to the calculation of τ_{pred} for nonlinear stochastic systems from deterministic models.

We now consider the analogy in more detail and give a geometrical interpretation of the inequality in (5). Consider one realization of (5):

$$\alpha_p [x_p(t) - z_p(t)]^2 \leq \varepsilon^2. \quad (6)$$

Imagine an ellipsoidal surface $S_t: \alpha_p [x_p - z_p(t)]^2 = \varepsilon^2$ moving in m -dimensional space. Here, $z_p(t)$ is the trajectory of the geometric center of the ellipsoid. The stochastic trajectory $x_p(t)$ can be within or outside of the ellipsoid. This trajectory will intersect the surface periodically. It is suggested that when the trajectory $x_p(t)$ is within the ellipsoid then the prediction is realized and when it is outside there is no prediction. If the trajectory leaves the ellipsoid then its intersection with the surface S_t denotes the prediction limit. In this case the prediction time is identical to the time the trajectory first passes S_t . This gives a direct analogy between our prediction problem and the classic first passage time problem. The difference is that in the present case S_t is a moving surface, while in the classic case it is motionless.

Using this analogy two types of predictions arise as a consequence of inequality (6). The first type (P1) is the prediction when the initial position of the trajectory starts and moves within the ellipsoid. For the analogous prediction problem introduce a probability function $P(t_0, \hat{x}_0, t - t_0)$ which gives the probability that the trajectory $\hat{x}(t)$, which was at the initial position \hat{x}_0 , will leave the ellipsoid at time $t - t_0$. Then the trajectory approaches the boundary from within. In this case τ_{pred}

agrees with the time of the first intersection of S_t and can be calculated from formula

$$\tau_{\text{pred1}}(\hat{x}_0) = \int_{t_0}^{\infty} (t - t_0) \frac{\partial P(t_0, \hat{x}_0, t - t_0)}{\partial t} dt. \tag{7}$$

We now discuss the second type of the prediction (P2). Here the stochastic trajectory $\hat{x}(t)$ is outside the ellipsoid at the initial position. However, it will later intersect the ellipsoidal boundary from outside for some time τ_{pred2} . It is suggested that τ_{pred2} is the time predictions can start. For the description of this type of prediction introduce the function $W(t, \hat{x}_0, t - t_0)$. This function gives the probability that the trajectory will intersect S_t from outside for time equal to $t - t_0$. In this case τ_{pred2} can be calculated from the following formula:

$$\tau_{\text{pred2}}(\hat{x}_0) = \int_{t_0}^{\infty} (t - t_0) \frac{\partial W(t_0, \hat{x}_0, t - t_0)}{\partial t} dt. \tag{8}$$

Of course, more complex types of the prediction can be formulated. For example, the stochastic trajectory can leave the ellipsoid and return several times for some time period. Such cases are not discussed here, however.

4 Equations for the calculation of $P(t_0, \hat{x}_0, t - t_0)$ and $W(t_0, \hat{x}_0, t - t_0)$

The functions $P(t_0, \hat{x}_0, t - t_0)$ and $W(t_0, \hat{x}_0, t - t_0)$ play a central role in this study. The method for calculating $P(t_0, \hat{x}_0, t - t_0)$ is discussed here in detail. Since the determination of W is quite similar only the final result will be given.

The determination of $P(t_0, \hat{x}_0, t - t_0)$ is facilitated by introducing $\hat{\xi}_0$ and $\hat{\xi}$ by

$$\begin{aligned} \hat{x}_0 &= \hat{\xi}_0 + \hat{z}(t_0) \\ \hat{x} &= \hat{\xi} + \hat{z}(t) \end{aligned} \tag{9}$$

Then define

$$\begin{aligned} P(t_0, \hat{x}_0, t - t_0) &= P[t_0, \hat{\xi}_0 + \hat{z}(t_0), t - t_0] \\ &\equiv \Phi(t_0, \hat{\xi}_0, t - t_0). \end{aligned} \tag{10}$$

Using L and T as space and time scales, respectively, the following nondimensional quantities are defined:

$$\begin{aligned} \hat{\xi}' &= L^{-1} \hat{\xi} \\ t' &= T^{-1} t \\ L'_q &= L^{-1} T L_q \\ \alpha'_{rs} &= L^{-2} T \alpha_{rs}. \end{aligned} \tag{11}$$

Hereafter (1) and (3) will be regarded as nondimensionalized and so the primes will be omitted.

The function $\Phi(t_0, \hat{\xi}_0, t - t_0)$ is the probability that the stochastic trajectory which starts at \hat{x}_0 can leave the fixed ellipsoid bounded by the surface $S_{\hat{\xi}}$:

$$\alpha_p \xi_p^2 = \varepsilon^2. \tag{12}$$

We now formulate a boundary value problem to determine $\Phi(t_0, \hat{\xi}_0, t - t_0)$. Taking into account that $\hat{\xi}(t)$ is a Markovian diffusion process it can be shown that $\Phi(t_0, \hat{\xi}_0, t - t_0)$ must satisfy Kolmogorov's equations (Stratonovich, 1961):

$$\mathcal{L}\Phi(t_0, \hat{\xi}_0, t - t_0) = 0 \tag{13}$$

$$\Phi(t, \hat{\xi}_0, 0) = 0 \tag{14}$$

$$\Phi(t_0, \hat{\xi}_0, t - t_0) |_{\xi_0 \in S_{\hat{\xi}}} = 1. \tag{15}$$

Here

$$\begin{aligned} \mathcal{L} &= \frac{\partial}{\partial t_0} + \{L_q[\hat{\xi}_0 + \hat{z}(t_0), t_0] \\ &\quad - M_q(t_0)\} \nabla_q + \alpha_{rs}[\hat{\xi}_0 + \hat{z}(t_0), t_0] \nabla_r \nabla_s \end{aligned}$$

and

$$\begin{aligned} \nabla_q &= \frac{\partial}{\partial \xi_0^q} \\ r, s, q &= 1, \dots, m. \end{aligned}$$

There is an elegant method due to Pogorzelski (1958) that can be used to solve (13)–(15). With this method, the solution of (13)–(15) is represented as a single-layer potential. This is in contrast to the classical theory of differential equations for the Dirichlet problem (e.g., Ladyzhenskaya et al., 1967) where a double-layer potential usually is used. The use of double-layer potentials for this problem causes unnecessary mathematical complexity.

The first stage of the determination of the solution (13)–(15) by Pogorzelski's method (Pogorzelski, 1958) is to find the solution to an auxiliary problem. This is formulated as a Neumann problem for part of m -dimensional space unrestricted by the elliptic surface $S_{\hat{\xi}}$:

$$\mathcal{L}\Phi'(t_0, \hat{\xi}_0, t - t_0) = 0 \tag{16}$$

$$\Phi'(t, \hat{\xi}_0, 0) = 0 \tag{17}$$

$$\frac{\partial \Phi'}{\partial \hat{n}} |_{\xi_0 \in S_{\hat{\xi}}} = V. \tag{18}$$

Here \hat{n} is normal to the surface $S_{\hat{\xi}}$ and V is some function which will be specified later. In addition to (18) note that if $\hat{\xi}_0$ tends to any point on the surface $S_{\hat{\xi}}$ from the interior of the ellipsoid, than Φ' will approach 1.

The solution of (16)–(18) can be written as a single-layer potential generated by some density φ at the surface $S_{\hat{\xi}}$:

$$\Phi'(t_0, \hat{\xi}_0, t - t_0) = \int_{t_0}^t d\tau \int_{S_{\hat{\xi}}} p(t_0, \hat{\xi}_0, \tau, \hat{\eta}) \varphi(\hat{\eta}, \tau) dS_{\hat{\eta}}. \quad (19)$$

Here $p(t_0, \hat{\xi}_0, \tau, \hat{\eta})$ is the fundamental solution for the operator \mathcal{L} and $\varphi(\hat{\eta}, \tau)$ satisfies the integral equation

$$\begin{aligned} \varphi(\hat{\eta}, \tau) = & -2 \int_{t_0}^t d\tau' \int_{S_{\hat{\eta}}} M(\tau, \hat{\eta}, \tau', \hat{\theta}) \varphi(\hat{\theta}, \tau) dS_{\hat{\theta}} \\ & + 2V(\hat{\eta}, \tau). \end{aligned} \quad (20)$$

Also, $M(\tau, \hat{\eta}, \tau', \hat{\theta}) = \sum_{r=1}^m \alpha_{rs}(\hat{\eta}, \tau) \nabla_s p(\tau, \hat{\eta}, \tau', \hat{\theta})$ is the kernel of the integral equation. In the theory of parabolic equations, Ladyzhenskaya et al. (1967) have argued that (20) are the Volterra equations which can be solved by the iterative method.

We now proceed to the second stage of the solution. Note that $\Phi'(t_0, \hat{\xi}_0, t - t_0)$ is a solution of (13) and satisfies (14) within an ellipsoid bounded by the surface $S_{\hat{\xi}}$. Moreover, as $\hat{\xi}_0$ tends to any point on $S_{\hat{\xi}}$ then Φ' will tend to 1. But this result is independent of direction: a general property of single-layer potentials. Therefore,

$$\Phi(t, \hat{\xi}_0, t - t_0) = \Phi'(t_0, \hat{\xi}_0, t - t_0) \quad (21)$$

within the domain bounded by $S_{\hat{\xi}}$. Thus, the solution of (13)–(15) is known from (21).

Obviously the calculation of the function $W(t, \hat{\xi}_0, t - t_0)$ proceeds in a similar manner. In that case, the result is

$$W(t, \hat{\xi}_0, t - t_0) = \int_{t_0}^t d\tau \int_{S_{\hat{\xi}}} p(t_0, \hat{\xi}_0, \tau, \hat{\eta}) \bar{\varphi}(\hat{\eta}, \tau) dS_{\hat{\eta}} \quad (22)$$

and

$$\begin{aligned} \bar{\varphi}(\hat{\eta}, \tau) = & 2 \int_{t_0}^t d\tau' \int_{S_{\hat{\eta}}} \\ & M(\tau, \hat{\eta}, \tau', \hat{\theta}) \bar{\varphi}(\hat{\theta}, \tau) dS_{\hat{\theta}} - 2V(\hat{\eta}, \tau). \end{aligned} \quad (23)$$

The final equations for $W(t_0, \hat{x}_0, t - t_0)$ and $P(t_0, \hat{x}_0, t - t_0)$ can be easily found from (19) and (22) through the reversible transformation back to the variables \hat{x}_0, \hat{x} . The mathematical structure is identical to the structure of (19) and (22).

Let us briefly discuss the functional dependencies of $W(t, \hat{x}_0, t - t_0)$ and $P(t, \hat{x}_0, t - t_0)$. The predictability times, $\tau_{\text{pred}1}$ and $\tau_{\text{pred}2}$ depend upon the accuracy of prediction, the intensity of turbulent noise, the accuracy of the initial conditions, the physics contained in the mathematical models, the measurement errors, and number of prediction variables. Of special interest here is how $\tau_{\text{pred}1}$ and $\tau_{\text{pred}2}$ functionally depend on the measurement errors. The latter are not shown explicitly in the equations for $W(t, \hat{x}_0, t - t_0)$ and $P(t, \hat{x}_0, t - t_0)$. The effect of measurement errors occurs in the nonlinear-differential operator $\hat{\mathcal{L}}$ solution of the prediction problem since this employs the Galerkin-Bubnov method. This method reduces the system of partial differential equations to an infinite dimensional system of ordinary differential equations. For practical applications only m -equations are used. Clearly, the value of m depends upon the measurement errors.

5 Approximate solutions for $P(t_0, \hat{x}, t - t_0)$ and $W(t_0, \hat{x}_0, t - t_0)$

The mathematical structure of (19) and (22) suggests the following geometric interpretation. The probability that the stochastic trajectory can leave or return to the ellipsoid bounded by the surface S_t is the composition of the probabilities of the intersection of the surface for different time period $\delta_l \epsilon [t_0, t]$, where l is number of the time period that the particle intersects the surface S_t . This interpretation can be used to simplify the calculation of $P(t_0, \hat{x}, t - t_0)$ and $W(t_0, \hat{x}, t - t_0)$. We shall also make use of the fact that ϵ is a small parameter for an exterior Dirichlet problem.

Starting from Eq. (22) we introduce a new variable

$$\hat{\eta}' = \epsilon^{-1} \hat{A}_{\hat{\eta}} \hat{\eta}. \quad (24)$$

Here $\hat{A}_{\hat{\eta}}$ is the linear mapping which transforms the differential operator $\alpha_{rs} \nabla_r \nabla_s$ to $\delta_{rs} \nabla'_r \nabla'_s$; $\nabla'_r = \partial / \partial \eta'_r$ and δ_{rs} is the Kronecker symbol.

Taking into account that $\epsilon \ll 1$, W , P , and φ can be represented as:

$$\bar{\varphi}(\hat{\eta}', \tau) = \bar{\varphi}^0(\hat{\eta}', \tau) + \epsilon \bar{\varphi}^1(\hat{\eta}', \tau) + \dots \quad (25)$$

$$\begin{aligned} W(t_0, \hat{x}_0, t - t_0) = & W^0(t_0, \hat{x}_0, t - t_0) + \epsilon W^1(t_0, \hat{x}_0, t - t_0) + \dots \end{aligned} \quad (26)$$

$$p(t_0, \hat{x}_0, \tau, \hat{\eta}) |_{\hat{\eta} \in S_{\hat{\xi}}} \simeq p[t_0, \hat{x}_0, \tau, \hat{z}(\tau)]. \quad (27)$$

Then, substituting (25) into (22) the ϵ^{m-2} order equation is found to be

$$W^0(t, \hat{x}_0, t - t_0) \simeq 2\epsilon^{m-2} \int_{t_0}^t d\tau p[t_0, \hat{x}_0, \tau, \hat{z}(\tau)] \int_{S_{\hat{\eta}'}} \frac{\partial \bar{\varphi}^0}{\partial \hat{\eta}'} d\hat{A}_{\hat{\eta}'}^{-1} S_{\hat{\eta}}. \quad (28)$$

Here $\bar{\varphi}^0(\hat{\eta}', \tau)$ is a harmonic function satisfying the equation

$$\Delta \bar{\varphi}^0(\hat{\eta}', \bar{\tau}) = 0, \quad (29)$$

with the boundary conditions

$$\bar{\varphi}^0(\hat{\eta}', \tau) |_{\hat{\eta}' \in S_{\hat{\eta}'}} = 1 \quad (30)$$

$$\bar{\varphi}^0(\hat{\eta}', \tau) \rightarrow 0 \text{ if } \hat{\eta}' \rightarrow \infty. \quad (31)$$

It can be shown that the integral in (28) over the elliptical surface $S_{\hat{\eta}'}$ depends only upon the coefficients of diffusion, which characterize the influence of turbulent noise on the trajectory.

A problem similar to (28) was previously solved in the theory of the optimal control (Mishenko and Pontryagin, 1961). They represented the solution to (22) by a double-layer potential. That solution has a very complex mathematical structure and cannot be used easily in application. Kolmogorov et al. (1962) used probabilistic ideas to postulate a solution to (22), which was similar to (28). However, that solution and ours are different. This difference is manifested in the integral multiplier $\int_{S_{\hat{\eta}'}}$ of (28). When $\mathfrak{x}_{r,s}$ does not depend on time this factor is not important and the two solutions are in good agreement. However, in the nonstationary case this multiplier becomes very important. Unfortunately, the Kolmogorov et al. (1962) solution cannot be checked because the result was postulated and not calculated. However, Kolmogorov et al. (1962) demonstrated that their solution and the solution of Mishenko and Pontryagin (1961) have identical asymptotics. It should be noted that our solution also has the same asymptotics.

It is important to note that a simplification to (19) is possible even when ε is not a small parameter. To demonstrate this introduce L_q , $\mathfrak{x}_{r,s}$ and φ in the following form:

$$\varphi(\hat{\eta}', \tau) = \varphi^0(\hat{\eta}', \tau) + \varphi^1(\hat{\eta}', \tau) + \dots \quad (32)$$

$$L_q[\hat{\xi}_0' + \hat{z}(t_0), t_0] \simeq L_q[\hat{z}(t_0)] \quad (33)$$

$$\mathfrak{x}_{r,s}[\hat{\xi}_0' + \hat{z}(t_0), t_0] \simeq \mathfrak{x}_{r,s}[\hat{z}(t_0)] \quad (34)$$

$$p(t_0, \hat{x}_0, \tau, \hat{\eta}) \simeq p[t_0, \hat{x}_0, \tau, \hat{z}(\tau)] \quad (35)$$

The convergence of the series (32) was shown by Ladyzhenskaya et al. (1967). Equations (33)–(34) are a consequence of the fact that transport coefficients change slowly on the spatial scales compared with value ε . In this case $\varphi^0(\hat{\eta}', \tau)$ is the solution of

$$\frac{\partial \varphi^0}{\partial t} + [L_q - M_q] \nabla_q' \varphi^0 - \mathfrak{x}_{r,s} \nabla_r' \nabla_s' \varphi^0 = 0 \quad (36)$$

$$\varphi^0(\hat{\eta}', t_0) = 0 \quad (37)$$

$$\varphi^0(\hat{\eta}', t_0) |_{\hat{\eta}' \in S_{\hat{\eta}'}} = 1. \quad (38)$$

The mathematical structure of Eq. (36) is simpler than the structure of (19). In this case, $P(t_0, \hat{x}_0, t - t_0)$ satisfies the following equation:

$$P(t_0, \hat{x}_0, t - t_0) \simeq \int_{t_0}^t d\tau p[t_0, \hat{x}_0, \tau, \hat{z}(\tau)] \int_{S_{\hat{\eta}'}} \frac{\partial \varphi^0}{\partial \hat{n}} \cdot d\hat{A}_{\hat{\eta}'}^{-1} S_{\hat{\eta}'}. \quad (39)$$

In the next section we use (39) to estimate τ_{pred1} for a small-mode number geophysical flow.

6 Application

In this section the theory is applied to the calculation of τ_{pred1} for a three-mode model of zonal flow.

Consider zonal, barotropic, and nondivergent flow in channel. The β -effect, bottom topography, dissipation (friction in the channel) and Ekman pumping are all included. This flow is prescribed by three modes ($m = 3$). The streamfunction Ψ , the wind stress $\hat{\tau} = (\tau_x, \tau_y) = (\tau_x, 0)$ and the bottom topography H can be represented as series expansions of the eigenfunctions ψ_p of the linearized flow stability problem:

$$\Psi = \sum_{p=1}^3 A_p(t) \psi_p(x, y) \quad (40)$$

$$\tau_x = \sum_{p=1}^3 B_p(t) \psi_p(x, y) \quad (41)$$

$$H = \sum_{p=1}^3 C_p(t) \psi_p(x, y). \quad (42)$$

Here,

$$\psi_1 = \sqrt{2} \cos(y/L); \quad \psi_2 = 2 \cos(2x/L) \sin(y/L);$$

$$\psi_3 = 2 \sin(2x/L) \sin(y/L)$$

and L is the channel width. Periodic and nonslip conditions were used on open and solid boundaries of the channel, respectively. The equations for the description of the evolution of mode amplitudes $A_p(t)$ can be easily obtained by the Galerkin-Bubnov method (e.g., Gledzer et al., 1981).

Consider a special case where $B_1 = B_3 = 0$; $B_2 \neq 0$; $C_1 \neq 0$; $C_2 \neq 0$; $C_3 = 0$. In this case the equations for A_p are

$$\frac{dA_1}{dt} = -\gamma A_1 + C_1 \tilde{H} A_2 \quad (43)$$

$$\frac{dA_2}{dt} = -\gamma A_2 - (rA_1 - \beta)A_3 + \eta B_2 \quad (44)$$

$$\frac{dA_3}{dt} = -\gamma A_3 - C_2 \tilde{H} A_1 + (rA_1 - \beta)A_2. \quad (45)$$

Here γ is a coefficient of the channel friction, η is intensity of the Ekman pumping, and r , C_1 , C_2 , \tilde{H} are all numerical multipliers.

For present purposes the following parameters were chosen (Eremeev et al., 1988):

$$\frac{L}{d} = 0.1; C_1 = \frac{8\sqrt{2}}{3\pi}; C_2 = \frac{L}{5}C_1; r = 64\sqrt{2}/15\pi;$$

$$\beta = \frac{2}{5} \frac{L}{d} ctg\theta = 0.04; \theta = 45^\circ;$$

$$\gamma = \frac{1}{H_0} \left(\frac{\nu_F}{2f_0} \right)^{1/2} = 0.015 - 0.06;$$

$$\nu_F = 10^4 - 10^5 \text{ cm}^2/\text{s};$$

$$H_0 = 10^5 \text{ cm}; B_2 = 1; \eta = 0.01; \tilde{H} = 0.01. \quad (46)$$

Here H_0 is the average depth in the channel, d is the radius of the Earth and ν_F is the turbulent viscosity. These values are typical of large scale processes in the ocean.

The following set of nondimensional quantities are used:

$$\begin{aligned} t' &= f_0 t; \\ x' &= L^{-1} x; \\ y' &= L^{-1} y; \\ \psi' &= L^{-2} f_0^{-1} \psi; \\ \tau'_x &= L^{-2} f_0^{-2} \tau_x; \\ H' &= H_0^{-1} H. \end{aligned} \quad (47)$$

In (43)–(45) and later, the primes of all nondimensional quantities are dropped. Initially, the fluid is at rest so

$$(A_1, A_2, A_3)|_{t=0} = 0. \quad (48)$$

If the wind is a deterministic then the solution is straightforward (e.g., see Fig. 1). Of more interest is the case when the wind is stochastic. Then (43) through (45) are changed to include stochastic additions on the right-hand side. In this case the dynamical Eqs. (43)–(45) become:

$$\frac{dA_1}{dt} = -\gamma A_1 + C_1 \tilde{H} A_2 + \eta_1 f_1(t) \quad (49)$$

$$\frac{dA_2}{dt} = -\gamma A_2 - (rA_1 - \beta)A_3 + \eta + \eta_2 f_2(t) \quad (50)$$

$$\frac{dA_3}{dt} = -\gamma A_3 - C_2 \tilde{H} A_1 + (rA_1 - \beta)A_2 + \eta_3 f_3(t) \quad (51)$$

$$(A_1, A_2, A_3)|_{t=0} = 0. \quad (52)$$

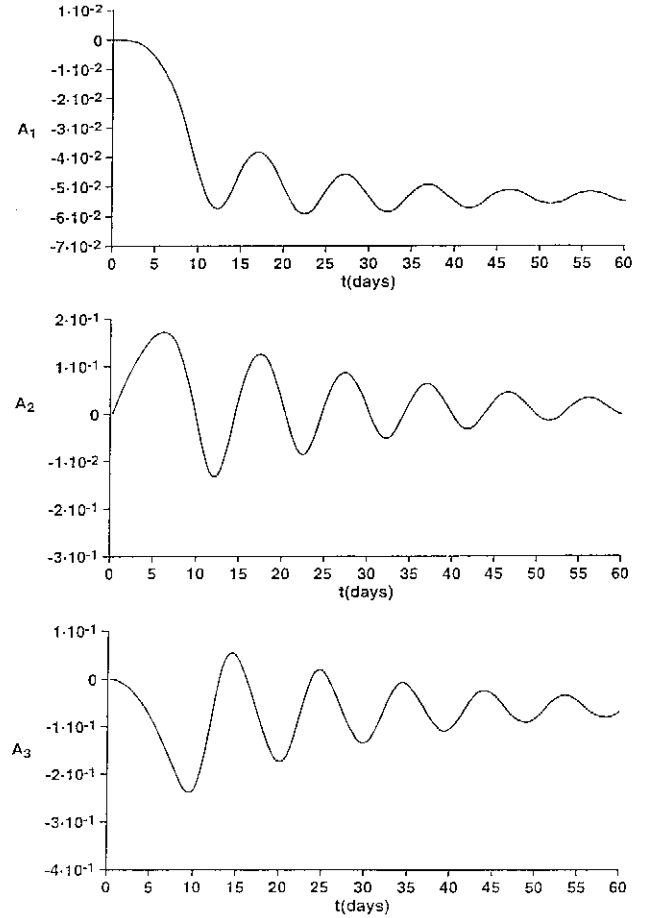


Fig. 1. Temporal evolution of mode amplitudes; $\gamma = 0.015$

Here f_r is a δ -correlated vector process with the statistical characteristics

$$\langle f_r \rangle = 0; \langle f_r(t) f_s(t_1) \rangle = \delta_{rs} \delta(t - t_1). \quad (53)$$

δ in (53) is the delta or unit impulse function and η_1 , η_2 , η_3 characterize the intensity of the stochastic wind pumping. Here we take ν_r as a percent of η . Predictions of the temporal evolution of the A_i use the following model:

$$\frac{d \langle A_1 \rangle}{dt} = -\tilde{\gamma} \langle A_1 \rangle + C_1 \tilde{H} \langle A_2 \rangle \quad (54)$$

$$\begin{aligned} \frac{d \langle A_2 \rangle}{dt} = \\ -\tilde{\gamma} \langle A_2 \rangle - (r \langle A_1 \rangle - \beta) \langle A_3 \rangle + \eta \end{aligned} \quad (55)$$

$$\begin{aligned} \frac{d \langle A_3 \rangle}{dt} = \\ -\tilde{\gamma} \langle A_3 \rangle - C_2 \tilde{H} \langle A_1 \rangle \\ + (r \langle A_1 \rangle - \beta) \langle A_2 \rangle \end{aligned} \quad (56)$$

$$\langle A_1 \rangle|_{t=0} = A_1(\delta t)$$

$$\langle A_2 \rangle|_{t=0} = A_2(\delta t)$$

$$\langle A_3 \rangle|_{t=0} = A_3(\delta t). \quad (57)$$

Here $\langle \dots \rangle$ is the ensemble average operator; $\delta t = 2$ days is the difference between starting times of the model system (54)–(57) and the stochastic system (49)–(52); $\tilde{\gamma}$ is the renormalized coefficient of the friction in the channel. The τ_{pred1} was found as the solution of the following inequality:

$$\alpha_p I_p(\tau_{\text{pred1}}) \leq \varepsilon^2 \quad (58)$$

$$\text{with } I_p = \overline{[A_p(t) - \langle A_p(t) \rangle]^2}.$$

For construction of the statistical ensemble one thousand realizations were utilized. Calculation τ_{pred1} by inequality (58) is called the direct calculation. Another way of calculating τ_{pred1} is to use equations (7) and (39). The fundamental solution $p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{x}_0, \hat{\mathbf{A}})$ plays a critical role in this case. This is found by solving the following Fokker-Plank equations:

$$\begin{aligned} & \frac{\partial p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{\mathbf{A}})}{\partial t} + L_q \nabla_q p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{\mathbf{A}}) \\ & = \alpha_{rs} \nabla_r \nabla_s p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{\mathbf{A}}) \quad ; r, s = 1, 2, 3 \end{aligned} \quad (59)$$

$$p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{\mathbf{A}})|_{t=0} = \delta(\hat{\mathbf{A}}) = \delta(A_1)\delta(A_2)\delta(A_3). \quad (60)$$

Here

$$\nabla_r = \frac{\partial}{\partial A_r}$$

$$L_1 = -\gamma A_1 + C_1 \tilde{H} A_2 \quad (61)$$

$$L_2 = -\gamma A_2 - (rA_1 - \beta)A_3 + \eta \quad (62)$$

$$L_3 = -\gamma A_3 - C_2 \tilde{H} A_1 + (rA_1 - \beta)A_2 \quad (63)$$

$$\alpha_{rs} = \eta_S^2 \delta_{rs}. \quad (64)$$

The solution can be easily obtained for (49)–(52) by standard statistical methods (e.g., Stratonovich, 1961).

Equations (49)–(52) and (54)–(57) were integrated numerically with a fourth order accurate variable time step integrater (Nordsieck, 1962). The solution of the Fokker-Plank equation was obtained by a special flux corrected transport (FCT) method (Boris and Book, 1976). The principle advantage of this method is that it works well with small values of viscosity. The calculation was performed on a $30 \times 30 \times 30$ grid with space steps $\Delta A_1 = \Delta A_2 = \Delta A_3 = 0.02$ and a time step of $\Delta t = 0.1$. On the boundaries $A_1 = A_2 = A_3 = 0$ were assumed. The calculation was controlled by the conservation of

$$\int_v p(t_0, \hat{\mathbf{A}}^\circ, t, \hat{\mathbf{A}}) d^3 \hat{\mathbf{A}} = 1. \quad (65)$$

Here v is the calculated volume.

The solution of (23) used an iterative method with a typical number of iterations ranging from 5 to 10. The function $\varphi^\circ(\hat{\eta}, t)$ was calculated from the Green's function for the three-dimensional Laplace's equation in elliptical coordinates (Morse and Feshbach, 1953).

A typical evolution of the functionals $\langle A_p \rangle$ and I_p is shown in Fig. 2 and Fig. 3, respectively. Fig. 4 is typical for the fundamental solution $p[t_0, \hat{\mathbf{A}}^\circ, t, \langle \hat{\mathbf{A}}(t) \rangle]$. Note that after 4 days this function can oscillate when $\gamma = 0.045$. This oscillation is an artifact of the calculation procedure and is related to the FCT method when the function $p[t_0, \hat{\mathbf{A}}^\circ, t, \langle \hat{\mathbf{A}}(t) \rangle]$ is small. However, even in this case there are no negative values for the fundamental solution.

This model of the circulation allows one to investigate the functional dependence of τ_{pred1} on three parameters: ε , η_S , and γ . The results of this calculation are shown in Fig. 5 along with results of the calculation from (19) and estimates using the direct method. The difference in solutions for large values γ can be explained by the numerical viscosity of the model. This can reduce somewhat the value of $p[t_0, \hat{\mathbf{A}}^\circ, t, \langle \hat{\mathbf{A}}(t) \rangle]$. Aside from this the agreement is quite satisfactory.

7 Generalization

This method is readily generalized to include general Markovian and non-Markovian processes. The Markovian process $\hat{x}(t)$ used as the model of reality should satisfy the following equation:

$$\frac{d\hat{x}}{dt} = \hat{\mathcal{L}}(\hat{x}, t) + \hat{\mathcal{L}}_s(\hat{\xi}, t). \quad (66)$$

Here $\hat{\xi}$ is a δ -correlation stochastic vector m -dimensional process. In principle $\hat{\xi}$ can be a Markovian processes with a finite correlation time. To generalize our approach to this case we use the fact that nearly all Markov stochastic processes with finite correlation times can be represented in the form (Klyatskin, 1980)

$$\frac{d\hat{\xi}}{dt} = \hat{\mathcal{L}}'_s(\hat{\xi}', t). \quad (67)$$

In (67) $\hat{\xi}'$ is a δ -correlation stochastic m' -dimensional process. In the expanded space of vectors with dimension of $m + m'$ the vector $(\hat{x}, \hat{\xi})$ can be considered as a Markovian diffusion process.

Similarly, it is possible to consider stochastic systems with non-Markovian behavior, if they satisfy

$$\frac{d\hat{x}}{dt} = \hat{\mathcal{L}}(\hat{x}, \hat{\xi}, t). \quad (68)$$

Here $\hat{\xi}$ is the Markovian stochastic process with an arbitrary correlation time.

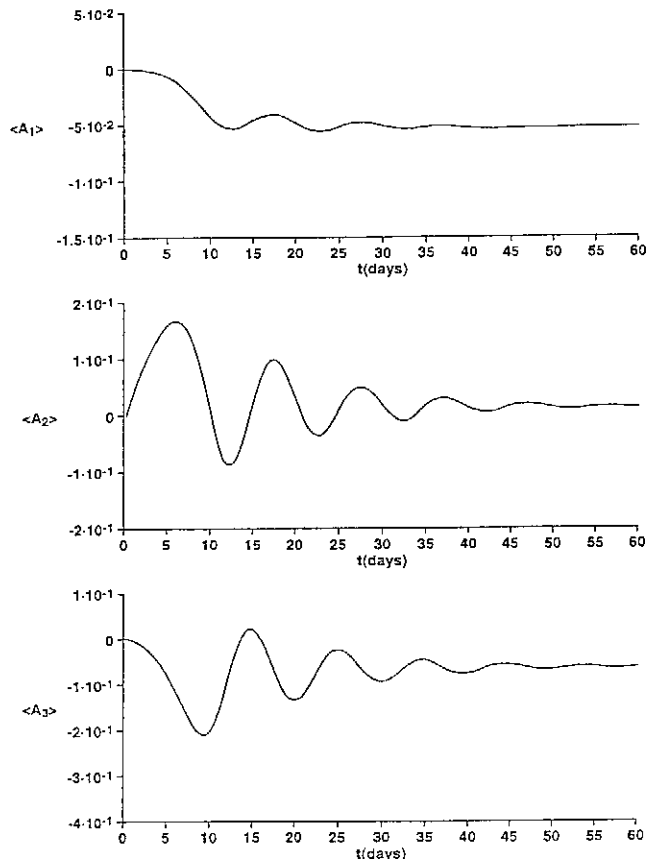


Fig. 2. Temporal evolution of average mode amplitudes; $\eta_S = 20\%$; $\gamma = 0.015$

8 Conclusions

In this paper an approach for calculating the prediction time $\tau_{\text{pred}1}$ for nonlinear stochastic systems was described. This approach can be extended in at least two ways. One is to analyze problems such as the comparison of the Lagrangian and Eulerian prediction times or the determination of the dependence between correlation and prediction times.

However, the principle application of the approach would be to maximize the prediction time using Pontryagin's principle (Moiseev, 1975). This problem should have numerous practical applications in hydrodynamics, plasma physics, geophysics, astrophysics, etc.

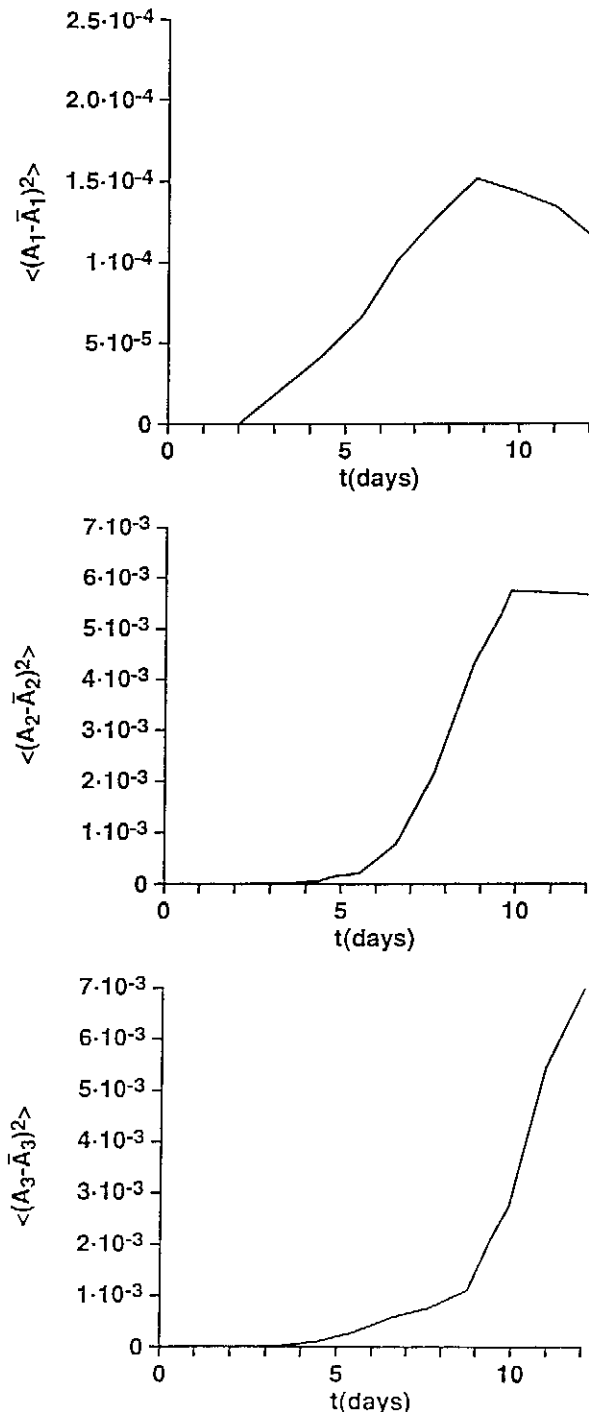


Fig. 3. Temporal evolution of functionals $I_p = [A_p(t) - \langle A_p(t) \rangle]$ ($p = 1, 2, 3$): $\gamma = 0.015$; $\eta_S = 20\%$; $\bar{\gamma} = 0.02$

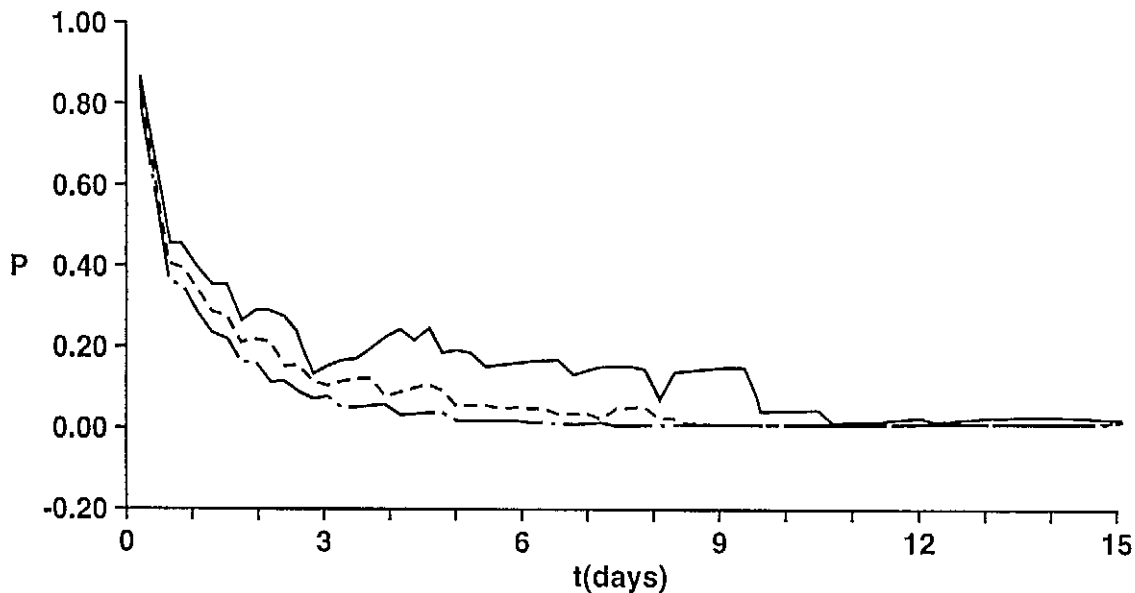


Fig. 4. Fundamental solution $p[t_0, \hat{A}^\circ, t, \langle \hat{A}(t) \rangle]$: $\eta_S = 40\%$; — $\gamma = 0.015$; - - - $\gamma = 0.030$; - · - · - $\gamma = 0.045$

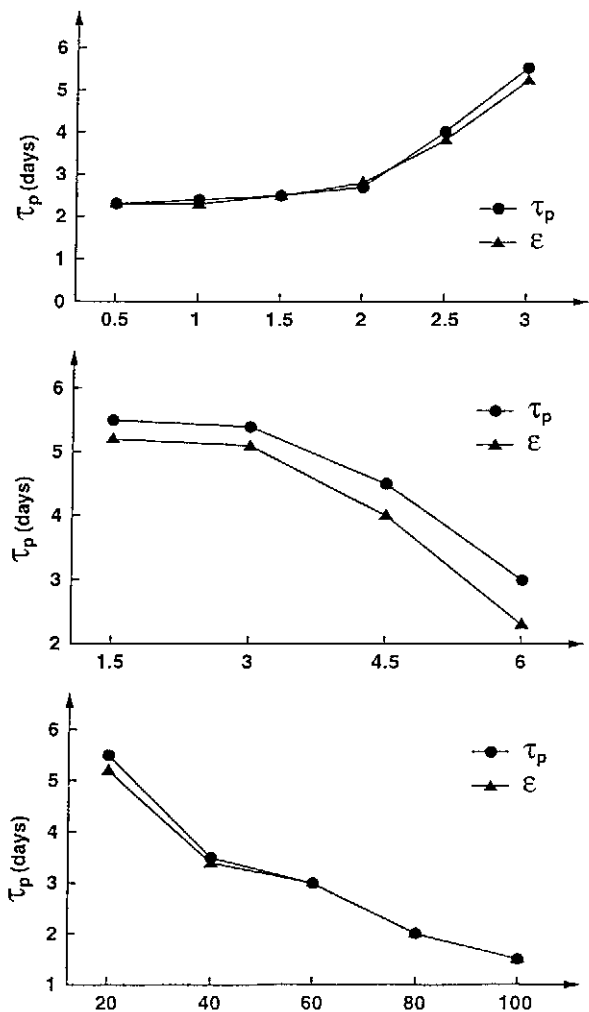


Fig. 5. a) Functional dependence of τ_{pred1} upon values ε : $\gamma = 0.015$; $\eta_S = 20\%$; $\alpha_1 = \alpha_2 = 1$; $\alpha_3 = 4$; $\tilde{\gamma} = 0.02$. Circles indicate the direct method; triangles from Eq. (39). b) Functional dependence of τ_{pred1} upon values γ : $\varepsilon = 3.10^{-2}$; $\eta_S = 20\%$; $\alpha_1 = \alpha_2 = 1$; $\alpha_3 = 4$; $\tilde{\gamma} = 0.02$. Circles indicate the direct method; triangles from Eq. (39). c) Functional dependence of τ_{pred1} upon values η_S : $\gamma = 0.015$; $\varepsilon = 3.10^{-2}$; $\alpha_1 = \alpha_2 = 1$; $\alpha_3 = 4$. Circles indicate the direct method; triangles from Eq. (39)

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